On canonical modules of idealizations

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Abstracts. 1 Let \((R, \mathfrak{m})\) be a Noetherian local ring which is a quotient of a Gorenstein local ring. Let \(M\) be a finitely generated \(R\)-module. In this paper, we study the structure of the canonical module \(K(R \ltimes M)\) of the idealization \(R \ltimes M\) via the polynomial type introduced by N. T. Cuong [5]. In particular, we give a characterization for \(K(R \ltimes M)\) being Cohen-Macaulay and generalized Cohen-Macaulay.

1 Introduction

Throughout this paper, \((R, \mathfrak{m})\) denotes an \(r\)-dimensional Noetherian local ring with maximal ideal \(\mathfrak{m}\) and \(M\) a finitely generated \(R\)-module with dimension \(d\). The concept of principle of idealization was introduced by M. Nagata [12]. In the cartesian product \(R \times M\), we introduce the componentwise addition and the multiplication defined by \((a, x)(b, y) = (ab, ay + bx)\). These operations give a structure of a commutative ring to \(R \times M\). This ring is called the idealization of \(M\) and denoted by \(R \ltimes M\). The purpose of idealization is to put \(M\) inside the commutative ring \(R \ltimes M\) so that the structure of \(M\) as an \(R\)-module is essentially the same as that of \(M\) as an ideal of \(R \ltimes M\). The notion of principle of idealization plays an important role in the study of Noetherian rings and modules. Idealization is useful for reducing results concerning submodules to the ideal case; generalizing results from rings to modules and constructing examples of commutative rings with zero divisors, cf. [1], [12], [17].

The notion of a canonical module of a Noetherian local ring is due to A. Grothendieck, who called it a module of dualizing differentials (cf. [6]). The term “a canonical module” was first adopted by J. Herzog, E. Kunz et al. [7], in which they defined the notion of a canonical module for general local rings. We note that a local ring \(R\) has a canonical module if and only if \(R\) is a

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homomorphic image of a Gorenstein local ring. P. Schenzel [15] has introduced
the canonical module $K(M)$ of an $R$-module $M$.

The polynomial type introduced by N. T. Cuong [5] makes an important
role in the study of finitely generated modules, cf. [5]. Let $\underline{a} = (a_1, \ldots, a_d)$
be a system of parameters of $M$ and $\underline{n} = (n_1, \ldots, n_d)$ a $d$-tuple of positive
integers. Set $\underline{a}(\underline{n}) = (a_1^{n_1}, \ldots, a_d^{n_d})$. Then the difference between length
and multiplicity $I(\underline{a}(\underline{n}) ; M) = \ell(M/(a_1^{n_1}, \ldots, a_d^{n_d})M) - n_1 \ldots n_d e(\underline{a}; M)$ can be con-
sidered as a function in $\underline{n}$. It is well-known that $M$ is Cohen-Macaulay (resp.
generalized Cohen-Macaulay) if and only if $I(\underline{a}(\underline{n}) ; M) = 0$ (resp. there exists a constant $C$
such that $I(\underline{a}(\underline{n}) ; M) \leq C$) for all $\underline{a}$ and $\underline{n}$. In general,
$I(\underline{a}(\underline{n}) ; M)$ is not a polynomial for $n_1, \ldots, n_d \gg 0$, but it takes non negative
values and bounded above by polynomials. The least degree of all polynomi-
als bounding above this function does not depend on the choice of $\underline{a}$, cf. [5,
Theorem 2.3]. This least degree is called the polynomial type of $M$ and denot-
ed by $p(M)$. It should be mentioned that $p(M)$ gives a lot of information
on the structure of $M$. For example, if we stipulate the degree of the zero
polynomial to be $-\infty$ then $M$ is Cohen-Macaulay if and only if $p(M) = -\infty$,
and $M$ is generalized Cohen-Macaulay if and only if $p(M) \leq 0$. We denote
by $\hat{R}$ and $\hat{M}$ the $m$-adic completion of $R$ and $M$ respectively. In general,
$p(M) = p(\hat{M}) = \max_{i \leq d} \dim \hat{R}/\text{Ann}_{\hat{R}} H_{m \hat{R}}^i(\hat{M})$. And if $R$ is a quotient of a Goren-
stein local ring and $M$ is equidimensional then $p(M) = \dim \text{nCM}(M)$, cf. [5,
Theorem 3.1, 3.3], where $\text{nCM}(M)$ is the non Cohen-Macaulay locus of $M$.

The purpose of this paper is to study the polynomial type of the canonical
module of the idealization $R \ltimes M$. Especially, we give a criterion for the
canonical module $K(R \ltimes M)$ being Cohen-Macaulay (resp. generalized Cohen-
Macaulay). Techniques used in this paper are the associativity formula of
multiplicity of local cohomology modules given by M. Brodmann and R.Y.
Sharp [3] (see also [14]) and the extension of idealization introduced by K.
Yamagishi [17]. The main result of this paper is the following theorem.

**Theorem 1.1.** The following statements are true:

(i) If $\dim M = \dim R$ then $p(K(R \ltimes M)) = \max\{p(K(R)), p(K(M))\}$;

(ii) If $\dim M < \dim R$ then $p(K(R \ltimes M)) = p(K(R))$.

In Section 2, we shall outline some properties of polynomial type and ide-
alization which will be needed later. The proof of Theorem 1.1 will be shown
in Section 3 (see Theorem 3.3).

## 2 Preliminaries

Firstly, we recall the notion of polynomial type which introduced by N.T.
Cuong [5]. Let $\underline{a} = (a_1, \ldots, a_d)$ be a system of parameters of $M$ and $\underline{n} =
(n_1, \ldots, n_d)$ a $d$-tuple of positive integers. Set $\underline{a}(\underline{n}) = (a_1^{n_1}, \ldots, a_d^{n_d})$ and

$$I(\underline{a}(\underline{n}) ; M) = \ell(M/(a_1^{n_1}, \ldots, a_d^{n_d})M) - n_1 \ldots n_d e(\underline{a}; M).$$
Then $I(\underline{a}(\underline{n}); M)$ can be considered as a function in $\underline{n}$. Note that this function is non-negative and ascending, i.e., $I(\underline{a}(\underline{n}); M) \geq I(\underline{a}(\underline{m}); M)$ for $\underline{n} = (n_1, \ldots, n_d)$, $\underline{m} = (m_1, \ldots, m_d)$ with $n_i \geq m_i$, $i = 1, \ldots, d$. This function is bounded above by a polynomial in $\underline{n}$. Moreover, we have the following important property.

**Lemma 2.1.** ([5, Theorem 2.3]) The least degree of all polynomials in $\underline{n}$ bounding above the function $I(\underline{a}(\underline{n}); M)$ does not depend on the choice of $\underline{a}$.

**Definition 2.2.** ([5, Definition 2.4]) The numerical invariant of $M$ given in Theorem 2.1 is called the polynomial type of $M$ and denote it by $p(M)$.

**Lemma 2.3.** ([5, Lemma 2.6]) The polynomial type is preserved by $\mathfrak{m}$-adic completion, i.e., $p(M) = p(\hat{M})$.

Next, we recall the concept of principle of idealization introduced by M. Nagata [12]. We make the cartesian product $R \times M$ to become a commutative ring under the componentwise addition and the multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$. This ring is called the idealization of $M$ over $R$ and denoted by $R \ltimes M$.

Note that the idealization $R \ltimes M$ is again a Noetherian local ring with the unique maximal ideal $\mathfrak{m} \times M$ and $\text{dim } R \ltimes M = \text{dim } R$. Moreover the $\mathfrak{m} \times M$-adic completion $\hat{R} \ltimes \hat{M}$ of $R \ltimes M$ is naturally isomorphic to $\hat{R} \ltimes \hat{M}$, cf. [1]. In particular, $(0, x_1)(0, x_2) = (0, 0)$, for all $x_1, x_2 \in M$ and hence $0 \times M$ is an ideal whose square is zero. Furthermore $R \ltimes M/0 \times M \cong R$.

There are a canonical projection $\rho : R \ltimes M \to R$ defined by $\rho((a, x)) = a$ and a canonical inclusion $\sigma : R \to R \ltimes M$ defined by $\sigma(a) = (a, 0)$. Note that $\rho$ and $\sigma$ are local homomorphisms and we can regard any $R$-module (resp. $R \ltimes M$-module) as an $R \times M$-module (resp. $R$-module) by $\rho$ (resp. $\sigma$). Moreover, the structure of $R$-modules induced by the composition $\rho \sigma$ coincides with the original one. Let $\epsilon : M \to R \ltimes M$ be the canonical inclusion defined by $\epsilon(x) = (0, x)$. Then we have an exact sequence of $R \ltimes M$-modules

$$0 \to M \xrightarrow{\epsilon} R \ltimes M \xrightarrow{\epsilon} R \to 0.$$

### 3 The proof of Theorem 1.1

Before proving the main result of this paper, we need to recall notions of canonical module and idealization. Let $R$ be a quotient of a $n$-dimensional Gorenstein local ring $(R', \mathfrak{m}')$. We denote by $K^i(M) = \text{Ext}_{R'}^{n-i}(M, R')$. Then $K^i(M)$ is a finitely generated $R$-module. Following P. Schenzel [16], $K^i(M)$ is called the $i^{th}$ deficiency module of $M$ for $i = 0, \ldots, d - 1$, and $K(M) = K^d(M)$ is called the canonical module of $M$. By the local duality (cf. [2, 11.2.6]), we have an isomorphism

$$H^i_{\mathfrak{m}}(M) \cong \text{Hom}_R(K^i(M), E(R/\mathfrak{m})).$$

for all $i$, where $E(R/\mathfrak{m})$ is the injective hull of $R/\mathfrak{m}$. 


Definition 3.1. ([16], [10]) An $R$-module $M$ is called a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) module if the canonical $R$-module $K(M)$ of $M$ is Cohen-Macaulay (resp. generalized Cohen-Macaulay). If $R$ itself is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) module then it is called a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) ring.

The Cohen-Macaulay canonical property is related to some important questions. For example, if $M$ is Cohen-Macaulay canonical then the monomial conjecture raised by M. Hochster [8] is valid for the ring $R/\text{Ann}_R M$. Furthermore, if $R$ is a domain then $R$ is Cohen-Macaulay canonical if and only if $R$ possesses a birational Macaulayfication $R_1$, i.e. an extension ring $R \subseteq R_1 \subseteq Q$ (where $Q$ is the field of fractions of $R$) such that $R_1$ is finitely generated as an $R$-module and $R_1$ is a Cohen-Macaulay ring, cf. [16, Theorem 1.1].

Remark 3.2. (i) It is easy to see that $\hat{K}(M) \cong K(\hat{M})$ as $\hat{R}$-module. Therefore $M$ is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) $R$-module if and only if $\hat{M}$ is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) $\hat{R}$-module.

(ii) Let $E$ and $F$ be $R$-modules. K. Yamagishi [17] extended the concept of the idealization as follows: given an $R$-linear map $\phi : M \otimes_R E \to F$, it can make the Cartesian product $E \times F$ into an $R \times M$-module with respect to componentwise addition and multiplication defined by

$$(a, x)(e, f) = (ae, af + \phi(x \otimes e)).$$

We denote this $R \times M$-module by $E^\phi \times F$.

Theorem 3.3. The following statements are true:

(i) If $\dim M = \dim R$ then $p(K(R \times M)) = \max\{p(K(R)), p(K(M))\}$;

(ii) If $\dim M < \dim R$ then $p(K(R \times M)) = p(K(R))$.

Proof. Note that $\hat{R} \times \hat{M}$ is isomorphic to the $m \times M$-adic completion of $R \times M$. Moreover, the polynomial type is preserved by the completion, i.e. $p(K(M)) = p(K(\hat{M})), p(K(R)) = p(K(\hat{R}))$ and $p(R \times M) = p(\hat{R} \times \hat{M})$ (see Lemma 2.3). Therefore without any loss of generality, we may assume that $R$ is complete with respect to $m$-adic completion.

Let $\mathfrak{Q}$ be an ideal of $R \times M$ and put $q = \rho(\mathfrak{Q})$, where $\rho : R \times M \to R$ is the map defined by $\rho(a, x) = a$ for all $(a, x) \in R \times M$. Note that, $\mathfrak{Q}$ is $m \times M$-primary if and only if $q$ is $m$-primary, cf. [17, Remark 2.1].

Firstly, we claim the following fact.

Claim 1. Let $q$ be an $m$-primary ideal of $R$. Then we have

$$e(q; K(M)) = e(q; M).$$

Proof of Claim 1. To prove this claim, we need recall some notions and facts on multiplicities for Artinian module. Suppose that $A$ is an Artinian
Let \( a \) be an ideal of \( R \) such that \( \ell(0 :_A a) < \infty \). Then \( \ell(0 :_A a^n) \) is a polynomial with rational coefficients for \( n \gg 0 \). Since \( R \) is complete, the degree of this polynomial is equal to \( t := \dim R/\text{Ann}_RA, \text{ cf. D. Kirby} \ [9] \). Following M. Brodmann and R. Y. Sharp \[3\], the multiplicity of \( A \) with respect to \( a \), denoted by \( \varepsilon'(a; A) \), is defined by the formula \( \varepsilon'(a; A) = a_1! \) where \( a_1 \) is the leading coefficient of the polynomial \( \ell(0 :_A a^n) \).

Let \( D(\_\) be the Matlis dual functor. Since \( A \) is Artinian and \( R \) is complete, \( D(A) \) is a finitely generated \( R \)-module. Since \( \ell_R(0 :_A a^n) < \infty \) with notice that \( D(0 :_A a^n) \cong D(A)/a^nD(A) \), we have \( \ell_R(0 :_A a^n) = \ell_R(D(A)/a^nD(A)) \) for all \( n \in \mathbb{N} \). It follows that \( \varepsilon'(a; A) = \varepsilon(a; D(A)) \). Now, we apply this fact for the Artinian module \( H^n_m(M) \) and the \( \mathfrak{m} \)-primary ideal \( q \). As \( R \) is complete, we have \( K(M) \cong D(H^n_m(M)) \). Now we get

\[
\varepsilon'(q; H^n_m(M)) = \varepsilon(q; K(M)).
\]

For each integer \( i \geq 0 \), let \( \text{Psupp}_{R^i}(M) = \{ p \in \text{Spec}R \mid H^i_{p^{R^i}}(M_p) \neq 0 \} \) be the \( i \)-th pseudo-support of \( M \) defined by M. Brodmann and R. Y. Sharp \[3\]. Then we get by \[14, \text{Corollary 3.4} \] that

\[
e'(q; H^n_m(M)) = \sum_{p \in \text{Psupp}_{R^n}(M) \atop \dim R/p = d} \ell_R(p^{R^n}(M_p)) \varepsilon(q; R/p).
\]

Since \( R \) is complete, \( R \) is catenary. Therefore, we get by \[14, \text{Corollary 3.4} \] that

\[
\text{Psupp}_{R^n}(M) = \{ p \in \text{Supp}(M) \mid \exists p' \in \text{Ass}_R(M), \dim R/p' = d, p' \subseteq p \}.
\]

Hence \( \{ p \in \text{Psupp}_{R^n}(M) \mid \dim R/p = d \} = \{ p \in \text{Supp}_R M \mid \dim R/p = d \} \). So by the associativity formula for multiplicity of \( M \) with respect to \( q \), cf. \[11, 14.7 \], we have

\[
e'(q; H^n_m(M)) = \sum_{p \in \text{Supp}_R M \atop \dim R/p = d} \ell_R(p^{R^n}(M_p)) \varepsilon(q; R/p)
\]

\[
= \sum_{p \in \text{Supp}_R M \atop \dim R/p = d} \ell_R(M_p) \varepsilon(q; R/p)
\]

\[
= \varepsilon(q; M).
\]

Therefore \( \varepsilon(q; K(M)) = \varepsilon(q; M) \), the claim is proved.

From now on, let \( \underline{a} = (a_1, \ldots, a_r) \) be a system of parameters of \( R \). Set \( \underline{u} = (u_1, \ldots, u_r) \) with \( u_i = (a_i, 0) \) for \( i = 1, \ldots, r \). It is easy to see that \( \underline{u} \) is a system of parameters of \( R \times M \). Set \( q = (a_1, \ldots, a_r)R \) and \( \Omega = \sum_{i=1} u_i(R \times M) \subseteq R \times M \). Then \( q \) is an \( \mathfrak{m} \)-primary ideal of \( R \) and \( \Omega \) is an \( \mathfrak{m} \times M \)-primary ideal of \( R \times M \). Moreover \( \Omega = q \times qM \) and \( q = \rho(\Omega) \).
Claim 2. With the above notations, if \( d = r \) (i.e. \( \dim M = \dim R \)) then

\[
\ell_{R\times M}(K(R\times M)/\Omega K(R\times M)) = \ell_R(K(R)/qK(R)) + \ell_R(K(M)/qK(M)).
\]

Otherwise, we have

\[
\ell_{R\times M}(K(R\times M)/\Omega K(R\times M)) = \ell_R(K(R)/qK(R)).
\]

Proof of Claim 2. By [7, 5.14], we have an isomorphism

\[
K(R\times M) \cong \text{Hom}_R(R\times M, K(R))
\]

of \( R\times M \)-modules. Moreover, there is an isomorphism of \( R \)-modules

\[
\text{Hom}_R(R\times M, K(R)) \rightarrow \text{Hom}_R(M, K(R)) \oplus K(R)
\]

defined by \( \alpha \mapsto (\alpha\epsilon, \alpha((1,0))) \) for each \( \alpha \in \text{Hom}_R(R\times M, K(R)) \), where \( \epsilon : M \rightarrow R\times M \) is defined by \( \epsilon(x) = (0, x) \) for all \( x \in M \). Then by Remark 3.2, (ii) we can make the \( R \)-module \( \text{Hom}_R(M, K(R)) \oplus K(R) \) into an \( R\times M \)-module, which is denoted by

\[
\text{Hom}_R(M, K(R)) \times K(R)
\]

with respect to the \( R \)-linear map \( \phi : M \otimes_R \text{Hom}_R(M, K(R)) \rightarrow K(R) \) such that \( \phi(x \otimes f) = f(x) \) for every \( x \in M \) and \( f \in \text{Hom}_R(M, K(R)) \). Therefore

\[
K(R\times M) \cong \text{Hom}_R(M, K(R)) \times K(R)
\]
as \( R\times M \)-modules. By [4, 3.5.10], there is an isomorphism

\[
\text{Hom}_R(H_m^R(M), E_R(R/m)) \cong \text{Hom}_R(M, K(R))
\]
of \( R \)-modules. Now, suppose \( d = r \). Then \( \text{Hom}_R(H_m^R(M), E_R(R/m)) \cong K(M) \) as \( R \) is complete, and hence \( \text{Hom}_R(M, K(R)) \cong K(M) \). Therefore, we get an isomorphism \( K(R\times M) \cong K(M) \times K(R) \) as \( R\times M \)-modules. It follows that

\[
\Omega K(R\times M) \cong (q \times qM)(K(M) \times K(R))
\]
\[
\cong qK(M) \times (qK(R) + \phi(qM \otimes K(M)))
\]
\[
\cong qK(M) \times (qK(R) + q\phi(M \otimes K(M)))
\]
\[
\cong qK(M) \times (qK(R) + q\Im\phi)
\]
\[
\cong qK(M) \times qK(R).
\]

Then we obtain that

\[
\ell_{R\times M}(K(R\times M)/\Omega K(R\times M)) = \ell_{R\times M}((K(M) \times K(R))/qK(M) \times qK(R))
\]
\[
= \ell_R(K(M)/qK(M)) + \ell_R(K(R)/qK(R)).
\]
Suppose that $d < r$. Then $\text{Hom}_R(H^n_m(M), E_R(R/m)) = 0$. Therefore we have $\text{Hom}_R(M, K(R)) = 0$. It follows that $K(R \times M) \cong 0 \leftarrow K(R)$ as $R \times M$-modules and therefore $\mathcal{Q} K(R \times M) \cong 0 \times qK(R)$. Then we get that
\[
\ell_{R \times M}(K(R \times M)/\mathcal{Q} K(R \times M)) = \ell_{R \times M}(0 \leftarrow K(R)/0 \times qK(R)) = \ell_R(K(R)/qK(R)),
\]
and Claim 2 is proved.

Now, we consider the exact sequence

\[ 0 \to M \xrightarrow{i} R \times M \xrightarrow{\rho} R \to 0. \]

If $d = r$ then $e(\mathcal{Q}; R \times M) = e(q; R) + e(q; M)$, and therefore

\[ e(\mathcal{Q}; K(R \times M)) = e(q; K(R)) + e(q; K(M)) \]

by Claim 1. On the other hand, if $d < r$ then $e(\mathcal{Q}; K(R \times M)) = e(q; R) = e(q; K(R))$ by Claim 1.

Let $\underline{n} = (n_1, \ldots, n_r)$ be a set of positive integers, let $a(\underline{n}) := (a_1^{n_1}, \ldots, a_r^{n_r})$ and $\underline{u}(\underline{n}) := (u_1^{n_1}, \ldots, u_r^{n_r}) = ((a_1^{n_1}, 0), \ldots, (a_r^{n_r}, 0))$. Set $\mathcal{Q}(\underline{n}) = \sum_{i=1}^{r} u_i^{n_i}(R \times M) \subseteq R \times M$ and $q(\underline{n}) = a(\underline{n})R$.

(i) If $d = r$ then $a(\underline{n})$ is a system of parameters of $R$, $K(R)$, $M$ and $K(M)$. Moreover, $\underline{u}(\underline{n})$ is a system of parameters of $R \times M$ and $K(R \times M)$. Therefore we get by Claim 2 and the above facts that

\[ I(\mathcal{Q}(\underline{n}); K(R \times M)) = I(q(\underline{n}); K(R)) + I(q(\underline{n}); K(M)). \]

So, we get by Lemma 2.1 that

\[ p(K(R \times M)) = \max\{p(K(R)), p(K(M))\}. \]

(ii) Suppose $d < r$. Then by Claim 2 with notice that $e(\mathcal{Q}; K(R \times M)) = e(q; K(R))$ we obtain

\[ I(\mathcal{Q}; K(R \times M)) = I(q; K(R)). \]

Thus $p(K(R \times M)) = p(K(R))$ by Lemma 2.1.

Note that $M$ is Cohen-Macaulay if and only if $p(M) = -\infty$ and $M$ is generalized Cohen-Macaulay if and only if $p(M) \leq 0$. Therefore we have the following characterization for $R \times M$ being Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay).

**Corollary 3.4.** The following statements are true:

(i) If $\dim M = \dim R$ then $R \times M$ is Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) if and only if so are $R$ and $M$.

(ii) If $\dim M < \dim R$ then $R \times M$ is Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) if and only if so is $R$.

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References


