Semigroups generated by non-local boundary conditions: Well-posednesss and asymptotics

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Abstract. We investigate a second order elliptic differential operator $A_{\beta,\mu}$ on a bounded, open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary subject to a nonlocal boundary condition of Robin type. More precisely we have $\beta \in L^\infty(\partial\Omega)$ and $\mu\colon \partial\Omega \to \mathscr{M}(\overline{\Omega})$, and boundary conditions of the form

$$\partial_\nu^{\mathscr{A}} u(z) + \beta(z)u(z) = \int_{\overline{\Omega}} u(x)\, \mu(z)(\mathrm{d}x), \; z \in \partial\Omega,$$

where $\partial_\nu^{\mathscr{A}}$ denotes the weak conormal derivative with respect to our differential operator. Under suitable conditions on the coefficients of the differential operator and the function μ we show that $A_{\beta,\mu}$ generates a holomorphic semigroup $T_{\beta,\mu}$ on $L^\infty(\Omega)$ which enjoys the strong Feller property. In particular, it takes values in $C(\overline{\Omega})$. Its restriction to $C(\overline{\Omega})$ is strongly continuous. We also establish positivity and contractivity of the semigroup under additional assumptions and study the asymptotic behavior of the semigroup.

1. Introduction

The aim of this paper is to prove existence and uniqueness of solutions to diffusion equations with non-local Robin boundary conditions. Let us describe this in more detail. We consider a bounded, open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. As far as our boundary condition is concerned, we make the following assumptions.

Hypothesis 1.1. We are given a real-valued function $\beta \in L^\infty(\partial\Omega)$, where $\partial\Omega$ is endowed with surface measure σ. Moreover, we are given a map $\mu : \partial\Omega \to \mathscr{M}(\overline{\Omega})$, the space of complex-valued measures on $\overline{\Omega}$, which satisfies the following 3 conditions.

(a) For every function $f \in B_b(\overline{\Omega})$, the space of all bounded and Borel measurable functions on $\overline{\Omega}$, the map $z \mapsto \langle f, \mu(z)\rangle := \int_{\overline{\Omega}} f(x)\mu(z)(\mathrm{d}x)$ is measurable;

(b) for some $p > d-1$ with $p \geq 2$ we have $\int_{\partial\Omega} \|\mu(z)\|^p \,\mathrm{d}\sigma(z) < \infty$ and

(c) there exists a positive and bounded measure τ on $\overline{\Omega}$ such that for every $z \in \partial\Omega$ the measure $\mu(z)$ is absolutely continuous with respect to τ.

In (a) it actually suffices to assume that the map $z \mapsto \langle f, \mu(z)\rangle$ is measurable for all $f \in C(\overline{\Omega})$. The measurability for those f which are merely bounded and measurable follows by a monotone class argument, cf. the proof of Lemma 6.1 in [21]. We will see later on that if instead of (a) we assume

(a′) For every $f \in B_b(\overline{\Omega})$ the map $z \mapsto \langle f, \mu(z)\rangle$ is continuous

then parts (b) and (c) in Hypothesis 1.1 are automatically satisfied.

Assuming Hypothesis 1.1 we can define the operator $\Delta_{\beta,\mu}$ on $L^\infty(\Omega)$ by

$$\begin{aligned} D(\Delta_{\beta,\mu}) := \{u \in H^1(\Omega) \cap C(\overline{\Omega}) :\; & \Delta u \in L^\infty(\Omega), \\ & \partial_\nu u(z) + \beta u(z) = \langle u, \mu(z)\rangle \;\forall\, z \in \partial\Omega\} \\ \Delta_{\beta,\mu} u = \Delta u. & \end{aligned}$$

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Here we have used the following weak definition of the *normal derivative* $\partial_\nu u$.

Definition 1.2. For a function $u \in H^1(\Omega)$, we write $\operatorname{tr} u$ for its trace in $L^2(\partial\Omega)$. Let $u \in H^1(\Omega)$ be such that $\Delta u \in L^2(\Omega)$ and let $h \in L^2(\partial\Omega)$. We say that $\partial_\nu u = h$ if Green's formula

$$\int_\Omega \Delta u v \,\mathrm{d}x + \int_\Omega \nabla u \nabla v \,\mathrm{d}x = \int_{\partial\Omega} h \operatorname{tr} v \,\mathrm{d}\sigma$$

holds for all $v \in H^1(\Omega)$.

In what follows we will not distinguish between a function $u \in H^1(\Omega)$ and its trace $\operatorname{tr} u$ in integrals over the boundary $\partial\Omega$.

With this definition of the normal derivative the operator $\Delta_{\beta,\mu}$ is well-defined. Indeed, if $u \in D(\Delta_{\beta,\mu})$ then $u \in C(\overline{\Omega})$ whence

$$h(z) := \langle u, \mu(z)\rangle - \beta(z)u(z)$$

defines a function $h \in L^2(\partial\Omega)$. Since furthermore $u \in H^1(\Omega)$ and $\Delta u \in L^\infty(\Omega) \subset L^2(\Omega)$ it makes sense to say that $\partial_\nu u = h$. This condition is the *Robin boundary condition* we are interested in with *local* part $\beta \operatorname{tr} u$ and *non-local* part $\langle u, \mu(\cdot)\rangle$.

We also consider the part $\Delta^C_{\beta,\mu}$ of $\Delta_{\beta,\mu}$ in $C(\overline{\Omega})$ given by

$$\begin{aligned} D(\Delta^C_{\beta,\mu}) &:= \{u \in H^1(\Omega) \cap C(\overline{\Omega}) : \ \Delta u \in C(\overline{\Omega}), \\ &\qquad \partial_\nu u + \beta u|_{\partial\Omega} = \langle u, \mu(\cdot)\rangle\} \\ \Delta^C_{\beta,\mu} u &= \Delta u. \end{aligned}$$

We will prove the following generation theorem.

Theorem 1.3. *Assuming Hypothesis 1.1, the operator $\Delta_{\beta,\mu}$ generates a holomorphic semigroup $(T_{\beta,\mu}(t))_{t>0}$ on $L^\infty(\Omega)$ which satisfies the strong Feller property. In particular, this semigroup leaves the space $C(\overline{\Omega})$ invariant. Its restriction to $C(\overline{\Omega})$ is a strongly continuous semigroup whose generator is $\Delta^C_{\beta,\mu}$.*

We refer to Section 2 for the definition of holomorphic semigroups which are not strongly continuous at 0 and for an explanation of the strong Feller property. We will actually prove Theorem 1.3 in more generality, replacing the Laplacian with a general second order strictly elliptic differential operator with measurable coefficients.

We will also establish positivity and contractivity of the semigroup $T_{\beta,\mu}$ under additional assumptions on β and μ. In the case of Theorem 1.3, where we consider the Laplacian, the conditions are as follows. If the measures $\mu(z)$ are positive for all $z \in \partial\Omega$ then the semigroup $T_{\beta,\mu}$ is *positive*; i.e. each $T_{\beta,\mu}(t)$ leaves the positive cone $L^\infty(\Omega)_+$ of $L^\infty(\Omega)$ invariant. If additionally we have that

$$\mu(z, \overline{\Omega}) \le \beta(z) \quad \text{for almost all } z \in \partial\Omega, \tag{1.1}$$

then the semigroup $T_{\beta,\mu}$ is *sub-Markovian*, i.e. $T_{\beta,\mu}$ is positive and $T_{\beta,\mu}(t)\mathbb{1} \le \mathbb{1}$ for all $t > 0$. If equality holds in (1.1) then $T_{\beta,\mu}$ is *Markovian*, i.e. $T_{\beta,\mu}(t)$ is positive and $T_{\beta,\mu}(t)\mathbb{1} = \mathbb{1}$. In these situations we will also study the asymptotic behavior of the semigroup $T_{\beta,\mu}$. In the sub-Markovian, non-Markovian case the semigroup converges in operator norm to 0, whereas in the Markovian case the orbits converge to an equilibrium.

The proof of Theorem 1.3 is based on a perturbation result by Greiner which we explain in Section 2. We actually present a slight generalization of Greiner's result which establishes additional properties of the perturbed semigroup. In a previous paper [8] we had treated non-local Dirichlet boundary conditions. It should be said that the techniques we use here are different from the case of Dirichlet boundary conditions where the maximum principle plays an essential role and Greiner's perturbation result cannot be applied.

Non-local Robin boundary conditions of the above form occur in several concrete situations, for example in heat control, where the heat is measured in the interior and the control is via the boundary.

The present article is part of an ongoing project to understand boundary conditions which yield realizations of differential operators that generate Markovian semigroups. This project was initiated by Feller [17, 18] in dimension one and Ventcel [29] in higher dimensions. Boundary conditions of the form considered here were studied by Galakhov and Skubachevskiĭ [19] and Taira [28] with different techniques and under more restrictive assumptions.

The structure of this article is as follows. In Section 2 we present Greiner's boundary perturbation theory along with our modifications. Section 3 contains results on elliptic differential operators with local Robin boundary conditions which are needed later on. In Section 4 we prove our main generation result. Section 5 contains our results concerning the asymptotic behavior of the semigroups and Section 6 is devoted to the special situation where all measures $\mu(z)$ are absolutely continuous with respect to Lebesgue measure. In the concluding Section 7 we give some examples where Hypothesis 1.1 is satisfied, in particular, we prove that it is satisfied whenever condition (a′) is fulfilled. In the appendix we present some general results on the asymptotic behavior of positive semigroups, which we use in Section 5.

2. Greiner's boundary perturbation revisited

An important tool in this article is boundary perturbation of the generator of an analytic semigroup, established by Greiner in his seminal article [20]. As a matter of fact, we need some extensions of Greiners results whose proofs follow along the lines of Greiners article with minor modifications. More precisely, we will consider semigroups which are not necessarily strongly continuous. Besides being interesting in its own right, this will allow us to establish under appropriate assumptions the strong Feller property for the perturbed semigroup. Likewise, other modifications allow us to prove compactness, positivity and domination for the perturbed semigroup. In an effort of being self contained and for the convenience of the reader we provide complete proofs.

Before addressing the perturbation results themselves, let us recall some properties of not necessarily strongly continuous analytic semigroups. For more information we refer the reader to [6, Section 3.7] or [24, Section 2.1].

A *semigroup* on a Banach space X is a strongly continuous mapping $T : (0,\infty) \to \mathscr{L}(X)$ such that (i) $T(t+s) = T(t)T(s)$ for all $t, s > 0$, (ii) $\sup_{0<t<1} \|T(t)\| < \infty$ and (iii) if $T(t)f = 0$ for all $t > 0$, then $f = 0$. If $T(t)f \to f$ as $t \to 0$ for every $f \in X$, then T is called *strongly continuous*. In this case, if we set $T(0) = I$, the identity operator, we obtain a strongly continuous mapping even on the interval $[0,\infty)$. The semigroup is called *analytic* if the map T has a holomorphic extension to some sector $\Sigma_\vartheta := \{z \in \mathbb{C} : |\arg z| < \vartheta\}$ which is bounded on $\{z \in \Sigma_\vartheta : |z| \leq 1\}$.

It follows that given a semigroup T we find constants $M, \omega > 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t > 0$. It can be proved, see [6, Equation (3.13)], that there exists a unique operator G such that $(\omega,\infty) \subset \rho(G)$ and

$$R(\lambda, G)f = \int_0^\infty e^{-\lambda t} T(t)f\,dt$$

whenever $\lambda > \omega$. The operator G is called the *generator* of the semigroup T.

An operator G on X generates an analytic semigroup if and only if there is an $\omega \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subset \rho(G)$ and the *holomorphic estimate*

$$\sup_{\operatorname{Re}\lambda > \omega} \|\lambda R(\lambda, G)\| < \infty$$

holds true. A proof of this fact can be found in [24, Proposition 2.1.11] or [6, Corollary 3.7.12 and Proposition 3.7.4].

For an analytic semigroup it follows from [24, Proposition 2.1.4] that $T(t)f \to f$ as $t \to 0$ if and only if $f \in \overline{D(G)}$. In particular, T is strongly continuous if and only if its generator is densely defined. Moreover, since the semigroup T leaves $D(G)$ invariant, it also leaves $\overline{D(G)}$ invariant and its restriction to $\overline{D(G)}$ is a strongly continuous semigroup.

We now return to our main topic, Greiners boundary perturbation. Throughout this section, we make the following assumption.

Hypothesis 2.1. We are given complex Banach spaces $(X, \|\cdot\|_X)$, $(D, \|\cdot\|_D)$ and $(\partial X, \|\cdot\|_{\partial X})$, where D is continuously embedded into X. We identify D with its image in X and frequently consider the closure $\overline{D}$ of D in X. Moreover, we are given a continuous *maximal operator* $A : D \to X$, a continuous *boundary operator* $B : D \to \partial X$ and a *boundary perturbation* $\Phi : \overline{D} \to \partial X$. We assume that all of these mappings are linear and continuous. Moreover, we assume the following.

(a) The boundary operator B is surjective;
(b) the boundary perturbation Φ is compact;
(c) the operator $A_0 := A|_{\ker B}$ generates an analytic semigroup on X and we have $\overline{D(A_0)} = \overline{D}$. We denote by ω a real number such that any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ belongs to $\rho(A_0)$.

In comparison to Greiner's original work, the main difference in our assumption is that we do not assume the operator A_0 to be densely defined in X. Consequently, the semigroup T generated by A_0 need not be strongly continuous. However, since $\overline{D(A_0)} = \overline{D}$, for every $f \in \overline{D}$ the orbit $t \mapsto T(t)f$ is strongly continuous on $[0, \infty)$ and T restricts to a strongly continuous analytic semigroup on $\overline{D}$. For us, the main motivation to allow semigroups which are not strongly continuous lies in the fact that we can treat semigroups on the space $L^\infty(\Omega)$ where Ω is a bounded open subset of $\mathbb{R}^d$. This is important to establish the strong Feller property of semigroups. By a result of Lotz [23] (see also [6, Corollary 4.3.19]), a strongly continuous semigroup on $L^\infty(\Omega)$ necessarily has a bounded generator. Thus, to study semigroups generated by differential operators, one has to allow for semigroups which are not strongly continuous.

Given the above maps, we define the perturbed operator A_Φ by

$$D(A_\Phi) := \{u \in D : Bu = \Phi u\}, \quad A_\Phi u = Au.$$

We can now formulate our version of Greiner's result.

Theorem 2.2. *Assuming Hypothesis 2.1, the operator A_Φ generates an analytic semigroup on X which restricts to a strongly continuous semigroup on $\overline{D}$.*

We prepare the proof of Theorem 2.2 with some preliminary results.

Lemma 2.3. *Assume that $\lambda \in \rho(A_0)$. Then $D = D(A_0) \oplus \ker(\lambda - A)$.*

Proof. If $u \in D(A_0) \cap \ker(\lambda - A)$, then $u \in D(A_0)$ satisfies $A_0 u = \lambda u$. As $\lambda \in \rho(A_0)$ we must have $u = 0$. Now let $u \in D$ be arbitrary. As $\lambda - A_0$ is surjective, we find $u_0 \in D(A_0)$ with $(\lambda - A)u = (\lambda - A_0)u_0$. Consequently $u - u_0 \in \ker(\lambda - A)$ whence $u = u_0 + (u - u_0) \in D(A_0) + \ker(\lambda - A)$. $\square$

In our framework we can formulate well-posedness of the following boundary value problem (2.1).

Lemma 2.4. *Let $\lambda \in \rho(A_0)$. Then for every $h \in \partial X$ the problem*

$$\begin{cases} \lambda u - Au &= 0 \\ Bu &= h \end{cases} \tag{2.1}$$

has a unique solution $u =: S_\lambda h$ in D. The operator $S_\lambda : \partial X \to D$ is continuous, $BS_\lambda = I_{\partial X}$ and $S_\lambda B$ is the projection onto $\ker(\lambda - A)$ *along* $D(A_0)$.

Proof. By Lemma 2.3 the map B defines a continuous bijection between $\ker(\lambda - A)$ and ∂X. As a consequence of the open mapping theorem $S_\lambda := (B|_{\ker(\lambda - A)})^{-1}$ is a continuous linear operator from ∂X to $\ker(\lambda - A)$. Obviously, $u := S_\lambda h$ solves (2.1). If $\tilde{u}$ was another solution, we must have $u - \tilde{u} \in \ker B \cap \ker(\lambda - A) = \{0\}$ by Lemma 2.3. This proves uniqueness. The last assertions are obvious from the definition. $\square$

Lemma 2.5. *Let $\lambda \in \rho(A_0)$. Then for $u \in \overline{D}$ one has $u \in D(A_\Phi)$ if and only if $(I - S_\lambda \Phi)u \in D(A_0)$. In this case*

$$(\lambda - A_\Phi)u = (\lambda - A_0)(I - S_\lambda \Phi)u$$

for every $u \in D(A_\Phi)$. In particular, if $(I - S_\lambda \Phi) : \overline{D} \to \overline{D}$ is invertible, we have $\lambda \in \rho(A_\Phi)$ and

$$R(\lambda, A_\Phi) = (I - S_\lambda \Phi)^{-1} R(\lambda, A_0). \tag{2.2}$$

Proof. Let us first assume that $u \in D(A_\Phi)$, i.e. $u \in D$ and $Bu = \Phi u$. Since $BS_\lambda = I_{\partial X}$ by Lemma 2.4, we find $B(I - S_\lambda \Phi)u = Bu - BS_\lambda \Phi u = Bu - \Phi u = 0$. Thus $(I - S_\lambda \Phi)u \in \ker B$ and consequently $(I - S_\lambda \Phi)u \in D(A_0)$.

Conversely, if we assume that $u - S_\lambda \Phi u \in D(A_0)$, then $u = (I - S_\lambda \Phi)u + S_\lambda \Phi u \in D$, as S_λ takes values in D, and $Bu = BS_\lambda \Phi u = \Phi u$ since $BS_\lambda = I_{\partial X}$. Thus $u \in D(A_\Phi)$.

Let us now assume that $u \in D(A_\Phi)$ or, equivalently, that $(I - S_\lambda \Phi)u \in D(A_0)$. Then

$$(\lambda - A_0)(I - S_\lambda \Phi)u = (\lambda - A)u - (\lambda - A)S_\lambda \Phi u = (\lambda - A)u$$

since S_λ takes values in $\ker(\lambda - A)$. This implies (2.2). $\square$

We now obtain the following criterion to prove that A_Φ generates an analytic semigroup.

Proposition 2.6. *Assume that there is some $\rho > \omega$ such that for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \rho$ the map $I - S_\lambda \Phi$ is invertible with*

$$C := \sup_{\operatorname{Re}\lambda > \rho} \|(I - S_\lambda \Phi)^{-1}\|_{\mathscr{L}(\overline{D})} < \infty.$$

Then A_Φ generates an analytic semigroup on X.

Proof. Set

$$M := \sup_{\operatorname{Re}\lambda > \rho} \|\lambda R(\lambda, A_0)\| < \infty$$

since A_0 generates an analytic semigroup. As a consequence of Lemma 2.5, for $\operatorname{Re}\lambda > \rho$ we have $\lambda \in \rho(A_\Phi)$ and

$$\|\lambda R(\lambda, A_\Phi)\| = \|\lambda (I - S_\lambda \Phi)^{-1} R(\lambda, A_0)\| \le \|(I - S_\lambda \Phi)^{-1}\| \|\lambda R(\lambda, A_0)\| \le CM.$$

This implies that A_Φ generates an analytic semigroup on X. $\square$

We can now prove the main result of this section.

Proof of Theorem 2.2. In view of Proposition 2.6, making use of the Neumann series, it suffices to prove that $S_\lambda \Phi \to 0$ in $\mathscr{L}(\overline{D})$ as $\operatorname{Re}\lambda \to \infty$. Since $\Phi : \overline{D} \to \partial X$ is compact it suffices to prove that $S_\lambda h \to 0$ as $\operatorname{Re}\lambda \to \infty$ for every $h \in \partial X$.

To prove this, let $h \in \partial X$ and fix $\mu \in \rho(A_0)$. We put

$$u_\lambda := S_\lambda h, \quad u_\mu := S_\mu h \quad \text{and} \quad u = u_\lambda - u_\mu.$$

Note that $u_\lambda, u_\mu, u \in D$ and that $u \in D(A_0)$. Since $u_\lambda \in \ker(\lambda - A)$ and $u_\mu \in \ker(\mu - A)$ we have

$$(\lambda - A_0)u = -(\lambda - A)u_\mu = (\mu - \lambda)u_\mu$$

and hence $u = (\mu - \lambda)R(\lambda, A_0)u_\mu$. Consequently,

$$u_\lambda = u_\mu - \lambda R(\lambda, A_0)u_\mu + \mu R(\lambda, A_0)u_\mu \to u_\mu - u_\mu + 0 = 0$$

as $\operatorname{Re}\lambda \to \infty$, since $\lambda R(\lambda, A_0)f \to f$ for every $f \in \overline{D(A_0)} = \overline{D}$. $\square$

We can now establish some additional properties of the operator A_Φ and the semigroup generated by it. We start with compactness.

Corollary 2.7. *In the situation of Theorem 2.2, if A_0 has compact resolvent, then so does A_Φ.*

Proof. This follows immediately from the identity (2.2) and the ideal property of compact operators. $\square$

Next we address positivity of the semigroup. Most often we will be concerned with Banach lattices such as $C(\overline{\Omega})$ or $L^\infty(\Omega)$. However, we will occasionally (for example in the following corollaries) also consider closed subspaces of such spaces and therefore need the notion of positivity also in a more general setting. To that end, we assume that our Banach space X is the complexification of a real ordered Banach space $X_\mathbb{R}$. This means that in the real Banach space $X_\mathbb{R}$ a positive, proper, closed cone X_+ is given, i.e. we have $X_+ + X_+ \subset X_+$, $\mathbb{R}_+ \cdot X_+ \subset X_+$ and $X_+ \cap (-X_+) = \{0\}$. For $u \in X$ we write $u \geq 0$ if $u \in X_+$. An operator $S : X \to X$ is called *positive* if $SX_+ \subset X_+$, we write $S \geq 0$. Given two operators $S_1, S_2 : X \to X$, we write $S_1 \leq S_2$ if $S_2 - S_1 \geq 0$. A semigroup T on X is called *positive* if $T(t) \geq 0$ for all $t > 0$.

If $Y \subset X$ is a closed subspace of X, then $Y_+ := Y \cap X_+$ is a closed, proper cone, such that $Y_\mathbb{R} := Y \cap X_\mathbb{R}$ becomes an ordered Banach space. Note that we do not assume that our cone is generating, i.e. we do not necessarily have that $X_+ - X_+ = X_\mathbb{R}$.

Corollary 2.8. *Assume in addition to Hypothesis 2.1 that X is the complexification of a real ordered Banach space and that A_0 generates a positive semigroup. If there is a $\rho > \omega$ such that for $\lambda \in \mathbb{R}$ with $\lambda > \rho$ the operator $S_\lambda\Phi$ is positive, then also the semigroup generated by A_Φ is positive.*

Proof. If the semigroup T generated by A_0 is positive then we have $R(\lambda, A_0) \geq 0$ for $\lambda > \omega$, as the resolvent is given as the Laplace transform of the semigroup. For sufficiently large $\lambda \in \mathbb{R}$ we have $\|S_\lambda\Phi\| < 1$ and $S_\lambda\Phi$ positive. Thus, by the Neumann series,

$$(I - S_\lambda\Phi)^{-1} = \sum_{n=0}^{\infty}(S_\lambda\Phi)^n$$

is a positive operator. It follows from (2.2) that $R(\lambda, A_\Phi)$ is positive for sufficiently large λ. It follows from the Post–Widder inversion formula [6, Theorem 1.7.7] that the semigroup generated by A_Φ is positive. $\square$

Next we want to compare different perturbations of our operator A. We can obtain different perturbations by either using different boundary operators B or by using different boundary perturbations Φ.

Corollary 2.9. *Let $X, D, \partial X$ and A be as in Hypothesis 2.1 and assume that X and ∂X are complexifications of real ordered Banach spaces. Moreover, assume that we are given maps $B_1, B_2 : D \to \partial X$ and $\Phi_1, \Phi_2 : \overline{D} \to \partial X$ such that Hypothesis 2.1 is satisfied for the operators A, B_1, Φ_1 and the operators A, B_2, Φ_2. We write $A_0^j := A|_{\ker B_j}$ and $S_\lambda^j := (B_j|_{\ker(\lambda - A)})^{-1}$ for $j = 1, 2$. Finally, we assume that*

(a) *The semigroup generated by* A_0^j *is positive for* $j = 1, 2$*;*
(b) $0 \le \Phi_1 \le \Phi_2$*;*
(c) *For some* $\rho > \omega$ *and all* $\lambda > \rho$ *we have* $0 \le S_\lambda^1 \le S_\lambda^2$*;*
(d) *If* $u \in D$ *is positive, then* $B_2 u \le B_1 u$*.*

Then for the semigroups T_1 *generated by* $A^1_{\Phi_1}$ *and* T_2 *generated by* $A^2_{\Phi_2}$ *we have* $0 \le T_1(t) \le T_2(t)$ *for all* $t > 0$*.*

Proof. Let us first note that since the operators Φ_j and S_λ^j are positive for $\lambda > \rho$ and $j = 1, 2$, it follows from Corollary 2.8 that T_1 and T_2 are positive semigroups. It follows from (b) and (c) that

$$(I - S_\lambda^1\Phi_1)^{-1} = \sum_{n=0}^{\infty}(S_\lambda^1\Phi_1)^n \le \sum_{n=0}^{\infty}(S_\lambda^2\Phi_2)^n = (I - S_\lambda^2\Phi_2)^{-1}$$

for all $\lambda > \omega$. Now fix $f \ge 0$ and $\lambda > \rho$. We put $u_j := R(\lambda, A_0^j)f$. Then $(\lambda - A)(u_1 - u_2) = 0$ and $B_1 u_1 = B_2 u_2 = 0$. Using our assumption (d) and the fact that $u_1 \ge 0$, we see that

$$B_2(u_1 - u_2) = B_2 u_1 - B_1 u_1 \le 0.$$

Consequently, as $u_1 - u_2 = S_\lambda^2(B_2(u_1 - u_2))$ and S_λ^2 is positive $u_1 - u_2 \le 0$. This proves $R(\lambda, A_0^1) \le R(\lambda, A_0^2)$. Combining this with the above and Equation (2.2), we find

$$R(\lambda, A^1_{\Phi_1}) = (I - S_\lambda^1\Phi_1)^{-1}R(\lambda, A_0^1) \le (I - S_\lambda^2\Phi_2)^{-1}R(\lambda, A_0^2) = R(\lambda, A^2_{\Phi_2})$$

for all sufficiently large λ. By the Post–Widder inversion formula [6, Theorem 1.7.7] it follows that $T_1 \le T_2$. $\square$

We next address the strong Feller property. To that end, we consider the situation where $X = L^\infty(\Omega)$ for some bounded open set $\Omega \subset \mathbb{R}^d$ and $\overline{D} = C(\overline{\Omega})$. A *strong Feller operator (on* $L^\infty(\Omega)$*)* is a bounded linear operator T on $L^\infty(\Omega)$ taking values in $C(\overline{\Omega})$ such that whenever f_n is a bounded sequence in $L^\infty(\Omega)$ converging pointwise almost everywhere to f, we have $Tf_n \to Tf$ pointwise. Classically, given a Polish state space E, a strong Feller operator is defined as an operator on $B_b(E)$, the space of all bounded Borel measurable functions on E, which are given through a transition kernel and take values in the space $C_b(E)$. It follows from [8, Lemma 5.5] that a strong Feller operator on $L^\infty(\Omega)$ can be extended to a classical strong Feller operator $\tilde{T}$ on the state space $E := \overline{\Omega}$ by setting $\tilde{T}f = T[f]$, where $[f]$ denotes the equivalence class modulo equality almost everywhere. We refer to [8, Section 5] for a more thorough discussion of the relationship of these two concepts. In connection with differential operators it is more natural to identify functions which are equal almost everywhere, whence in this article we will work exclusively with the concept of a strong Feller operator on $L^\infty(\Omega)$.

Corollary 2.10. *Assume in addition to Hypothesis 2.1 that* $X = L^\infty(\Omega)$ *and* $\overline{D} = C(\overline{\Omega})$*. If* A_0 *generates a strong Feller semigroup on* X*, then so does* A_Φ*.*

Proof. By the proof of [8, Corollary 5.8] it suffices to prove that for sufficiently large $\operatorname{Re}\lambda$ the operator $R(\lambda, A_\Phi)$ is a strong Feller operator. But this follows from (2.2): The hypothesis implies that $R(\lambda, A_0)$ is a strong Feller operator, in particular it maps $L^\infty(\Omega)$ to $C(\overline{\Omega})$. Since $U := (I - S_\lambda\Phi)^{-1}$ is a bounded linear operator on $C(\overline{\Omega})$ also $R(\lambda, A_\Phi)$ maps $L^\infty(\Omega)$ to $C(\overline{\Omega})$. Moreover, if f_n is a bounded sequence in $L^\infty(\Omega)$ converging pointwise almost everywhere to f, then $R(\lambda, A_0)f_n$ is a bounded sequence which converges pointwise to $R(\lambda, A_0)f$. Since U is bounded on $C(\overline{\Omega})$ we have for $x \in \overline{\Omega}$

$$\begin{aligned} R(\lambda, A_\Phi)f_n(x) &= \langle UR(\lambda, A_0)f_n, \delta_x\rangle = \langle R(\lambda, A_0)f_n, U^*\delta_x\rangle \\ &\to \langle R(\lambda, A_0)f, U^*\delta_x\rangle = R(\lambda, A_\Phi)f(x), \end{aligned}$$

where we have used dominated convergence. $\square$

3. Local Robin boundary conditions

In this section we collect some results on elliptic operators with local Robin boundary conditions which we will need in the next section when we establish our results concerning non-local boundary conditions.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. As we are talking about positive semigroups, we will consider real-valued spaces $L^p(\Omega), C(\overline{\Omega}), C_b(\Omega)$ and $B_b(\Omega)$ throughout. Only when we are concerned with analytic semigroups we need spaces of complex-valued functions, in which case we pass to the complexification of these spaces. Concerning the coefficients of our operator we make the following assumptions.

Hypothesis 3.1. We are given bounded, real-valued, measurable functions a_{ij}, b_j, c_j, d_0 on Ω for $i, j = 1, \ldots d$. The diffusion coefficients $a = (a_{ij})$ are assumed to be bounded and *strictly elliptic*, i.e. there is a constant $\eta > 0$ such that for all $\xi \in \mathbb{R}^d$ and almost all $x \in \Omega$ we have

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \eta|\xi|^2.$$

With these assumptions we define the operator $\mathscr{A} : H^1(\Omega) \to \mathscr{D}(\Omega)'$ by

$$\mathscr{A}u := -\sum_{i,j=1}^{d} D_i(a_{ij}D_ju) - \sum_{j=1}^{d} D_j(b_ju) + \sum_{j=1}^{d} c_jD_ju + d_0u.$$

Here, $\mathscr{D}(\Omega) = C_c^\infty(\Omega)$ is the space of all test functions and $\mathscr{D}(\Omega)'$ is the space of all distributions. We introduce the continuous bilinear form $\mathfrak{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$\mathfrak{a}[u, v] := \sum_{i,j=1}^{d} \int_\Omega a_{ij}D_iuD_jv \, \mathrm{d}x + \sum_{j=1}^{d} \int_\Omega b_juD_jv + c_j(D_ju)v \, \mathrm{d}x + \int_\Omega d_0uv \, \mathrm{d}x$$

for $u, v \in H^1(\Omega)$. Thus $\langle \mathscr{A}u, \varphi\rangle = \mathfrak{a}[u, \varphi]$ for all $u \in H^1(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$.

If $u \in H^1(\Omega)$, we say that $\mathscr{A}u \in L^2(\Omega)$ if there exists a function $f \in L^2(\Omega)$ such that $\langle \mathscr{A}u, \varphi\rangle = [f, \varphi]$ for all $\varphi \in \mathscr{D}(\Omega)$. Here, and in what follows,

$$[f, g] := \int_\Omega fg \, \mathrm{d}x$$

denotes the scalar product in $L^2(\Omega)$. If $\mathscr{A}u \in L^2(\Omega)$ the function f above is unique and we identify $\mathscr{A}u$ and f.

Next we define the *weak conormal derivative* by testing against functions in $H^1(\Omega)$ rather than functions in $\mathscr{D}(\Omega)$ only.

Definition 3.2. Let $u \in H^1(\Omega)$ be such that $\mathscr{A}u \in L^2(\Omega)$. For a function $h \in L^2(\partial\Omega)$ we say that h is the *weak conormal derivative of* u and write $\partial_\nu^{\mathscr{A}}u := h$ if the Green formula

$$\mathfrak{a}[u, v] - [\mathscr{A}u, v] = \int_{\partial\Omega} hv \, \mathrm{d}\sigma$$

holds for all $v \in H^1(\Omega)$.

Under our assumptions on the coefficients the weak conormal derivative, if it exists, is unique. It depends on the operator $\mathscr{A}$ only through the coefficients $a = (a_{ij})$ and b_j. Moreover, if the coefficients and the boundary of Ω are smooth enough the weak conormal derivative coincides with the usual conormal derivative

$$\partial_\nu^{\mathscr{A}}u = \sum_{j=1}^{d} \Big(\sum_{i=1}^{d} a_{ij}D_iu + \operatorname{tr} b_ju \Big)\nu_j$$

where $\nu = (\nu_1, \ldots, \nu_d)$ is the unit outer normal of Ω. In particular, $\partial_\nu^{\mathscr{A}} \mathbb{1} = \sum_{j=1}^d \operatorname{tr} b_j \nu_j$. For a proof of these facts and more information we refer to [1, Section 8.1].

Next we endow our differential operator with Robin boundary conditions, given through a real function $\beta \in L^\infty(\partial\Omega)$ as in Hypothesis 1.1. To that end, we employ the theory of bilinear forms, defining $\mathfrak{a}_\beta : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$\mathfrak{a}_\beta[u,v] := \mathfrak{a}[u,v] + \int_{\partial\Omega} \beta uv \,\mathrm{d}\sigma.$$

The associated operator $\mathscr{A}_\beta^2$ on $L^2(\Omega)$ is given by

$$\begin{aligned} D(\mathscr{A}_\beta^2) &:= \{u \in D(\mathfrak{a}_\beta) : \exists\, f \in L^2(\Omega) \text{ with } \mathfrak{a}_\beta[u,v] = [f,v] \;\forall\, v \in H^1(\Omega)\} \\ \mathscr{A}_\beta^2 u &:= f. \end{aligned}$$

Testing against test functions we see that $\mathscr{A}_\beta^2 u = \mathscr{A} u$ for all $u \in D(\mathscr{A}_\beta^2)$. By the definition of the weak conormal derivative we obtain the following description of the domain:

$$D(\mathscr{A}_\beta^2) = \{u \in H^1(\Omega) : \mathscr{A} u \in L^2(\Omega) \text{ and } \partial_\nu^{\mathscr{A}} u + \beta \operatorname{tr} u = 0\}.$$

Thus $\mathscr{A}_\beta^2$ is the realization of $\mathscr{A}$ with Robin boundary condition.

We next prove that the operator $-\mathscr{A}_\beta^2$ generates a strongly continuous semigroup T_β^2 on $L^2(\Omega)$ which leaves the space $L^\infty(\Omega)$ invariant. We will also prove that the restriction to $L^\infty(\Omega)$ is a holomorphic semigroup, by which we mean that the $\mathbb{C}$-linear extension of $T_\beta^2|_{L^\infty(\Omega)}$ to the complexification $L^\infty(\Omega;\mathbb{C})$ of $L^\infty(\Omega)$ is holomorphic.

Theorem 3.3. *Assume Hypothesis 3.1. The operator $-\mathscr{A}_\beta^2$ generates a positive, strongly continuous semigroup T_β^2 on $L^2(\Omega)$. Its restriction T_β to $L^\infty(\Omega)$ is a holomorphic semigroup on $L^\infty(\Omega)$. Each operator $T_\beta(t)$, $t > 0$, is compact and enjoys the strong Feller property. In particular, $C(\overline{\Omega})$ is invariant. The restriction T_β^C of T_β^2 to $C(\overline{\Omega})$ is a strongly continuous semigroup.*

Proof. By standard results from the theory of quadratic forms ([27, Section 1.4]) $-\mathscr{A}_\beta^2$ generates a holomorphic semigroup T_β^2. The positivity of T_β^2 follows from [27, Theorem 2.6] noting that $\mathfrak{a}_\beta[u^+, u^-] = 0$ for all $u \in H^1(\Omega)$. It was proved in [13, Corollary 6.1] (see also [11, Theorem 4.9]) that the semigroup T_β^2 has Gaussian estimates so that T_β^2 extrapolates to a consistent family of semigroups T_β^q on $L^q(\Omega)$ for $q \in [1,\infty]$. In particular, T_β^2 leaves the space $L^\infty(\Omega)$ invariant and restricts to a semigroup T_β on this space. By [11, Theorem 5.3] the semigroup T_β is holomorphic on $L^\infty(\Omega)$. Moreover, by the proof of [26, Theorem 4.3] $T_\beta(t)L^\infty(\Omega) \subset C(\overline{\Omega})$ for all $t > 0$. It was also seen in that theorem that $T_\beta(t)$ is compact for all $t > 0$. We now show that $T_\beta(t)$ is strongly Feller for $t > 0$. Since T_β^2 is ultracontractive by [3, 7.3 Criterion (v)] it follows that $T_\beta^2(t)L^q(\Omega) \subset L^\infty(\Omega)$ and hence $T_\beta^2(t)L^q(\Omega) \subset T_\beta^2(t/2)L^\infty(\Omega) \subset C(\overline{\Omega})$ for some $q \in (2,\infty)$. By the closed graph theorem, $T_\beta^2(t)$ is a bounded operator from $L^q(\Omega)$ to $C(\overline{\Omega})$. Now the strong Feller property, as defined before Corollary 2.10, follows from the dominated convergence theorem. It follows from [26, Theorem 4.3] that the restriction of the semigroup to $C(\overline{\Omega})$ is strongly continuous. $\square$

Of course the generator of T_β is the part A_β of $-\mathscr{A}_\beta^2$ in $L^\infty(\Omega)$, i.e.

$$\begin{aligned} D(A_\beta) &= \{u \in H^1(\Omega) \cap L^\infty(\Omega) : \mathscr{A} u \in L^\infty(\Omega), \partial_\nu^{\mathscr{A}} u + \beta u = 0\} \\ A_\beta u &= -\mathscr{A} u. \end{aligned}$$

Similarly, the generator of T_β^C is the part A_β^C of $-\mathscr{A}_\beta^2$ in $C(\Omega)$, i.e.

$$\begin{aligned} D(A_\beta^C) &= \{u \in H^1(\Omega) \cap C(\overline{\Omega}) : \mathscr{A} u \in C(\overline{\Omega}), \partial_\nu^{\mathscr{A}} u + \beta \operatorname{tr} u = 0\} \\ A_\beta^C u &= -\mathscr{A} u. \end{aligned}$$

As a consequence of the strong continuity of T_β^C we find that $D(A_\beta^C)$ is dense in $C(\overline{\Omega})$.

We next investigate when the semigroup T_β^2 generated by $-\mathscr{A}_\beta^2$ is sub-Markovian. If this is the case, it follows that the restriction T_β of T_β^2 to $L^\infty(\Omega)$ is contractive. We will use the following lemma.

Lemma 3.4. *Let $g \in L^2(\Omega)$ and $h \in L^2(\partial\Omega)$ be such that*

$$\int_\Omega gv \,\mathrm{d}x + \int_{\partial\Omega} hv \,\mathrm{d}\sigma \geq 0 \tag{3.1}$$

for all $0 \leq v \in H^1(\Omega)$. Then $g \geq 0$ a.e. on Ω and $h \geq 0$ a.e. on $\partial\Omega$. Moreover, if in (3.1) *identity holds for all $v \in H^1(\Omega)$, then $g = 0$ a.e. on Ω and $h = 0$ a.e. on $\partial\Omega$.*

Proof. By (3.1) we have $\int_\Omega gv \,\mathrm{d}x \geq 0$ for all $0 \leq v \in C_c^\infty(\Omega)$. Thus $g \geq 0$ almost everywhere on Ω. Given a function $\varphi \in C(\partial\Omega)$, we find a sequence $v_n \in C^\infty(\overline{\Omega})$ such that $v_n|_{\partial\Omega} \to \varphi$ in $C(\partial\Omega)$, $0 \leq v_n \leq \|\varphi\|_\infty$ in Ω and such that v_n is supported in a relatively open set $U_n \subset \overline{\Omega}$ with $U_n \supset U_{n+1}$ and $\bigcap_{n\in\mathbb{N}} U_n = \partial\Omega$. Choosing $v = v_n$ in (3.1) and letting $n \to \infty$, we infer from dominated convergence that $\int_{\partial\Omega} h\varphi \,\mathrm{d}\sigma \geq 0$. As $\varphi \in C(\partial\Omega)$ was arbitrary, the claim follows. □

Proposition 3.5. *Assume in addition to Hypothesis 3.1 that $b_j \in W^{1,\infty}(\Omega)$ for $j = 1, \ldots, d$.*

(a) The semigroup T_β^2 is sub-Markovian if and only if

$$\sum_{j=1}^d D_j b_j \leq d_0 \qquad \textit{almost everywhere on } \Omega \textit{ and} \tag{3.2}$$

$$\sum_{j=1}^d \operatorname{tr}(b_j)\nu_j + \beta \geq 0 \qquad \textit{almost everywhere on } \partial\Omega. \tag{3.3}$$

(b) The semigroup T_β^2 is Markovian if and only if

$$\sum_{j=1}^d D_j b_j = d_0 \qquad \textit{almost everywhere on } \Omega \textit{ and} \tag{3.4}$$

$$\sum_{j=1}^d \operatorname{tr}(b_j)\nu_j + \beta = 0 \qquad \textit{almost everywhere on } \partial\Omega. \tag{3.5}$$

Proof. (a) The semigroup T_β^2 is sub-Markovian if and only if the Beurling–Deny–Ouhabaz

$$\mathfrak{a}_\beta[u \wedge 1, (u-1)^+] \geq 0$$

for all $u \in H^1(\Omega)$, see [27, Chapter 2] and [25, Corollary 2.8] or [16] for the case where the form is not necessarily accretive. Recall that for $u \in H^1(\Omega)$ the functions $u \wedge 1$ and $(u-1)^+$ also belong to $H^1(\Omega)$ and

$$D_j(u \wedge 1) = \mathbb{1}_{\{u<1\}} D_j u \quad \text{and} \quad D_j(u-1)^+ = \mathbb{1}_{\{u>1\}} D_j u.$$

Thus $D_i(u\wedge 1)D_j(u-1)^+ = (u-1)^+ D_j(u\wedge 1) = 0$. We see that

$$\begin{aligned}
&\mathfrak{a}_\beta[u\wedge 1,(u-1)^+]\\
&= \int_\Omega \sum_{j=1}^d b_j D_j (u-1)^+ \,\mathrm{d}x + \int_{\{u>1\}} d_0 (u-1)^+ \,\mathrm{d}x + \int_{\partial\Omega} \beta (u-1)^+ \,\mathrm{d}\sigma\\
&= -\int_\Omega \sum_{j=1}^d (D_j b_j)(u-1)^+ \,\mathrm{d}x + \int_{\partial\Omega} \sum_{j=1}^d b_j \nu_j (u-1)^+ \,\mathrm{d}\sigma\\
&\qquad\qquad + \int_\Omega d_0 (u-1)^+ \,\mathrm{d}x + \int_{\partial\Omega} \beta (u-1)^+ \,\mathrm{d}\sigma.
\end{aligned}$$

The latter is positive if (3.2) and (3.3) hold whence T^2_β is sub-Markovian in this case. This shows sufficiency of these two conditions.

Conversely, if the semigroup T^2_β is sub-Markovian, the Beurling–Deny–Ouhabaz criterion yields

$$\int_\Omega \Big(d_0 - \sum_{j=1}^d D_j b_j\Big)(u-1)^+ \,\mathrm{d}x + \int_{\partial\Omega} \Big(\sum_{j=1}^d b_j\nu_j + \beta\Big)(u-1)^+ \,\mathrm{d}\sigma \geq 0$$

for all $u \in H^1(\Omega)$. Choosing $u = \mathbb{1} + v$ with $0 \leq v \in H^1(\Omega)$, Lemma 3.4 shows that (3.2) and (3.3) are valid.

(b) A Markovian semigroup is in particular sub-Markovian whence the inequalities (3.2) and (3.3) are satisfied. If T^2_β is sub-Markovian, then it is Markovian if and only if $\mathbb{1} \in \ker(-\mathscr{A}^2_\beta)$. Note that

$$-\mathscr{A}\mathbb{1} = \sum_{j=1}^d D_j b_j - d_0.$$

Thus (3.4) is necessary for T^2_β to be Markovian. If (3.4) holds, then for $v \in H^1(\Omega)$ we have

$$\mathfrak{a}[\mathbb{1}, v] - [\mathscr{A}\mathbb{1}, v] = \sum_{j=1}^d \int_\Omega (b_j D_j v + d_0 v)\,\mathrm{d}x = \sum_{j=1}^d \int_{\partial\Omega} b_j \nu_j v \,\mathrm{d}\sigma,$$

where we used an integration by parts. Thus saying $\partial^{\mathscr{A}}_\nu \mathbb{1} + \beta = 0$, i.e. $\mathbb{1} \in D(-\mathscr{A}^2_\beta)$, is equivalent to

$$\sum_{j=1}^d \int_{\partial\Omega} b_j\nu_j v\,\mathrm{d}\sigma = -\int_{\partial\Omega} \beta v\,\mathrm{d}\sigma$$

for all $v \in H^1(\Omega)$ and hence to (3.5). $\square$

In order to apply the abstract results of Section 2, we need some results about the following elliptic problem, which were also used implicitly in Theorem 3.3.

$$\text{(3.6)}\qquad \begin{cases} \lambda u + \mathscr{A} u = f \text{ on } \Omega\\ \partial^{\mathscr{A}}_\nu u + \beta u = h \text{ on } \partial\Omega. \end{cases}$$

Obviously, $\mathfrak{a}_\beta$ defines a continuous sesquilinear mapping on $H^1(\Omega)$. By [14, Corollary 2.5] it is also *elliptic*, i.e. there are some $\omega, \alpha > 0$ such that $\mathfrak{a}_\beta[u,u] + \omega\|u\|^2_{L^2(\Omega)} \geq \alpha\|u\|^2_{H^1(\Omega)}$. With this information at hand, one can prove existence and uniqueness of solutions to (3.6) by means of the Lax–Milgram Theorem. Indeed, considering the continuous functional F on $H^1(\Omega)$, given by $F(v) = \int_\Omega fv\,\mathrm{d}x + \int_{\partial\Omega} hv\,\mathrm{d}\sigma$, it follows from the Lax–Milgram Theorem that for $\lambda > \omega$ there is a unique $u \in H^1(\Omega)$ such that

$$\mathfrak{a}_\beta[u,v] + \lambda[u,v] = F(v)$$

for all $v \in H^1(\Omega)$. From [26, Theorem 3.14(iv)] we obtain the following result concerning regularity of the solution.

Proposition 3.6. *Fix $q > d$ and $\lambda > \omega$. Then there exist constants $\gamma > 0$ and $C > 0$ such that whenever $f \in L^{q/2}(\Omega)$ and $h \in L^{q-1}(\partial\Omega)$ the unique solution u of* (3.6) *belongs to $C^\gamma(\overline{\Omega})$ and we have*

$$\|u\|_{C^\gamma(\overline{\Omega})} \le C\big(\|f\|_{L^{\frac{q}{2}}(\Omega)} + \|h\|_{L^{q-1}(\partial\Omega)}\big).$$

The following lemma is easy to prove, see e.g. [9, Lemma 2.3].

Lemma 3.7. *Let X_1, X_2, X_3 be Banach spaces such that X_1 is reflexive. Let $T : X_1 \to X_3$ be compact, $S : X_1 \to X_2$ be injective. Then, given $\varepsilon > 0$ there exists a constant $c > 0$ such that*

$$\|Tx\|_{X_3} \le \varepsilon\|x\|_{X_1} + c\|Sx\|_{X_2}$$

for all $x \in X_1$.

We use this lemma to prove the following domination result.

Proposition 3.8. *Let $\beta_1, \beta_2 \in L^\infty(\partial\Omega)$ be such that $\beta_1 \le \beta_2$. There exists ω so that both $\mathfrak{a}_{\beta_1} + \omega$ and $\mathfrak{a}_{\beta_2} + \omega$ are coercive and such that for $\lambda > \omega$ the following holds. Let $0 \le f \in L^2(\Omega)$, $0 \le h \in L^2(\partial\Omega)$. For $j = 1, 2$, let $u_j \in H^1(\Omega)$ be the unique solution of*

$$\begin{cases} \lambda u + \mathscr{A} u = f \ \text{ on } \Omega \\ \partial_\nu^{\mathscr{A}} u + \beta_j u = h \ \text{ on } \partial\Omega. \end{cases}$$

Then $0 \le u_2 \le u_1$.

Proof. We first show positivity for weak solutions u of (3.6). To that end consider $f \le 0$ and $h \le 0$ for now. Since u solves (3.6) we have

$$\lambda[u, v] + \mathfrak{a}_\beta[u, v] = [f, v] + \int_{\partial\Omega} hv \, \mathrm{d}\sigma$$

for all $v \in H^1(\Omega)$. Setting $v := u^+$ and noting that $\mathfrak{a}_\beta[u, u^+] = \mathfrak{a}_\beta[u^+, u^+]$ by the locality of $\mathfrak{a}_\beta$, we find

$$\lambda[u^+, u^+] + \mathfrak{a}_\beta[u^+, u^+] = [f, u^+] + \int_{\partial\Omega} hu^+ \, \mathrm{d}\sigma \le 0.$$

As $\mathfrak{a}_\beta + \omega$ is coercive we have that $\mathfrak{a}_\beta[u^+, u^+] + \omega\|u^+\|^2_{L^2(\Omega)} \ge \alpha\|u^+\|^2_{H^1(\Omega)}$ for some $\alpha > 0$. Together with $\lambda > \omega$ it follows that $\|u^+\|_{H^1(\Omega)} \le 0$, whence $u \le 0$.

We can prove the domination similarly. This time we fix $f \ge 0$ and $h \ge 0$. The solution u_j $(j = 1, 2)$ satisfies the equation

$$\lambda[u_j, v] + \mathfrak{a}_{\beta_j}[u_j, v] = [f, v] + \int_{\partial\Omega} hv \, \mathrm{d}\sigma$$

for all $v \in H^1(\Omega)$. Subtracting these equations we find for a positive v that

$$\lambda[u_2 - u_1, v] + \mathfrak{a}[u_2 - u_1, v] = \int_{\partial\Omega} (\beta_1 u_1 - \beta_2 u_2) v \, \mathrm{d}\sigma \le \int_{\partial\Omega} \beta_2 (u_1 - u_2) v \, \mathrm{d}\sigma,$$

since $u_1 \ge 0$ by the above. Testing against $v := (u_2 - u_1)^+$, we find

$$\begin{aligned} &\lambda[(u_2 - u_1)^+, (u_2 - u_1)^+] + \mathfrak{a}[(u_2 - u_1)^+, (u_2 - u_1)^+] \\ &\le -\int_{\partial\Omega} \beta_2\big((u_2 - u_1)^+\big)^2 \le \|\beta_2\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \big((u_2 - u_1)^+\big)^2 \, \mathrm{d}\sigma. \end{aligned}$$

Applying Lemma 3.7 with $X_1 = H^1(\Omega)$, $X_2 = L^2(\Omega)$ and $X_3 = L^2(\partial\Omega)$ where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator (which is compact) and $S : H^1(\Omega) \to L^2(\Omega)$ is the natural embedding, given $\varepsilon > 0$ we find a constant $c > 0$ such that

$$\begin{aligned}\|\beta_2\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \big((u_2-u_1)^+\big)^2 \,\mathrm{d}\sigma &\le \varepsilon \|(u_2-u_1)^+\|_{H^1(\Omega)}^2 + c\|(u_2-u_1)^+\|_{L^2(\Omega)}^2 \\ &= \varepsilon \int_\Omega \big|\nabla(u_2-u_1)^+\big|^2 \,\mathrm{d}x + (c+\varepsilon)\int_\Omega \big((u_2-u_1)^+\big)^2 \,\mathrm{d}x.\end{aligned}$$

Using the ellipticity of $\mathfrak{a}$ we deduce that, for a suitable constant $\alpha > 0$, we have

$$\begin{aligned}&(\lambda+\alpha-\omega)\|(u_2-u_1)^+\|_{L^2(\Omega)}^2 + \alpha\int_\Omega \big|\nabla(u_2-u_1)^+\big|^2 \,\mathrm{d}x \\ \le&\varepsilon\int_\Omega \big|\nabla(u_2-u_1)^+\big|^2 \,\mathrm{d}x + (c+\varepsilon)\|(u_2-u_1)^+\|_{L^2(\Omega)}^2\end{aligned}$$

Choosing $\varepsilon = \alpha/2$ and $\lambda_0 > \omega + c + \varepsilon + 1$, it follows that for $\lambda > \lambda_0$ we have $(u_2 - u_1)^+ = 0$, i.e. $u_2 \le u_1$. □

Proposition 3.8 yields in particular the following monotonicity property.

Corollary 3.9. *Let $\beta_1, \beta_2 \in L^\infty(\Omega)$ be such that $\beta_1 \le \beta_2$. Then $0 \le T^2_{\beta_2}(t) \le T^2_{\beta_1}(t)$ for all $t \ge 0$.*

Proof. Proposition 3.8 shows that for large λ we have $0 \le (\lambda+\mathscr{A}^2_{\beta_2})^{-1} \le (\lambda+\mathscr{A}^2_{\beta_1})^{-1}$. This implies the claim in view of Euler's formula. □

4. Non-local boundary conditions

We are now prepared to prove the main results of this article. We begin by setting up the framework in which we apply Greiner's boundary perturbation. In contrast to the last section, in this section only consider complex Banach spaces in order to handle (possibly) complex valued functions $\mu : \partial\Omega \to \mathscr{M}(\Omega)$.

We assume throughout Hypotheses 1.1 and 3.1. We then define

$$D := \{u \in C(\overline{\Omega}) \cap H^1(\Omega) : \mathscr{A}u \in L^\infty(\Omega), \partial_\nu^{\mathscr{A}} u \in L^p(\partial\Omega)\},$$

where $p > d - 1$ is as in Hypothesis 1.1(b). Endowed with the norm

$$\|u\|_D := \|u\|_{C(\overline{\Omega})} + \|u\|_{H^1(\Omega)} + \|\mathscr{A}u\|_{L^\infty(\Omega)} + \|\partial_\nu^{\mathscr{A}} u\|_{L^p(\partial\Omega)}$$

D is a Banach space which is continuously embedded into $X = L^\infty(\Omega)$. Since $D \subset D(A^C_\beta)$, it follows that D is dense in $C(\overline{\Omega})$. We define our maximal operator $A : D \to X$ by $Au := -\mathscr{A}u$ which is linear and continuous. We set $\partial X := L^p(\partial\Omega)$ and consider the boundary operator $B : D \to \partial\Omega$ defined via $Bu = \partial_\nu^{\mathscr{A}} u + \beta u$ where β is as in Hypothesis 1.1. Finally, given μ as in Hypothesis 1.1, the function $\Phi : \overline{D} \to \partial X$ is given by

$$(\Phi u)(z) := \int_\Omega u(x)\mu(z)(\,\mathrm{d}x).$$

Making use of the results of Section 2 we can now prove our main generation result for the operator $A_{\beta,\mu}$, defined by

$$\begin{aligned}D(A_{\beta,\mu}) &= \big\{u \in C(\overline{\Omega}) \cap H^1(\Omega) : \mathscr{A}u \in L^\infty(\Omega), \partial_\nu^{\mathscr{A}} u + \beta u = \langle u, \mu(\cdot)\rangle\big\} \\ A_{\beta,\mu} &= -\mathscr{A}u.\end{aligned}$$

The following result contains Theorem 1.3 from the introduction as a special case.

Theorem 4.1. *Assume Hypotheses 1.1 and 3.1. Then the operator $A_{\beta,\mu}$ generates a holomorphic semigroup $T_{\beta,\mu}$ on $L^\infty(\Omega)$ which satisfies the strong Feller property. In particular, it leaves the space $C(\overline{\Omega})$ invariant. Its restriction to this space is a strongly continuous semigroup whose generator is $A^C_{\beta,\mu}$, the part of $A_{\beta,\mu}$ in $C(\overline{\Omega})$.*

Proof. Noting that the operator $A_{\beta,\mu}$ is exactly the perturbed operator A_Φ, where A and Φ are as defined above, the claim follows immediately from Theorem 2.2 and Corollary 2.10 once we verified that the maps A, B and Φ satisfy Hypothesis 2.1.

(a) The operator $B : D \to \partial X$ is surjective.

Fix $\lambda > \omega$. Given $h \in \partial X = L^p(\partial\Omega)$, it follows from Proposition 3.6 that the unique solution $u \in H^1(\Omega)$ of the problem

$$\begin{cases} \lambda u + \mathscr{A} u &= 0 \\ \partial_\nu^{\mathscr{A}} u + \beta u &= h \end{cases}$$

belongs to $C(\overline{\Omega})$. Moreover, $\mathscr{A} u = -\lambda u \in C(\overline{\Omega}) \subset L^\infty(\Omega)$. Thus, $u \in D$ and $Bu = h$, proving that B is surjective.

(b) The boundary map Φ is compact.

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C(\overline{\Omega})$, say $\|u_n\|_{C(\overline{\Omega})} \le M$ for all $n \in \mathbb{N}$. Since $\mu(z) \ll \tau$ by Hypothesis 1.1(c), for every $z \in \partial\Omega$ we find a Radon–Nikodym density $\varphi_z \in L^1(\Omega, \tau)$ of $\mu(z)$ with respect to τ, i.e. we have

$$\int_{\overline{\Omega}} f(x)\, \mu(z)(\mathrm{d}x) = \int_{\overline{\Omega}} f \varphi_z \, \mathrm{d}\tau$$

for all $f \in C(\overline{\Omega})$. In particular, $(\Phi u_n)(z) = \langle u_n, \varphi_z \rangle_{L^\infty(\tau), L^1(\tau)}$. Since the sequence u_n is bounded in $L^\infty(\tau)$ and $L^1(\tau)$ is separable, it follows from the Banach–Alaoglu theorem that we find a weak*-convergent subsequence, say $u_{n_k} \rightharpoonup^* u$ for some $u \in L^\infty(\tau)$. In particular,

$$(\Phi u_{n_k})(z) = \int_{\overline{\Omega}} u_{n_k} \varphi_z \, \mathrm{d}\tau \to \int_{\overline{\Omega}} u \varphi_z \, \mathrm{d}\tau$$

for all $z \in \partial\Omega$, i.e. Φu_n has a subsequence which converges pointwise. Note that we have

$$|(\Phi u_n)(z)| \le M \|\mu(z)\|.$$

As a consequence of Hypothesis 1.1(b) the functions Φu_n have a p-integrable majorant and it follows from the dominated convergence theorem that Φu_n has a subsequence which converges in $L^p(\partial\Omega)$.

(c) The operator A_0 is exactly the part of $-\mathscr{A}_\beta^2$ in $L^\infty(\Omega)$. It follows from Theorem 3.3 that A_0 generates an analytic semigroup on $X = L^\infty(\Omega)$ which enjoys the strong Feller property and whose domain is dense in $C(\overline{\Omega})$. □

For $\mu \equiv 0$ we have $T_{\beta,0}(t) = T_\beta(t)$, where T_β is the semigroup on $L^\infty(\Omega)$, defined in Section 3 for local Robin boundary conditions.

We next prove some additional properties of the semigroup $T_{\beta,\mu}$ making use of the corollaries to Theorem 2.2.

Proposition 4.2. *Assume Hypotheses 1.1 and 3.1 and let $T_{\beta,\mu}$ be the semigroup generated by $A_{\beta,\mu}$ according to Theorem 4.1.*

(a) $T_{\beta,\mu}$ is compact.

(b) If $\mu(z)$ is a positive measure for almost every $z \in \partial\Omega$, then the semigroup $T_{\beta,\mu}$ is positive.

Proof. (a) Follows immediately from Corollary 2.7, noting that the semigroup generated by A_0 is compact as a consequence of Theorem 3.3.

(b) By Theorem 3.3, the semigroup generated by A_0 is positive. If $\mu(z)$ is positive for almost every $z \in \partial\Omega$, then the map Φ is positive. Note that for the solution map S_λ the function $S_\lambda h$ is the unique solution of the boundary value problem

$$\begin{cases} \lambda u + \mathscr{A} u &= 0 \\ \partial_\nu^{\mathscr{A}} u + \beta u &= h. \end{cases}$$

Thus, by Proposition 3.8, S_λ is positive for $\lambda > \omega$. Altogether $S_\lambda\Phi$ is positive and it follows from Corollary 2.8 that $T_{\beta,\mu}$ is positive. □

Next we characterize when $T_{\beta,\mu}$ is Markovian.

Proposition 4.3. *Assume in addition to Hypotheses 1.1 and 3.1 that $\mu(z)$ is a positive measure for almost every $z \in \partial\Omega$. The following are equivalent.*

(i) The semigroup $T_{\beta,\mu}$ is Markovian.
(ii) We have

$$\sum_{j=1}^{d} D_j b_j = d_0 \qquad \textit{almost everywhere on } \Omega \textit{ and} \tag{4.1}$$

$$\mu(z)(\overline{\Omega}) = \beta(z) + \sum_{j=1}^{d} \nu_j(z) b_j(z) \qquad \textit{for almost all } z \in \partial\Omega. \tag{4.2}$$

Proof. Since $T_{\beta,\mu}$ is positive, (i) is equivalent to $\mathbb{1} \in \ker A_{\beta,\mu}$. Observe that $-\mathscr{A}\mathbb{1} = \sum_{j=1}^{d} D_j b_j - d_0$. Thus $-\mathscr{A}\mathbb{1} = 0$ if and only if (4.1) holds. In that case, integration by parts yields for $v \in H^1(\Omega)$ that

$$\mathfrak{a}[\mathbb{1}, v] - [\mathscr{A}\mathbb{1}, v] = \sum_{j=1}^{d} \int_\Omega b_j D_j v + d_0 v \,\mathrm{d}x = \sum_{j=1}^{d} \int_{\partial\Omega} b_j \nu_j v \,\mathrm{d}\sigma.$$

Thus $\mathbb{1} \in D(A_{\beta,\mu})$ if and only if

$$\sum_{j=1}^{d} \int_{\partial\Omega} b_j(z)\nu_j(z)v(z)\,\mathrm{d}\sigma(z) = \int_{\partial\Omega} \big(-\beta(z) + \langle \mu(z), \mathbb{1}\rangle\big) v(z)\,\mathrm{d}\sigma$$

for all $v \in H^1(\Omega)$. This is equivalent to (4.2). □

If we merely have inequalities in (4.1) and (4.2), then the semigroup is sub-Markovian as we show next. In the proof, we use the following monotonicity result.

Proposition 4.4. *Assume Hypothesis 3.1 and let $\beta_1, \beta_2 \in L^\infty(\partial\Omega)$ with $\beta_2 \le \beta_1$. Moreover, let functions $\mu_1, \mu_2 : \partial\Omega \to \mathscr{M}(\overline{\Omega})$ be given such that $0 \le \mu_1(z) \le \mu_2(z)$ for almost all $z \in \partial\Omega$ and such that μ_1, μ_2 satisfy Hypothesis 1.1 with the same p. Then*

$$0 \le T_{\beta_1,\mu_1}(t) \le T_{\beta_2,\mu_2}(t)$$

for all $t \ge 0$.

Proof. The semigroups T_{β_1,μ_1} and T_{β_2,μ_2} are obtained from the same maximal operator A but using different boundary perturbations $\Phi_j \colon u \mapsto \langle \mu_j(\cdot), u\rangle$ and boundary operators $B_j \colon u \mapsto \partial_\nu^{\mathscr{A}} u + \beta_j u$. We clearly have $B_2 u \le B_1 u$ and $0 \le \Phi_1 u \le \Phi_2 u$ for $u \ge 0$. Moreover, if we write $S_\lambda^j := (B_j|_{\ker(\lambda - A)})^{-1}$, then we have $S_\lambda^1 \le S_\lambda^2$ by Proposition 3.8. Thus Corollary 2.9 yields the claim. □

Proposition 4.5. *Assume in addition to Hypotheses 1.1 and 3.1 that $\mu(z)$ is positive for almost all $z \in \partial\Omega$ and that $b_j \in W^{1,\infty}(\Omega)$ for $j = 1, \dots, d$. If*

$$\sum_{j=1}^{d} D_j b_j \le d_0 \qquad \textit{almost everywhere on } \Omega \textit{ and} \tag{4.3}$$

$$\mu(z)(\overline{\Omega}) \le \beta(z) + \sum_{j=1}^{d} \mathrm{tr}(b_j)(z)\nu_j(z) \qquad \textit{for almost all } z \in \partial\Omega \tag{4.4}$$

then the semigroup $T_{\beta,\mu}$ is sub-Markovian.

Proof. Assume at first that $\sum_{j=1}^{d} D_j b_j = d_0$. Let us define $\beta_0(z) := \mu(z)(\overline{\Omega}) - \sum_{j=1}^{d} \operatorname{tr} b_j(z)\nu_j(z)$. By Proposition 4.3 the semigroup $T_{\beta_0,\mu}$ is Markovian. As a consequence of Proposition 4.4 we have $0 \leq T_{\beta,\mu}(t) \leq T_{\beta_0,\mu}(t)$ for all $t > 0$ which clearly implies that $T_{\beta,\mu}$ is sub-Markovian. That $T_{\beta,\mu}$ is still sub-Markovian when $\sum_{j=1}^{d} D_j b_j \leq d_0$ follows from a standard perturbation result:

Denote by $\tilde{A}_{\beta,\mu}$ the operator where d_0 is replaced by $\tilde{d}_0 := \sum_{j=1}^{d} D_j b_j$. Then the semigroup $\tilde{T}_{\beta,\mu}$ generated by $\tilde{A}_{\beta,\mu}$ is sub-Markovian by what has been proved so far. Note that $A_{\beta,\mu} + (d_0 - \tilde{d}_0) = \tilde{A}_{\beta,\mu}$, so that $\tilde{A}_{\beta,\mu}$ is a bounded and positive perturbation of $A_{\beta,\mu}$. Using a perturbation result for resolvent positive operators [6, Proposition 3.11.12] we find that $R(\lambda, A_{\beta,\mu}) \leq R(\lambda, \tilde{A}_{\beta,\mu})$ for large enough λ and the domination of the semigroups follows from the Post–Widder inversion formula [6, Theorem 1.7.7]. Alternatively, the domination property can be inferred from the Dyson–Phillips formula for the perturbed semigroup, see [22, Example 3.4] for a version which covers our setting. □

As a further consequence of Proposition 4.4 we have

$$0 \leq T_{\beta,0}(t) \leq T_{\beta,\mu}(t) \tag{4.5}$$

for all $t > 0$ in the case where $\mu(z)$ is a positive measure for almost every $z \in \partial\Omega$. It thus follows from Proposition 3.5 that condition (4.3) is necessary for $T_{\beta,\mu}$ to be sub-Markovian. It seems not so easy to show that also condition (4.4) is necessary for this. Also concerning the positivity of the semigroup $T_{\beta,\mu}$ it seems unclear if the condition that $\mu(z)$ is a positive measure for almost every $z \in \partial\Omega$ is necessary. However, in Section 7 we will give a proof of necessity in the special case where every measure $\mu(z)$ is absolutely continuous with respect to the Lebesgue measure.

5. Asymptotic behavior

Throughout this section we assume Hypotheses 1.1 and 3.1 so that $T_{\beta,\mu}$ is a semigroup on $L^\infty(\Omega)$. It is our aim to describe its asymptotic behavior as $t \to \infty$. Since $T_{\beta,\mu}(t)L^\infty(\Omega) \subset C(\overline{\Omega})$ for all $t > 0$ it suffices to study $T^C_{\beta,\mu}$, the restriction to $C(\overline{\Omega})$, which is a strongly continuous semigroup. We also assume throughout that $\mu(z) \geq 0$ for almost all $z \in \partial\Omega$ so that the semigroup is positive.

For the definition of spectral bound and irreducibility we refer to Appendix A. The asymptotic behavior of $T^C_{\beta,\mu}$ is determined by the spectral bound $s(A^C_{\beta,\mu})$ of its generator (see Appendix A). We first show that the spectrum is not empty.

Proposition 5.1. *One has $s(A^C_{\beta,\mu}) > -\infty$. Moreover, $s(A^C_{\beta,\mu})$ is an eigenvalue of $A^C_{\beta,\mu}$ with positive eigenfunction.*

Proof. We first show that $s(A^C_{\beta,0}) \leq s(A^C_{\beta,\mu})$. As a consequence of Proposition 4.4 we have $0 \leq T^C_{\beta,0}(t) \leq T^C_{\beta,\mu}(t)$. Taking Laplace transforms, it follows that $0 \leq R(\lambda, A^C_{\beta,0}) \leq R(\lambda, A^C_{\beta,\mu})$ for all large enough λ. By [6, Theorem 5.3.1] for a positive semigroup the abscissa of the Laplace transform coincides with the spectral bound. Thus, if we assume that $s(A^C_{\beta,0}) > s(A^C_{\beta,\mu})$ we have $0 \leq R(\lambda, A^C_{\beta,0}) \leq R(\lambda, A^C_{\beta,\mu})$ for all $\lambda > s(A^C_{\beta,0})$. By [6, Proposition 3.11.2] we have $s(A^C_{\beta,0}) \in \sigma(A^C_{\beta,0})$ and hence $\sup_{\lambda > s(A^C_{\beta,0})} \|R(\lambda, A^C_{\beta,0})\| = \infty$. Consequently, also $\|R(\lambda, A^C_{\beta,\mu})\|$ is unbounded as $\lambda \downarrow s(A^C_{\beta,0})$. It thus follows that $s(A^C_{\beta,0}) \in \sigma(A^C_{\beta,\mu})$, a contradiction to our assumption $s(A^C_{\beta_0}) > s(A^C_{\beta,\mu})$.

The operator $A^C_{\beta,0}$ is the part of $-\mathscr{A}^2_\beta$ in $C(\overline{\Omega})$, defined before Theorem 3.3. It follows from Proposition A.4 that the semigroup generated by $-\mathscr{A}^2_\beta$ is irreducible. Since the resolvent of that operator is compact, it follows from de Pagter's Theorem (see [15, Theorem 3] or [7, C-III, Theorem 3.7.(c)]) that $s(-\mathscr{A}^2_\beta) > -\infty$. But we have $s(A^C_{\beta,0}) = s(-\mathscr{A}^2_\beta)$ since the resolvents are compact and consistent, see [2, Proposition 2.6]. □

Note that the semigroup $T^C_{\beta,\mu}$ is compact and hence immediately norm continuous whence spectral bound and growth bound coincide. Thus, if $s(A_{\beta,\mu}) < 0$, then $\|T^C_{\beta,\mu}(t)\| \leq Me^{-\varepsilon t}$ for all $t > 0$ and suitable constants $M > 0, \varepsilon > 0$, i.e. the semigroup is exponentially stable. If, on the other hand, $s(A^C_{\beta,\mu}) > 0$ then there exists $\varepsilon > 0$ $M > 0$ such that $\|T^C_{\beta,\mu}(t)\| \geq Me^{\varepsilon t}$ for all $t > 0$. Finally, if $s(A_{\beta,\mu}) = 0$, then the semigroup converges if it is bounded. This is not easy to decide, though. However, we have a precise criterion for the semigroup to be sub-Markovian. In that case, we obtain the following result from Theorem A.1.

Proposition 5.2. *Assume that* $\mu(z) \geq 0$ *and*

$$\mu(z)(\overline{\Omega}) \leq \beta(z) + \sum_{j=1}^{d} \operatorname{tr} b_j \nu_j(z) \tag{5.1}$$

for almost every $z \in \partial\Omega$ *and*

$$\sum_{j=1}^{d} D_j b_j \leq d_0 \tag{5.2}$$

almost everywhere. Then there exist a positive projection $P \in \mathscr{L}(C(\overline{\Omega}))$ *with finite rank and* $M > 0$, $\varepsilon > 0$ *such that*

$$\|T^C_{\mu,\beta}(t) - P\|_{\mathscr{L}(C(\overline{\Omega}))} \leq Me^{-\varepsilon t}$$

for all $t > 0$.

In the situation of Proposition 5.2, if $s(A^C_{\beta,\mu}) = 0$, there exists a function $0 < u = Pu$, i.e. a positive function in the kernel of $A^C_{\beta,\mu}$. If the semigroup is Markovian, then $\mathbb{1}$ is such a function. It is interesting to know when it is the only one (up to a scalar multiple). If $T^C_{\beta,\mu}$ is irreducible, then this is the case. Unfortunately, it is not easy to prove irreducibility on $C(\overline{\Omega})$. However, it follows from the domination property (4.5) that $T^C_{\beta,\mu}$ is irreducible whenever $T^C_{\beta,0}$ is so. As for the latter semigroup, a particular case will be settled in Theorem 6.3. We also remark that in a forthcoming paper [10] it will be shown that $T^C_{\beta,0}$ is irreducible whenever Ω is connected, $b_j = 0$ and $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, d$.

Theorem 5.3. *Assume that* $\mu(z) \geq 0$ *and*

$$0 \leq \mu(z)(\overline{\Omega}) = \beta(z) + \sum_{j=1}^{d} \operatorname{tr} b_j \nu_j(z)$$

for almost all $z \in \partial\Omega$ *and* $\sum_{j=1}^d D_j b_j = d_0$. *Assume further that* $T^C_{\beta,0}$ *is irreducible. Then there exist a strictly positive measure* ρ *on* $\overline{\Omega}$ *and constants* $\varepsilon, M > 0$ *such that for* $P \in \mathscr{L}(C(\overline{\Omega}))$, *given by*

$$Pf = \int_{\overline{\Omega}} f \, \mathrm{d}\rho \cdot \mathbb{1}$$

for all $f \in C(\overline{\Omega})$, *we have*

$$\|T^C_{\beta,\mu}(t) - P\|_{\mathscr{L}(C(\overline{\Omega}))} \leq Me^{-\varepsilon t}$$

for all $t > 0$.

Proof. By Proposition 4.2 the semigroup $T^C_{\beta,\mu}$ is Markovian and hence $\mathbb{1}$ is a fixed vector of the semigroup. As a consequence of (4.5), $T^C_{\beta,\mu}$ is irreducible. Now the claim follows from Theorem A.2. □

We next prove exponential stability in the sub-Markovian case.

Theorem 5.4. *Assume that $\mu(z) \geq 0$ for almost all $z \in \partial\Omega$ and that* (5.1) *and* (5.2) *hold. Moreover, assume that $T^C_{\beta,0}$ is irreducible. If in* (5.1) *or* (5.2) *the inequality is strict on some set of positive measure, then there exist $\varepsilon, M > 0$ such that*

$$\|T^C_{\beta,\mu}(t)\|_{\mathscr{L}(C(\overline{\Omega}))} \leq Me^{-\varepsilon t}$$

for all $t > 0$.

Proof. Let us put

$$\tilde{\beta}(z) := \mu(z)(\overline{\Omega}) - \sum_{j=1}^{d} \operatorname{tr} b_j(z)\nu_j(z)$$

and $\tilde{d}_0(x) = \sum_{j=1}^d (D_j b_j)(x)$. Replace d_0 with $\tilde{d}_0$ and β with $\tilde{\beta}$ and denote by $\tilde{T}_{\beta,\mu}$ the corresponding semigroup on $C(\overline{\Omega})$. We denote the generator of $\tilde{T}^C_{\tilde{\beta},\mu}$ Then $0 \leq T^C_{\beta,\mu}(t) \leq \tilde{T}^C_{\tilde{\beta},\mu}(t)$ for all $t > 0$ by Proposition 4.4 and a perturbation argument, cf. the proof of Proposition 4.5. By Proposition 4.3 the semigroup $\tilde{T}$ is Markovian so that its generator has spectral bound 0. However, the generators of these two semigroups are different. To see this, let us first assume that $\beta \neq \tilde{\beta}$ in $L^\infty(\partial\Omega)$. Note that the conormal derivative $\partial^{\mathscr{A}}_\nu$ does depend on the zero order term d_0 resp. $\tilde{d}_0$. We find

$$\langle \mathbb{1}, \mu(z)\rangle = \partial^{\tilde{\mathscr{A}}}_\nu \mathbb{1} + \tilde{\beta}\mathbb{1} \neq \partial^{\mathscr{A}}_\nu \mathbb{1} + \beta\mathbb{1}.$$

Thus $\mathbb{1} \notin D(A^C_{\beta,\mu})$ but $\mathbb{1} \in D(\tilde{A}^C_{\tilde{\beta},\mu})$. If, on the other hand, $\beta = \tilde{\beta}$ in $L^\infty(\partial\Omega)$, then we have $d_0 \neq \tilde{d}_0$ in $L^\infty(\Omega)$. Note that $A_{\beta,\mu}\mathbb{1} = \tilde{d}_0 - d_0$. If $\tilde{d}_0 - d_0 \in C(\overline{\Omega})$, it follows that $\mathbb{1} \in D(A^C_{\beta,\mu})$ but $A^C_{\beta,\mu}\mathbb{1} \neq \tilde{A}^C_{\tilde{\beta},\mu}\mathbb{1}$. If $\tilde{d}_0 - d_0 \notin C(\overline{\Omega})$, then $\mathbb{1} \notin D(A^C_{\beta,\mu})$. In any case we have $\tilde{A}^C_{\tilde{\beta},\mu} \neq A^C_{\beta,\mu}$. Thus the claim follows from Theorem A.3. □

Next we show a blow-up result in the case where we perturb a Markovian semigroup $T_{\beta,0}$ by a positive μ. Recall from Proposition 3.5 that $T_{\beta,0}$ is Markovian if and only if the identities (3.4) and (3.5) hold.

Theorem 5.5. *Assume the identities* (3.4) *and* (3.5) *and that Ω is connected. If $\mu(z) \geq 0$ for almost all $z \in \partial\Omega$ but not identically* 0 *almost everywhere, then there exist $\omega, M > 0$ such that*

$$\|T^C_{\beta,\mu}(t)\|_{\mathscr{L}(C(\overline{\Omega}))} \geq Me^{\omega t}$$

for all $t > 0$.

Proof. The semigroup $T^C_{\beta,0}$ is Markovian (by Proposition 3.5) and has an extension to $L^2(\Omega)$ which is irreducible (as a consequence of Proposition A.4). From Proposition A.5, it follows that $T^C_{\beta,0}$ is irreducible. By Proposition 4.4 we have $T^C_{\beta,0}(t) \leq T^C_{\beta,\mu}(t)$ for all $t > 0$. Since $\partial_\nu\mathbb{1} + \beta\mathbb{1} = 0 < \mu(z)(\Omega)$ for z in a set of positive measure, one has $\mathbb{1} \notin D(A^C_{\beta,\mu})$. Thus the two semigroups are different and it follows from Theorem A.3 that $0 = s(A^C_{\beta,0}) < s(A^C_{\beta,\mu}) =: \omega$. Thus there exists $u \in C(\overline{\Omega})$ such that $u \geq \mathbb{1}$ with $A^C_{\beta,\mu}u = \omega u$. But this implies $T^C_{\beta,\mu}(t)u = e^{\omega t}u$ which, in turn, yields the claim. □

Remark 5.6. In particular, it follows from Theorem 5.5 that the only realization of our operator with non-local *Neumann* boundary conditions (i.e. where $\beta = 0$) which generates a sub-Markovian semigroup is that with classical (local) Neumann boundary conditions (i.e. $\beta = 0$ and $\mu = 0$).

6. Absolutely continuous measures $\mu(z)$

In this section we consider the case where the measures $\mu(z)$ are absolutely continuous with respect to the Lebesgue measure on Ω. More precisely, we assume that we are given a function $h \in L^2(\partial\Omega \times \Omega)$ such that

$$\mu(z)(A) = \int_A h(z,x)\,\mathrm{d}x.$$

In this situation we can use form methods to show that the semigroup $T_{\beta,\mu}$, defined on $L^\infty(\Omega)$, has an extension to $L^2(\Omega)$. This allows us to establish irreducibility of $T^C_{\beta,\mu}$ via Propositions A.4 and A.5 in the Markovian case, provided Ω is connected. On the other hand, we can use form methods to show that our assumptions to infer positivity resp. sub-Markovianity are close to optimal.

We consider the form $\mathfrak{a}_{\beta,h} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, given by

$$\mathfrak{a}_{\beta,h}[u,v] := \mathfrak{a}_\beta[u,v] - \int_{\partial\Omega}\int_\Omega h(z,x)u(x)\,\mathrm{d}x\, v(z)\,\mathrm{d}\sigma(z).$$

Then the form $\mathfrak{a}_{\beta,h}$ is elliptic and continuous. Denote by $\mathscr{A}^2_{\beta,h}$ the associated operator on $L^2(\Omega)$. Then $-\mathscr{A}^2_{\beta,h}$ generates a holomorphic, strongly continuous semigroup $T^2_{\beta,h}$ on $L^2(\Omega)$. It is easy to see that if in addition

$$\int_{\partial\Omega}\Big(\int_\Omega |h(z,x)|\,\mathrm{d}x\Big)^p\,\mathrm{d}\sigma, \tag{6.1}$$

for some $p > d-1$ with $p \geq 2$, then the measures $\mu(z) = h(z,x)\,\mathrm{d}x$ satisfy Hypothesis 1.1 whence we obtain a semigroup $T_{\beta,\mu}$ on $L^\infty(\Omega)$ with generator $A_{\beta,\mu}$. Using the definition of the co-normal derivative one sees that the part of $-\mathscr{A}^2_{\beta,h}$ in $L^\infty(\Omega)$ is precisely the operator $A_{\beta,\mu}$. It follows that $T^2_{\beta,h}$ leaves the space $L^\infty(\Omega)$ invariant and the restriction of that semigroup to $L^\infty(\Omega)$ is $T_{\beta,\mu}$.

Proposition 6.1. *With the notation above, we have:*

- *(a) The semigroup $T^2_{\beta,h}$ is positive if and only if $h \geq 0$ almost everywhere.*
- *(b) Assume that $b_j \in W^{1,\infty}(\Omega)$ for $j = 1,\ldots,d$. Then $T^2_{\beta,h}$ is sub-Markovian if and only if (4.3) holds, $h \geq 0$ almost everywhere and $0 \leq \int_\Omega h(z,x)\,\mathrm{d}x \leq \beta(z) + \sum_{j=1}^d \operatorname{tr} b_j(z)\nu_j(z)$ for almost every $z \in \partial\Omega$.*

Proof. (a) By the first Beurling–Deny criterion [25, Corollary 2.6] $T^2_{\beta,\mu}$ is positive if and only if $\mathfrak{a}_{\beta,\mu}[u^+,u^-] \leq 0$ for all $u \in H^1(\Omega)$. If $h \geq 0$ almost everywhere this is clearly fulfilled.

Conversely assume that $T^2_{\beta,\mu}(t) \geq 0$ for all $t > 0$. Then

$$\int_{\partial\Omega}\int_\Omega h(z,x)u^+(x)\,\mathrm{d}x\, u^-(z)\,\mathrm{d}\sigma(z) = -\mathfrak{a}_{\beta,h}[u^+,u^-] \geq 0$$

for all $u \in H^1(\Omega)$. Now let functions $0 \leq v \in \mathscr{D}(\Omega)$ and $0 \leq \varphi \in C(\partial\Omega)$ be given. We find a sequence $w_n \in \mathscr{D}(\mathbb{R}^d)$ with $0 \leq w_n \leq \|\varphi\|_\infty$ such that $\operatorname{supp} w_n \cap \operatorname{supp} v = \emptyset$ and $w_n(z) \to \varphi(z)$ for all $z \in \partial\Omega$. Inserting $u = v - w_n$ in the above inequality and using dominated convergence, we obtain that

$$\int_{\partial\Omega}\int_\Omega h(z,x)v(x)\,\mathrm{d}x\,\varphi(z)\,\mathrm{d}\sigma(z) \geq 0$$

As $0 \leq \varphi \in C(\partial\Omega)$ was arbitrary, we conclude that

$$\int_\Omega h(z,x)v(x)\,\mathrm{d}x \geq 0$$

for almost all $z \in \partial\Omega$. As $0 \leq v \in \mathscr{D}(\Omega)$ was arbitrary, it follows that for almost all $z \in \partial\Omega$ we have $h(z,x) = 0$ for almost all $x \in \Omega$. Now Fubini's theorem implies that $h \geq 0$ with respect to the product measure, proving the necessity of the condition.

(b) The sufficiency of the inequality above was already established in Proposition 4.5, so we only need to prove its necessity. If the semigroup is sub-Markovian, it is positive and thus $h \geq 0$ almost everywhere by (a).

By the Beurling–Deny–Ouhabaz criterion [25, Corollary 2.8], for $u \in H^1(\Omega)$ we have

$$
\begin{aligned}
0 &\leq \mathfrak{a}_{\beta,h}[u \wedge 1, (u-1)^+] \\
&= -\sum_j \int_\Omega (D_j b_j)(u-1)^+ \,\mathrm{d}x + \int_\Omega d_0 (u-1)^+ \,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} \Big(\sum_j b_j \nu_j (u-1)^+ + \beta(z) - \int_\Omega (u \wedge 1)(x) h(z,x) \,\mathrm{d}x \Big) (u-1)^+(z) \,\mathrm{d}\sigma(z).
\end{aligned}
$$

Now let $v \in H^1(\Omega)$ such that $v \geq 0$. Inserting $u = v + \mathbb{1}$ in the above inequality, the desired inequalities follow from Lemma 3.4. □

Remark 6.2. We have already noted after Proposition 4.5 that Condition (4.3) is necessary for $T_{\beta,\mu}$ to be sub-Markovian.

We now consider the case where the semigroup is Markovian. Then we can prove irreducibility via Proposition A.4 and deduce convergence of the semigroup to an equilibrium.

Theorem 6.3. *Assume that Ω is connected, and that $h \geq 0$ almost everywhere satisfies Equation (6.1). Moreover, assume that $\sum_{j=1}^d D_j b_j = d_0$ almost everywhere on Ω and*

$$\sum_{j=1}^d b_j(z)\nu_j(z) + \beta(z) = \int_\Omega h(z,x)\,\mathrm{d}x$$

almost everywhere on $\partial\Omega$. Then the semigroup $T^C_{\beta,\mu}$ on $C(\overline{\Omega})$ is irreducible and Markovian. Consequently, there exist $0 \ll \varphi \in L^2(\Omega)$ such that $\int_\Omega \varphi(x)\,\mathrm{d}x = 1$ and constants $\varepsilon, M > 0$ such that

$$\|T^C_{\beta,\mu}(t) - \varphi \otimes \mathbb{1}\|_{\mathscr{L}(C(\overline{\Omega}))} \leq M e^{-\varepsilon t}$$

for all $t > 0$.

7. Measures satisfying Hypothesis 1.1

In this brief section we give some examples of maps μ for which Hypothesis 1.1 is satisfied.

Example 7.1. Assume that for every Borel set $A \subset \overline{\Omega}$ the complex-valued map $z \mapsto \mu(z)(A)$ is continuous. Then μ satisfies conditions (a), (b) and (c) in Hypothesis 1.1.

Proof. It is obvious that (a) holds. As for (b), we note that by continuity and compactness of $\partial\Omega$ we have $\sup_{z\in\partial\Omega} |\mu(z)(A)| < \infty$ for every $A \in \mathscr{B}(\overline{\Omega})$. Now [12, Corollary 4.6.4] yields $\sup_{z\in\partial\Omega} \|\mu(z)\| < \infty$. To prove (c), pick a dense sequence z_n in $\partial\Omega$. We set

$$\tau := \sum_{n\in\mathbb{N}} \frac{1}{2^n} |\mu(z_n)|,$$

where $|\mu(z)|$ denotes the total variation of $\mu(z)$. Then τ is a finite positive measure and we have $\mu(z_n) \ll \tau$ for every $n \in \mathbb{N}$. Let $A \in \mathscr{B}(\overline{\Omega})$ with $\tau(A) = 0$ be given. Consider the function $\varphi(z) := \mu(z)(A)$. By the above $\varphi(z_n) = 0$ for all $n \in \mathbb{N}$. Moreover, φ is continuous by assumption. Thus $\varphi \equiv 0$, proving that in fact $\mu(z) \ll \tau$ for all $z \in \partial\Omega$. □

Similarly, we can consider maps μ which only take countably many values.

Example 7.2. Assume that $\mu(z) = \sum_{n\in J} \mathbb{1}_{A_n}(z)\mu_j$ where $(A_n)_{j\in J} \subset \mathscr{B}(\partial\Omega)$ and $(\mu_j)_{n\in J} \subset \mathscr{M}(\overline{\Omega})$ and J is a finite or countably infinite index set. Then μ satisfies Hypothesis 1.1 provided $\sum_{n\in J} \sigma(A_n)|\mu_n|(\overline{\Omega})^p < \infty$ where p is as in Hypothesis 1.1(b).

Proof. Part (a) is obvious and (b) was assumed. Part (c) is fulfilled with $\tau = \sum_{n\in J} 2^{-n}|\mu_n|$. □

Appendix A. Irreducible semigroups

In this appendix we collect some known facts on positive, irreducible semigroups. In some cases we present some variations or adapt results to our special situation.

Let E be a real Banach lattice. In our context E will be $C(\overline{\Omega})$ or $L^q(\Omega)$. Let T be a strongly continuous semigroup on E which is positive, i.e. for $f \in E_+$ we have $T(t)f \in E_+$ for all $t \geq 0$. We denote the generator of T by A. The *spectral bound* of A is defined by

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A_{\mathbb{C}})\}$$

where $\sigma(A_{\mathbb{C}})$ is the spectrum of the generator $A_{\mathbb{C}}$ of the complexification of T. In what follows, we will not distinguish between an operator and its complexification. In particular, when we talk about the spectrum, resolvent, etc. of an operator, we always mean the spectrum/resolvent, etc. of its complexification.

By [7, C-III Theorem 1.1], $s(A) \in \sigma(A)$ whenever $\sigma(A) \neq \emptyset$. If A has compact resolvent, then $\sigma(A)$ consists of isolated points which are all eigenvalues.

Theorem A.1. *Assume that $T(t)$ is compact for all $t > 0$, that $s(A) = 0$ and that T is bounded. Then there exist a positive projection $P \neq 0$ of finite rank, $\varepsilon > 0$ and $M > 0$ such that*

$$\|T(t) - P\|_{\mathscr{L}(E)} \leq Me^{-\varepsilon t}$$

for all $t > 0$.

Proof. Since $T(t)$ is compact for all $t > 0$, T is immediately norm continuous and it follows from [7, C-III Corollary 2.13] that there is some $\delta > 0$ such that $\operatorname{Re}\lambda \leq -2\delta < 0$ for all $\lambda \in \sigma(A) \setminus \{0\}$. Denote by P the spectral projection with respect to 0, i.e.

$$P := \frac{1}{2\pi i}\int_{|\lambda|=\delta} R(\lambda, A)\,\mathrm{d}\lambda.$$

As $T(t)$ is compact for all $t > 0$, so is the resolvent and thus also P, whence it has finite rank. The restriction of T to the range of P is a bounded semigroup on a finite dimensional vector space whose generator has spectrum $\{0\}$. It follows that the generator of the restriction is diagonalizable and is thus the zero operator. Consequently, $T(t)P = P$ for all $t > 0$. The space $F = (I - P)E$ is invariant under the semigroup and the generator A_F of the restriction has its spectrum in a strict left half plane. Since the semigroup is immediately norm continuous there exist $\varepsilon > 0$, $M > 0$ such that $\|T(t)|_F\|_{\mathscr{L}(F)} \leq Me^{-\varepsilon t}$ and hence $\|T(t) - P\|_{\mathscr{L}(E)} \leq Me^{-\varepsilon t}$ for all $t \geq 0$. □

Theorem A.1 implies in particular that there exists $u > 0$, i.e. $u \geq 0$ and $u \neq 0$, such that $T(t)u = u$ for all $t \geq 0$. Thus the Krein–Rutman Theorem which asserts that the largest eigenvalue (i.e. $s(A)$) has a positive eigenfunction is incorporated in Theorem A.1.

We next want to investigate when P has rank one and the positive eigenfunction is strictly positive. This will be done via the notion of *irreducibility*. A subspace J of E is called an *ideal* if

(i) $u \in J$ implies $|u| \in J$ and
(ii) if $u \in J$, then $0 \leq v \leq u$ implies $v \in J$.

A positive, strongly continuous semigroup T on E is called *irreducible* if the only invariant closed ideals are $J = \{0\}$ and $J = E$.

If $J = C(\overline{\Omega})$ then $J \subset E$ is a closed ideal if and only if there exists a closed subset K of $\overline{\Omega}$ such that

$$J = \{f \in C(\overline{\Omega}) : f|_K = 0\}.$$

If $E = L^q(\Omega)$ $(1 \leq q < \infty)$ then $J \subset E$ is a closed ideal if and only if there exists a measurable subset K of Ω such that

$$J = \{f \in L^q(\Omega) : f|_K = 0 \text{ a.e.}\}.$$

We say that $u \in E$ is a *quasi interior point* and write $u \gg 0$ if the principal ideal

$$E_u := \{v \in E : \exists\, c > 0 \text{ such that } |v| \leq cu\}$$

is dense in E.

If $E = C(\overline{\Omega})$ then $u \gg 0$ if and only if there is $\delta > 0$ such that $u(x) \geq \delta > 0$ for all $x \in \overline{\Omega}$. In this case u is actually an inner point of the positive cone. If $E = L^p(\Omega)$ then $u \gg 0$ if and only if $u(x) > 0$ for almost every x.

We call $\varphi \in E'$ a *strictly positive functional* if $\langle \varphi, f\rangle = 0$ implies $f = 0$ for all $f \in E_+$.

If $E = C(\overline{\Omega})$, then φ is strictly positive if and only if there exists a strictly positive Borel measure ν, i.e. $\nu(O) > 0$ for all non-empty open sets $O \subset \overline{\Omega}$, such that

$$\langle \varphi, f\rangle = \int_{\overline{\Omega}} f(x)\,\mathrm{d}\nu(x).$$

If $E = L^q(\Omega)$ for $\varphi \in L^{q'}(\Omega) \simeq (L^q(\Omega))'$ to be strictly positive is equivalent to that $\varphi(x) > 0$ almost everywhere, i.e. $\varphi \gg 0$.

The importance of these concepts in the study of asymptotic behavior stems from the fact that positive fixed points of positive, irreducible semigroups are *strictly positive*. More precisely, if T is a positive, irreducible, strongly continuous semigroup and $u > 0$ is such that $T(t)u = u$ for all $t > 0$, then $u \gg 0$ and if $0 < \varphi \in E'$ is such that $T(t)'\varphi = \varphi$ for all $t > 0$ then φ is strictly positive. Moreover, because of irreducibility, $s(A)$ cannot be a pole of order larger than 1, see [7, C-III Proposition 3.5]. This implies that $T(t)P = P$ for all $t > 0$ in the proof of Theorem A.1 even though the semigroup is not assumed to be bounded. We thus obtain the following result on asymptotic stability.

Theorem A.2. *Let T be a positive, irreducible strongly continuous semigroup on E with generator A. Assume that $T(t)$ is compact for $t > 0$ and $s(A) = 0$. Then there exist $0 \ll u \in \ker A$, a strictly positive $\varphi \in \ker A'$, $\varepsilon > 0$, $M > 0$ such that $\langle \varphi, u\rangle = 1$ and*

$$\|T(t) - \varphi \otimes u\|_{\mathscr{L}(E)} \leq M^{-\varepsilon t}$$

for all $t \geq 0$ where we have written $\varphi \otimes u$ for the projection defined by

$$(\varphi \otimes u)(f) = \langle \varphi, f\rangle u,$$

for all $f \in E$. In particular

$$\lim_{t\to\infty} T(t)f = \langle \varphi, f\rangle u,$$

i.e. the orbits of the semigroup converge to an equilibrium.

Theorems A.1 and A.2 lie at the heart of the Perron–Frobenius theory. We refer to [7] for more information.

We shall have occasion to use the strict monotonicity of the spectral bound.

Theorem A.3. *Let S and T be strongly continuous semigroups on E with generators B and A respectively. Assume that*

(i) $0 \le S(t) \le T(t)$ *for all* $t > 0$*;*
(ii) A *has compact resolvent, and*
(iii) T *is irreducible.*

If $A \neq B$*, then* $s(B) < s(A)$*.*

Proof. This is a version of [5, Theorem 1.3], see also [4, Theorem 10.2.10] in connection with [4, Theorems 10.6.3 and 10.6.1]. □

Next we describe ways to prove irreducibility. On $L^2(\Omega)$ this is very easy if the semigroup is associated with a form by virtue of the Beurling–Deny–Ouhabaz criterion for the invariance of closed convex sets. In particular the following holds true (see [27, Theorem 2.10]).

Proposition A.4. *Let* $V \subset H^1(\Omega)$ *be a closed subspace containing* $H^1_0(\Omega)$*, where* $\Omega \subset \mathbb{R}^d$ *is a connected, open set. Let* $\mathfrak{a} : V \times V \to \mathbb{R}$ *be a continuous and elliptic form such that the associated semigroup* T *is positive. Then* T *is irreducible.*

On $C(\overline{\Omega})$ irreducibility is a stronger notion than on $L^2(\Omega)$. However, the following result shows how irreducibility on $C(\overline{\Omega})$ can be deduced from irreducibility on $L^2(\Omega)$.

Proposition A.5. *Let* $\Omega \subset \mathbb{R}^d$ *be open and bounded and* T *be a positive, irreducible, strongly continuous semigroup on* $L^2(\Omega)$ *whose generator* A *has compact resolvent. Assume that* T *leaves* $C(\overline{\Omega})$ *invariant and that the restriction* T^C *of* T *to* $C(\overline{\Omega})$ *is strongly continuous and suppose that its generator* A^C *has compact resolvent. Assume that* $s(A) = 0$*. Then* T^C *is irreducible if and only if there exists* $u \in \ker A \cap C(\overline{\Omega})$ *such that* $u(x) \ge \delta > 0$ *for all* $x \in \overline{\Omega}$*.*

Proof. Assume that there exists $0 \ll u \in C(\overline{\Omega}) \cap \ker A$. Since T is irreducible 0 is a pole of order 1 and the residuum P is of the form

$$Pf = \Big(\int_\Omega \varphi f \,\mathrm{d}x \Big) \cdot u$$

for some $0 \ll \varphi \in L^2(\Omega)$, see [7, C-III Proposition 3.5]. Since $C(\overline{\Omega})$ is dense in $L^2(\Omega)$, it follows that the coefficients in the Laurent series expansion in $C(\overline{\Omega})$ around 0 (see [7, A-III, Equation (3.1)]) are the restriction of those in $L^2(\Omega)$. Thus 0 is also a pole of order 1 of the resolvent of A^C. The residuum

$$P^C = \frac{1}{2\pi i} \int_{|\lambda| = \varepsilon} R(\lambda, A^C) \,\mathrm{d}\lambda$$

is the same, i.e. $P^C = P|_{C(\overline{\Omega})}$. Now let $J = \{f \in C(\overline{\Omega}) : f|_K = 0\}$ be an invariant ideal. Then for $z \in K$, $f \in J$, $f \ge 0$ we have $(T(t)f)(z) = 0$ for all $t > 0$ and hence $(R(\lambda, A^C)f)(z) = 0$ for all $\lambda > 0$, since we suppose that $s(A) = 0$ and know that $s(A)$ is the abscissis of the Laplace transform of the semigroup [6, Theorem 5.3.1]. Thus

$$\int_\Omega f(x)\varphi(x) \,\mathrm{d}x \cdot u(z) = \lim_{\lambda \downarrow 0} (\lambda R(\lambda, A^C) f)(z) = 0.$$

Since $\varphi \gg 0$ in $L^2(\Omega)$ this implies $f = 0$. Consequently $J = \{0\}$. This proves the sufficiency.

To show the necessity, recall that 0 is also a pole of $R(\lambda, A^C)$. It follows that $s(A^C) = 0$. By Theorem A.2, there exists $0 \ll u \in \ker(A^C) \subset \ker(A)$. □

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