ON A NON-EXISTENCE RESULT FOR EINSTEIN CONSTRAINT-TYPE
SYSTEMS WITH NON-POSITIVE YAMABE CONSTANT

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ABSTRACT. We extend a recent non-existence result for an Einstein constraint-type sys-
tem due to Dahl–Gicquaud–Humbert (Class. Quantum Grav. 30, 075004) from non-
vvanishing, non-positive metrics to metrics with non-positive Yamabe constant.

1. INTRODUCTION

Of importance in general theory of relativity is the study of the initial value problem. Mathematically, this problem asks: for given a Riemannian manifold \((M, g)\) of dimension \(n\), whether we can construct a globally hyperbolic spacetime \((\mathcal{M}, \bar{g})\) of dimension \(n+1\) in which \((M, g)\) can be embedded as a spacelike hypersurface such that the vacuum Einstein equation

\[
\text{Ric}_{\bar{g}} - \frac{1}{2} \bar{g} \text{Scal}_{\bar{g}} = 0
\]

is satisfied. Consequently, it is natural to ask: (a) how could we construct a spacetime \((\mathcal{M}, \bar{g})\) from a suitable spacelike \((M, \hat{g})\) and (b) under what conditions for \((M, \hat{g})\) the answer for part (a) is positive. To answer part (a), we observe from the Gauss and Codazzi equations that the “induced” metric \(\hat{g}\) and the second fundamental form \(\hat{K}\) of the embedding must satisfy the so-called constraint equations. These equations can be formulated as follows

\[
\begin{align}
\text{Scal}_{\hat{g}} + (\text{tr}_{\hat{g}} \hat{K})^2 - |\hat{K}|_{\hat{g}}^2 &= 0, \\
\text{div}_{\hat{g}} \hat{K} - d(\text{tr}_{\hat{g}} \hat{K}) &= 0;
\end{align}
\]

see [BI04]. Concerning to the part (a), the answer is affirmative by the two celebrated papers by Choquet-Bruhat [FB52] and Choquet-Bruhat–Geroch [CBG69]. In these papers, the authors proved that if the pair \((\hat{g}, \hat{K})\) solves (1.1), then the Cauchy problem is well-posed and there do exist a unique, maximal spacetime \((\mathcal{M}, \bar{g})\) such that (E) holds. To answer the part (b), we first observe that (1.1) forms an under-determined system; hence it is difficult to solve (1.1).

The most efficient way to look for solutions of (1.1) is to make use of the conformal method developed by Lichnerowicz [Lic44] and Choquet-Bruhat and York [CBY80]. The idea of the conformal method is as follows: First we look for \(\bar{g} = \phi^{1/(n-2)} g\) in the conformal class represented by some fixed metric \(g\). Then we split \(\hat{K} = \tau \hat{g} / n + \phi^{-2}(\hat{\sigma} + \nabla W)\)

where \(\tau\) is the mean curvature of the hypersurface \(M\), \(\hat{\sigma}\) is some trace-free and divergence-free symmetric 2-tensor on \(M\), and \(W\) is a vector field on \(M\). Note that in the decomposition of \(\hat{K}\) above, by \(L\) we mean the Killing operator acting on vector fields through

\[
L W_{ij} := \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \nabla_k W_k g_{ij},
\]
where $\nabla$ is the Levi–Civita connection associated to the background metric $g$. In terms of $(\phi, W)$, we can reformulate (1.1) into the following system

\[
\begin{cases}
-\frac{4(n-1)}{n-2} \Delta_g \phi + \text{Scal}_g \phi = -\frac{n-1}{n} \tau^2 \phi^{\frac{n+2}{n}} + |\sigma + LW|_g^2 \phi^{-\frac{2n}{n-2}}, \\
\text{div}_g(LW) = -\frac{n}{n-1} \phi^{\frac{2n}{n-2}} d\tau,
\end{cases}
\tag{1.2a}
\tag{1.2b}
\]

where $\Delta_g = \text{div}_g(\nabla)$ is the Laplace–Beltrami operator. In general, the system (1.2) is coupled except in the case when $\tau$ is constant, called the CMC case. In such situation, a complete and beautiful existence result for solutions of (1.2) was obtained by Isenberg [Ise95]. When $\tau$ is no longer constant, much less is known for solutions for (1.2). The first set of existence results for near-CMC cases was obtained by Isenberg–Moncrief [IM96] for metrics with negative Yamabe constant and by Allen–Clausen–Isenberg [ACI08] for metrics with non-negative Yamabe constant.

In a slow progress of studies for non-CMC cases, two major breakthroughs for far-from-CMC cases were obtained. The first breakthrough was achieved by Holst–Nagy–Tsogtgerel [HNT09] for compact manifolds without boundary via introducing the notion of global super/sub-solutions for coupled systems. The analysis in [HNT09] was then refined by Maxwell in [Max09]. This remarkable method works also very well in other situations, for example, it can be applied for compact manifolds with boundary [HT13], asymptotically Euclidean manifolds in [DIMM14], for asymptotically cylindrical manifolds [Lea14], and for asymptotically Euclidean manifolds with boundary in [HM15].

The second breakthrough was found by Dahl–Gicquaud–Humbert [DGH12] for compact manifolds without boundary by introducing for the first time the limit equation associated to (1.2). This method was then simplified by Maxwell in [Max09]. This method was successfully adapted to other contexts such as asymptotically cylindrical manifolds in [DL14] and asymptotically hyperbolic manifolds [GS12].

Very recently, by an implicit function theorem argument, Gicquaud and the author [GN14] provided a constructive proof to offer another point of view on the methods in [HNT09, DGH12]. The idea in [GN14] has recently been used to study solutions to the Einstein-scalar field constraint equations, which offers some new perspectives; see [GN16].

In contrast to the existence results mentioned above, non-existence results for (1.2) in non-CMC cases are very few. As far as we know, the first non-existence result was obtained by Isenberg–Ó Murchadha in [IM04, Theorem 2] when $\sigma \equiv 0$ and $d\tau/\tau$ is relatively small, the near-CMC-case, for metrics of non-negative scalar curvature. Later, this result was enhanced in [DGH12, Theorem 1.7] and in [GN14, Theorem 2.4] for metrics of non-negative Yamabe constant, still assuming $\sigma \equiv 0$. It turns out that these findings give a clue for a question raised by Maxwell in [Max09, p. 630], which asks whether the requirement $\sigma \not\equiv 0$ is necessary for metrics of either positive Yamabe constant or zero Yamabe constant with non-vanishing $\tau$. In a very recent paper, Nguyen [Ngu15n, Corollary 1.3] disproved this fact. Some recent non-existence results can also be found in [Max11].

Knowing the lack of non-existence results for (1.2) in non-CMC cases, Dahl–Gicquaud–Humbert [DGH13] recently studied the following system similar to (1.2).

\[
\begin{cases}
-\frac{4(n-1)}{n-2} \Delta_g \phi + \text{Scal}_g \phi = -\frac{n-1}{n} \tau^2 \phi^{\frac{n+2}{n}} + |\sigma + LW|_g^2 \phi^{-\frac{2n}{n-2}}, \\
\text{div}_g(LW) = a \phi^{\frac{2n}{n-2}} \xi,
\end{cases}
\tag{1.3a}
\tag{1.3b}
\]

The only difference between (1.2) and (1.3) is that the 1-form $d\tau$ in (1.2b) was replaced by some non-vanishing Lipschitz 1-form $\xi$, up to a constant multiple. Loosely speaking, they
showed in [DGH13, Theorem 1.1] that for a manifold \((M, g)\) of dimension \(3 \leq n \leq 5\), if \(\text{Scal}_g \leq 0\), \(\text{Scal}_g \neq 0\), and if \(\xi\) is a non-zero Lipchitz 1-form, then there exists some constant \(a_0\) (depending mainly on \(\tau, \sigma, \text{Scal}_g, \xi\)) such that (1.3) has no solution \((\varphi, W)\) if \(a > a_0\).

As indicated in [DGH13, p. 3], the condition \(\text{Scal}_g \leq 0\) with \(\text{Scal}_g \neq 0\) implies that the metric \(g\) has a negative Yamabe constant. Then, it is natural to ask if the result in [DGH13] is still valid for any metric \(g\) having a negative Yamabe constant. In this note, we provide a positive answer for this question by exploding the conformally covariant property of Eq. (1.3a). In fact, we shall show that such a result still holds for metrics with zero Yamabe constant. The following is our main result.

**Theorem 1.** Let \((M, g)\) be a closed Riemannian manifold of dimension \(n \geq 3\). We assume further that (i) either \(n \leq 5\) if \(g \in C^2(M)\) has a negative Yamabe constant; (ii) or \(n \leq 4\) if \(g \in C^2(M)\) has zero Yamabe constant. Let \(\tau \in L^\infty(M)\) be a positive function and \(\sigma \in L^{2n/(n-2)}(M)\) a symmetric traceless divergence-free \((2,0)\)-tensor on \(M\). Assume that \(\xi\) is a Lipchitz 1-form which does not vanish anywhere on \(M\). Then there is a constant \(a_0\) such that there does not exist any solution to the system (1.3) whenever \(a > a_0\).

As in [DGH13], the constant \(a_0\) depends on a Sobolev constant, a constant appearing in a Schauder estimate, \(\max |\xi|, \min |\xi|, \|L\xi\|_2, \|L\xi\|_\infty, \|\tau\|_{4/(n-2)}, \min \tau, \|\text{Scal}_g\|_n, \|\sigma\|_{2n/(n-2)}, \|\phi_\cdot\|_{2n/(n-2)}\), and on \(\psi\) where \(\phi_\cdot\) is the (unique) solution of the prescribed scalar curvature equation, see Proposition 1 below,

\[
-\frac{4(n-1)}{n-2} \Delta_g \phi + \text{Scal}_g \phi = -\frac{n-1}{n} \tau^2 \psi^{\frac{n+2}{n}} \tag{1.4}
\]

and \(\psi \in C^\infty(M)\) is a positive function such that the scalar curvature of the conformal metric \(\psi^{4/(n-2)}g\) is non-positive.

Clearly, one wishes to strengthen Theorem 1 by assuming that the metric \(g\) has a positive Yamabe constant. Unfortunately, the prescribed scalar curvature equation (1.4) has no solution in this setting. The role of metrics with non-positive Yamabe constant is to make sure that the solution \(\phi_\cdot\) of the constraint equations (1.3) has a strictly positive lower bound independent of \(W\); see Proposition 1 below. Such an important property is the key step in the proof of Claim 2.1 in [DGH13]. In general, strictly positive lower bounds for solutions \((\phi_\cdot, W_\cdot)\) of the constraint equations are barely understood.

Simply by observing the change in the dimension \(n\) in [DGH13, Theorem 1.1] and our Theorem 1 above, it is likelihood that a similar result could hold for metrics with positive Yamabe constant on 3-manifolds which is not conformally diffeomorphic to 3-sphere; see [ES86, Theorem 2.3].

## 2. Proof of Theorem 1

Since our proof basically follows the lines in [DGH13, Theorem 1.1], we only indicate the difference. First, we assume that \((\phi, W, \sigma)\) solves (1.3). Then by using

\[
\gamma_g = \int_M |\sigma + LW|^2 dvol_g
\]

and transforming \(\phi, W, \sigma\ via \gamma_g\ through

\[
\tilde{\phi} = \gamma_g^{-\frac{n+2}{4n}} \phi, \quad \tilde{W} = \gamma_g^{-\frac{1}{2}} W, \quad \tilde{\sigma} = \gamma_g^{\frac{1}{4}} \sigma, \quad (2.1)
\]
the system (1.3) becomes
\[
\begin{cases}
\gamma \frac{1}{4} \left( - \frac{4(n-1)}{n-2} \Delta g \phi + \text{Scal}_g \phi \right) = - \frac{n-1}{n} \tau^2 \phi^{\frac{n+2}{n-2}} + |\tilde{\sigma} + L \tilde{W}|_g^2 \phi^{-\frac{4n-2}{n-2}}, \\
\text{div}_g (L \tilde{W}) = a \phi^{\frac{2n}{n-2}} \xi.
\end{cases}
\tag{2.2a}
\tag{2.2b}
\]

Before going further, we prove the following simple observation.

**Proposition 1.** Assume that the background metric \( g \) satisfies the assumptions in Theorem 1, then there exists a solution \( \phi^- \) of Eq. (1.4). Furthermore, if \((\phi, W)\) is a solution of the system (1.3), then \( \phi^- \leq \phi \) holds.

**Proof of Proposition 1.** For the existence part, depending on the sign of the Yamabe constant for \( g \), we have two cases: For the case of zero Yamabe constant, we make use of a general existence result due to Escobar–Schoen [ES86, Theorem 3.1] in which the dimension restriction \( 3 \leq n \leq 4 \) and the fact \( \int_M \tau^2 \text{dvol}_g > 0 \) are crucial; see also [NX15, HNX16]. For the case of negative Yamabe constant, the argument is standard and it is well-known that Eq. (1.4) always admits at least one positive solution for any \( n \geq 3 \). For the estimation part, we refer to a useful result due to Dahl–Gicquaud–Humbert [DGH12, Lemma 2.2] and omit the details. \( \square \)

The proof in [DGH13] consists of seven claims. In their proof for Claim 2.1, it made use of no information on Scal_g. The only issue we need to take care is the existence of \( \phi^- \) in (1.4). However, thanks to Proposition 1, we automatically obtain the same claim as theirs.

**Claim 2.1.** There exists a constant \( c_1 > 0 \) such that
\[
\gamma \geq c_1 a^2.
\]

For their Claim 2.2, since their proof made use of the sign of Scal_g, we need to modify that proof. However, the statement remains the same except up to constants.

**Claim 2.2.** There exists a constant \( c_2 > 0 \) such that
\[
\int_M \phi^2 \text{dvol}_g \leq c_2 a^{\frac{3n-2}{n-2}}.
\]

**Proof of Claim 2.2.** The only place in [DGH13] used the sign of Scal_g is to obtain the first inequality in (6), that is
\[
\int_M |\tilde{\sigma} + L \tilde{W}|^2_g \phi^{-\frac{4n-2}{n-2}} \text{dvol}_g \leq \frac{n-1}{n} \int_M \tau^2 \phi^{\frac{n+2}{n-2}} \text{dvol}_g
\]
where \( \tilde{\sigma}, \tilde{\phi}, \) and \( \tilde{W} \) are in (2.1). To obtain an inequality in the fashion of (2.3) without assuming a sign for Scal_g, we use the conformal change \( \tilde{g} = \psi^{4/(n-2)} g \) to transform (1.2a) into
\[
- \frac{4(n-1)}{n-2} \Delta \tilde{g}(\psi^{-1} \phi) + \text{Scal}_{\tilde{g}}(\psi^{-1} \phi) = - \frac{n-1}{n} \tau^2 (\psi^{-1} \phi)^{\frac{n+2}{n-2}} + |\psi^{-2} (\sigma + L \tilde{W})|_{\tilde{g}}^2 (\psi^{-1} \phi)^{-\frac{4n-2}{n-2}},
\tag{2.4}
\]
where the conformal factor \( \psi \) is chosen in such a way that the conformal metric \( \tilde{g} \) has non-positive scalar curvature, i.e. Scal_{\tilde{g}} \leq 0. From this, an integration over \( M \) with respect to
Note that under the conformal change $\tilde{g} = \psi^{4/(n-2)}g$, there holds
$$d\text{vol}_{\tilde{g}} = \psi^{\frac{2n}{n-2}} d\text{vol}_g,$$
and
$$|\psi^{-2}(\sigma + LW)|_{\tilde{g}}^2 = \psi^{-\frac{4n}{n-2}} |\sigma + LW|_{g}^2.$$
Using the two relations above, we can re-evaluate (2.5) in terms of $g$ as follows
$$\int_M |\psi^{-2}(\sigma + LW)|_g^2 \phi^{-\frac{3n-2}{n-2}} d\text{vol}_g \leq \frac{n-1}{n} \int_M \tau^2 \phi^{\frac{n+2}{n}} d\text{vol}_g,$$
(2.6)
Hence, dividing both sides of (2.6) by $\gamma_g^{(n+2)/(4n)}$ gives
$$\int_M |\tilde{\sigma} + \tilde{LW}|_{\tilde{g}}^2 \tilde{\phi}^{-\frac{3n-2}{n-2}} d\text{vol}_{\tilde{g}} \leq \frac{n-1}{n} \left(\frac{\max \psi}{\min \psi}\right) \int_M \tau^2 \phi^{\frac{n+2}{n}} d\text{vol}_g,$$
(2.7)
Once we have (2.7) in hand we can go through the rest of their proof for Claim 2.2 to get a new constant
$$c_2 = \left(\frac{1}{2}\right)^\frac{3n-2}{n} \left(\frac{n-1}{n} \max \psi \min \psi\right)^\frac{n+2}{n} \left(\frac{\|\xi\|_\infty}{\inf |\xi|^2}\right)^\frac{3n-2}{n}\left(\int_M \tau^2 \phi^{\frac{n+2}{n}} d\text{vol}_g\right)^{1/2}.$$
This completes our proof of Claim 2.2. \hfill \Box

After completing the proof of Claim 2.2, we obtain Claim 2.3 as in their paper.

Claim 2.3. There exists a constant $c_3 > 0$ such that
$$\int_M |\tilde{\sigma} + \tilde{LW}|_{\tilde{g}}^2 \tilde{\phi}^{-\frac{3n-2}{n-2}} d\text{vol}_{\tilde{g}} \leq c_3 a^{-\frac{(3n-2)(n-2)}{2n(n+2)}}.$$

For Claim 2.4, although some integrals involving $\text{Scal}_g$ were estimated in the proof, we find that the sign of $\text{Scal}_g$ in those estimates plays no role. This can be easily seen from their formula for $c_4$ where $|\text{Scal}_g|$ was used. Therefore, we can conclude the following result.

Claim 2.4. There exists a constant $c_4 > 0$ such that
$$\int_M \tilde{\phi}^{\frac{4n}{n-2}} d\text{vol}_{\tilde{g}} \leq c_4$$
provided $a$ is large enough.

Since Claims 2.5–2.7 in their paper involved no information on the sign of $\text{Scal}_g$ directly, we conclude that their claims still hold in our context. For the sake of completeness, we mention Claims 2.5–2.7 in [DGH13] below.

Claim 2.5. There exists a constant $c_5 > 0$ such that
$$\int_M |\tilde{\sigma} + \tilde{LW}|_{\tilde{g}}^\frac{2n}{n-1} d\text{vol}_{\tilde{g}} \leq c_5 a^{\frac{2n}{n-1}},$$
provided $a$ is large enough.

Claim 2.6. There exists a constant $c_6 > 0$ such that
$$\int_M |\tilde{\sigma} + \tilde{LW}|_{\tilde{g}}^\frac{2n}{n-2} d\text{vol}_{\tilde{g}} \leq c_6 a^{\frac{2n}{n-2}},$$
Claim 2.7. There exists a constant $c_7 > 0$ such that
\[ \int_M |\tilde{\sigma} + L\tilde{W}^2\tilde{\phi}^{\alpha-1} d\nu| \leq c_7 a^{-\frac{2(n+2)}{m-n+2}}. \]

Thus, one can obtain a non-existence result for large $a$ similar to their equation.

Proof of Theorem 1. To prove the theorem, we use (2.1) and Claims 2.6 and 2.7 above to obtain
\[ 1 \leq \left( \int_M |\tilde{\sigma} + L\tilde{W}^2\tilde{\phi}^{\alpha-1} d\nu| \right) \left( \int_M |\tilde{\sigma} + L\tilde{W}^2\tilde{\phi}^{\alpha-1} d\nu| \right) \leq c_6 c_7 a^{\frac{2(n-2)}{n+m-2}} \frac{2(n+2)}{m-n+2} \]
\[ = c_6 c_7 a^{\frac{2n-2a}{2n+2}}. \]
Thus in case $3 \leq n \leq 5$, we obtain a contradiction if $a$ is large enough. This provides us a non-existence result as claimed. \qed

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