

ON A NON-EXISTENCE RESULT FOR EINSTEIN CONSTRAINT-TYPE SYSTEMS WITH NON-POSITIVE YAMABE CONSTANT

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ABSTRACT. We extend a recent non-existence result for an Einstein constraint-type system due to Dahl–Gicquaud–Humbert (*Class. Quantum Grav.* **30**, 075004) from non-vanishing, non-positive metrics to metrics with non-positive Yamabe constant.

1. INTRODUCTION

Of importance in general theory of relativity is the study of the initial value problem. Mathematically, this problem asks: *for given a Riemannian manifold (M, g) of dimension n , whether we can construct a globally hyperbolic spacetime (\mathcal{M}, \bar{g}) of dimension $n + 1$ in which (M, g) can be embedded as a spacelike hypersurface such that the vacuum Einstein equation*

$$\text{Ric}_{\bar{g}} - \frac{1}{2}\bar{g}\text{Scal}_{\bar{g}} = 0 \tag{E}$$

holds. Consequently, it is natural to ask: (a) how could we construct a spacetime (\mathcal{M}, \bar{g}) from a suitable spacelike (M, \hat{g}) and (b) under what conditions for (M, \hat{g}) the answer for part (a) is positive. To answer part (a), we observe from the Gauss and Codazzi equations that the “induced” metric \hat{g} and the second fundamental form \hat{K} of the embedding must satisfy the so-called constraint equations. These equations can be formulated as follows

$$\begin{cases} \text{Scal}_{\hat{g}} + (\text{tr}_{\hat{g}} \hat{K})^2 - |\hat{K}|_{\hat{g}}^2 = 0, & (1.1a) \\ \text{div}_{\hat{g}} \hat{K} - d(\text{tr}_{\hat{g}} \hat{K}) = 0; & (1.1b) \end{cases}$$

see [BI04]. Concerning to the part (a), the answer is affirmative by the two celebrated papers by Choquet-Bruhat [FB52] and Choquet-Bruhat–Geroch [CBG69]. In these papers, the authors proved that if the pair (\hat{g}, \hat{K}) solves (1.1), then the Cauchy problem is well-posed and there do exist a unique, maximal spacetime (\mathcal{M}, \bar{g}) such that (E) holds. To answer the part (b), we first observe that (1.1) forms an under-determined system; hence it is difficult to solve (1.1).

The most efficient way to look for solutions of (1.1) is to make use of the conformal method developed by Lichnerowicz [Lic44] and Choquet-Bruhat and York [CBY80]. The idea of the conformal method is as follows: First we look for $\hat{g} = \phi^{4/(n-2)}g$ in the conformal class represented by some fixed metric g . Then we split $\hat{K} = \tau\hat{g}/n + \phi^{-2}(\hat{\sigma} + \mathbb{L}W)$ where τ is the mean curvature of the hypersurface M , $\hat{\sigma}$ is some trace-free and divergence-free symmetric 2-tensor on M , and W is a vector field on M . Note that in the decomposition of \hat{K} above, by \mathbb{L} we mean the Killing operator acting on vector fields through

$$\mathbb{L}W_{ij} := \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \nabla^k W_k g_{ij},$$

Date: 22nd Mar, 2016 at 13:09.

Key words and phrases. Einstein constraint equations, non-CMC, conformal method, non-positive Yamabe invariant, non-existence result.

where ∇ is the Levi–Civita connection associated to the background metric g . In terms of (ϕ, W) , we can reformulate (1.1) into the following system

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta_g\phi + \text{Scal}_g\phi = -\frac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} + |\sigma + \mathbb{L}W|_g^2\phi^{-\frac{3n-2}{n-2}}, & (1.2a) \\ \text{div}_g(\mathbb{L}W) = \frac{n-1}{n}\phi^{\frac{2n}{n-2}}d\tau, & (1.2b) \end{cases}$$

where $\Delta_g = \text{div}_g(\nabla)$ is the Laplace–Beltrami operator. In general, the system (1.2) is coupled except in the case when τ is constant, called the *CMC* case. In such situation, a complete and beautiful existence result for solutions of (1.2) was obtained by Isenberg [Ise95]. When τ is no longer constant, much less is known for solutions for (1.2). The first set of existence results for *near-CMC* cases was obtained by Isenberg–Moncrief [IM96] for metrics with negative Yamabe constant and by Allen–Clausen–Isenberg [ACI08] for metrics with non-negative Yamabe constant.

In a slow progress of studies for *non-CMC* cases, two major breakthroughs for far-from-CMC cases were obtained. The first breakthrough was achieved by Holst–Nagy–Tsogetgerel [HNT09] for compact manifolds without boundary via introducing the notion of global super-/sub- solutions for coupled systems. The analysis in [HNT09] was then refined by Maxwell in [Max09]. This remarkable method works also very well in other situations, for example, it can be applied for compact manifolds with boundary [HT13], asymptotically Euclidean manifolds in [DIMM14], for asymptotically cylindrical manifolds [Lea14], and for asymptotically Euclidean manifolds with boundary in [HM15]. The second breakthrough was found by Dahl–Gicquaud–Humbert [DGH12] for compact manifolds without boundary by introducing for the first time the limit equation associated to (1.2). This method was then simplified by Nguyen in [Ngu15]. This method was successfully adapted to other contexts such as asymptotically cylindrical manifolds in [DL14] and asymptotically hyperbolic manifolds [GS12].

Very recently, by an implicit function theorem argument, Gicquaud and the author [GN14] provided a constructive proof to offer another point of view on the methods in [HNT09, DGH12]. The idea in [GN14] has recently been used to study solutions to the Einstein-scalar field constraint equations, which offers some new perspectives; see [GN16].

In contrast to the existence results mentioned above, non-existence results for (1.2) in non-CMC cases are very few. As far as we know, the first non-existence result was obtained by Isenberg–Ó Murchadha in [IM04, Theorem 2] when $\sigma \equiv 0$ and $d\tau/\tau$ is relatively small, the *near-CMC*-case, for metrics of non-negative scalar curvature. Later, this result was enhanced in [DGH12, Theorem 1.7] and in [GN14, Theorem 2.4] for metrics of non-negative Yamabe constant, still assuming $\sigma \equiv 0$. It turns out that these findings give a clue for a question raised by Maxwell in [Max09, p. 630], which asks whether the requirement $\sigma \neq 0$ is necessary for metrics of either positive Yamabe constant or zero Yamabe constant with non-vanishing τ . In a very recent paper, Nguyen [Ngu15n, Corollary 1.3] disproved this fact. Some recent non-existence results can also be found in [Max11].

Knowing the lack of non-existence results for (1.2) in non-CMC cases, Dahl–Gicquaud–Humbert [DGH13] recently studied the following system similar to (1.2).

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta_g\phi + \text{Scal}_g\phi = -\frac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} + |\sigma + \mathbb{L}W|_g^2\phi^{-\frac{3n-2}{n-2}}, & (1.3a) \\ \text{div}_g(\mathbb{L}W) = a\phi^{\frac{2n}{n-2}}\xi. & (1.3b) \end{cases}$$

The only difference between (1.2) and (1.3) is that the 1-form $d\tau$ in (1.2b) was replaced by some non-vanishing Lipschitz 1-form ξ , up to a constant multiple. Loosely speaking, they

showed in [DGH13, Theorem 1.1] that for a manifold (M, g) of dimension $3 \leq n \leq 5$, if $\text{Scal}_g \leq 0$, $\text{Scal}_g \not\equiv 0$, and if ξ is a non-zero Lipschitz 1-form, then there exists some constant a_0 (depending mainly on $\tau, \sigma, \text{Scal}_g, \xi$) such that (1.3) has no solution (φ, W) if $a > a_0$.

As indicated in [DGH13, p. 3], the condition $\text{Scal}_g \leq 0$ with $\text{Scal}_g \not\equiv 0$ implies that the metric g has a negative Yamabe constant. Then, it is natural to ask *if the result in [DGH13] is still valid for any metric g having a negative Yamabe constant*. In this note, we provide a positive answer for this question by exploding the conformally covariant property of Eq. (1.3a). In fact, we shall show that such a result still holds for metrics with zero Yamabe constant. The following is our main result.

Theorem 1. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. We assume further that (i) either $n \leq 5$ if $g \in C^2(M)$ has a negative Yamabe constant; (ii) or $n \leq 4$ if $g \in C^2(M)$ has zero Yamabe constant. Let $\tau \in L^\infty(M)$ be a positive function and $\sigma \in L^{2n/(n-2)}(M)$ a symmetric traceless divergence-free $(2, 0)$ -tensor on M . Assume that ξ is a Lipschitz 1-form which does not vanish anywhere on M . Then there is a constant a_0 such that there does not exist any solution to the system (1.3) whenever $a > a_0$.*

As in [DGH13], the constant a_0 depends on a Sobolev constant, a constant appearing in a Schauder estimate, $\max |\xi|, \min |\xi|, \|\mathbb{L}\xi\|_2, \|\mathbb{L}\xi\|_\infty, \|\tau\|_{4n/(n-2)}, \min \tau, \|\text{Scal}_g\|_n, \|\sigma\|_{2n/(n-2)}, \|\phi_-\|_{2n/(n-2)}$, and on ψ where ϕ_- is the (unique) solution of the prescribed scalar curvature equation, see Proposition 1 below,

$$-\frac{4(n-1)}{n-2}\Delta_g\phi + \text{Scal}_g\phi = -\frac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} \quad (1.4)$$

and $\psi \in C^\infty(M)$ is a positive function such that the scalar curvature of the conformal metric $\psi^{4/(n-2)}g$ is non-positive.

Clearly, one wishes to strengthen Theorem 1 by assuming that the metric g has a positive Yamabe constant. Unfortunately, the prescribed scalar curvature equation (1.4) has no solution in this setting. The role of metrics with non-positive Yamabe constant is to make sure that the solution ϕ of the constraint equations (1.3) has a strictly positive lower bound independent of W ; see Proposition 1 below. Such an important property is the key step in the proof of Claim 2.1 in [DGH13]. In general, strictly positive lower bounds for solutions (ϕ, W) of the constraint equations are barely understood.

Simply by observing the change in the dimension n in [DGH13, Theorem 1.1] and our Theorem 1 above, it is likelihood that a similar result could hold for metrics with positive Yamabe constant on 3-manifolds which is not conformally diffeomorphic to 3-sphere; see [ES86, Theorem 2.3].

2. PROOF OF THEOREM 1

Since our proof basically follows the lines in [DGH13, Theorem 1.1], we only indicate the difference. First, we assume that (ϕ, W) solves (1.3). Then by using

$$\gamma_g = \int_M |\sigma + \mathbb{L}W|_g^2 d\text{vol}_g$$

and transforming ϕ, W, σ via γ_g through

$$\tilde{\phi} = \gamma_g^{-\frac{n-2}{4n}}\phi, \quad \tilde{W} = \gamma_g^{-\frac{1}{2}}W, \quad \tilde{\sigma} = \gamma_g^{-\frac{1}{2}}\sigma, \quad (2.1)$$

the system (1.3) becomes

$$\begin{cases} \gamma_g^{-\frac{1}{n}} \left(-\frac{4(n-1)}{n-2} \Delta_g \tilde{\phi} + \text{Scal}_g \tilde{\phi} \right) = -\frac{n-1}{n} \tau^2 \tilde{\phi}^{\frac{n+2}{n-2}} + |\tilde{\sigma} + \mathbb{L}\tilde{W}|_g^2 \tilde{\phi}^{-\frac{3n-2}{n-2}}, & (2.2a) \\ \text{div}_g(\mathbb{L}\tilde{W}) = a \tilde{\phi}^{\frac{2n}{n-2}} \xi. & (2.2b) \end{cases}$$

Before going further, we prove the following simple observation.

Proposition 1. *Assume that the background metric g satisfies the assumptions in Theorem 1, then there exists a solution φ_- of Eq. (1.4). Furthermore, if (ϕ, W) is a solution of the system (1.3), then $\varphi_- \leq \varphi$ holds.*

Proof of Proposition 1. For the existence part, depending on the sign of the Yamabe constant for g , we have two cases: For the case of zero Yamabe constant, we make use of a general existence result due to Escobar–Schoen [ES86, Theorem 3.1] in which the dimension restriction $3 \leq n \leq 4$ and the fact $\int_M \tau^2 d\text{vol}_g > 0$ are crucial; see also [NX15, HNX16]. For the case of negative Yamabe constant, the argument is standard and it is well-known that Eq. (1.4) always admits at least one positive solution for any $n \geq 3$. For the estimation part, we refer to a useful result due to Dahl–Gicquaud–Humbert [DGH12, Lemma 2.2] and omit the details. \square

The proof in [DGH13] consists of seven claims. In their proof for Claim 2.1, it made use of no information on Scal_g . The only issue we need to take care is the existence of ϕ_- in (1.4). However, thanks to Proposition 1, we automatically obtain the same claim as theirs.

Claim 2.1. There exists a constant $c_1 > 0$ such that

$$\gamma_g \geq c_1 a^2.$$

For their Claim 2.2, since their proof made use of the sign of Scal_g , we need to modify that proof. However, the statement remains the same except up to constants.

Claim 2.2. There exists a constant $c_2 > 0$ such that

$$\int_M \tilde{\phi} d\text{vol}_g \leq c_2 a^{-\frac{3n-2}{n-2}}.$$

Proof of Claim 2.2. The only place in [DGH13] used the sign of Scal_g is to obtain the first inequality in (6), that is

$$\int_M |\tilde{\sigma} + \mathbb{L}\tilde{W}|_g^2 \tilde{\phi}^{-\frac{3n-2}{n-2}} d\text{vol}_g \leq \frac{n-1}{n} \int_M \tau^2 \tilde{\phi}^{\frac{n+2}{n-2}} d\text{vol}_g \quad (2.3)$$

where $\tilde{\sigma}$, $\tilde{\phi}$, and \tilde{W} are in (2.1). To obtain an inequality in the fashion of (2.3) without assuming a sign for Scal_g , we use the conformal change $\hat{g} = \psi^{4/(n-2)}g$ to transform (1.2a) into

$$\begin{aligned} & -\frac{4(n-1)}{n-2} \Delta_{\hat{g}}(\psi^{-1}\phi) + \text{Scal}_{\hat{g}}(\psi^{-1}\phi) \\ & = -\frac{n-1}{n} \tau^2 (\psi^{-1}\phi)^{\frac{n+2}{n-2}} + |\psi^{-2}(\sigma + \mathbb{L}W)|_{\hat{g}}^2 (\psi^{-1}\phi)^{-\frac{3n-2}{n-2}}, \end{aligned} \quad (2.4)$$

where the conformal factor ψ is chosen in such a way that the conformal metric \hat{g} has non-positive scalar curvature, i.e. $\text{Scal}_{\hat{g}} \leq 0$. From this, an integration over M with respect to

\widehat{g} yields

$$\int_M |\psi^{-2}(\sigma + \mathbb{L}W)|_{\widehat{g}}^2 (\psi^{-1}\phi)^{-\frac{3n-2}{n-2}} d\text{vol}_{\widehat{g}} \leq \frac{n-1}{n} \int_M \tau^2 (\psi^{-1}\phi)^{\frac{n+2}{n-2}} d\text{vol}_{\widehat{g}}. \quad (2.5)$$

Note that under the conformal change $\widehat{g} = \psi^{4/(n-2)}g$, there holds

$$d\text{vol}_{\widehat{g}} = \psi^{\frac{2n}{n-2}} d\text{vol}_g$$

and

$$|\psi^{-2}(\sigma + \mathbb{L}W)|_{\widehat{g}}^2 = \psi^{-\frac{4n}{n-2}} |\sigma + \mathbb{L}W|_g^2.$$

Using the two relations above, we can re-evaluate (2.5) in terms of g as follows

$$\int_M \psi |\sigma + \mathbb{L}W|_g^2 \phi^{-\frac{3n-2}{n-2}} d\text{vol}_g \leq \frac{n-1}{n} \int_M \tau^2 \psi \phi^{\frac{n+2}{n-2}} d\text{vol}_g. \quad (2.6)$$

Hence, dividing both sides of (2.6) by $\gamma_g^{(n+2)/(4n)}$ gives

$$\int_M |\widetilde{\sigma} + \mathbb{L}\widetilde{W}|_g^2 \widetilde{\phi}^{-\frac{3n-2}{n-2}} d\text{vol}_g \leq \frac{n-1}{n} \left(\frac{\max \psi}{\min \psi} \right) \int_M \tau^2 \widetilde{\phi}^{\frac{n+2}{n-2}} d\text{vol}_g. \quad (2.7)$$

Once we have (2.7) in hand we can go through the rest of their proof for Claim 2.2 to get a new constant

$$c_2 = \left(\frac{1}{2} \right)^{\frac{3n-2}{n-2}} \left(\frac{n-1}{n} \frac{\max \psi}{\min \psi} \right)^{\frac{n}{n-2}} \left(\frac{\|\mathbb{L}\xi\|_{\infty}}{\inf |\xi|^2} \right)^{\frac{3n-2}{n-2}} \left(\int_M \tau^{\frac{4n}{n-2}} d\text{vol}_g \right)^{1/2}.$$

This completes our proof of Claim 2.2. \square

After completing the proof of Claim 2.2, we obtain Claim 2.3 as in their paper.

Claim 2.3. There exists a constant $c_3 > 0$ such that

$$\int_M |\widetilde{\sigma} + \mathbb{L}\widetilde{W}|_g^2 \widetilde{\phi}^{-\frac{3n-2}{n-2}} d\text{vol}_g \leq c_3 a^{-\frac{(3n-2)(n+2)}{2n(n-2)}}.$$

For Claim 2.4, although some integrals involving Scal_g were estimated in the proof, we find that the sign of Scal_g in those estimates plays no role. This can be easily seen from their formula for c_4 where $|\text{Scal}_g|$ was used. Therefore, we can conclude the following result.

Claim 2.4. There exists a constant $c_4 > 0$ such that

$$\int_M \widetilde{\phi}^{\frac{4n}{n-2}} d\text{vol}_g \leq c_4$$

provided a is large enough.

Since Claims 2.5–2.7 in their paper involved no information on the sign of Scal_g directly, we conclude that their claims still hold in our context. For the sake of completeness, we mention Claims 2.5–2.7 in [DGH13] below.

Claim 2.5. There exists a constant $c_5 > 0$ such that

$$\int_M |\widetilde{\sigma} + \mathbb{L}\widetilde{W}|_g^{\frac{2n}{n-2}} d\text{vol}_g \leq c_5 a^{\frac{2n}{n-2}},$$

provided a is large enough.

Claim 2.6. There exists a constant $c_6 > 0$ such that

$$\int_M |\widetilde{\sigma} + \mathbb{L}\widetilde{W}|_g^2 \widetilde{\phi}^{\frac{4}{n-2}} d\text{vol}_g \leq c_6 a^{2 - \frac{2(3n-2)}{n(n-2)}}.$$

Claim 2.7. There exists a constant $c_7 > 0$ such that

$$\int_M |\tilde{\sigma} + \mathbb{L}\tilde{W}|_g^2 \tilde{\phi}^{-\frac{4}{n-2}} d\text{vol}_g \leq c_7 a^{-\frac{2(n+2)}{n(n-2)}}.$$

Thus, one can obtain a non-existence result for large a similar to their equation.

Proof of Theorem 1. To prove the theorem, we use (2.1) and Claims 2.6 and 2.7 above to obtain

$$\begin{aligned} 1 &\leq \left(\int_M |\tilde{\sigma} + \mathbb{L}\tilde{W}|_g^2 \tilde{\phi}^{-\frac{4}{n-2}} d\text{vol}_g \right) \left(\int_M |\tilde{\sigma} + \mathbb{L}\tilde{W}|_g^2 \tilde{\phi}^{-\frac{4}{n-2}} d\text{vol}_g \right) \\ &\leq c_6 c_7 a^{2-\frac{2(3n-2)}{n(n-2)}} a^{-\frac{2(n+2)}{n(n-2)}} \\ &= c_6 c_7 a^{6-2\frac{2n}{n-2}}. \end{aligned}$$

Thus in case $3 \leq n \leq 5$, we obtain a contradiction if a is large enough. This provides us a non-existence result as claimed. \square

ACKNOWLEDGMENTS

The author would like to thank the Vietnam Institute for Advanced Study in Mathematics (VIASM) for hosting and support where this note was carried.

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