On canonical modules of idealizations

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Abstracts. [1] Let $(R, \mathfrak{m})$ be a Noetherian local ring which is a quotient of a Gorenstein local ring. Let M be a finitely generated R-module. In this paper, we study the structure of the canonical module $K(R \ltimes M)$ of the idealization $R \ltimes M$ via the polynomial type introduced by N. T. Cuong [5]. In particular, we give a characterization for $K(R \ltimes M)$ being Cohen-Macaulay and generalized Cohen-Macaulay.

1 Introduction

Throughout this paper, $(R, \mathfrak{m})$ denotes an r-dimensional Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated R-module with dimension d. The concept of principle of idealization was introduced by M. Nagata [12]. In the cartesian product $R \times M$, we introduce the componentwise addition and the multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$. These operations give a structure of a commutative ring to $R \times M$. This ring is called the *idealization of* M and denoted by $R \ltimes M$. The purpose of idealization is to put M inside the commutative ring $R \ltimes M$ so that the structure of M as an R-module is essentially the same as that of M as an ideal of $R \ltimes M$. The notion of principle of idealization plays an important role in the study of Noetherian rings and modules. Idealization is useful for reducing results concerning submodules to the ideal case; generalizing results from rings to modules and constructing examples of commutative rings with zero divisors, cf. [1], [12], [17].

The notion of a canonical module of a Noetherian local ring is due to A. Grothendieck, who called it a module of dualizing differentials (cf. [6]). The term "a canonical module" was first adopted by J. Herzog, E. Kunz et al. [7], in which they defined the notion of a canonical module for general local rings. We note that a local ring R has a canonical module if and only if R is a

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homomorphic image of a Gorenstein local ring. P. Schenzel [15] has introduced the canonical module $K(M)$ of an R-module M.

The polynomial type introduced by N. T. Cuong [5] makes an important role in the study of finitely generated modules, cf. [5]. Let $\underline{a} = (a_1, \dots, a_d)$ be a system of parameters of M and $\underline{n} = (n_1, \dots, n_d)$ a d-tuple of positive integers. Set $\underline{a}(\underline{n}) = (a_1^{n_1}, \dots, a_d^{n_d})$. Then the difference between length and multiplicity $I(\underline{a}(\underline{n}); M) = \ell(M/(a_1^{n_1}, \dots, a_d^{n_d})M) - n_1 \dots n_d e(\underline{a}; M)$ can be considered as a function in $\underline{n}$. It is well-known that M is Cohen-Macaulay (resp. generalized Cohen-Macaulay) if and only if $I(\underline{a}(\underline{n}); M) = 0$ (resp. there exists a constant C such that $I(\underline{a}(\underline{n}); M) \leqslant C$) for all $\underline{a}$ and $\underline{n}$. In general, $I(\underline{a}(\underline{n}); M)$ is not a polynomial for $n_1, \dots, n_d \gg 0$, but it takes non negative values and bounded above by polynomials. The least degree of all polynomials bounding above this function does not depend on the choice of $\underline{a}$, cf. [5, Theorem 2.3]. This least degree is called *the polynomial type of* M and denoted by $p(M)$. It should be mentioned that $p(M)$ gives a lot of information on the structure of M. For example, if we stipulate the degree of the zero polynomial to be $-\infty$ then M is Cohen-Macaulay if and only if $p(M) = -\infty$, and M is generalized Cohen-Macaulay if and only if $p(M) \leqslant 0$. We denote by $\widehat{R}$ and $\widehat{M}$ the $\mathfrak{m}$-adic completion of R and M respectively. In general, $p(M) = p(\widehat{M}) = \max\limits_{i<d} \dim \widehat{R}/\mathrm{Ann}_{\widehat{R}} H^i_{\widehat{\mathfrak{m}}}(\widehat{M})$. And if R is a quotient of a Gorenstein local ring and M is equidimensional then $p(M) = \dim \mathrm{nCM}(M)$, cf. [5, Theorem 3.1, 3.3], where $\mathrm{nCM}(M)$ is the non Cohen-Macaulay locus of M.

The purpose of this paper is to study the polynomial type of the canonical module of the idealization $R \ltimes M$. Especially, we give a criterion for the canonical module $K(R \ltimes M)$ being Cohen-Macaulay (resp. generalized Cohen-Macaulay). Techniques used in this paper are the associativity formula of multiplicity of local cohomology modules given by M. Brodmann and R.Y. Sharp [3] (see also [14]) and the extension of idealization introduced by K. Yamagishi [17]. The main result of this paper is the following theorem.

Theorem 1.1. *The following statements are true:*
(i) If $\dim M = \dim R$ *then* $p(K(R \ltimes M)) = \max\{p(K(R)), p(K(M))\}$*;*
(ii) If $\dim M < \dim R$ *then* $p(K(R \ltimes M)) = p(K(R))$.

In Section 2, we shall outline some properties of polynomial type and idealization which will be needed later. The proof of Theorem 1.1 will be shown in Section 3 (see Theorem 3.3).

2 Preliminaries

Firstly, we recall the notion of polynomial type which introduced by N.T. Cuong [5]. Let $\underline{a} = (a_1, \dots, a_d)$ be a system of parameters of M and $\underline{n} = (n_1, \dots, n_d)$ a d-tuple of positive integers. Set $\underline{a}(\underline{n}) = (a_1^{n_1}, \dots, a_d^{n_d})$ and

$$I(\underline{a}(\underline{n}); M) = \ell(M/(a_1^{n_1}, \dots, a_d^{n_d})M) - n_1 \dots n_d e(\underline{a}; M).$$

Then $I(\underline{a}(\underline{n}); M)$ can be considered as a function in $\underline{n}$. Note that this function is non-negative and ascending, i.e., $I(\underline{a}(\underline{n}); M) \geq I(\underline{a}(\underline{m}); M)$ for $\underline{n} = (n_1, \ldots, n_d)$, $\underline{m} = (m_1, \ldots, m_d)$ with $n_i \geq m_i$, $i = 1, \ldots, d$. This function is bounded above by a polynomial in $\underline{n}$. Moreover, we have the following important property.

Lemma 2.1. ([5, Theorem 2.3]) *The least degree of all polynomials in $\underline{n}$ bounding above the function $I(\underline{a}(\underline{n}); M)$ does not depend on the choice of $\underline{a}$.*

Definition 2.2. ([5, Definition 2.4]) The numerical invariant of M given in Theorem 2.1 is called the *polynomial type of M* and denote it by $p(M)$.

Lemma 2.3. ([5, Lemma 2.6]) *The polynomial type is preserved by $\mathfrak{m}$-adic completion, i.e., $p(M) = p(\widehat{M})$.*

Next, we recall the concept of principle of idealization introduced by M. Nagata [12]. We make the cartesian product $R \times M$ to become a commutative ring under the componentwise addition and the multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$. This ring is called the *idealization* of M over R and denoted by $R \ltimes M$.

Note that the idealization $R \ltimes M$ is again a Noetherian local ring with the unique maximal ideal $\mathfrak{m} \times M$ and $\dim R \ltimes M = \dim R$. Moreover the $\mathfrak{m} \times M$-adic completion $\widehat{R \ltimes M}$ of $R \ltimes M$ is naturally isomorphic to $\widehat{R} \ltimes \widehat{M}$, cf. [1]. In particular, $(0, x_1)(0, x_2) = (0, 0)$, for all $x_1, x_2 \in M$ and hence $0 \times M$ is an ideal whose square is zero. Furthermore $R \ltimes M / 0 \times M \cong R$.

There are a canonical projection $\rho : R \ltimes M \to R$ defined by $\rho((a, x)) = a$ and a canonical inclusion $\sigma : R \to R \ltimes M$ defined by $\sigma(a) = (a, 0)$. Note that ρ and σ are local homomorphisms and we can regard any R-module (resp. $R \ltimes M$-module) as an $R \ltimes M$-module (resp. R-module) by ρ (resp. σ). Moreover, the structure of R-modules induced by the composition $\rho\sigma$ coincides with the original one. Let $\epsilon : M \to R \ltimes M$ be the canonical inclusion defined by $\epsilon(x) = (0, x)$. Then we have an exact sequence of $R \ltimes M$-modules

$$0 \to M \xrightarrow{\epsilon} R \ltimes M \xrightarrow{\rho} R \to 0.$$

3 The proof of Theorem 1.1

Before proving the main result of this paper, we need to recall notions of canonical module and idealization. Let R be a quotient of a n-dimensional Gorenstein local ring $(R', \mathfrak{m}')$. We denote by $K^i(M) = \operatorname{Ext}_{R'}^{n-i}(M, R')$. Then $K^i(M)$ is a finitely generated R-module. Following P. Schenzel [16], $K^i(M)$ is called the i^{th} *deficiency module* of M for $i = 0, \ldots, d-1$, and $K(M) = K^d(M)$ is called the *canonical module* of M. By the local duality (cf. [2, 11.2.6]), we have an isomorphism

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(K^i(M), E(R/\mathfrak{m})),$$

for all i, where $E(R/\mathfrak{m})$ is the injective hull of $R/\mathfrak{m}$.

Definition 3.1. ([16], [10]) An R-module M is called a *Cohen-Macaulay canonical* (resp. *generalized Cohen-Macaulay canonical*) *module* if the canonical R-module $K(M)$ of M is Cohen-Macaulay (resp. generalized Cohen-Macaulay). If R itself is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) module then it is called a *Cohen-Macaulay canonical* (resp. *generalized Cohen-Macaulay canonical*) *ring.*

The Cohen-Macaulay canonical property is related to some important questions. For example, if M is Cohen-Macaulay canonical then the monomial conjecture raised by M. Hochster [8] is valid for the ring $R/\operatorname{Ann}_R M$. Furthermore, if R is a domain then R is Cohen-Macaulay canonical if and only if R posseses a birational Macaulayfication R_1, i.e. an extension ring $R \subseteq R_1 \subseteq Q$ (where Q is the field of fractions of R) such that R_1 is finitely generated as an R-module and R_1 is a Cohen-Macaulay ring, cf. [16, Theorem 1.1].

Remark 3.2. (i) It is easy to see that $\widehat{K(M)} \cong K(\widehat{M})$ as $\widehat{R}$-module. Therefore M is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) R-module if and only if $\widehat{M}$ is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) $\widehat{R}$-module.

(ii) Let E and F be R-modules. K. Yamagishi [17] extended the concept of the idealization as follows: given an R-linear map $\phi : M \otimes_R E \to F$, it can make the Cartesian product $E \times F$ into an $R \ltimes M$-module with respect to componentwise addition and multiplication defined by

$$(a, x)(e, f) = (ae, af + \phi(x \otimes e)).$$

We denote this $R \ltimes M$-module by $E \overset{\phi}{\ltimes} F$.

Theorem 3.3. *The following statements are true:*

(i) If $\dim M = \dim R$ *then* $p(K(R \ltimes M)) = \max\{p(K(R)), p(K(M))\}$*;*

(ii) If $\dim M < \dim R$ *then* $p(K(R \ltimes M)) = p(K(R))$*.*

Proof. Note that $\widehat{R} \ltimes \widehat{M}$ is isomorphic to the $\mathfrak{m} \times M$-adic completion of $R \ltimes M$. Moreover, the polynomial type is preserved by the completion, i.e. $p(K(M)) = p(K(\widehat{M}))$, $p(K(R)) = p(K(\widehat{R}))$ and $p(R \ltimes M) = p(\widehat{R} \ltimes \widehat{M}))$ (see Lemma 2.3). Therefore without any loss of generality, we may assume that R is complete with respect to $\mathfrak{m}$-adic completion.

Let $\mathfrak{Q}$ be an ideal of $R \ltimes M$ and put $\mathfrak{q} = \rho(\mathfrak{Q})$, where $\rho : R \ltimes M \to R$ is the map defined by $\rho(a, x) = a$ for all $(a, x) \in R \ltimes M$. Note that, $\mathfrak{Q}$ is $\mathfrak{m} \times M-$primary if and only if $\mathfrak{q}$ is $\mathfrak{m}$-primary, cf. [17, Remark 2.1].

Firstly, we claim the following fact.

Claim 1. *Let* $\mathfrak{q}$ *be an* $\mathfrak{m}$*-primary ideal of* R*. Then we have*

$$e(\mathfrak{q}; K(M)) = e(\mathfrak{q}; M).$$

Proof of Claim 1. To prove this claim, we need recall some notions and facts on multiplicities for Artinian module. Suppose that A is an Artinian

R-module. Let $\mathfrak{a}$ be an ideal of R such that $\ell(0:_A \mathfrak{a}) < \infty$. Then $\ell(0:_A \mathfrak{a}^n)$ is a polynomial with rational coefficients for $n \gg 0$. Since R is complete, the degree of this polynomial is equal to $t := \dim R/\mathrm{Ann}_R A$, cf. D. Kirby [9]. Following M. Brodmann and R. Y. Sharp [3], the multiplicity of A with respect to $\mathfrak{a}$, denoted by $e'(\mathfrak{a}; A)$, is defined by the formula $e'(\mathfrak{a}; A) = a_t t!$ where a_t is the leading coefficient of the polynomial $\ell(0:_A \mathfrak{a}^n)$.

Let $D(-)$ be the Matlis dual functor. Since A is Artinian and R is complete, $D(A)$ is a finitely generated R-module. Since $\ell_R(0:_A \mathfrak{a}) < \infty$ with notice that $D(0:_A \mathfrak{a}^n) \cong D(A)/\mathfrak{a}^n D(A)$, we have $\ell_R(0:_A \mathfrak{a}^n) = \ell_R(D(A)/\mathfrak{a}^n D(A))$ for all $n \in \mathbb{N}$. It follows that $e'(\mathfrak{a}; A) = e(\mathfrak{a}; D(A))$. Now, we apply this fact for the Artinian module $H^d_{\mathfrak{m}}(M)$ and the $\mathfrak{m}$-primary ideal $\mathfrak{q}$. As R is complete, we have $K(M) \cong D(H^d_{\mathfrak{m}}(M))$. Now we get

$$e'(\mathfrak{q}; H^d_{\mathfrak{m}}(M)) = e(\mathfrak{q}; K(M)).$$

For each integer $i \geq 0$, let $\mathrm{Psupp}^i_R(M) = \{\mathfrak{p} \in \mathrm{Spec} R \mid H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}$ be the i-th pseudo-support of M defined by M. Brodmann and R. Y. Sharp [3]. Then we get by [14, Corollary 3.4] that

$$e'(\mathfrak{q}; H^d_{\mathfrak{m}}(M)) = \sum_{\substack{\mathfrak{p} \in \mathrm{Psupp}^d_R(M) \\ \dim R/\mathfrak{p} = d}} \ell_{R_{\mathfrak{p}}}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})) e(\mathfrak{q}; R/\mathfrak{p}).$$

Since R is complete, R is catenary. Therefore, we get by [14, Corollary 3.4] that

$$\mathrm{Psupp}^d_R(M) = \{\mathfrak{p} \in \mathrm{Supp}(M) \mid \exists \mathfrak{p}' \in \mathrm{Ass}_R(M), \dim R/\mathfrak{p}' = d, \mathfrak{p}' \subseteq \mathfrak{p}\}.$$

Hence $\{\mathfrak{p} \in \mathrm{Psupp}^d_R(M) \mid \dim R/\mathfrak{p} = d\} = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \dim R/\mathfrak{p} = d\}$. So by the associativity formula for multiplicity of M with respect to $\mathfrak{q}$, cf. [11, 14.7], we have

$$\begin{aligned} e'(\mathfrak{q}; H^d_{\mathfrak{m}}(M)) &= \sum_{\substack{\mathfrak{p} \in \mathrm{Supp}_R(M) \\ \dim R/\mathfrak{p} = d}} \ell_{R_{\mathfrak{p}}}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})) e(\mathfrak{q}; R/\mathfrak{p}) \\ &= \sum_{\substack{\mathfrak{p} \in \mathrm{Supp}_R(M) \\ \dim R/\mathfrak{p} = d}} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) e(\mathfrak{q}; R/\mathfrak{p}) \\ &= e(\mathfrak{q}; M). \end{aligned}$$

Therefore $e(\mathfrak{q}; K(M)) = e(\mathfrak{q}; M)$, the claim is proved.

From now on, let $\underline{a} = (a_1, \ldots, a_r)$ be a system of parameters of R. Set $\underline{u} = (u_1, \ldots, u_r)$ with $u_i = (a_i, 0)$ for $i = 1, \ldots, r$. It is easy to see that $\underline{u}$ is a system of parameters of $R \ltimes M$. Set $\mathfrak{q} = (a_1, \ldots, a_r)R$ and $\mathfrak{Q} = \sum_{i=1}^{r} u_i(R \ltimes M) \subseteq R \ltimes M$. Then $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of R and $\mathfrak{Q}$ is an $\mathfrak{m} \times M$-primary ideal of $R \ltimes M$. Moreover $\mathfrak{Q} = \mathfrak{q} \times \mathfrak{q}M$ and $\mathfrak{q} = \rho(\mathfrak{Q})$.

Claim 2. *With the above notations, if* $d = r$ *(i.e.* $\dim M = \dim R$*) then*

$$\ell_{R\ltimes M}(K(R\ltimes M)/\mathfrak{Q}K(R\ltimes M)) = \ell_R(K(R)/\mathfrak{q}K(R)) + \ell_R(K(M)/\mathfrak{q}K(M)).$$

Otherwise, we have

$$\ell_{R\ltimes M}(K(R\ltimes M)/\mathfrak{Q}K(R\ltimes M)) = \ell_R(K(R)/\mathfrak{q}K(R)).$$

Proof of Claim 2. By [7, 5.14], we have an isomorphism

$$K(R\ltimes M) \cong \mathrm{Hom}_R(R\ltimes M, K(R))$$

of $R\ltimes M$-modules. Moreover, there is an isomorphism of R-modules

$$\mathrm{Hom}_R(R\ltimes M, K(R)) \to \mathrm{Hom}_R(M, K(R)) \oplus K(R)$$

defined by $\alpha \mapsto (\alpha\epsilon, \alpha((1,0)))$ for each $\alpha \in \mathrm{Hom}_R(R\ltimes M, K(R))$, where $\epsilon : M \to R\ltimes M$ is defined by $\epsilon(x) = (0, x)$ for all $x \in M$. Then by Remark 3.2, (ii) we can make the R-module $\mathrm{Hom}_R(M, K(R))\oplus K(R)$ into an $R\ltimes M$-module, which is denoted by

$$\mathrm{Hom}_R(M, K(R))\overset{\phi}{\ltimes} K(R)$$

with respect to the R-linear map $\phi : M \otimes_R \mathrm{Hom}_R(M, K(R)) \to K(R)$ such that $\phi(x \otimes f) = f(x)$ for every $x \in M$ and $f \in \mathrm{Hom}_R(M, K(R))$. Therefore

$$K(R\ltimes M) \cong \mathrm{Hom}_R(M, K(R))\overset{\phi}{\ltimes} K(R)$$

as $R\ltimes M$-modules. By [4, 3.5.10], there is an isomorphism

$$\mathrm{Hom}_R(H^r_{\mathfrak{m}}(M), E_R(R/\mathfrak{m})) \cong \mathrm{Hom}_R(M, K(R))$$

of R-modules. Now, suppose $d = r$. Then $\mathrm{Hom}_R(H^r_{\mathfrak{m}}(M), E_R(R/\mathfrak{m})) \cong K(M)$ as R is complete, and hence $\mathrm{Hom}_R(M, K(R)) \cong K(M)$. Therefore, we get an isomorphism $K(R\ltimes M) \cong K(M)\overset{\phi}{\ltimes} K(R)$ as $R\ltimes M$-modules. It follows that

$$\begin{aligned}
\mathfrak{Q}K(R\ltimes M) &\cong (\mathfrak{q} \times \mathfrak{q}M)\big(K(M)\overset{\phi}{\ltimes} K(R)\big)\\
&\cong \mathfrak{q}K(M) \times \big(\mathfrak{q}K(R) + \phi(\mathfrak{q}M \otimes K(M))\big)\\
&\cong \mathfrak{q}K(M) \times \big(\mathfrak{q}K(R) + \mathfrak{q}\phi(M \otimes K(M))\big)\\
&\cong \mathfrak{q}K(M) \times \big(\mathfrak{q}K(R) + \mathfrak{q}\mathrm{Im}\phi\big)\\
&\cong \mathfrak{q}K(M) \times \mathfrak{q}K(R).
\end{aligned}$$

Then we obtain that

$$\begin{aligned}
\ell_{R\ltimes M}(K(R\ltimes M)/\mathfrak{Q}K(R\ltimes M)) &= \ell_{R\ltimes M}((K(M)\overset{\phi}{\ltimes} K(R))/(\mathfrak{q}K(M) \times \mathfrak{q}K(R)))\\
&= \ell_R(K(M)/\mathfrak{q}K(M)) + \ell_R(K(R)/\mathfrak{q}K(R)).
\end{aligned}$$

Suppose that $d < r$. Then $\mathrm{Hom}_R(H^r_{\mathfrak{m}}(M), E_R(R/\mathfrak{m})) = 0$. Therefore we have $\mathrm{Hom}_R(M, K(R)) = 0$. It follows that $K(R\ltimes M) \cong 0 \overset{\phi}{\ltimes} K(R)$ as $R\ltimes M$-modules and therefore $\mathfrak{Q}K(R\ltimes M) \cong 0 \times \mathfrak{q}K(R)$. Then we get that

$$\begin{aligned}\ell_{R\ltimes M}\big(K(R\ltimes M)/\mathfrak{Q}K(R\ltimes M)\big) &= \ell_{R\ltimes M}\big(0\overset{\phi}{\ltimes}K(R)/0 \times \mathfrak{q}K(R)\big)\\ &= \ell_R\big(K(R)/\mathfrak{q}K(R)\big),\end{aligned}$$

and Claim 2 is proved.

Now, we consider the exact sequence

$$0 \to M \overset{\epsilon}{\to} R\ltimes M \overset{\rho}{\to} R \to 0.$$

If $d = r$ then $e(\mathfrak{Q}; R\ltimes M) = e(\mathfrak{q}; R) + e(\mathfrak{q}; M)$, and therefore

$$e(\mathfrak{Q}; K(R\ltimes M)) = e(\mathfrak{q}; K(R)) + e(\mathfrak{q}; K(M))$$

by Claim 1. On the other hand, if $d < r$ then $e(\mathfrak{Q}; K(R\ltimes M)) = e(\mathfrak{q}; R) = e(\mathfrak{q}; K(R))$ by Claim 1.

Let $\underline{n} = (n_1, \ldots, n_r)$ be a set of positive integers, let $\underline{a}(\underline{n}) := (a_1^{n_1}, \ldots, a_r^{n_r})$ and $\underline{u}(\underline{n}) := (u_1^{n_1}, \ldots, u_r^{n_r}) = ((a_1^{n_1}, 0), \ldots, (a_r^{n_r}, 0))$. Set $\mathfrak{Q}(\underline{n}) = \sum_{i=1}^{r} u_i^{n_i}(R\ltimes M) \subseteq R\ltimes M$ and $\mathfrak{q}(\underline{n}) = \underline{a}(\underline{n})R$.

(i) If $d = r$ then $\underline{a}(\underline{n})$ is a system of parameters of R, $K(R), M$ and $K(M)$. Moreover, $\underline{u}(\underline{n})$ is a system of parameters of $R\ltimes M$ and $K(R\ltimes M)$. Therefore we get by Claim 2 and the above facts that

$$I(\mathfrak{Q}(\underline{n}); K(R\ltimes M)) = I(\mathfrak{q}(\underline{n}); K(R)) + I(\mathfrak{q}(\underline{n}); K(M)).$$

So, we get by Lemma 2.1 that

$$p(K(R\ltimes M)) = \max\{p(K(R)), p(K(M))\}.$$

(ii) Suppose $d < r$. Then by Claim 2 with notice that $e(\mathfrak{Q}; K(R\ltimes M)) = e(\mathfrak{q}; K(R))$ we obtain

$$I(\mathfrak{Q}; K(R\ltimes M)) = I(\mathfrak{q}; K(R)).$$

Thus $p(K(R\ltimes M)) = p(K(R))$ by Lemma 2.1. □

Note that M is Cohen-Macaulay if and only if $p(M) = -\infty$ and M is generalized Cohen-Macaulay if and only if $p(M) \leqslant 0$. Therefore we have the following characterization for $R\ltimes M$ being Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay).

Corollary 3.4. *The following statements are true:*

(i) If $\dim M = \dim R$ *then* $R\ltimes M$ *is Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) if and only if so are* R *and* M.

(ii) If $\dim M < \dim R$ *then* $R\ltimes M$ *is Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) if and only if so is* R.

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