# Matrix Perturbation

## 1 Basic Linear Algebra Results

**Fact 1** (Eigenvalues of a Hermitian matrix)**.** *Let $A$ be a Hermitian matrix. Then all the eigenvalues of $A$ are real. As a consequence the singular values are the absolute value of the eigenvalues.*

**Fact 2** (Continuity of the eigenvalues for Hermitian matrices)**.** *The eigenvalues of a Hermitian matrix are continuous with respect to the changes in the entries. The same is not true for non Hermitian matrices. For example consider the characteristic polynomial for the following matrices:*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & 0 \end{bmatrix}.$$

**Theorem 3** (Courant-Fisher Min-Max principle)**.** *Let $A$ be Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. Then the following holds:*

$$\lambda_k = \max_{S, \dim S = k} \ \min_{v \in S, |v|=1} v^* A v.$$

$$\lambda_k = \min_{S, \dim S = n-k+1} \ \max_{v \in S, |v|=1} v^* A v.$$

**Theorem 4** (Interlacing Property)**.** *Let $A$ be a symmetric $n \times n$ matrix and let $A_{n-1}$ be the top-left $n-1 \times n-1$ minor. Then the following holds:*

$$\lambda_n(A_n) \leq \lambda_{n-1}(A_{n-1}) \leq \lambda_{n-1}(A_n) \leq ... \leq \lambda_1(A_{n-1}) \leq \lambda_1(A_n).$$

*Proof.*

**Lemma 5.** *Two polynomials $f, g$ interlace if and only if $p_\alpha := f + \alpha g$ has real roots for all $\alpha \in \mathbb{R}$.*

*Proof of lemma.* Wlog assume that both $f$ and $g$ are monic. For simplicity let $x_1 < x_2 < ... < x_n$ be the roots of $f$.

"$\Rightarrow$". Let $\alpha > 0$. Then $(f + \alpha g)(x_i) \geq 0$ iff $i$ is odd and is $(f + \alpha g)(x_i) < 0$ iff $i$ is even. Similarly is $\alpha < 0$ or $\alpha = 0$.

"$\Leftarrow$". Suppose $f$ and $g$ do not interlace, then wlog assume $g$ has no roots in $[x_1, x_2]$ and both $f$ and $g$ are positive in $(x_1, x_2)$. Let $x' = (x_1 + x_2)/2$ so $f(x') > 0$, it means there exists $\epsilon > 0$ such that $p_{-\epsilon}(x') > 0$. Note that $p_{-\epsilon}$ is negative at $x_1$ and $x_2$, hence $p_{-\epsilon}$ must have at least two roots between $x_1$ and $x_2$. Since we have $p_{-t}$ is negative at $x_1$ and $x_2$ all the time, if we vary it from $-\epsilon$ to $-\infty$ the roots will move continuously and cannot leave the interval $(x_1, x_2)$. But $p_{-\infty}(x) < 0$ for all $x \in [x_1, x_2]$, so the roots disappeared - contradiction. $\square$

Write

$$A_n = \begin{bmatrix} A_{n-1} & b \\ b^* & d \end{bmatrix}.$$

Let $\alpha > 0$ be fixed. We have:

$$\begin{aligned}
\det(A_n - xI) + \alpha \det(A_{n-1} - xI) &= \det \begin{bmatrix} A_{n-1} - xI_{n-1} & b \\ b^* & d - x \end{bmatrix} + \alpha \det \begin{bmatrix} A_{n-1} - xI_{n-1} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} A_{n-1} - xI_{n-1} & b \\ b^* & d - x \end{bmatrix} + \det \begin{bmatrix} A_{n-1} - xI_{n-1} & 0 \\ 0 & \alpha \end{bmatrix} \\
&= \det\det \begin{bmatrix} A_{n-1} - xI_{n-1} & b \\ b^* & d - x \end{bmatrix} + \det \begin{bmatrix} A_{n-1} & b \\ 0 & \alpha \end{bmatrix} \\
&= \det \begin{bmatrix} A_{n-1} - xI_{n-1} & b \\ b^* & d - x + \alpha \end{bmatrix}.
\end{aligned}$$

Note that

$$\begin{bmatrix} A_{n-1} & b \\ b^* & d + \alpha \end{bmatrix}$$

is a Hermitian matrix so its characteristic polynomial has all the roots reals. By lemma we have that the eigenvalues of $A_n$ and $A_{n-1}$ interlace.

□

## 2 Deterministic perturbation

**Theorem 6** (Weyl's bound)**.** *Let $A$ and $b$ be two Hermitian matrices. Then the following holds:*

$$\lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B),$$

*valid whenever $i, j \geq 1$ and $i + j - i \leq n$.*

*Proof.*

$$\begin{aligned}
\lambda_{i+j-1}(A + B) &= \min_{S, \dim S = n-i-j} \max_{v \in S, |v|=1} v^*(A + B)v \\
&\leq \max_{v,w \in S'} v^*(A + Z)v,
\end{aligned}$$

where

$$S' = (\mathbb{R}^n \cap \operatorname{span}\{v_1(A), ...., v_{i-1}(A)\}) \cap \operatorname{span}\{v_1(B), ..., v_{j-1}(B)\}$$

as $\dim S' \geq n - i - j$. This implies:

$$\begin{aligned}
\lambda_{i+j-1}(A + B) &\leq \max_{v \in S', |v|=1} v^*Av + \max_{v \in S', |v|=1} v^*Bv \\
&\leq \lambda_i(A) + \lambda_j(B).
\end{aligned}$$

□

**Theorem 7** (Ky Fan theorem)**.** *For any $k$ the following holds:*

$$\sum_{i=1}^{k} \lambda_i(A + B) \leq \sum_{i=1}^{k} \lambda_i(A) + \lambda_i(B).$$

*Proof.* Proof is similar with the one of the Weyl's theorem. □

**Theorem 8** (Wielandt-Hoffman)**.** *Let $A$ and $B$ be two symmetric matrices, then*

$$\min_{\sigma \in S_n} \quad \sum_{i=1}^{n} (\lambda_i(A) - \lambda_{\sigma(i)}(B))^2 \leq ||A - B||_F^2.$$

*Proof.* Let $A = V^* A_1 V$ and $B = U^* B_1 U$ be the SVDs of $A$ and $B$. Then, since the Frobenius norm is invariant under orthogonal transformation (as it can be written as the trace of the square matrix), the RHS becomes:

$$\begin{aligned} \|A - B\|_F^2 &= \|A_1 V U^* - V U^* B_1\|_F^2 \\ &:= \|A_1 Q - Q B_1\|_F^2, \end{aligned}$$

where $Q := VU^*$ is unitary matrix. We can explicitly write

$$\begin{aligned} \|A_1 Q - Q B_1\|_F^2 &= \sum_{i,j} (\lambda_i(A) - \lambda_j(B))^2 Q_{i,j}^2 \\ &\leq \sum_{i,j} (\lambda_i(A) - \lambda_{\sigma(i)}(B))^2 Q_{i,j}^2 \\ &= \sum_{i} (\lambda_i(A) - \lambda_{\sigma(i)}(B))^2, \end{aligned}$$

as $Q$ is unitary. □

**Observation 9.** *The Wielandt-Hoffman can be interpreted as follows: Let $B$ be a small perturbation of $A$ (small enough that the permutation $\sigma$ will be the identity), then the following holds:*

$$\sum_{i=1}^{n} (\lambda_i(A + B) - \lambda_i(A))^2 \leq ||B||_F^2.$$

**Theorem 10** (Davis-Kahan, Wedin; sine theorem)**.**

- $\sin \angle(v_1, v_1') \leq \frac{||B||}{\lambda_1(A) - \lambda_2(A+B)}$
- $\sin \angle(v_1, v_1') \leq 2\frac{||B||}{\delta}$.

*Proof.* Let the SVD's of $A$ and $A + B$ be

$$A = \lambda_1 v_1 v_i^* + U^* \Lambda U$$

$$A + B = \lambda_1' v_1' v_1'^* + U'^* \Lambda' U'.$$

We have:

$$\begin{aligned} \|B\| \geq \|U' B v_1\| &= \|U'(A + B) v_1 - U' A v_1\| \\ &= \|U'(\lambda_1' v_1' v_1'^* + U'^* \Lambda' U') v_1 - U' \lambda_1 v_1\| \\ &= \|\Lambda' U' v_1 - \lambda_1 U' v_1\| \\ &\geq \|\Lambda' - \lambda_1 I\| \cdot \|U' v_1\| \\ &\geq |\lambda_2' - \lambda_1| \|U' v_1\|, \end{aligned}$$

which implies:

$$\sin(v_1, v_1') = \|U' v_1\| \leq \frac{\|B\|}{|\lambda_1(A) - \lambda_2(A+B)|}.$$

The second bound follows by the fact that

$$|\lambda_2(A+B) - \lambda_2(A)| \leq \|B\|.$$

□