ON A NON-EXISTENCE RESULT FOR EINSTEIN CONSTRAINT-TYPE SYSTEMS WITH NON-POSITIVE YAMABE CONSTANT

QUỐC ANH NGÔ

ABSTRACT. We extend a recent non-existence result for an Einstein constraint-type system due to Dahl–Gicquaud–Humbert (*Class. Quantum Grav.* **30**, 075004) from non-vanishing, non-positive metrics to metrics with non-positive Yamabe constant.

1. INTRODUCTION

Of importance in general theory of relativity is the study of the initial value problem. Mathematically, this problem asks: *for given a Riemannian manifold (M, g) of dimension n, whether we can construct a globally hyperbolic spacetime $(\mathscr{M}, \overline{g})$ of dimension $n+1$ in which (M, g) can be embedded as a spacelike hypersurface such that the vacuum Einstein equation*

$$\mathrm{Ric}_{\overline{g}} - \frac{1}{2}\overline{g}\,\mathrm{Scal}_{\overline{g}} = 0 \tag{E}$$

holds. Consequently, it is natural to ask: (a) how could we construct a spacetime $(\mathscr{M}, \overline{g})$ from a suitable spacelike $(M, \widehat{g})$ and (b) under what conditions for $(M, \widehat{g})$ the answer for part (a) is positive. To answer part (a), we observe from the Gauss and Codazzi equations that the "induced" metric $\widehat{g}$ and the second fundamental form $\widehat{K}$ of the embedding must satisfy the so-called constraint equations. These equations can be formulated as follows

$$\begin{cases} \mathrm{Scal}_{\widehat{g}} + (\mathrm{tr}_{\widehat{g}}\, \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}} = 0, & (1.1a) \\ \mathrm{div}_{\widehat{g}}\, \widehat{K} - d(\mathrm{tr}_{\widehat{g}}\, \widehat{K}) = 0; & (1.1b) \end{cases}$$

see [BI04]. Concerning to the part (a), the answer is affirmative by the two celebrated papers by Choquet-Bruhat [FB52] and Choquet-Bruhat–Geroch [CBG69]. In these papers, the authors proved that if the pair $(\widehat{g}, \widehat{K})$ solves (1.1), then the Cauchy problem is well-posed and there do exist a unique, maximal spacetime $(\mathscr{M}, \overline{g})$ such that (E) holds. To answer the part (b), we first observe that (1.1) forms an under-determined system; hence it is difficult to solve (1.1).

The most efficient way to look for solutions of (1.1) is to make use of the conformal method developed by Lichnerowicz [Lic44] and Choquet-Bruhat and York [CBY80]. The idea of the conformal method is as follows: First we look for $\widehat{g} = \phi^{4/(n-2}g$ in the conformal class represented by some fixed metric g. Then we split $\widehat{K} = \tau\widehat{g}/n + \phi^{-2}(\widehat{\sigma} + \mathbb{L}W)$ where τ is the mean curvature of the hypersurface M, $\widehat{\sigma}$ is some trace-free and divergence-free symmetric 2-tensor on M, and W is a vector field on M. Note that in the decomposition of $\widehat{K}$ above, by $\mathbb{L}$ we mean the Killing operator acting on vector fields through

$$\mathbb{L}W_{ij} := \nabla_i W_j + \nabla_j W_i - \frac{2}{n}\nabla^k W_k g_{ij},$$

Date: **22nd Mar, 2016 at 13:09.**

Key words and phrases. Einstein constraint equations, non-CMC, conformal method, non-positive Yamabe invariant, non-existence result.

where ∇ is the Levi–Civita connection associated to the background metric g. In terms of (ϕ, W), we can reformulate (1.1) into the following system

$$
\left\{
\begin{aligned}
-\frac{4(n-1)}{n-2}\Delta_g\phi + \mathrm{Scal}_g\,\phi &= -\frac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} + |\sigma + \mathbb{L}W|_g^2\phi^{-\frac{3n-2}{n-2}}, && \text{(1.2a)}\\
\mathrm{div}_g(\mathbb{L}W) &= \frac{n-1}{n}\phi^{\frac{2n}{n-2}}d\tau, && \text{(1.2b)}
\end{aligned}
\right.
$$

where $\Delta_g = \mathrm{div}_g(\nabla)$ is the Laplace–Beltrami operator. In general, the system (1.2) is coupled except in the case when τ is constant, called the *CMC* case. In such situation, a complete and beautiful existence result for solutions of (1.2) was obtained by Isenberg [Ise95]. When τ is no longer constant, much less is known for solutions for (1.2). The first set of existence results for *near-CMC* cases was obtained by Isenberg–Moncrief [IM96] for metrics with negative Yamabe constant and by Allen–Clausen–Isenberg [ACI08] for metrics with non-negative Yamabe constant.

In a slow progress of studies for *non-CMC* cases, two major breakthroughs for far-from-CMC cases were obtained. The first breakthrough was achieved by Holst–Nagy–Tsogtgerel [HNT09] for compact manifolds without boundary via introducing the notion of global super-/sub- solutions for coupled systems. The analysis in [HNT09] was then refined by Maxwell in [Max09]. This remarkable method works also very well in other situations, for example, it can be applied for compact manifolds with boundary [HT13], asymptotically Euclidean manifolds in [DIMM14], for asymptotically cylindrical manifolds [Lea14], and for asymptotically Euclidean manifolds with boundary in [HM15]. The second breakthrough was found by Dahl–Gicquaud–Humbert [DGH12] for compact manifolds without boundary by introducing for the first time the limit equation associated to (1.2). This method was then simplified by Nguyen in [Ngu15]. This method was successfully adapted to other contexts such as asymptotically cylindrical manifolds in [DL14] and asymptotically hyperbolic manifolds [GS12].

Very recently, by an implicit function theorem argument, Gicquaud and the author [GN14] provided a constructive proof to offer another point of view on the methods in [HNT09, DGH12]. The idea in [GN14] has recently been used to study solutions to the Einstein-scalar field constraint equations, which offers some new perspectives; see [GN16].

In contrast to the existence results mentioned above, non-existence results for (1.2) in non-CMC cases are very few. As far as we know, the first non-existence result was obtained by Isenberg–Ó Murchadha in [IM04, Theorem 2] when $\sigma \equiv 0$ and $d\tau/\tau$ is relatively small, the near-CMC-case, for metrics of non-negative scalar curvature. Later, this result was enhanced in [DGH12, Theorem 1.7] and in [GN14, Theorem 2.4] for metrics of non-negative Yamabe constant, still assuming $\sigma \equiv 0$. It turns out that these findings give a clue for a question raised by Maxwell in [Max09, p. 630], which asks whether the requirement $\sigma \not\equiv 0$ is necessary for metrics of either positive Yamabe constant or zero Yamabe constant with non-vanishing τ. In a very recent paper, Nguyen [Ngu15n, Corollary 1.3] disproved this fact. Some recent non-existence results can also be found in [Max11].

Knowing the lack of non-existence results for (1.2) in non-CMC cases, Dahl–Gicquaud–Humbert [DGH13] recently studied the following system similar to (1.2).

$$
\left\{
\begin{aligned}
-\frac{4(n-1)}{n-2}\Delta_g\phi + \mathrm{Scal}_g\,\phi &= -\frac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} + |\sigma + \mathbb{L}W|_g^2\phi^{-\frac{3n-2}{n-2}}, && \text{(1.3a)}\\
\mathrm{div}_g(\mathbb{L}W) &= a\phi^{\frac{2n}{n-2}}\xi. && \text{(1.3b)}
\end{aligned}
\right.
$$

The only difference between (1.2) and (1.3) is that the 1-form $d\tau$ in (1.2b) was replaced by some non-vanishing Lipschitz 1-form ξ, up to a constant multiple. Loosely speaking, they showed in [DGH13, Theorem 1.1] that for a manifold (M, g) of dimension $3 \leqslant n \leqslant 5$, if $\mathrm{Scal}_g \leqslant 0$, $\mathrm{Scal}_g \not\equiv 0$, and if ξ is a non-zero Lipchitz 1-form, then there exists some constant a_0 (depending mainly on τ, σ, Scal_g, ξ) such that (1.3) has no solution (φ, W) if $a > a_0$.

As indicated in [DGH13, p. 3], the condition $\mathrm{Scal}_g \leqslant 0$ with $\mathrm{Scal}_g \not\equiv 0$ implies that the metric g has a negative Yamabe constant. Then, it is natural to ask *if the result in [DGH13] is still valid for any metric g having a negative Yamabe constant*. In this note, we provide a positive answer for this question by exploding the conformally covariant property of Eq. (1.3a). In fact, we shall show that such a result still holds for metrics with zero Yamabe constant. The following is our main result.

Theorem 1. *Let (M, g) be a closed Riemannian manifold of dimension $n \geqslant 3$. We assume further that (i) either $n \leqslant 5$ if $g \in C^2(M)$ has a negative Yamabe constant; (ii) or $n \leqslant 4$ if $g \in C^2(M)$ has zero Yamabe constant. Let $\tau \in L^\infty(M)$ be a positive function and $\sigma \in L^{2n/(n-2)}(M)$ a symmetric traceless divergence-free $(2,0)$-tensor on M. Assume that ξ is a Lipchitz 1-form which does not vanish anywhere on M. Then there is a constant a_0 such that there does not exist any solution to the system (1.3) whenever $a > a_0$.*

As in [DGH13], the constant a_0 depends on a Sobolev constant, a constant appearing in a Schauder estimate, $\max|\xi|$, $\min|\xi|$, $\|\mathbb{L}\xi\|_2$, $\|\mathbb{L}\xi\|_\infty$, $\|\tau\|_{4n/(n-2)}$, $\min\tau$, $\|\mathrm{Scal}_g\|_n$, $\|\sigma\|_{2n/(n-2)}$, $\|\phi_-\|_{2n/(n-2)}$, and on ψ where ϕ_- is the (unique) solution of the prescribed scalar curvature equation, see Proposition 1 below,

$$-\frac{4(n-1)}{n-2}\Delta_g\phi + \mathrm{Scal}_g\,\phi = -\frac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} \tag{1.4}$$

and $\psi \in C^\infty(M)$ is a positive function such that the scalar curvature of the conformal metric $\psi^{4/(n-2)}g$ is non-positive.

Clearly, one wishes to strengthen Theorem 1 by assuming that the metric g has a positive Yamabe constant. Unfortunately, the prescribed scalar curvature equation (1.4) has no solution in this setting. The role of metrics with non-positive Yamabe constant is to make sure that the solution ϕ of the constraint equations (1.3) has a strictly positive lower bound independent of W; see Proposition 1 below. Such an important property is the key step in the proof of Claim 2.1 in [DGH13]. In general, strictly positive lower bounds for solutions (ϕ, W) of the constraint equations are barely understood.

Simply by observing the change in the dimension n in [DGH13, Theorem 1.1] and our Theorem 1 above, it is likelihood that a similar result could hold for metrics with positive Yamabe constant on 3-manifolds which is not conformally diffeomorphic to 3-sphere; see [ES86, Theorem 2.3].

2. Proof of Theorem 1

Since our proof basically follows the lines in [DGH13, Theorem 1.1], we only indicate the difference. First, we assume that (ϕ, W) solves (1.3). Then by using

$$\gamma_g = \int_M |\sigma + \mathbb{L}W|_g^2 \,\mathrm{dvol}_g$$

and transforming ϕ, W, σ via γ_g through

$$\widetilde{\phi} = \gamma_g^{-\frac{n-2}{4n}}\phi, \quad \widetilde{W} = \gamma_g^{-\frac{1}{2}}W, \quad \widetilde{\sigma} = \gamma_g^{-\frac{1}{2}}\sigma, \tag{2.1}$$

the system (1.3) becomes

$$\begin{cases} \gamma_g^{-\frac{1}{n}}\Big(-\dfrac{4(n-1)}{n-2}\Delta_g\widetilde{\phi}+\mathrm{Scal}_g\,\widetilde{\phi}\Big)=-\dfrac{n-1}{n}\tau^2\widetilde{\phi}^{\frac{n+2}{n-2}}+|\widetilde{\sigma}+\mathbb{L}\widetilde{W}|_g^2\widetilde{\phi}^{-\frac{3n-2}{n-2}}, & (2.2a)\\ \mathrm{div}_g(\mathbb{L}\widetilde{W})=a\widetilde{\phi}^{\frac{2n}{n-2}}\xi. & (2.2b)\end{cases}$$

Before going further, we prove the following simple observation.

Proposition 1. *Assume that the background metric g satisfies the assumptions in Theorem 1, then there exists a solution φ_- of Eq. (1.4). Furthermore, if (ϕ, W) is a solution of the system (1.3), then $\varphi_- \leqslant \varphi$ holds.*

Proof of Proposition 1. For the existence part, depending on the sign of the Yamabe constant for g, we have two cases: For the case of zero Yamabe constant, we make use of a general existence result due to Escobar–Schoen [ES86, Theorem 3.1] in which the dimension restriction $3 \leqslant n \leqslant 4$ and the fact $\int_M \tau^2\, d\mathrm{vol}_g > 0$ are crucial; see also [NX15, HNX16]. For the case of negative Yamabe constant, the argument is standard and it is well-known that Eq. (1.4) always admits at least one positive solution for any $n \geqslant 3$. For the estimation part, we refer to a useful result due to Dahl–Gicquaud–Humbert [DGH12, Lemma 2.2] and omit the details. $\square$

The proof in [DGH13] consists of seven claims. In their proof for Claim 2.1, it made use of no information on Scal_g. The only issue we need to take care is the existence of ϕ_- in (1.4). However, thanks to Proposition 1, we automatically obtain the same claim as theirs.

Claim 2.1. There exists a constant $c_1 > 0$ such that

$$\gamma_g \geqslant c_1 a^2.$$

For their Claim 2.2, since their proof made use of the sign of Scal_g, we need to modify that proof. However, the statement remains the same except up to constants.

Claim 2.2. There exists a constant $c_2 > 0$ such that

$$\int_M \widetilde{\phi}\, d\mathrm{vol}_g \leqslant c_2 a^{-\frac{3n-2}{n-2}}.$$

Proof of Claim 2.2. The only place in [DGH13] used the sign of Scal_g is to obtain the first inequality in (6), that is

$$\int_M |\widetilde{\sigma}+\mathbb{L}\widetilde{W}|_g^2\widetilde{\phi}^{-\frac{3n-2}{n-2}}\, d\mathrm{vol}_g \leqslant \frac{n-1}{n}\int_M \tau^2\widetilde{\phi}^{\frac{n+2}{n-2}}\, d\mathrm{vol}_g \tag{2.3}$$

where $\widetilde{\sigma}$, $\widetilde{\phi}$, and $\widetilde{W}$ are in (2.1). To obtain an inequality in the fashion of (2.3) without assuming a sign for Scal_g, we use the conformal change $\widehat{g} = \psi^{4/(n-2)}g$ to transform (1.2a) into

$$\begin{aligned}-\frac{4(n-1)}{n-2}&\Delta_{\widehat{g}}(\psi^{-1}\phi)+\mathrm{Scal}_{\widehat{g}}(\psi^{-1}\phi)\\ &=-\frac{n-1}{n}\tau^2(\psi^{-1}\phi)^{\frac{n+2}{n-2}}+|\psi^{-2}(\sigma+\mathbb{L}W)|_{\widehat{g}}^2(\psi^{-1}\phi)^{-\frac{3n-2}{n-2}},\end{aligned} \tag{2.4}$$

where the conformal factor ψ is chosen in such a way that the conformal metric $\widehat{g}$ has non-positive scalar curvature, i.e. $\mathrm{Scal}_{\widehat{g}} \leqslant 0$. From this, an integration over M with respect to

$\widehat{g}$ yields

$$\int_M |\psi^{-2}(\sigma+\mathbb{L}W)|^2_{\widehat{g}}(\psi^{-1}\phi)^{-\frac{3n-2}{n-2}}\,d\mathrm{vol}_{\widehat{g}} \leqslant \frac{n-1}{n}\int_M \tau^2(\psi^{-1}\phi)^{\frac{n+2}{n-2}}\,d\mathrm{vol}_{\widehat{g}}. \tag{2.5}$$

Note that under the conformal change $\widehat{g}=\psi^{4/(n-2)}g$, there holds

$$d\mathrm{vol}_{\widehat{g}} = \psi^{\frac{2n}{n-2}}\,d\mathrm{vol}_g$$

and

$$|\psi^{-2}(\sigma+\mathbb{L}W)|^2_{\widehat{g}} = \psi^{-\frac{4n}{n-2}}|\sigma+\mathbb{L}W|^2_g.$$

Using the two relations above, we can re-evaluate (2.5) in terms of g as follows

$$\int_M \psi|\sigma+\mathbb{L}W|^2_g\phi^{-\frac{3n-2}{n-2}}\,d\mathrm{vol}_g \leqslant \frac{n-1}{n}\int_M \tau^2\psi\phi^{\frac{n+2}{n-2}}\,d\mathrm{vol}_g. \tag{2.6}$$

Hence, dividing both sides of (2.6) by $\gamma_g^{(n+2)/(4n)}$ gives

$$\int_M |\widetilde{\sigma}+\mathbb{L}\widetilde{W}|^2_g\widetilde{\phi}^{-\frac{3n-2}{n-2}}\,d\mathrm{vol}_g \leqslant \frac{n-1}{n}\Big(\frac{\max\psi}{\min\psi}\Big)\int_M \tau^2\widetilde{\phi}^{\frac{n+2}{n-2}}\,d\mathrm{vol}_g. \tag{2.7}$$

Once we have (2.7) in hand we can go through the rest of their proof for Claim 2.2 to get a new constant

$$c_2=\Big(\frac{1}{2}\Big)^{\frac{3n-2}{n-2}}\Big(\frac{n-1}{n}\frac{\max\psi}{\min\psi}\Big)^{\frac{n}{n-2}}\Big(\frac{\|\mathbb{L}\xi\|_\infty}{\inf|\xi|^2}\Big)^{\frac{3n-2}{n-2}}\Big(\int_M \tau^{\frac{4n}{n-2}}\,d\mathrm{vol}_g\Big)^{1/2}.$$

This completes our proof of Claim 2.2. □

After completing the proof of Claim 2.2, we obtain Claim 2.3 as in their paper.

Claim 2.3. There exists a constant $c_3>0$ such that

$$\int_M |\widetilde{\sigma}+\mathbb{L}\widetilde{W}|^2_g\widetilde{\phi}^{-\frac{3n-2}{n-2}}\,d\mathrm{vol}_g \leqslant c_3 a^{-\frac{(3n-2)(n+2)}{2n(n-2)}}.$$

For Claim 2.4, although some integrals involving Scal_g were estimated in the proof, we find that the sign of Scal_g in those estimates plays no role. This can be easily seen from their formula for c_4 where $|\,\mathrm{Scal}_g\,|$ was used. Therefore, we can conclude the following result.

Claim 2.4. There exists a constant $c_4>0$ such that

$$\int_M \widetilde{\phi}^{\frac{4n}{n-2}}\,d\mathrm{vol}_g \leqslant c_4$$

provided a is large enough.

Since Claims 2.5–2.7 in their paper involved no information on the sign of Scal_g directly, we conclude that their claims still hold in our context. For the sake of completeness, we mention Claims 2.5–2.7 in [DGH13] below.

Claim 2.5. There exists a constant $c_5>0$ such that

$$\int_M |\widetilde{\sigma}+\mathbb{L}\widetilde{W}|^{\frac{2n}{n-2}}_g\,d\mathrm{vol}_g \leqslant c_5 a^{\frac{2n}{n-2}},$$

provided a is large enough.

Claim 2.6. There exists a constant $c_6>0$ such that

$$\int_M |\widetilde{\sigma}+\mathbb{L}\widetilde{W}|^2_g\widetilde{\phi}^{\frac{4}{n-2}}\,d\mathrm{vol}_g \leqslant c_6 a^{2-\frac{2(3n-2)}{n(n-2)}}.$$

Claim 2.7. There exists a constant $c_7 > 0$ such that

$$\int_M |\widetilde{\sigma} + \mathbb{L}\widetilde{W}|_g^2 \widetilde{\phi}^{-\frac{4}{n-2}}\, d\mathrm{vol}_g \leqslant c_7 a^{-\frac{2(n+2)}{n(n-2)}}.$$

Thus, one can obtain a non-existence result for large a similar to their equation.

Proof of Theorem 1. To prove the theorem, we use (2.1) and Claims 2.6 and 2.7 above to obtain

$$\begin{aligned}1 \leqslant & \Big(\int_M |\widetilde{\sigma} + \mathbb{L}\widetilde{W}|_g^2 \widetilde{\phi}^{\frac{4}{n-2}}\, d\mathrm{vol}_g\Big)\Big(\int_M |\widetilde{\sigma} + \mathbb{L}\widetilde{W}|_g^2 \widetilde{\phi}^{-\frac{4}{n-2}}\, d\mathrm{vol}_g\Big)\\ \leqslant & c_6 c_7 a^{2-\frac{2(3n-2)}{n(n-2)}} a^{-\frac{2(n+2)}{n(n-2)}}\\ = & c_6 c_7 a^{6-2\frac{2n}{n-2}}.\end{aligned}$$

Thus in case $3 \leqslant n \leqslant 5$, we obtain a contradiction if a is large enough. This provides us a non-existence result as claimed. $\square$

ACKNOWLEDGMENTS

The author would like to thank the Vietnam Institute for Advanced Study in Mathematics (VIASM) for hosting and support where this note was carried.

REFERENCES

[ACI08] P.T. ALLEN, A. CLAUSEN, J. ISENBERG, Near-constant mean curvature solutions of the Einstein constraint equations with non-negative Yamabe metrics, *Class. Quantum Grav.* **25** (2008), no. 7075009. 2

[BI04] R. BARTNIK AND J. ISENBERG, *The constraint equations*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 1–38. 1

[ES86] J.F. ESCOBAR AND R. SCHOEN, Conformal metrics with prescribed scalar curvature, *Invent. Math.* **86** (1986), pp. 243–254. 3, 4

[FB52] Y. FOURÈS-BRUHAT, Théoréme d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires, *Acta Math.* **88** (1952), pp. 141–225. 1

[CBG69] Y. CHOQUET-BRUHAT, R. GEROCH, Global aspects of the Cauchy problem in general relativity, *Comm. Math. Phys.* **14** 1969, pp. 329–335. 1

[CBY80] Y. CHOQUET-BRUHAT, J.W. YORK, JR., The Cauchy problem, *General Relativity and Gravitation*, Vol. 1, Plenum, New York, 1980, pp. 99–172. 1

[DGH12] M. DAHL, R. GICQUAUD, E. HUMBERT, A limit equation associated to the solvability of the vacuum Einstein constraint equations by using the conformal method, *Duke Math. J.* **161** (2012), pp. 2669–2697. 2, 4

[DGH13] ———, A non-existence result for a generalization of the equations of the conformal method in general relativity, *Class. Quantum Grav.* **30** (2013), 075004 (8pp). 2, 3, 4, 5

[DIMM14] J. DILTS, J. ISENBERG, R. MAZZEO, C. MEIER, Non-CMC solutions of the Einstein constraint equations on asymptotically Euclidean manifolds, *Class. Quantum Grav.* **31** (2014), no. 6, 065001, 10. 2

[DL14] J. DILTS, J. LEACH, A limit equation criterion for solving the Einstein constraint equations on manifolds with ends of cylindrical type, *Ann. Henri Poincaré* **16** (2015), no. 7, pp. 1583–1607. 2

[GN14] R. GICQUAUD, Q.A. NGÔ, A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small TT-tensor, *Class. Quantum Grav.* **31** (2014), 195014. 2

[GN16] R. GICQUAUD, T.C. NGUYEN, Solutions to the Einstein-scalar field constraint equations with a small TT-tensor, *Calc. Var.* **55** (2016), Art. 29. 2

[GS12] R. GICQUAUD, A. SAKOVICH, A large class of non-constant mean curvature solutions of the Einstein constraint equations on an asymptotically hyperbolic manifold, *Comm. Math. Phys.* **310** (2012), no. 3, 705–763. 2

[HT13] M. J. HOLST, G. TSOGTGEREL, The Lichnerowicz equation on compact manifolds with boundary, *Class. Quantum Grav.* **30** (2013), 205011. 2

[HM15] M. J. HOLST, C. MEIER, Non-CMC solutions to the Einstein constraint equations on asymptotically Euclidean manifolds with apparent horizon boundaries, *Class. Quantum Grav.* **32** (2015), 025006. 2

[HNT09] M. HOLST, G. NAGY, G. TSOGTGEREL, Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions, *Comm. Math. Phys.* **288** (2009), pp. 547–613. 2

[HNX16] L. HONG, Q.A. NGÔ, X. XU, On the scalar curvature functions on closed manifolds with positive Yamabe constant, preprint, 2016. 4

[Ise95] J. ISENBERG, Constant mean curvature solutions of the Einstein constraint equations on closed manifolds, *Class. Quantum Grav.* **12** (1995), pp. 2249–2274. 2

[IM96] J. ISENBERG, V. MONCRIEF, A set of nonconstant mean curvature solutions of the Einstein constraint equations on closed manifolds, *Class. Quantum Grav.* **13** (1996), no. 7, pp. 1819–1847. 2

[IM04] J. ISENBERG, N. Ó MURCHADHA, Non-CMC conformal data sets which do not produce solutions of the Einstein constraint equations, *Class. Quantum Grav.* **21** (2004), S233–S241. 2

[Lea14] J. LEACH, A far-from-CMC existence result for the constraint equations on manifolds with ends of cylindrical type, *Class. Quantum Grav.* **31** (2014), 035003. 2

[Lic44] A. LICHNEROWICZ, L'intégration des équations de la gravitation relativiste et le problème des n corps, *J. Math. Pures Appl.* (9) **23** (1944), pp. 37–63. 1

[Max09] D. MAXWELL, A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature, *Math. Res. Lett.* **16** (2009), pp. 627–645. 2

[Max11] ______, A model problem for conformal parameterizations of the Einstein constraint equations, *Comm. Math. Phys.* **302** (2011), pp. 697–736. 2

[NX15] Q.A. NGÔ, X. XU, A flow approach to the prescribed scalar curvature problem with vanishing Yamabe constant, preprint, 2015. 4

[Ngu15] T.C. NGUYEN, Applications of fixed point theorems to the vacuum Einstein constraint equations with non-constant mean curvature, *Ann. Henri Poincaré* (2015), doi: 10.1007/s00023-015-0446-5. 2

[Ngu15n] ______, Nonexistence and nonuniqueness results for solutions to the vacuum Einstein conformal constraint equations, arXiv: 1507.01081. 2

(Q.A. Ngô) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, VIÊT NAM NATIONAL UNIVERSITY, HÀ NÔI, VIÊT NAM.

E-mail address: nqanh@vnu.edu.vn

E-mail address: bookworm_vn@yahoo.com