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MODULES WHICH ARE INVARIANT UNDER IDEMPOTENTS OF THEIR ENVELOPES

BY

LE VAN THUYET (Hue), PHAN DAN (Ho Chi Minh City) and
TRUONG CONG QUYNH (DaNang)

Abstract. We study the class of modules which are invariant under idempotents of their envelopes. We say that a module M is $\mathcal{X}$-idempotent-invariant if there exists an $\mathcal{X}$-envelope $u : M \to X$ such that for any idempotent $g \in \mathrm{End}(X)$ there exists an endomorphism $f : M \to M$ such that $uf = gu$. The properties of this class of modules are discussed. We prove that M is $\mathcal{X}$-idempotent-invariant if and only if for every decomposition $X = \bigoplus_{i\in I} X_i$, we have $M = \bigoplus_{i\in I}(u^{-1}(X_i) \cap M)$. Moreover, some generalizations of $\mathcal{X}$-idempotent-invariant modules are considered.

1. Introduction. Recently, some generalizations of quasi-injective modules have been studied and several interesting results have been obtained. In 1961, Johnson and Wong [10] proved that a module is quasi-injective if it is invariant under endomorphisms of its injective envelope. This is one of the most interesting characterizations of quasi-injective modules. It shows that quasi-injectivity can be checked by means of an intrinsic property of the module. Other characterizations of quasi-injective modules have been studied and generalized to other classes of modules.

Let us begin by discussing the class of modules which are invariant under automorphisms of their envelopes.

Let $\mathcal{C}$ be a class of right R-modules closed under isomorphisms. An R-homomorphism $g : M \to E$ is a $\mathcal{C}$-*preenvelope* of the module M provided that $E \in \mathcal{C}$ and each diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "g'"'] & E \arrow[dl, dotted, "\alpha"'] \\ E' & \end{tikzcd}$$

with $E' \in \mathcal{C}$ can be completed by a homomorphism $\alpha : E \to E'$ to a

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commutative diagram. If, moreover, the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "g"'] & E \arrow[dl, dotted, "\alpha"'] \\ E & \end{tikzcd}$$

can only be completed by automorphisms α, we call g a $\mathcal{C}$*-envelope* of M. It is easy to see that the $\mathcal{C}$-envelope is unique up to isomorphisms. Dualizing, one defines the notions of a $\mathcal{C}$-precover and a $\mathcal{C}$-cover of a module M.

It is well known that when $\mathcal{C}$ is the class of all injective modules, a $\mathcal{C}$-(pre)cover of each module exists if and only if R is a right noetherian ring (see e.g. [5, 5.4.1]). Moreover, $\mathcal{C}$-envelope is usually called injective hull.

Dickson–Fuller [3] proved in 1969 that if R is any algebra over a field F with more than two elements, then an indecomposable module M is quasi-injective iff M is invariant under automorphisms of the injective hull $E(M)$. In 2013 this concept was extended to modules over a general ring by Lee and Zhou [11]. They defined a module M to be automorphism-invariant if $\varphi(M) \leq M$ for every $\varphi \in \mathrm{Aut}(E(M))$. They also obtained several characterizations and applications of such modules. Er, Sing and Srivastava [6] proved that a module is automorphism-invariant if and only if every monomorphism from a submodule of M to M can be extended to an endomorphism of M (modules M with this property are called *pseudo-injective*). And Guil Asensio, Keskin Tütüncü and Srivastava [9] proved that the endomorphism ring of every automorphism-invariant module is semiregular. These results are interesting because of their applications to the structure of modules.

Along this paper, we will always assume that $\mathcal{X}$ is a class of modules which is closed under isomorphisms.

The concept of automorphism-invariant modules was generalized in 2014 by Guil Asensio, Keskin Tütüncü and Srivastava [9]. A module M is called $\mathcal{X}$*-automorphism-invariant* if there exists an $\mathcal{X}$-envelope $u : M \to X$ such that for any automorphism $g : X \to X$, there exists an endomorphism $f : M \to M$ such that $uf = gu$. Various properties of $\mathcal{X}$-automorphism-invariant modules have been studied.

Next, we discuss the class of modules which are invariant under an idempotent endomorphism of their envelopes. Let us consider the following conditions:

- (C1) Every submodule of M is essential in a direct summand of M.
- (C2) If a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M.
- (C3) If M_1 and M_2 are direct summands of M and $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

A module M is called *continuous* if it satisfies (C1) and (C2); and M is called *quasi-continuous* if it enjoys (C1) and (C3). A module M satisfying (C1) is usually called an *extending* (or *CS*) *module*.

In 1978, Goel and Jain [7] proved that M is quasi-continuous if and only if M is invariant under all idempotent endomorphisms of $E(M)$. Moreover, M is quasi-continuous if and only if for any decomposition $E(M) = \bigoplus_{i\in I} E_i$, we have $M = \bigoplus_{i\in I}(E_i \cap M)$. In this paper, we generalize this result to $\mathcal{X}$-idempotent-invariant modules. Let $u : M \to X$ be a monomorphic $\mathcal{X}$-envelope (i.e., an $\mathcal{X}$-envelope in which u is a monomorphism). Then M is $\mathcal{X}$-idempotent-invariant if and only if for every decomposition $X = \bigoplus_{i\in I} X_i$, we have $M = \bigoplus_{i\in I}(u^{-1}(X_i) \cap M)$ (Theorem 2.7). This definition suggests that one can extend the concept of quasi-continuous modules by considering different enveloping classes of modules.

On the other hand, we also extend the concept of extending modules. We will say that M is $\mathcal{X}$-*extending-invariant* (or $\mathcal{X}$-*extending*) if there exists an $\mathcal{X}$-envelope $u : M \to X$ such that for any idempotent $g \in \mathrm{End}(X)$ there exists an idempotent $f : M \to M$ such that $g(X) \cap u(M) = uf(M)$ or $uf = guf$. It is clear that if $\mathcal{X}$ is the class of all injective modules, then the concept of extending module and $\mathcal{X}$-extending-invariant module coincide. Let $u : M \to X$ be a monomorphic $\mathcal{X}$-envelope. Then we prove that M is an $\mathcal{X}$-extending-invariant module if and only if $u^{-1}(U)\cap M$ is a direct summand of M whenever U is a direct summand of X (Theorem 2.11). Moreover, for enveloping classes $\mathcal{C}$ satisfying certain special properties, we also show that M is an $\mathcal{C}$-idempotent-invariant module if and only if M is an $\mathcal{C}$-extending-invariant module such that whenever $M = M_1 \oplus M_2$ is a direct sum of submodules, then M_1 and M_2 are relatively $\mathcal{C}$-injective (Theorem 3.2).

Throughout this article all rings are associative rings with unit, and all modules are right unital modules. The notation $N \le M$ (resp. $N < M$) will mean that N is a submodule of a module M (resp. proper submodule). And we will write $N \le^e M$ to indicate that N is an essential submodule of M. Let M be an arbitrary module. We denote by $I(M)$ the set of all idempotent elements of $\mathrm{End}(M)$. Recall that $Z(M) = \{m \in M \mid \mathrm{ann}(m) \le^e R_R\}$ is called the *singular submodule* of M. And M is called *singular* (resp. *nonsingular*) if $Z(M) = M$ (resp. $Z(M) = 0$).

General background material can be found in [1], [4], [12].

2. Classes of modules via their envelopes

Definition 2.1. Let M be a right R-module. We will say that M is $\mathcal{X}$-*idempotent-invariant* if there exists an $\mathcal{X}$-envelope $u : M \to X$ such that for any idempotent $g \in \mathrm{End}(X)$ there exists an endomorphism $f : M \to M$ such that $uf = gu$.

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, dotted, "f"'] & X \arrow[d, "g"] \\ M \arrow[r, "u"] & X \end{tikzcd}$$

REMARK 2.2. (1) Assume that M is an $\mathcal{X}$-idempotent-invariant module. Let $u : M \to X$ be a monomorphic $\mathcal{X}$-envelope and $g \in I(X)$. Then there exists an $f \in I(M)$ with $uf = gu$ (see a detailed proof in Proposition 2.5). Moreover, f is unique since u is a monomorphism. Thus, we get a map

$$\nabla : I(X) \to I(M), \quad g \mapsto f,$$

between idempotents in $\mathrm{End}(X)$ and in $\mathrm{End}(M)$.

(2) If $\mathcal{X}$ is the class of all injective modules, then the $\mathcal{X}$-idempotent-invariant modules are precisely the quasi-continuous modules.

EXAMPLE 2.3. (i) If $\mathcal{X} = \text{Mod-}R$, then each right R-module is trivially $\mathcal{X}$-idempotent-invariant.

(ii) Let M be an R-S-bimodule such that M is linearly compact as a left R-module, but it is not quasi-continuous as a right module (for example, a left artinian ring R which is not right quasi-continuous) and let $\mathcal{X}$ be the class of all pure-injective right R-modules. Then M is right pure-injective by [13] and thus $\mathcal{X}$-idempotent-invariant, but it is not quasi-continuous (see also [14, 52.4(ii)]).

(iii) Let R be a local ring and let $\mathcal{X}$ be the class of right cotorsion R-modules. Then the cotorsion envelope of the right regular module R is indecomposable by [8, Theorem 21], and thus R is clearly $\mathcal{X}$-idempotent-invariant. However, R need not be a right cotorsion ring.

(iv) Let R be a ring and

$$\mathcal{X} = \{X \text{ injective} \mid \mathrm{Im}(f) \text{ is orthogonal to } \mathrm{Ker}(f), \forall f = f^2 \in \mathrm{End}(X)\}.$$

In particular, we can choose $\mathcal{X} = \{X \text{ is uniform injective nonsingular}\}$. Then a right R-module M is $\mathcal{X}$-idempotent-invariant if and only if M is a TS-module with property T_3 (see [15]).

DEFINITION 2.4. Let M be a right R-module. We will say that M is $\mathcal{X}$-*extending-invariant* (or $\mathcal{X}$-*extending*) if there exists an $\mathcal{X}$-envelope $u : M \to X$ such that for any idempotent $g \in \mathrm{End}(X)$ there exists an idempotent $f : M \to M$ with $g(X) \cap u(M) = uf(M)$. In this case, $uf = guf$.

$$\begin{tikzcd} M \arrow[r, "uf"] \arrow[d, dotted, "f"'] & X \arrow[d, "g"] \\ M \arrow[r, "u"] & X \end{tikzcd}$$

PROPOSITION 2.5. *Let $u : M \to X$ be a monomorphic $\mathcal{X}$-envelope. If M is $\mathcal{X}$-idempotent-invariant then it is $\mathcal{X}$-extending-invariant.*

Proof. Choose $g \in \mathrm{End}(X)$ such that $g^2 = g$. Since M is $\mathcal{X}$-idempotent-invariant, there exists $f : M \to M$ such that $uf = gu$. Furthermore, $1 - g \in I(X)$, and thus there is $f' : M \to M$ such that $uf' = (1-g)u$. Then $uf'f = 0$, and so $f'f = 0$. It follows that $u = gu + (1-g)u = uf + uf' = u(f + f')$. Then $f + f' = \mathrm{id}$ (since u is a monomorphism). Thus, $f^2 = f$ and $g(X) \cap u(M) = uf(M)$. ■

NOTE. (1) If $\mathcal{X}$ is the class $\mathcal{I}$ of all injective modules, then the $\mathcal{I}$-extending-invariant modules are precisely the extending modules.

(2) Let $\mathbb{F}$ be a field and $R = \mathbb{T}_2(\mathbb{F})$, the 2×2 upper triangular matrix ring over $\mathbb{F}$. It is easy to check that R is $\mathcal{I}$-extending-invariant. Let us show that, however, R_R is not $\mathcal{I}$-idempotent-invariant. Let $E_{ij} \in R$ be the matrix with 1 in the (i, j)-position and 0 elsewhere, and set $E = E_{12} + E_{22}$ and $F = F_{22}$. Then E and F are idempotents in R and $ER \cap FR = 0$. But it can be checked that $ER_R \oplus FR_R$ is not a direct summand of R_R. Therefore R_R is not quasi-continuous, and thus not $\mathcal{I}$-idempotent-invariant.

LEMMA 2.6. *Let M be a module, and N a direct summand of M.*

(1) *If M is $\mathcal{X}$-idempotent-invariant and N has an $\mathcal{X}$-envelope, then N is also $\mathcal{X}$-idempotent-invariant.*
(2) *If M is $\mathcal{X}$-extending-invariant and N has an $\mathcal{X}$-envelope and is invariant under all idempotents of* $\mathrm{End}(M)$, *then N is also $\mathcal{X}$-extending-invariant.*

Proof. (1) Let $u : M \to X$ and $u_1 : N \to X_1$ be $\mathcal{X}$-envelopes, $\pi : M \to N$ the projection map and $\iota : N \to M$ the inclusion. Let g_2 be an idempotent endomorphism of X_1. We claim that there exists $f_2 : N \to N$ such that $u_1 f_2 = g_2 u_1$.

By the $\mathcal{X}$-envelope property, there exist $h_1 : X \to X_1$ and $h_2 : X_1 \to X$ such that $h_1 u = u_1 \pi$ and $h_2 u_1 = u\iota$. It follows that $h_1 h_2 u_1 = u_1$, and so $h_1 h_2$ is an isomorphism. There exists $h : X_1 \to X_1$ such that $(h_1 h_2)h = \mathrm{id}_{X_1}$. Let $g_1 = h_2(hg_2)h_1 : X \to X$. Then g_1 is an idempotent endomorphism of X. As M is an $\mathcal{X}$-idempotent-invariant module, there exists $f_1 : M \to M$ such that $uf_1 = g_1 u$. Let $f_2 = \pi f_1 \iota$. Then

$$\begin{aligned} u_1 f_2 &= u_1 \pi f_1 \iota = h_1 u f_1 \iota = h_1 g_1 u \iota = h_1 h_2 h g_2 h_1 u \iota \\ &= g_2 h_1 u \iota = g_2 u_1 \pi \iota = g_2 u_1. \end{aligned}$$

Thus, N is $\mathcal{X}$-idempotent-invariant.

(2) Let $u, u_1, \pi, \iota, g_2, h_1, h_2$ be as above. We will show that there exists an idempotent $f_2 : N \to N$ such that $g_2(X_1) \cap u_1(N) = u_1 f_2(M)$.

Since $h_1 h_2$ is an isomorphism, there exists $h : X_1 \to X_1$ such that $h(h_1 h_2) = \mathrm{id}_{X_1}$. It follows that h_2 splits. This means that X_1 is isomorphic to a direct summand of X. We may assume that X_1 is a direct summand of X and $u_1 = u\iota$. Let $g_1 = \iota_0 g_2 \pi_0 : X \to X$ with $\pi_0 : X \to X_1$ the canonical projection and $\iota_0 : X_1 \to X$ the inclusion. Then g_1 is an idempotent of $\mathrm{End}(X)$. As M is an $\mathcal{X}$-idempotent-invariant module, there exists an idempotent $f_1 : M \to M$ such that $g_1(X) \cap u(M) = uf_1(M)$. As N is invariant under all idempotent endomorphisms of M, we have $f_1(N) \leq N$. Define $f_2 = f_1|_N : N \to N$. Then f_2 is an idempotent endomorphism of N. It is easy to see that $g_2(X_1) \cap u_1(N) = u_1 f_2(N)$.

Thus, N is $\mathcal{X}$-extending-invariant. ■

THEOREM 2.7. *Let $u : M \to X$ be a monomorphic $\mathcal{X}$-envelope. The following conditions are equivalent:*

(1) *M is $\mathcal{X}$-idempotent-invariant.*
(2) *If $X = \bigoplus_{i\in I} X_i$, then $M = \bigoplus_{i\in I}(u^{-1}(X_i) \cap M)$.*
(3) *If $X = X_1 \oplus X_2$, then $M = (u^{-1}(X_1) \cap M) \oplus (u^{-1}(X_2) \cap M)$.*

Proof. (1)⇒(2). Assume that $X = \bigoplus_{i\in I} X_i$. For each $m \in M$, there is a finite subset $F \subseteq I$ such that $u(m) \in \bigoplus_{k\in F} X_k$. But there also exists a set $\{g_k : k \in F\}$ of orthogonal idempotents of $\mathrm{End}(X)$ such that $g_k(X) = X_k$. We know that M is $\mathcal{X}$-idempotent-invariant, and thus there exists a family $\{f_k : k \in F\} \subseteq \mathrm{End}(M)$ such that $uf_k = g_k u$ for all $k \in F$. On the other hand,

$$u(m) = \sum_F g_k u(m) = \sum_F uf_k(m) \quad \text{or} \quad m = \sum_F f_k(m).$$

Note $f_k(m) \in M \cap u^{-1}(X_k)$ for every $k \in F$. Therefore, $m \in \sum_F (M \cap u^{-1}(X_i))$, and thus $M = \bigoplus_{i\in I}(u^{-1}(X_i) \cap M)$.

(2)⇒(3). This is obvious.

(3)⇒(1). Let g be an idempotent endomorphism of X. Then $X = g(X) \oplus (1-g)(X)$. By (3), we get $M = (M \cap u^{-1}(g(X)) \oplus u^{-1}((1-g)(X))$. Let $f : M \to M \cap u^{-1}(g(X))$ be the projection. Then, for every $m = x + y \in M$, where $x \in M \cap u^{-1}(g(X))$ and $y \in M \cap u^{-1}((1-g)(X))$, we obtain $uf(m) = u(x)$. But $x \in M \cap u^{-1}(g(X))$, and so $u(x) = g(m_0)$ for some $m_0 \in M$. On the other hand,

$$gu(m) = gu(x+y) = gu(x) + gu(y) = g(g(m_0)) + gu(y) = g(m_0) + gu(y).$$

We have $y \in M \cap u^{-1}((1-g)(X))$, and thus $gu(y) = 0$. It follows that $uf(m) = gu(m)$. Therefore, $uf = gu$. ■

Recall that if M is a module and $N \leq M$, then a submodule C of M is a *complement* of N if it is maximal with respect to $C \cap N = 0$.

PROPOSITION 2.8. *Let $u : M \to X$ be a monomorphic $\mathcal{X}$-envelope with $u(M)$ essential in X. Consider the following conditions:*

(1) *$M = U \oplus V$ for any U, V which are complements of each other.*
(2) *M is $\mathcal{X}$-idempotent-invariant.*

Then (1) *always implies* (2). *Moreover, if X is a quasi-continuous module, then M is also quasi-continuous and so* (2)$\Rightarrow$(1).

Proof. (1)$\Rightarrow$(2). Let g be an idempotent endomorphism of X and let $A_1 = M \cap u^{-1}(g(X))$ and $A_2 = M \cap u^{-1}((1-g)(X))$. Let B_1 be a complement of A_2 in M containing A_1, and B_2 a complement of B_1 in M containing A_2. It follows that B_1 and B_2 are complements of each other in M. Therefore, $M = B_1 \oplus B_2$. Let $\pi : B_1 \oplus B_2 \to B_1$ be the projection.

Let us now show that $u\pi = gu$. Assume that $u\pi \neq gu$. As $u(M) \leq^e X$, there exist $m = b_1 + b_2 \in M$ with $b_1 \in B_1$, $b_2 \in B_2$, and $m_0 \in M$ such that $u(m_0) = (u\pi - gu)(m) \neq 0$. It follows that $u(m_0 + \pi(m)) = gu(m)$ or $m_0 + \pi(m) \in A_1$. Furthermore, $gu(m_0 + \pi(m)) = gu(m)$ or $m_0 + \pi(m) - m \in A_2$. Then $m_0 + \pi(m) - b_1 - b_2 \in A_2$ and $m_0 + \pi(m) - b_1 \in B_1 \cap B_2 = 0$. Note that $\pi(m) - b_1 = 0$, and thus $m_0 = 0$, a contradiction.

(2)$\Rightarrow$(1). Assume that X is quasi-continuous. Let $A \leq M$. There exists $H \leq X$ such that $X = H \oplus K$ and $u(A) \leq^e H$. We have $A \leq^e u^{-1}(H) \cap M$ (since u is a monomorphism). By (3), we get

$$M = (u^{-1}(H) \cap M) \oplus (u^{-1}(K) \cap M).$$

Thus M satisfies (C1).

Next, we show that M also satisfies (C3). Assume that U, V are direct summands of M such that $U \cap V = 0$. There exist decompositions $X = X_1 \oplus Y_1 = X_2 \oplus Y_2$ such that $u(U) \leq^e X_1$ and $u(V) \leq^e X_2$. We have $U \cap V = 0$, and so $X_1 \cap X_2 = 0$. On the other hand, since X satisfies (C3), we have $X = (X_1 \oplus X_2) \oplus X_3$. Thus

$$M = (u^{-1}(X_1) \cap M) \oplus (u^{-1}(X_2) \cap M) \oplus (u^{-1}(X_3) \cap M).$$

Since U, V are direct summands of M, $U \leq^e u^{-1}(X_1) \cap M$ and $V \leq^e u^{-1}(X_2) \cap M$, we deduce that $U = u^{-1}(X_1) \cap M$ and $V = u^{-1}(X_2) \cap M$. Therefore, $M = (U \oplus V) \oplus (u^{-1}(X_3) \cap M)$. ■

Recall that if M is a module, then $E(M)$ denotes the injective hull of M, that is, the minimal injective extension of M.

COROLLARY 2.9 ([2, Proposition 2.1.25]). *Let M be a module. The following conditions are equivalent:*

(1) *M is quasi-continuous.*
(2) *$M = X \oplus Y$ for any pair of submodules X and Y which are complements of each other.*
(3) *$f(M) \leq M$ for every idempotent $f \in \mathrm{End}(E(M))$.*
(4) *For every decomposition $E(M) = \bigoplus_{i \in \Lambda} E_i$, we have*

$$M = \bigoplus_{i \in \Lambda} (M \cap E_i).$$

Proof. Apply Theorem 2.7 and Proposition 2.8 in the special case where $\mathcal{C}$ is the class $\mathcal{I}$ of all injective modules. ∎

Let us recall that a *closed* submodule of M has the form $X \cap M$ for some direct summand X of $E(M)$.

DEFINITION 2.10. Let $u : M \to X$ be an $\mathcal{X}$-envelope, and A a submodule of M. Then A is said to be *$\mathcal{X}$-closed* in M if there exists an idempotent endomorphism g of X such that $A = u^{-1}(g(X)) \cap M$.

THEOREM 2.11. *Assume that $u : M \to X$ is a monomorphic $\mathcal{X}$-envelope. The following conditions are equivalent:*

(1) *M is an $\mathcal{X}$-extending-invariant module.*
(2) *Every $\mathcal{X}$-closed submodule is a direct summand of M.*

Proof. (1)⇒(2). Let $U = g(X)$ with $g \in I(X)$. There exists $f \in I(M)$ such that $g(X) \cap u(M) = uf(M)$. It follows that $u^{-1}(U) \cap M = f(M)$ is a direct summand of M.

(2)⇒(1). Let g be an idempotent of $\mathrm{End}(X)$. By hypothesis, $u^{-1}(g(X)) \cap M$ is a direct summand of M. If $\pi : M \to u^{-1}(g(X)) \cap M$ is the canonical projection, then $\pi \in I(M)$ and $g(X) \cap u(M) = u\pi(M)$. Thus, M is an $\mathcal{X}$-extending-invariant module. ∎

Specializing the above theorem to the case $\mathcal{C} = \mathcal{I}$, we get:

COROLLARY 2.12 ([2, Proposition 2.1.15]). *Let M be a module. Then the following conditions are equivalent:*

(1) *M is an extending module.*
(2) *Every closed submodule of M is a direct summand of M.*

Recall that M is *$\mathcal{X}$-endomorphism-invariant* (resp. *$\mathcal{X}$-automorphism-invariant*) if there exists an $\mathcal{X}$-envelope $u : M \to X$ such that for any endomorphism (resp. automorphism) g of X there is an endomorphism $f : M \to M$ such that $uf = gu$.

DEFINITION 2.13. Let M_1 and M_2 be modules. We will say that M_2 is *$\mathcal{X}$-M_1-injective* if there exist $\mathcal{X}$-envelopes $u_1 : M_1 \to X_1$, $u_2 : M_2 \to X_2$ such that for any homomorphism $g : X_1 \to X_2$ there is a homomorphism $f : M_1 \to M_2$ such that $gu_1 = u_2 f$.

$$\begin{tikzcd} M_1 \arrow[r, "u_1"] \arrow[d, dotted, "f"'] & X_1 \arrow[d, "g"] \\ M_2 \arrow[r, "u_2"] & X_2 \end{tikzcd}$$

It is easy to see that M is $\mathcal{X}$-endomorphism-invariant if and only if it is $\mathcal{X}$-M-injective.

Two modules M_1 and M_2 are called *relatively $\mathcal{X}$-injective* if M_1 is $\mathcal{X}$-M_2-injective and M_2 is $\mathcal{X}$-M_1-injective.

LEMMA 2.14. *Let M_1 and M_2 be relatively $\mathcal{X}$-injective and let $u_i : M_i \to X_i$ be the relevant $\mathcal{X}$-envelopes. If $X_1 \simeq X_2$ then $M_1 \simeq M_2$.*

Proof. Assume that $g : X_1 \to X_2$ is an isomorphism. There exist homomorphisms $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_1$ such that $u_2 f_1 = g u_1$ and $u_1 f_2 = g^{-1} u_2$. Then $u_1 f_2 f_1 = u_1$ and $u_2 f_1 f_2 = u_2$. It follows that $f_1 f_2 = \mathrm{id}_{M_2}$ and $f_2 f_1 = \mathrm{id}_{M_1}$. ∎

LEMMA 2.15. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a module and $u_i : M_i \to X_i$ be $\mathcal{X}$-envelopes. The following are equivalent:*

(1) *M is $\mathcal{X}$-endomorphism-invariant.*
(2) *M_i is $\mathcal{X}$-M_j-injective for all $i, j \in \{1, \ldots, n\}$.*

Proof. We will prove this for $n = 2$.

(1)⇒(2). Let $g : X_i \to X_j$ be a homomorphism. Denote by $\pi_i : X_1 \oplus X_2 \to X_i$ and $\iota_i : X_i \to X_1 \oplus X_2$ the canonical projections and injections, for $i = 1, 2$. By (1), there exists $f : M_1 \oplus M_2 \to M_1 \oplus M_2$ such that $(u_1 \oplus u_2) f = \iota_j g \pi_i (u_1 \oplus u_2)$. Let $k = q_j f v_i$, where $q_j : M_1 \oplus M_2 \to M_j$ is the canonical projection and $v_i : M_i \to M_1 \oplus M_2$ the canonical injection. Then $u_j k = g u_i$.

(2)⇒(1). Assume that M_i is $\mathcal{X}$-M_j-injective for every $i, j \in \{1, 2\}$ and $u_1 \oplus u_2 : M_1 \oplus M_2 \to X_1 \oplus X_2$ is an $\mathcal{X}$-envelope. Let $g \in \mathrm{End}(X_1 \oplus X_2)$, $\iota_1 : X_1 \to X_1 \oplus X_2$, $\iota_2 : X_2 \to X_1 \oplus X_2$ and $\pi_1 : X_1 \oplus X_2 \to X_1$, $\pi_2 : X_1 \oplus X_2 \to X_2$ be the canonical maps. For $i, j \in \{1, 2\}$, there exist $f_{ij} : M_i \to M_j$ such that $\pi_j g \iota_i u_i = u_j f_{ij}$. Let $f : M_1 \oplus M_2 \to M_1 \oplus M_2$ be defined via $f(m_1 + m_2) = f_{11}(m_1) + f_{21}(m_1) + f_{12}(m_2) + f_{22}(m_2)$. Then $g(u_1 \oplus u_2) = (u_1 \oplus u_2) f$. Thus, $M = M_1 \oplus M_2$ is $\mathcal{X}$-endomorphism-invariant. ∎

COROLLARY 2.16. *A module M is $\mathcal{X}$-endomorphism-invariant if and only if M^n is.*

In particular, when $\mathcal{X} = \mathcal{I}$, we obtain the following two corollaries.

COROLLARY 2.17 ([2, Proposition 2.2.2]). *Let $M_1, \ldots, M_n$ be modules. Then the following conditions are equivalent:*

(1) *$M_1 \oplus \cdots \oplus M_n$ is quasi-injective.*
(2) *M_i and M_j are relatively injective for $i, j = 1, \ldots, n$.*

COROLLARY 2.18 ([2, Proposition 2.2.3]). *Let M be a module and n a positive integer. Then M is quasi-injective if and only if M^n is quasi-injective.*

Lemma 2.19. *Assume that* $M = M_1 \oplus M_2$, *and* $u_i : M_i \to X_i$ *and* $u_1 \oplus u_2 : M \to X_1 \oplus X_2$ *are* $\mathcal{X}$*-envelopes. If* M *is* $\mathcal{X}$*-idempotent-invariant, then* M_i *is* $\mathcal{X}$*-*M_j*-injective for all* $i \neq j$.

Proof. Let $g : X_1 \to X_2$ be a homomorphism. Define $g' : X_1 \oplus X_2 \to X_1 \oplus X_2$ by $g'(x_1 + x_2) = x_1 + g(x_1)$. Then g' is an idempotent endomorphism. Since M is $\mathcal{X}$-idempotent-invariant, there exists $f' : M \to M$ such that $uf' = g'u$. For any $m_1 \in M_1$, there exist $m_1' \in M_1$ and $m_2 \in M_2$ such that $f'(m_1) = m_1' + m_2$. Define $f : M_1 \to M_2$ by $f(m_1) = m_2$; note that f is well-defined. Furthermore, for every $m_1 \in M_1$, we have

$$g'u(m_1) = g'(u_1(m_1)) = u_1(m_1) + gu_1(m_1)$$

and

$$uf'(m_1) = u(m_1' + m_2) = u_1(m_1') + u_2(m_2) = u_1(m_1') + u_2 f(m_1).$$

It follows that $gu_1(m_1) = u_2 f(m_1)$. Thus, $gu_1 = u_2 f$. ∎

Corollary 2.20. M *is* $\mathcal{X}$*-endomorphism-invariant if and only if* $M \oplus M$ *is* $\mathcal{X}$*-idempotent-invariant.*

Proof. One implication follows from Lemma 2.19, and the other from Corollary 2.16. ∎

Corollary 2.21. *Let* M *be a module. Then* M *is quasi-injective if and only if* $M \oplus M$ *is quasi-continuous.*

Proof. Apply Lemma 2.19 and Corollary 2.16 to the case $\mathcal{X} = \mathcal{I}$. ∎

A module M is called *purely infinite* if $M \cong M \oplus M$, and *directly finite* if M is is not isomorphic to a proper summand of itself.

Proposition 2.22. *Assume that* M *is an* $\mathcal{X}$*-idempotent-invariant module and every direct summand of* M *has an* $\mathcal{X}$*-envelope. Let* $u : M \to X$ *be a monomorphic* $\mathcal{X}$*-envelope with* X *a direct sum of a directly finite module and a purely infinite module. Then*

(1) M *is purely infinite if and only if* X *is.*
(2) M *is directly finite if and only if* X *is.*

Proof. (1) ($\Rightarrow$) Assume that M is purely infinite. Then $M = M_1 \oplus M_2$ with $M_1 \simeq M_2 \simeq M$. Let $u_1 : M_1 \to X_1$ and $u_2 : M_2 \to X_2$ be $\mathcal{X}$-envelopes. We deduce that $X \simeq X_1 \oplus X_2$ and $X \simeq X_1 \simeq X_2$. Thus, X is purely infinite.

($\Leftarrow$) Assume that $X = X_1 \oplus X_2$ with $X_1 \simeq X_2 \simeq X$. By Theorem 2.7,

$$M = [M \cap u^{-1}(X_1)] \oplus [M \cap u^{-1}(X_2)].$$

Furthermore, $M \cap u^{-1}(X_1)$ and $M \cap u^{-1}(X_2)$ are relatively injective. Let $M_1 = M \cap u^{-1}(X_1)$ and $M_2 = M \cap u^{-1}(X_2)$. Then $M = M_1 \oplus M_2$ and $u_1 = u|_{M_1} : M \cap u^{-1}(X_1) \to X_1$, $u_2 = u|_{X_2} : M \cap X_2 \to u^{-1}(X_2)$ are $\mathcal{X}$-envelopes. Indeed, $X_1, X_2 \in \mathcal{X}$. Let $f : M_1 \to U$ be a homomorphism with

$U \in \mathcal{X}$. There exists $h : X \to U$ such that $hu = f\pi_1$, where $\pi_1 : M \to M_1$ is the canonical projection. Note that $u(M_1) \leq X_1$. Let $k = h\pi_{X_1} : X \to X_1$ be the canonical projection. Then $ku_1 = f$. On the other hand, assume that $\alpha u_1 = u_1$ with $\alpha : X_1 \to X_1$. Let $\beta = \alpha \oplus \mathrm{id}_{X_2} : X \to X$. Then $\beta u = u$. So β is an isomorphism. Therefore so is α. This shows that $u_1 : M_1 \to X_1$ is an $\mathcal{X}$-envelope. Similarly, $u_2 : M_2 \to X_2$ is an $\mathcal{X}$-envelope. It is easy to check that M_i is $\mathcal{X}$-M-injective. It follows now from Lemma 2.14 that $M_1 \simeq M_2 \simeq M$. Thus, M is purely infinite.

(2) ($\Leftarrow$) Assume that M is not directly finite. Then $M = M_1 \oplus M_2$ with $M_1 \simeq M$. It is easy to see that $X \simeq X_1 \oplus X_2$ and $X_1 \simeq X$. Thus X is not directly finite either.

($\Rightarrow$) Now assume that X is not directly finite. There exist submodules X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $X_1 \simeq X_1^2$. It follows that $M = (M \cap u^{-1}(X_1)) \oplus (M \cap u^{-1}(X_2))$. Set $M_1 = M \cap u^{-1}(X_1)$. It is easy to check that $u_1 = u|_{M_1} : M \cap u^{-1}(X_1) \to X_1$ is an $\mathcal{X}$-envelope, and thus $M_1 \simeq M_1^2$ by (1). Thus M is not directly finite. ∎

Corollary 2.23. *Assume that M is $\mathcal{X}$-idempotent-invariant and every direct summand of M has an $\mathcal{X}$-envelope. If $u : M \to X$ is an $\mathcal{X}$-envelope and X is a direct sum of a directly finite module and a purely infinite module, then M has a decomposition $M = M_1 \oplus M_2$ such that M_1 is directly finite, M_2 is purely infinite and M_1, M_2 are relatively $\mathcal{X}$-injective.*

3. Relationships between $\mathcal{X}$-idempotent-invariant modules and $\mathcal{X}$-extending-invariant modules for some special classes $\mathcal{X}$

Lemma 3.1. *Let $A \leq M$, and let $u_A : A \to X_A$ and $u : M \to X$ be monomorphic $\mathcal{X}$-envelopes with $u_A(A) \leq^e X_A$. Then there exists a monomorphic $\mathcal{X}$-envelope $v : A \to Y$ such that $u|_A = v$.*

Proof. As u_A is an $\mathcal{X}$-envelope, $u|_A$ factors through u_A. So there exists $h : X_A \to X$ such that $hu_A = u|_A$. Let us factor $h = w \circ p$ with $p : X_A \to Y$ an epimorphism and $w : Y \to X$ the inclusion. As u_A is an essential monomorphism, $p : X_A \to Y$ must be a monomorphism, and thus an isomorphism. Thus, $v = pu_A : A \to Y$ is an $\mathcal{X}$-envelope. ∎

Next we are going to consider the situation where M has a $\mathcal{C}$-envelope $u : M \to X$, for a certain class of modules $\mathcal{C}$, satisfying the following conditions:

(1) $\mathcal{C}$ is closed under isomorphisms and finite direct sums.
(2) Every submodule A of M has a $\mathcal{C}$-envelope $u_A : A \to X_A$ with u_A an essential monomorphism.
(3) If $A \leq B \leq M$ and $u_1 : A \to X_1$ and $u_2 : B \to X_2$ are $\mathcal{C}$-envelopes, then X_1 is a direct summand of X_2.

Note that, in particular, the class $\mathcal{I}$ of all injective modules satisfies the above conditions.

Then by Lemma 3.1, for every submodule A of M with envelope $u : M \to X$, we can choose a $v : A \to X_A$ such that $u|_A = v$ and X_A is a direct summand of X. We can prove the following result.

Theorem 3.2. *Under the above assumptions, the following conditions are equivalent:*

(1) *M is a $\mathcal{C}$-idempotent-invariant module.*
(2) *M is a $\mathcal{C}$-extending-invariant module such that whenever $M = M_1 \oplus M_2$ is a direct sum of submodules, then M_1 and M_2 are relatively $\mathcal{C}$-injective.*

Proof. (1)$\Rightarrow$(2) by Lemmas 2.5, 2.6(1) and 2.19.

(2)$\Rightarrow$(1). Let $u : M \to X$ be a $\mathcal{C}$-envelope. Assume $g \in I(X)$. Since M is $\mathcal{C}$-extending-invariant, $u^{-1}((1-g)(X)) \cap M$ is a direct summands of M by Theorem 2.11. Let $A_1 = u^{-1}(g(X)) \cap M$ and $B_1 = u^{-1}((1-g)(X)) \cap M$. Then $A_1 \cap B_1 = 0$ and $M = B_1 \oplus B_2$ for some $B_2 \leq M$.

We claim that there exists $M' \leq M$ such that $M = B_1 \oplus M'$ and $A_1 \leq M'$.

Let $\pi_i : M \to B_i$ be the projections. Then $A_1' := \pi_2(A_1) \simeq A_1$ and $\pi_1' : \pi_2(A_1) \to B_1$ via $\pi_1'(\pi_2(a_1)) = \pi_1(a_1)$ for all $a_1 \in A_1$ is a monomorphism. Assume that $u|_{A_1'} : A_1' \to X_1'$ is a $\mathcal{C}$-envelope and $X = X_1' \oplus X_2'$ with $\pi_{X_1'} : X \to X_1'$ is the projection. We have the diagram

$$\begin{tikzcd}
A_1' \arrow[r, "u|_{A_1'}"] \arrow[d, "\pi_1'"'] & X_1' \arrow[ddl, dashed, "h"] & X \arrow[l, "\pi_{X_1'}"'] & X_2 \arrow[l] \\
B_1 \arrow[d, "u|_{B_1}"'] & & & \\
X_1 & & &
\end{tikzcd}$$

where $u|_{B_i} : B_i \to X_i$ are $\mathcal{C}$-envelopes and X_i are the direct summands of X.

By definition of envelope, there exists $h : X_1' \to X_1$ such that $h \circ (u|_{A_1'}) = (u|_{B_1}) \circ \pi_1'$. Let $k = (h\pi_{X_1'})|_{X_2}$.

$$\begin{tikzcd}
B_2 \arrow[r, "u|_{B_2}"] \arrow[d, dotted, "v"'] & X_2 \arrow[d, "k"] \\
B_1 \arrow[r, "u|_{B_1}"] & X_1
\end{tikzcd}$$

Since B_2 is $\mathcal{C}$-B_1-injective, there exists $v : B_2 \to B_1$ such that

$$(u|_{B_1}) \circ v = k \circ u|_{B_2}.$$

Let $M' = \{b_2 + v(b_2) \mid b_2 \in B_2\}$. For every $a_1 \in A_1$, we have $a_1 = \pi_1(a_1) + \pi_2(a_1)$. Hence

$$\begin{aligned} uv(\pi_2(a_1)) &= ku(\pi_2(a_1)) = h\pi_{X_1'}u(\pi_2(a_1)) = hu(\pi_2(a_1)) \\ &= u\pi_1'(\pi_2(a_1)) = u(\pi_1(a_1)). \end{aligned}$$

It follows that $v(\pi_2(a_1)) = \pi_1(a_1)$, and so $a_1 \in M'$. Then $A_1 \leq M'$. It is easy to see that $M = B_1 \oplus M'$.

Now we will show that there exists $f^2 = f \in \mathrm{End}(M)$ such that $uf = gu$. Thus M is $\mathcal{C}$-idempotent-invariant. In fact, let $\pi : M \to M'$ be the projection. Then $\pi = \pi^2$. For any $u(m_1) = u\pi(m_2) - gu(m_2) \in u(M) \cap (u\pi - gu)(M)$ with $m_1, m_2 \in M$, we have

$$(*) \qquad \pi(m_2) - m_1 \in A_1 \leq M'$$

and

$$u(m_1 - \pi(m_2) + m_2) = (1 - g)u(m_2).$$

It follows that

$$m_1 - \pi(m_2) + m_2 \in B_1.$$

We obtain $\pi(m_1 - \pi(m_2) + m_2) = 0$ or $\pi(m_1) - \pi(m_2) + \pi(m_2) = 0$. It follows that

$$(**) \qquad m_1 \in B_1.$$

From $(*)$ and $(**)$, we have $m_1 = 0$, which implies $u(M) \cap (u\pi - gu)(M) = 0$. It follows that $u\pi = gu$ (since $u(M) \leq^e X$). ∎

Corollary 3.3. *The following conditions are equivalent:*

(1) *M is a quasi-continuous module.*
(2) *M is an extending module such that whenever $M = M_1 \oplus M_2$, then M_1 and M_2 are relatively injective.*

Proof. Apply Theorem 3.2 for $\mathcal{C} = \mathcal{I}$. ∎

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Le Van Thuyet
Department of Mathematics
Hue University
3 Le Loi
Hue, Vietnam
E-mail: lvthuyet@hueuni.edu.vn

Phan Dan
Banking University of Ho Chi Minh City
39 Ham Nghi
Ho Chi Minh City, Vietnam
E-mail: mrphandan@yahoo.com

Truong Cong Quynh
Department of Mathematics
Danang University
459 Ton Duc Thang
DaNang, Vietnam
E-mail: tcquynh@live.com, tcquynh@dce.udn.vn