

On canonical modules of idealizations

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Abstracts. ¹ Let (R, \mathfrak{m}) be a Noetherian local ring which is a quotient of a Gorenstein local ring. Let M be a finitely generated R -module. In this paper, we study the structure of the canonical module $K(R \times M)$ of the idealization $R \times M$ via the polynomial type introduced by N. T. Cuong [5]. In particular, we give a characterization for $K(R \times M)$ being Cohen-Macaulay and generalized Cohen-Macaulay.

1 Introduction

Throughout this paper, (R, \mathfrak{m}) denotes an r -dimensional Noetherian local ring with maximal ideal \mathfrak{m} and M a finitely generated R -module with dimension d . The concept of principle of idealization was introduced by M. Nagata [12]. In the cartesian product $R \times M$, we introduce the componentwise addition and the multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$. These operations give a structure of a commutative ring to $R \times M$. This ring is called the *idealization of M* and denoted by $R \times M$. The purpose of idealization is to put M inside the commutative ring $R \times M$ so that the structure of M as an R -module is essentially the same as that of M as an ideal of $R \times M$. The notion of principle of idealization plays an important role in the study of Noetherian rings and modules. Idealization is useful for reducing results concerning submodules to the ideal case; generalizing results from rings to modules and constructing examples of commutative rings with zero divisors, cf. [1], [12], [17].

The notion of a canonical module of a Noetherian local ring is due to A. Grothendieck, who called it a module of dualizing differentials (cf. [6]). The term “a canonical module” was first adopted by J. Herzog, E. Kunz et al. [7], in which they defined the notion of a canonical module for general local rings. We note that a local ring R has a canonical module if and only if R is a

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homomorphic image of a Gorenstein local ring. P. Schenzel [15] has introduced the canonical module $K(M)$ of an R -module M .

The polynomial type introduced by N. T. Cuong [5] makes an important role in the study of finitely generated modules, cf. [5]. Let $\underline{a} = (a_1, \dots, a_d)$ be a system of parameters of M and $\underline{n} = (n_1, \dots, n_d)$ a d -tuple of positive integers. Set $\underline{a}(\underline{n}) = (a_1^{n_1}, \dots, a_d^{n_d})$. Then the difference between length and multiplicity $I(\underline{a}(\underline{n}); M) = \ell(M/(a_1^{n_1}, \dots, a_d^{n_d})M) - n_1 \dots n_d e(\underline{a}; M)$ can be considered as a function in \underline{n} . It is well-known that M is Cohen-Macaulay (resp. generalized Cohen-Macaulay) if and only if $I(\underline{a}(\underline{n}); M) = 0$ (resp. there exists a constant C such that $I(\underline{a}(\underline{n}); M) \leq C$) for all \underline{a} and \underline{n} . In general, $I(\underline{a}(\underline{n}); M)$ is not a polynomial for $n_1, \dots, n_d \gg 0$, but it takes non negative values and bounded above by polynomials. The least degree of all polynomials bounding above this function does not depend on the choice of \underline{a} , cf. [5, Theorem 2.3]. This least degree is called *the polynomial type of M* and denoted by $p(M)$. It should be mentioned that $p(M)$ gives a lot of information on the structure of M . For example, if we stipulate the degree of the zero polynomial to be $-\infty$ then M is Cohen-Macaulay if and only if $p(M) = -\infty$, and M is generalized Cohen-Macaulay if and only if $p(M) \leq 0$. We denote by \widehat{R} and \widehat{M} the \mathfrak{m} -adic completion of R and M respectively. In general, $p(M) = p(\widehat{M}) = \max_{i < d} \dim \widehat{R}/\text{Ann}_{\widehat{R}} H_{\mathfrak{m}}^i(\widehat{M})$. And if R is a quotient of a Gorenstein local ring and M is equidimensional then $p(M) = \dim \text{nCM}(M)$, cf. [5, Theorem 3.1, 3.3], where $\text{nCM}(M)$ is the non Cohen-Macaulay locus of M .

The purpose of this paper is to study the polynomial type of the canonical module of the idealization $R \times M$. Especially, we give a criterion for the canonical module $K(R \times M)$ being Cohen-Macaulay (resp. generalized Cohen-Macaulay). Techniques used in this paper are the associativity formula of multiplicity of local cohomology modules given by M. Brodmann and R.Y. Sharp [3] (see also [14]) and the extension of idealization introduced by K. Yamagishi [17]. The main result of this paper is the following theorem.

Theorem 1.1. *The following statements are true:*

- (i) *If $\dim M = \dim R$ then $p(K(R \times M)) = \max\{p(K(R)), p(K(M))\}$;*
- (ii) *If $\dim M < \dim R$ then $p(K(R \times M)) = p(K(R))$.*

In Section 2, we shall outline some properties of polynomial type and idealization which will be needed later. The proof of Theorem 1.1 will be shown in Section 3 (see Theorem 3.3).

2 Preliminaries

Firstly, we recall the notion of polynomial type which introduced by N.T. Cuong [5]. Let $\underline{a} = (a_1, \dots, a_d)$ be a system of parameters of M and $\underline{n} = (n_1, \dots, n_d)$ a d -tuple of positive integers. Set $\underline{a}(\underline{n}) = (a_1^{n_1}, \dots, a_d^{n_d})$ and

$$I(\underline{a}(\underline{n}); M) = \ell(M/(a_1^{n_1}, \dots, a_d^{n_d})M) - n_1 \dots n_d e(\underline{a}; M).$$

Then $I(\underline{a}(\underline{n}); M)$ can be considered as a function in \underline{n} . Note that this function is non-negative and ascending, i.e., $I(\underline{a}(\underline{n}); M) \geq I(\underline{a}(\underline{m}); M)$ for $\underline{n} = (n_1, \dots, n_d)$, $\underline{m} = (m_1, \dots, m_d)$ with $n_i \geq m_i$, $i = 1, \dots, d$. This function is bounded above by a polynomial in \underline{n} . Moreover, we have the following important property.

Lemma 2.1. ([5, Theorem 2.3]) *The least degree of all polynomials in \underline{n} bounding above the function $I(\underline{a}(\underline{n}); M)$ does not depend on the choice of \underline{a} .*

Definition 2.2. ([5, Definition 2.4]) The numerical invariant of M given in Theorem 2.1 is called the *polynomial type* of M and denote it by $p(M)$.

Lemma 2.3. ([5, Lemma 2.6]) *The polynomial type is preserved by \mathfrak{m} -adic completion, i.e., $p(M) = p(\widehat{M})$.*

Next, we recall the concept of principle of idealization introduced by M. Nagata [12]. We make the cartesian product $R \times M$ to become a commutative ring under the componentwise addition and the multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$. This ring is called the *idealization* of M over R and denoted by $R \times M$.

Note that the idealization $R \times M$ is again a Noetherian local ring with the unique maximal ideal $\mathfrak{m} \times M$ and $\dim R \times M = \dim R$. Moreover the $\mathfrak{m} \times M$ -adic completion $\widehat{R \times M}$ of $R \times M$ is naturally isomorphic to $\widehat{R} \times \widehat{M}$, cf. [1]. In particular, $(0, x_1)(0, x_2) = (0, 0)$, for all $x_1, x_2 \in M$ and hence $0 \times M$ is an ideal whose square is zero. Furthermore $R \times M / 0 \times M \cong R$.

There are a canonical projection $\rho : R \times M \rightarrow R$ defined by $\rho((a, x)) = a$ and a canonical inclusion $\sigma : R \rightarrow R \times M$ defined by $\sigma(a) = (a, 0)$. Note that ρ and σ are local homomorphisms and we can regard any R -module (resp. $R \times M$ -module) as an $R \times M$ -module (resp. R -module) by ρ (resp. σ). Moreover, the structure of R -modules induced by the composition $\rho\sigma$ coincides with the original one. Let $\epsilon : M \rightarrow R \times M$ be the canonical inclusion defined by $\epsilon(x) = (0, x)$. Then we have an exact sequence of $R \times M$ -modules

$$0 \rightarrow M \xrightarrow{\epsilon} R \times M \xrightarrow{\rho} R \rightarrow 0.$$

3 The proof of Theorem 1.1

Before proving the main result of this paper, we need to recall notions of canonical module and idealization. Let R be a quotient of a n -dimensional Gorenstein local ring (R', \mathfrak{m}') . We denote by $K^i(M) = \text{Ext}_{R'}^{n-i}(M, R')$. Then $K^i(M)$ is a finitely generated R -module. Following P. Schenzel [16], $K^i(M)$ is called the i^{th} *deficiency module* of M for $i = 0, \dots, d-1$, and $K(M) = K^d(M)$ is called the *canonical module* of M . By the local duality (cf. [2, 11.2.6]), we have an isomorphism

$$H_{\mathfrak{m}}^i(M) \cong \text{Hom}_R(K^i(M), E(R/\mathfrak{m})),$$

for all i , where $E(R/\mathfrak{m})$ is the injective hull of R/\mathfrak{m} .

Definition 3.1. ([16], [10]) An R -module M is called a *Cohen-Macaulay canonical* (resp. *generalized Cohen-Macaulay canonical*) module if the canonical R -module $K(M)$ of M is Cohen-Macaulay (resp. generalized Cohen-Macaulay). If R itself is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) module then it is called a *Cohen-Macaulay canonical* (resp. *generalized Cohen-Macaulay canonical*) ring.

The Cohen-Macaulay canonical property is related to some important questions. For example, if M is Cohen-Macaulay canonical then the monomial conjecture raised by M. Hochster [8] is valid for the ring $R/\text{Ann}_R M$. Furthermore, if R is a domain then R is Cohen-Macaulay canonical if and only if R possesses a birational Macaulayfication R_1 , i.e. an extension ring $R \subseteq R_1 \subseteq Q$ (where Q is the field of fractions of R) such that R_1 is finitely generated as an R -module and R_1 is a Cohen-Macaulay ring, cf. [16, Theorem 1.1].

Remark 3.2. (i) It is easy to see that $\widehat{K(M)} \cong K(\widehat{M})$ as \widehat{R} -module. Therefore M is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) R -module if and only if \widehat{M} is a Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) \widehat{R} -module.

(ii) Let E and F be R -modules. K. Yamagishi [17] extended the concept of the idealization as follows: given an R -linear map $\phi : M \otimes_R E \rightarrow F$, it can make the Cartesian product $E \times F$ into an $R \times M$ -module with respect to componentwise addition and multiplication defined by

$$(a, x)(e, f) = (ae, af + \phi(x \otimes e)).$$

We denote this $R \times M$ -module by $E \times_{\phi} F$.

Theorem 3.3. *The following statements are true:*

- (i) *If $\dim M = \dim R$ then $p(K(R \times M)) = \max\{p(K(R)), p(K(M))\}$;*
- (ii) *If $\dim M < \dim R$ then $p(K(R \times M)) = p(K(R))$.*

Proof. Note that $\widehat{R \times M}$ is isomorphic to the $\mathfrak{m} \times M$ -adic completion of $R \times M$. Moreover, the polynomial type is preserved by the completion, i.e. $p(K(M)) = p(K(\widehat{M}))$, $p(K(R)) = p(K(\widehat{R}))$ and $p(R \times M) = p(\widehat{R \times M})$ (see Lemma 2.3). Therefore without any loss of generality, we may assume that R is complete with respect to \mathfrak{m} -adic completion.

Let \mathfrak{Q} be an ideal of $R \times M$ and put $\mathfrak{q} = \rho(\mathfrak{Q})$, where $\rho : R \times M \rightarrow R$ is the map defined by $\rho(a, x) = a$ for all $(a, x) \in R \times M$. Note that, \mathfrak{Q} is $\mathfrak{m} \times M$ -primary if and only if \mathfrak{q} is \mathfrak{m} -primary, cf. [17, Remark 2.1].

Firstly, we claim the following fact.

Claim 1. *Let \mathfrak{q} be an \mathfrak{m} -primary ideal of R . Then we have*

$$e(\mathfrak{q}; K(M)) = e(\mathfrak{q}; M).$$

Proof of Claim 1. To prove this claim, we need recall some notions and facts on multiplicities for Artinian module. Suppose that A is an Artinian

R -module. Let \mathfrak{a} be an ideal of R such that $\ell(0 :_A \mathfrak{a}) < \infty$. Then $\ell(0 :_A \mathfrak{a}^n)$ is a polynomial with rational coefficients for $n \gg 0$. Since R is complete, the degree of this polynomial is equal to $t := \dim R/\text{Ann}_R A$, cf. D. Kirby [9]. Following M. Brodmann and R. Y. Sharp [3], the multiplicity of A with respect to \mathfrak{a} , denoted by $e'(\mathfrak{a}; A)$, is defined by the formula $e'(\mathfrak{a}; A) = a_t t!$ where a_t is the leading coefficient of the polynomial $\ell(0 :_A \mathfrak{a}^n)$.

Let $D(-)$ be the Matlis dual functor. Since A is Artinian and R is complete, $D(A)$ is a finitely generated R -module. Since $\ell_R(0 :_A \mathfrak{a}) < \infty$ with notice that $D(0 :_A \mathfrak{a}^n) \cong D(A)/\mathfrak{a}^n D(A)$, we have $\ell_R(0 :_A \mathfrak{a}^n) = \ell_R(D(A)/\mathfrak{a}^n D(A))$ for all $n \in \mathbb{N}$. It follows that $e'(\mathfrak{a}; A) = e(\mathfrak{a}; D(A))$. Now, we apply this fact for the Artinian module $H_{\mathfrak{m}}^d(M)$ and the \mathfrak{m} -primary ideal \mathfrak{q} . As R is complete, we have $K(M) \cong D(H_{\mathfrak{m}}^d(M))$. Now we get

$$e'(\mathfrak{q}; H_{\mathfrak{m}}^d(M)) = e(\mathfrak{q}; K(M)).$$

For each integer $i \geq 0$, let $\text{Psupp}_R^i(M) = \{\mathfrak{p} \in \text{Spec} R \mid H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-\dim R/\mathfrak{p}}(M_{\mathfrak{p}}) \neq 0\}$ be the i -th pseudo-support of M defined by M. Brodmann and R. Y. Sharp [3]. Then we get by [14, Corollary 3.4] that

$$e'(\mathfrak{q}; H_{\mathfrak{m}}^d(M)) = \sum_{\substack{\mathfrak{p} \in \text{Psupp}_R^d(M) \\ \dim R/\mathfrak{p}=d}} \ell_{R_{\mathfrak{p}}}(H_{\mathfrak{p}R_{\mathfrak{p}}}^0(M_{\mathfrak{p}}))e(\mathfrak{q}; R/\mathfrak{p}).$$

Since R is complete, R is catenary. Therefore, we get by [14, Corollary 3.4] that

$$\text{Psupp}_R^d(M) = \{\mathfrak{p} \in \text{Supp}(M) \mid \exists \mathfrak{p}' \in \text{Ass}_R(M), \dim R/\mathfrak{p}' = d, \mathfrak{p}' \subseteq \mathfrak{p}\}.$$

Hence $\{\mathfrak{p} \in \text{Psupp}_R^d(M) \mid \dim R/\mathfrak{p} = d\} = \{\mathfrak{p} \in \text{Supp}_R M \mid \dim R/\mathfrak{p} = d\}$. So by the associativity formula for multiplicity of M with respect to \mathfrak{q} , cf. [11, 14.7], we have

$$\begin{aligned} e'(\mathfrak{q}; H_{\mathfrak{m}}^d(M)) &= \sum_{\substack{\mathfrak{p} \in \text{Supp}_R(M) \\ \dim R/\mathfrak{p}=d}} \ell_{R_{\mathfrak{p}}}(H_{\mathfrak{p}R_{\mathfrak{p}}}^0(M_{\mathfrak{p}}))e(\mathfrak{q}; R/\mathfrak{p}) \\ &= \sum_{\substack{\mathfrak{p} \in \text{Supp}_R(M) \\ \dim R/\mathfrak{p}=d}} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})e(\mathfrak{q}; R/\mathfrak{p}) \\ &= e(\mathfrak{q}; M). \end{aligned}$$

Therefore $e(\mathfrak{q}; K(M)) = e(\mathfrak{q}; M)$, the claim is proved.

From now on, let $\underline{a} = (a_1, \dots, a_r)$ be a system of parameters of R . Set $\underline{u} = (u_1, \dots, u_r)$ with $u_i = (a_i, 0)$ for $i = 1, \dots, r$. It is easy to see that \underline{u} is a system of parameters of $R \times M$. Set $\mathfrak{q} = (a_1, \dots, a_r)R$ and $\mathfrak{Q} = \sum_{i=1}^r u_i(R \times M) \subseteq R \times M$. Then \mathfrak{q} is an \mathfrak{m} -primary ideal of R and \mathfrak{Q} is an $\mathfrak{m} \times M$ -primary ideal of $R \times M$. Moreover $\mathfrak{Q} = \mathfrak{q} \times \mathfrak{q}M$ and $\mathfrak{q} = \rho(\mathfrak{Q})$.

Claim 2. *With the above notations, if $d = r$ (i.e. $\dim M = \dim R$) then*

$$\ell_{R \times M}(K(R \times M)/\Omega K(R \times M)) = \ell_R(K(R)/\mathfrak{q}K(R)) + \ell_R(K(M)/\mathfrak{q}K(M)).$$

Otherwise, we have

$$\ell_{R \times M}(K(R \times M)/\Omega K(R \times M)) = \ell_R(K(R)/\mathfrak{q}K(R)).$$

Proof of Claim 2. By [7, 5.14], we have an isomorphism

$$K(R \times M) \cong \text{Hom}_R(R \times M, K(R))$$

of $R \times M$ -modules. Moreover, there is an isomorphism of R -modules

$$\text{Hom}_R(R \times M, K(R)) \rightarrow \text{Hom}_R(M, K(R)) \oplus K(R)$$

defined by $\alpha \mapsto (\alpha\epsilon, \alpha((1, 0)))$ for each $\alpha \in \text{Hom}_R(R \times M, K(R))$, where $\epsilon : M \rightarrow R \times M$ is defined by $\epsilon(x) = (0, x)$ for all $x \in M$. Then by Remark 3.2, (ii) we can make the R -module $\text{Hom}_R(M, K(R)) \oplus K(R)$ into an $R \times M$ -module, which is denoted by

$$\text{Hom}_R(M, K(R)) \overset{\phi}{\times} K(R)$$

with respect to the R -linear map $\phi : M \otimes_R \text{Hom}_R(M, K(R)) \rightarrow K(R)$ such that $\phi(x \otimes f) = f(x)$ for every $x \in M$ and $f \in \text{Hom}_R(M, K(R))$. Therefore

$$K(R \times M) \cong \text{Hom}_R(M, K(R)) \overset{\phi}{\times} K(R)$$

as $R \times M$ -modules. By [4, 3.5.10], there is an isomorphism

$$\text{Hom}_R(H_{\mathfrak{m}}^r(M), E_R(R/\mathfrak{m})) \cong \text{Hom}_R(M, K(R))$$

of R -modules. Now, suppose $d = r$. Then $\text{Hom}_R(H_{\mathfrak{m}}^r(M), E_R(R/\mathfrak{m})) \cong K(M)$ as R is complete, and hence $\text{Hom}_R(M, K(R)) \cong K(M)$. Therefore, we get an isomorphism $K(R \times M) \cong K(M) \overset{\phi}{\times} K(R)$ as $R \times M$ -modules. It follows that

$$\begin{aligned} \Omega K(R \times M) &\cong (\mathfrak{q} \times \mathfrak{q}M)(K(M) \overset{\phi}{\times} K(R)) \\ &\cong \mathfrak{q}K(M) \times (\mathfrak{q}K(R) + \phi(\mathfrak{q}M \otimes K(M))) \\ &\cong \mathfrak{q}K(M) \times (\mathfrak{q}K(R) + \mathfrak{q}\phi(M \otimes K(M))) \\ &\cong \mathfrak{q}K(M) \times (\mathfrak{q}K(R) + \mathfrak{q}\text{Im}\phi) \\ &\cong \mathfrak{q}K(M) \times \mathfrak{q}K(R). \end{aligned}$$

Then we obtain that

$$\begin{aligned} \ell_{R \times M}(K(R \times M)/\Omega K(R \times M)) &= \ell_{R \times M}((K(M) \overset{\phi}{\times} K(R))/(\mathfrak{q}K(M) \times \mathfrak{q}K(R))) \\ &= \ell_R(K(M)/\mathfrak{q}K(M)) + \ell_R(K(R)/\mathfrak{q}K(R)). \end{aligned}$$

Suppose that $d < r$. Then $\text{Hom}_R(H_{\mathfrak{m}}^r(M), E_R(R/\mathfrak{m})) = 0$. Therefore we have $\text{Hom}_R(M, K(R)) = 0$. It follows that $K(R \times M) \cong 0 \times K(R)$ as $R \times M$ -modules and therefore $\mathfrak{Q}K(R \times M) \cong 0 \times \mathfrak{q}K(R)$. Then we get that

$$\begin{aligned} \ell_{R \times M}(K(R \times M)/\mathfrak{Q}K(R \times M)) &= \ell_{R \times M}(0 \times K(R)/0 \times \mathfrak{q}K(R)) \\ &= \ell_R(K(R)/\mathfrak{q}K(R)), \end{aligned}$$

and Claim 2 is proved.

Now, we consider the exact sequence

$$0 \rightarrow M \xrightarrow{\epsilon} R \times M \xrightarrow{\rho} R \rightarrow 0.$$

If $d = r$ then $e(\mathfrak{Q}; R \times M) = e(\mathfrak{q}; R) + e(\mathfrak{q}; M)$, and therefore

$$e(\mathfrak{Q}; K(R \times M)) = e(\mathfrak{q}; K(R)) + e(\mathfrak{q}; K(M))$$

by Claim 1. On the other hand, if $d < r$ then $e(\mathfrak{Q}; K(R \times M)) = e(\mathfrak{q}; R) = e(\mathfrak{q}; K(R))$ by Claim 1.

Let $\underline{n} = (n_1, \dots, n_r)$ be a set of positive integers, let $\underline{a}(\underline{n}) := (a_1^{n_1}, \dots, a_r^{n_r})$ and $\underline{u}(\underline{n}) := (u_1^{n_1}, \dots, u_r^{n_r}) = ((a_1^{n_1}, 0), \dots, (a_r^{n_r}, 0))$. Set $\mathfrak{Q}(\underline{n}) = \sum_{i=1}^r u_i^{n_i}(R \times M) \subseteq R \times M$ and $\mathfrak{q}(\underline{n}) = \underline{a}(\underline{n})R$.

(i) If $d = r$ then $\underline{a}(\underline{n})$ is a system of parameters of R , $K(R)$, M and $K(M)$. Moreover, $\underline{u}(\underline{n})$ is a system of parameters of $R \times M$ and $K(R \times M)$. Therefore we get by Claim 2 and the above facts that

$$I(\mathfrak{Q}(\underline{n}); K(R \times M)) = I(\mathfrak{q}(\underline{n}); K(R)) + I(\mathfrak{q}(\underline{n}); K(M)).$$

So, we get by Lemma 2.1 that

$$p(K(R \times M)) = \max\{p(K(R)), p(K(M))\}.$$

(ii) Suppose $d < r$. Then by Claim 2 with notice that $e(\mathfrak{Q}; K(R \times M)) = e(\mathfrak{q}; K(R))$ we obtain

$$I(\mathfrak{Q}; K(R \times M)) = I(\mathfrak{q}; K(R)).$$

Thus $p(K(R \times M)) = p(K(R))$ by Lemma 2.1. \square

Note that M is Cohen-Macaulay if and only if $p(M) = -\infty$ and M is generalized Cohen-Macaulay if and only if $p(M) \leq 0$. Therefore we have the following characterization for $R \times M$ being Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay).

Corollary 3.4. *The following statements are true:*

(i) *If $\dim M = \dim R$ then $R \times M$ is Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) if and only if so are R and M .*

(ii) *If $\dim M < \dim R$ then $R \times M$ is Cohen-Macaulay canonical (resp. generalized Cohen-Macaulay canonical) if and only if so is R .*

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