# VIASM Summer School in Differential Geometry 2023 Course Outline

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**Title:** Convergence of Riemannian Manifolds and Scalar Curvature

**Abstract**

During this lecture series we will introduce Gromov-Hausdorff (GH) convergence and Sormani-Wenger Intrinsic Flat (SWIF) convergence of Riemannian manifolds including various methods for estimating these notions of convergence. Theorems which relate these notions of convergence to Ricci curvature and scalar curvature will be introduced and several open geometric stability conjectures involving scalar curvature will round out the course.

**Course 1:** Gromov-Huasdorff (GH) Distance [BBI, Pet06]

1. Metric Spaces and Length Spaces (2.1-2.5 of [BBI])
2. Hausdorff Distance and Convergence (7.3 of [BBI], 10.1.1 of [Pet06])
3. GH Distance and Convergence (7.3-7.4 of [BBI], 10.1.1 of [Pet06])
4. Estimating GH Distance (7.4 of [BBI], 10.1.1 of [Pet06])
5. Regularity of Limits under GH Convergence (7.5 of [BBI])

**Course 2:** Ricci Curvature and GH Convergence [BBI, Pet06]

1. Gromov's Compactness Theorem (10.1.4 of [Pet06])
2. Ricci Curvature and Volume of Balls (9.1 of [Pet06])

3. Ricci Curvature Compactness Theorem (10.1.4 of [Pet06])
4. Ricci Limit Spaces (10.7 of [BBI], Prof. Wei will discuss this topic in more detail)

**Course 3:** Sormani-Wenger Intrinsic Flat (SWIF) Convergence [Sor12, Sor17]

1. Examples of Sequences without Ricci Curvature Bounds
2. Flat Distance on $\mathbb{R}^n$ [SW11, Sor12]
3. Sormani-Wenger Intrinsic Flat Distance [SW11, Sor12, Sor17]
4. Wenger's Compactness Theorem [Wen11]
5. Gromov-Lawson Tunnels and Sewing Examples [GL80]

**Course 4:** Estimating GH/SWIF Convergence of Riemannian Manifolds [AS19, AS20, APS20, AP20]

1. Examples showing necessity of control from below [AS19, AS20]
2. Quantitative SWIF Distance Estimate [APS20, AP20]
3. VADB Theorem [APS20, AP20]
4. VADB Open Problems
5. Examples with Blow Up [AS20]

**Course 5:** Scalar Curvature Geometric Stability Conjectures [SCC21]

1. Scalar Curvature Characterization (Section 2 of [SCC21])
2. Geometric Stability of Scalar Torus Rigidity Conjecture (Section 7 of [SCC21], [Gro14])
3. Geometric Stability of Larrull 's Theorem ([HKKZ22])
4. Geometric Stability of the Positive Mass Theorem Conjecture (Section 10 of [SCC21])
5. Geometric Stability of Scalar Prism Rigidity (Section 8 of [SCC21])

**Problem Sessions:**

*Petersen Riemannian Geometry Text:*

1. Prop 42
2. Ex 55
3. Ex 56
4. Prop 43

*Burago-Burago-Ivanov Metric Geometry Text:*

1. Ex 7.3.11
2. Exercise 7.3.13
3. Prop 7.3.16
4. Exercise 7.4.6
5. Exercise 7.4.7
6. Ex 7.4.9
7. Exercise 7.4.14
8. Exercise 7.4.16
9. Exercise 7.4.17

*Sormani-Wenger https://arxiv.org/abs/1002.1073 Appendix:*

1. Lem A1
2. Lem A2
3. Prop A3
4. Ex A4
5. Ex A7

## References


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[AS19] Brian Allen and Christina Sormani. Contrasting various notions of convergence in geometric analysis. *Pacific Journal of Mathematics*, 303(1):1–46, 2019.

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