

VIASM SUMMER SCHOOL IN DIFFERENTIAL GEOMETRY 2023 COURSE OUTLINE

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TITLE: SOME RECENT RESULTS FOR SPACES WITH RICCI CURVATURE LOWER BOUNDS

ABSTRACT. Recently there are tremendous developments in the study of manifolds with Ricci curvature lower bounds, their Gromov-Hausdorff limits (Ricci limit spaces), and/or RCD spaces. We will first recall the basic tools like Bochner formula, Bishop-Gromov volume comparison and some generalizations, and Cheng-Yau's gradient estimate, then go on to the almost volume rigidity. Then we will present examples showing the Busemann function of manifolds with nonnegative Ricci curvature may not be proper (which was open since the seventies), the Hausdorff dimension of singular set of Ricci limit spaces may be bigger than the Hausdorff dimension of the regular set, which answer a question of Cheeger-Colding more than twenty years ago. In the end we will present the topological result that Ricci limit space/RCD spaces are semi-locally simply connected.

For general references about manifolds with Ricci curvature bounds, see [2, 10]. For topics below without references, a note will be provided before the lecture.

1. LECTURE 1: BASIC TOOLS FOR RICCI CURVATURE

- 1.1. **Bochner formula.**
- 1.2. **Laplacian and volume comparison.**
- 1.3. **Extension to integral Ricci curvature.** [8]
- 1.4. **Extension to Bakry-Emery Ricci curvature.** [12]

2. LECTURE 2: ALMOST VOLUME CONE RIGIDITY

- 2.1. **Metric cone.**
- 2.2. **Cheng-Yau's gradient estimate.**
- 2.3. **Segement inequality.**
- 2.4. **Integral estimate of Hessian.**

3. LECTURE 3: BUSEMANN FUNCTION OF MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

- 3.1. **Busemann function.**
- 3.2. **Nabonnand's/Wei's example of manifolds with positive Ricci curvature.** [4, 11]

3.3. Example of manifolds with positive Ricci curvature and non-proper Busemann function. [6]

4. LECTURE 4: HAUSDORFF DIMENSION OF RICCI LIMIT SPACES

4.1. Hausdorff dimension.

4.2. Rectifiable dimension of Ricci limit spaces.

4.3. Examples of Ricci limit space with Hausdorff dimension different from the rectifiable dimension. [5] [1]

5. LECTURE 5: TOPOLOGY OF RICCI LIMIT/RCD SPACES

5.1. Relative δ -covers. [7]

5.2. Universal cover of Ricci limit/RCD space exists. [3,8]

5.3. Ricci limit/RCD spaces are semi-locally simply connected. [9]

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