Introduction to statistical learning

3.3 Logistic regression

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- Explain the presence/absence of cardiovascular disease (CHD) by age X of patients.
- Predict the status of a machine tool (on/off) as a function of its age, for predictive maintenance purposes, for example.
- Predict the probability of repayment of a loan (credit scoring).
- More generally, predict a qualitative variable (binary here) based on quantitative variables.

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An example: cardiovascular disease

Y : with or without cardiovascular disease (CHD)

X : patient's age

$n = 100$ observations

Id	age	chd
1	20	healthy
2	23	healthy
3	24	healthy
⋮		⋮
98	64	diseased
99	65	diseased
100	69	diseased

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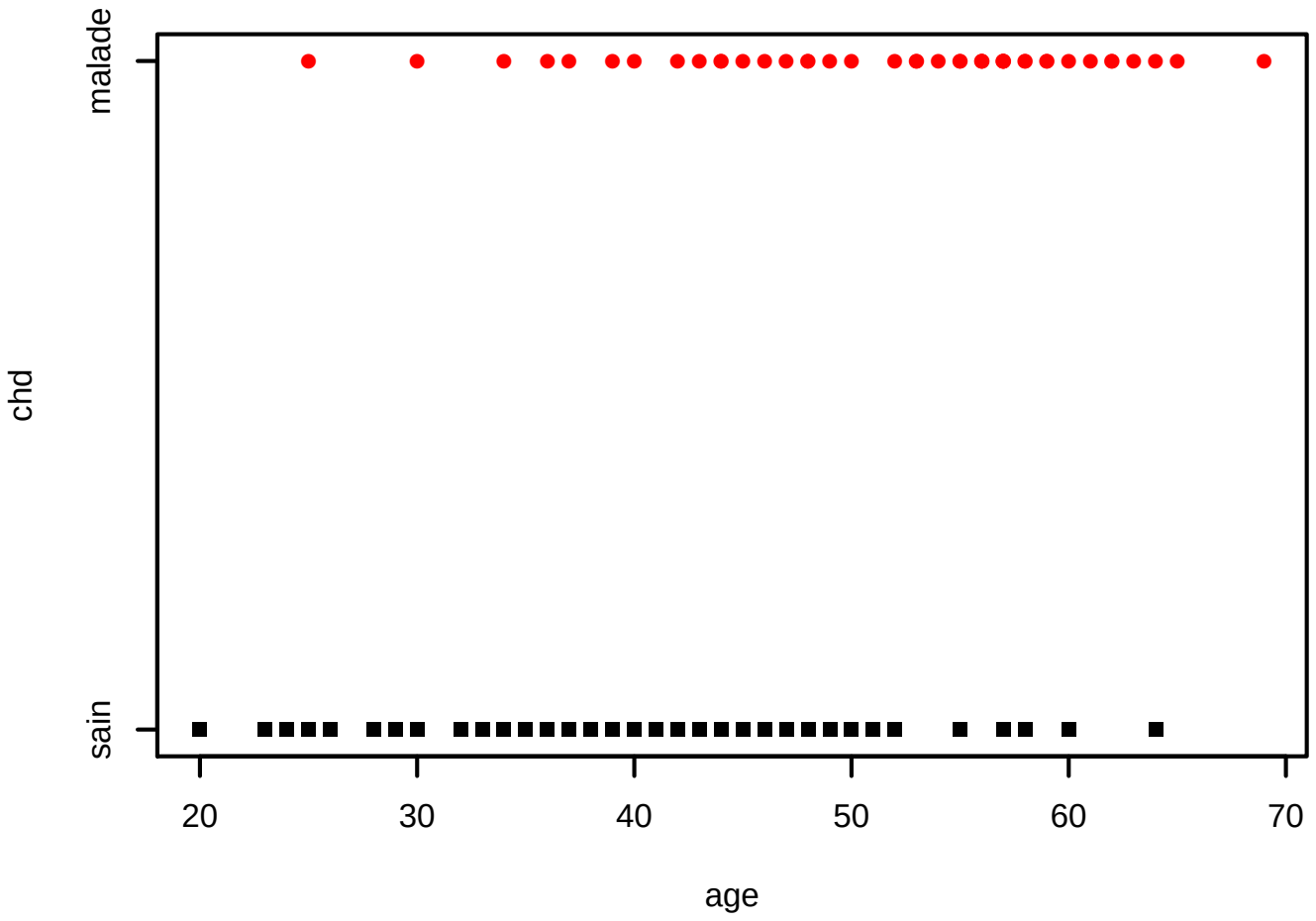
Graphical representation



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Y binary variable

Here the variable Y takes 2 values, we model it by:

$$Y\,/X = x \sim \mathcal{B}\left(p(x)\right) \ .$$

Here:

$$\mathbb{P}\left(Y = 1\,/X = x\right) = p(x) \ ,$$
$$\mathbb{P}\left(Y = 0\,/X = x\right) = 1 - p(x) \ .$$

So:

$$\mathbb{E}\left(Y\,/X = x\right) = p(x) \ ,$$
$$\mathrm{Var}\left(Y\,/X = x\right) = p(x)\left[1 - p(x)\right] \ .$$

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A first idea: the division into age classes

Age	n	Healthy	Diseased	Average
$[19, 29[$	10	9	1	.10
$[29, 34[$	15	13	2	.13
$[34, 39[$	12	9	3	.25
$[39, 44[$	15	10	5	.33
$[44, 49[$	13	7	6	.46
$[49, 54[$	8	3	5	.625
$[54, 59[$	17	4	13	.76
$[59, 69[$	10	2	8	.8

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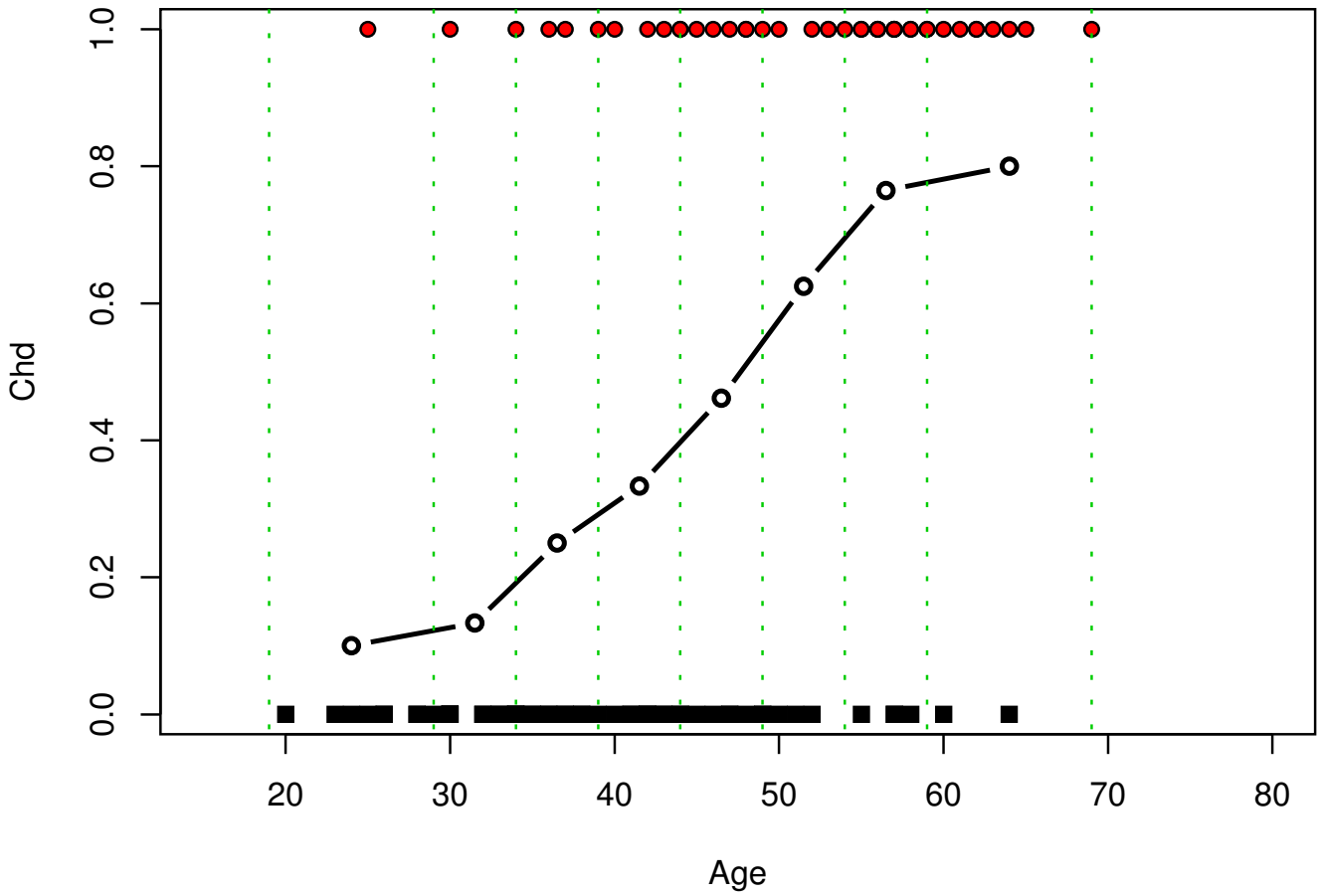

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To go further

We would like to find a function:

- smoother,
- that uses all the data.

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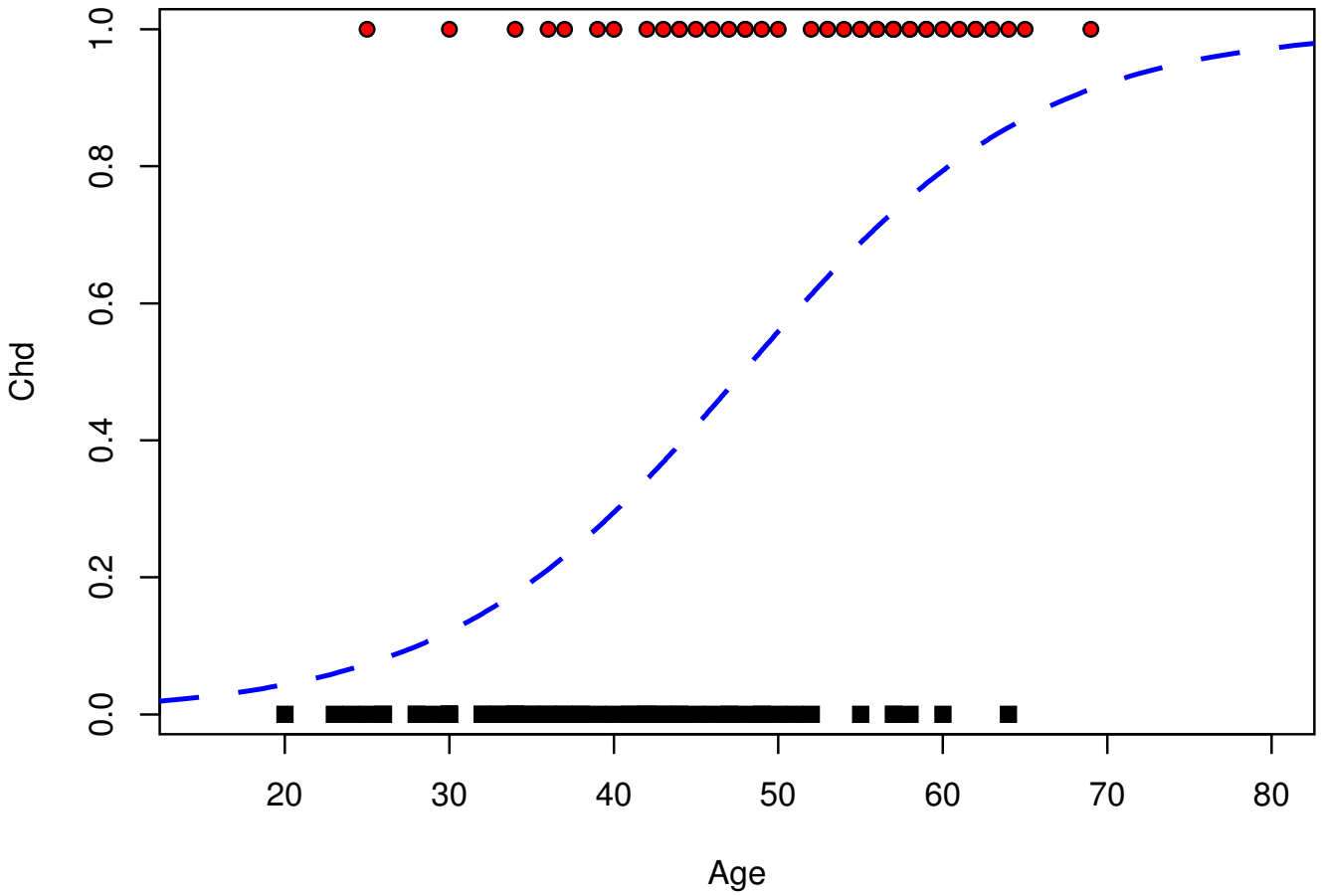
Expected function



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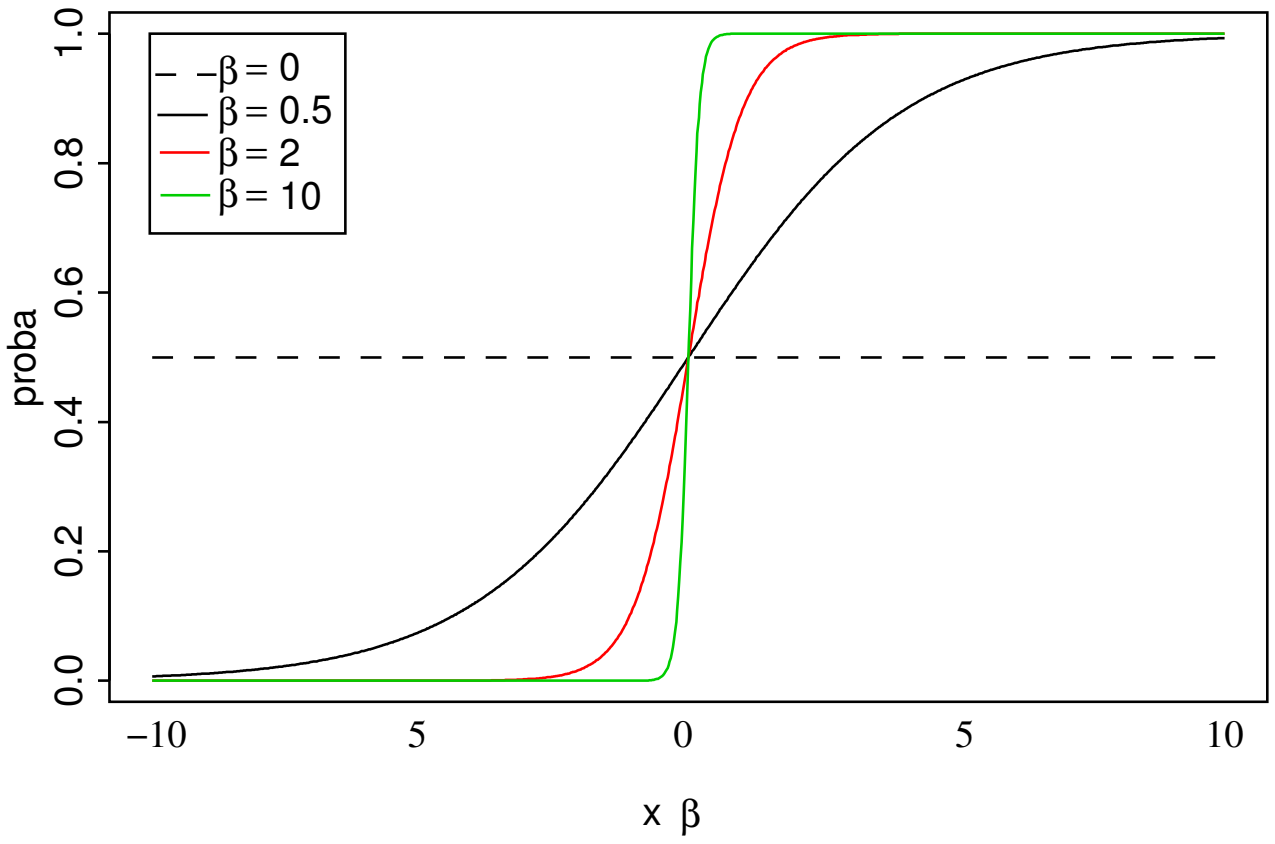
Equation of an S curve

A first way to get a S curve is to consider the function :

$$x \mapsto \frac{\exp\left(x^\top \beta\right)}{1 + \exp\left(x^\top \beta\right)} \ .$$



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Comparison with the linear model

In the linear model:

$$\mathbb{E}\left(Y / X = x\right) = x^\top \beta \ .$$

When Y is binary:

$$\mathbb{E}\left(Y / X = x\right) = p(x) \in [0, 1]$$

but it is possible to consider g link functions such as:

$$g\left(p(x)\right) = x^\top \beta \ .$$

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The logit function

Let's consider:

$$\mathbb{E}\left(Y\,/X=x\right)=p(x)=\frac{\exp\left(x^\top\beta\right)}{1+\exp\left(x^\top\beta\right)}\ .$$

The following function, named logit:

$$g(p)=\ln\left(\frac{p}{1-p}\right)$$

is bijective (derivable) and:

$$g\left(p(x)\right)=\ln\left(\frac{p(x)}{1-p(x)}\right)=x^\top\beta\ .$$

This function is used in logistic regression.
The outstanding issue is the estimation of the *beta* parameter.

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Maximum likelihood estimate I

Let's consider the likelihood:

$$p\left(y_1, \ldots, y_n; \beta\right) = \prod_{i=1}^{n} \mathbb{P}\left(Y = y_i / X = x_i\right) \ .$$

So:

$$p\left(y_1, \ldots, y_n; \beta\right) = \prod_{i=1}^{n} p\left(x_i\right)^{y_i} \left(1 - p\left(x_i\right)\right)^{1-y_i} \ .$$

The log-vraisemblance is:

$$\ell\left(y_1, \ldots, y_n; \beta\right) = \sum_{i=1}^{n} \left\{y_i \ln\left[p\left(x_i\right)\right] + \left(1 - y_i\right) \ln\left[1 - p\left(x_i\right)\right]\right\} \ .$$

We obtain:

$$\ell\left(y_1, \ldots, y_n; \beta\right) = \sum_{i=1}^{n} \left\{y_i x_i^\top \beta - \ln\left[1 + \exp\left(x_i^\top \beta\right)\right]\right\} \ .$$

Maximum likelihood estimate II

- Partial derivatives are calculated and cancelled to obtain the normal equations.
- Unfortunately, there are no explicit solutions for $\widehat{\beta}$.
- Likelihood has (usually) a single maximum, and there are iterative numerical algorithms to obtain this maximum, including :
  - Newton's algorithm,
  - Fisher's score algorithm.

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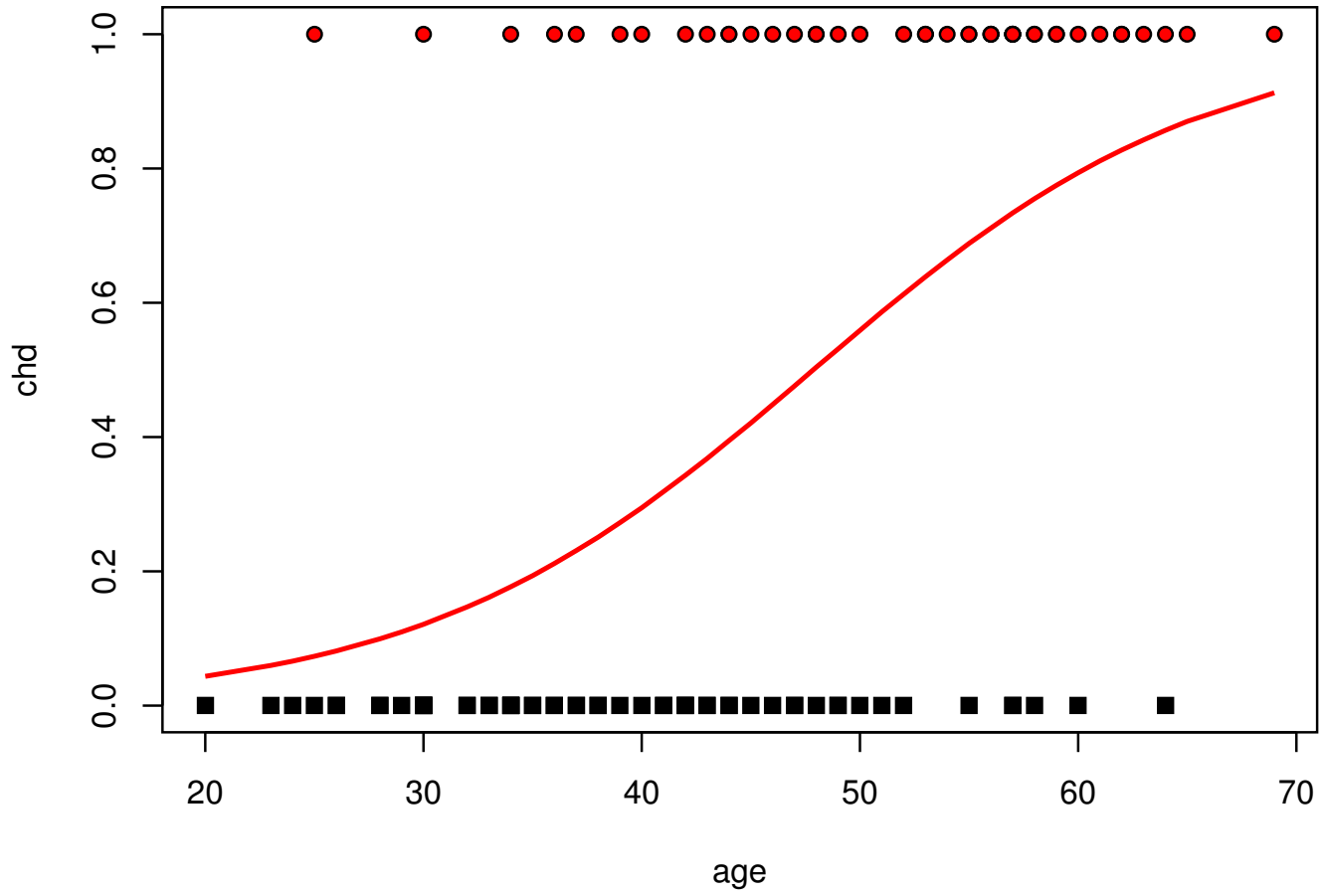
Back to introductory example

$$\widehat{\mathbb{P}}(CHD = 1 \,/\, age) = \frac{\exp(-5.30945 + 0.11092 \times age)}{1 + \exp(-5.30945 + 0.11092 \times age)} \; .$$



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Direct interpretation

When β_j associated with the variable X_j is:

- Positive : if X_j increases then p increases.
- Negative : if X_j increases then p decreases.

In the example $\widehat{\beta}_{age} = 0.11$, so the probability of CHD increases with age.

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Interpretation by odd-ratios I

The odds (of an event) is the ratio of the probability of an event to its complementary.

Let's imagine a game whose probability of winning is $p = \frac{1}{20}$. The odd is worth:

$$\frac{p}{1-p} = \frac{\frac{1}{20}}{1-\frac{1}{20}} = \frac{1}{19} \ .$$

On average, there is 1 winner for 19 losers.

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Interpretation by odd-ratios II

For an individual x, the odd to get the answer $Y = 1$ is defined by:

$$odd(x) = \frac{p(x)}{1 - p(x)}$$

where $p(x) = \mathbb{P}\left(Y = 1 \,/ X = x\right)$.

The odds ratio (OR) between 2 individuals x and x' is:

$$OR = \frac{odd(x)}{odd(x')} = \frac{\frac{p(x)}{1-p(x)}}{\frac{p(x')}{1-p(x')}}$$

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Interpretation by odd-ratios II

For 2 categories j and j':

1. Interpretation rules of odd-ratios

$$
\begin{aligned}
OR_{j,j'} > 1 &\Leftrightarrow p_j > p_{j'} \ , \\
OR_{j,j'} = 1 &\Leftrightarrow p_j = p_{j'} \ , \\
OR_{j,j'} < 1 &\Leftrightarrow p_j < p_{j'} \ .
\end{aligned}
$$

2. Odd-ratios and relative risk (or risk ratio: RR)
   If $p_{j'} = p_j \approx 0$ then:

$$
OR_{j,j'} \approx p_j / p_{j'} := RR_{j,j'} \ .
$$

3. Odd-ratios and parameters:

$$
\begin{aligned}
OR_{j,\mathrm{ref}} &= \exp\left(\beta_j \left(x_j - x_{\mathrm{ref}}\right)\right) \\
OR_{j,j'} &= \exp\left(\beta_j - \beta_{j'}\right)
\end{aligned}
$$

References

Hastie, T., Tibshirani, R., and Friedman, J. (2009). *The elements of statistical learning. Data Mining, inference, and prediction*. Springer Series in Statistics. Springer, 2 edition.

McCullagh, P. and Nelder, J. A. (1989). *Generalized linear models*. Monographs on Statistics & Applied Probability (37). Chapman & Hall.

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