Lecture notes

1 Perturbation of low rank matrices

Let $r \geq s \geq 0$ be fixed integers and let $\theta_1 \geq \theta_2 \geq ... \geq \theta_s > 0 > \theta_{s+1} \geq ... \geq \theta_r$. Define $P = \operatorname{diag}(\theta_1, \theta_2, ..., \theta_r, 0, 0, ..., 0)$.

Let X_n be an $n \times n$ symmetric (Hermitian) unitary matrix, μ_n be its empirical eigenvalues measure, i.e.

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(X_n)}$$

and let G_{μ_n} be its Cauchy transform of μ_n, defined by:

$$G_{\mu_n} := \int \frac{d\mu_n(t)}{z-t}, \text{ for } z \notin \operatorname{supp}\mu.$$

Assume that μ_n, converge almost surely to a non-randomly compact supported probability measure μ. Suppose also that $\lambda_i(X_n)$ and $\lambda_{n-i}(X_n)$ converge almost surely to b respectively a for every $i = 1, .., r$, where $-\infty < a \leq b < \infty$ are fixed.

Define

$$\tilde{X}_n = X_n + P_n.$$

Theorem 1 (Eigenvalues)**.** *[Benaych-Georges, Nadakuditi] For $1 \leq i \leq s$ and $\theta_i > 1/G_\mu(b^+)$,*

$$\lambda_i(\tilde{X}_n) \longrightarrow_{a.s} G_\mu^{-1}(1/\theta_i)$$

For $1 \leq i \leq s$ and $\theta_i \leq 1/G_\mu(b^+)$,

$$\lambda_i(\tilde{X}_n) \longrightarrow_{a.s} b$$

For $0 \leq j < r-s$ and $\theta_i \geq 1/G_\mu(a^-)$,

$$\lambda_{n-j}(\tilde{X}_n) \longrightarrow_{a.s} a$$

For $0 \leq j < r-s$ and $\theta_i < 1/G_\mu(a^-)$,

$$\lambda_{n-j}(\tilde{X}_n) \longrightarrow_{a.s} G_\mu^{-1}(1/\theta_{r-j}).$$

Example 2. If $X = \frac{1}{\sqrt{n}} Z$, where is Z is GUE, then, we have $a = -2$, $b = 2$, $G_\mu(b^+) = 1$, $G_\mu(z) = \frac{z - \sqrt{4-z^2}}{2}$, $G_\mu(a^-) = -1$ and $G_\mu^{-1}(1/\theta_i) = \theta_i + \frac{1}{\theta_i}$.

Theorem 3 (Eigenvectors)**.** *[Benaych-Georges, Nadakuditi] Consider $i \in \{1, .., r\}$ such that $1/\theta_i \in (G_\mu(a^-), G_\mu(b^+))$. Let $\tilde{u}_i$ be the unit-norm vector of $\tilde{X}$ associated with the eigenvalue $\lambda_i(\tilde{X})$.*

- *If $\theta_i > 0$ we have :*

$$|\langle \tilde{u}_i, \ker(\theta_i I - P)\rangle|^2 \longrightarrow_{n\to\infty} \frac{-1}{\theta_i^2 G'_\mu(\rho)},$$

  where $\rho = G_\mu^{-1}(1/\theta_i)$ is the limit of $\tilde{\lambda}_i$. Moreover,

$$\left\langle \tilde{u}_i, \bigoplus_{j\neq i} \ker(\theta_j I - P)\right\rangle \longrightarrow 0.$$

- *If $\theta_i < 0$ we have:*

$$|\langle \tilde{u}_{n-r+i}, \ker(\theta_i I - P)\rangle|^2 \longrightarrow_{n\to\infty} \frac{-1}{\theta_i^2 G'_\mu(\rho)},$$

  where $\rho = G_\mu^{-1}(1/\theta_i)$ is the limit of $\tilde{\lambda}_{n-r+i}$.
  Moreover,

$$\left\langle \tilde{u}_{n-r+i}, \bigoplus_{j\neq i} \ker(\theta_j I - P)\right\rangle \longrightarrow 0.$$

Example 4. If $X = \frac{1}{\sqrt{n}}Z$, where Z is GUE, we have for $z > 0$, $G_\mu(z) = \frac{z-\sqrt{z^2-4}}{2}$, so $G'_\mu(z) = \frac{1}{2}\left(1 - \frac{z}{\sqrt{z^2-4}}\right)$ which implies that $\frac{-1}{\theta_i^2 G'_\mu(\theta+\frac{1}{\theta})} = \frac{\theta_i^2-1}{\theta_i^2}$. Then, if all the θ_i's are distinct we have:

$$|\langle \tilde{u}_i, u_i\rangle|^2 \to \begin{cases} 1 - \frac{1}{\theta_i^2} & \text{if } \theta_i > 1 \\ 0 & \text{if } 0 < \theta_i \leq 1 \end{cases}$$

$$|\langle \tilde{u}_i, \text{span}\{u_1, u_2, ..., u_{i-1}, u_{i+1}, .., u_r\}\rangle|^2 \to 0.$$

Proof of Theorem **??**.

Using Weyl's interlacing inequalities imply that:

$$\lambda_{k+(r-s)} \leq \lambda_k(\tilde{X}) \leq \lambda_{k-s},$$

(we use the convention that $\lambda_k = -\infty$, if $k > n$ and $+\infty$ if $k < 0$) and so, since $\lambda_r \longrightarrow b$ it follows that if $\lambda_i(\tilde{X})$ does not converge to a limit outside $[a, b]$ for some $1 \leq i \leq s$, then it has to go to b. Similarly, if $0 < j \leq r-s$ and $\lambda_{n-i}(\tilde{X})$ does not converge to a limit outside $[a, b]$, then it has to go to a.

Intuition from rank 1

Assume $r = 1$. Since X is unitary invariant wlog we can assume that $X = \text{diag}(\lambda_1, ..., \lambda_n)$ and $P = \theta_1 u^* u$, where u is randomly distributed on the unit sphere.

Suppose $\lambda'_1 > b$ is an eigenvalue of $\tilde{X}$ which is not an eigenvalue of X. Then:

$$0 = \det(\lambda'_1 I - (X + P)) = \det(\lambda'_1 I - X) \cdot \det(I - (\lambda'_1 I - X)^{-1}P)$$

It follows that:

$$\det(I - (\lambda'_1 I - X)^{-1}P) = 0,$$

which translates as 1 is an eigenvalue for $(\lambda'_1 I - X)^{-1}P$. But $\text{rank}(P) = 1$, so it follows that

$$\text{Trace}\left((\lambda'_1 I - X)^{-1}P\right) = 1. \tag{1}$$

Since X is unitary invariant, wlog we can assume that $X = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ and $P = \theta u u^*$, where $u = (u_1, ..., u_n)$ is a random vector uniformly distributed on the unit sphere. Hence relation (**??**) becomes:

$$1 = \theta \sum_{k=1}^{n} \frac{|u_i|^2}{\lambda'_1 - \lambda_k}.$$

Since u is uniformly distributed on the unit sphere, it follows that, if $\lambda'_1 \geq \lambda_1 + \epsilon$, where $\epsilon > 0$ is fixed, then:

$$\sum_{k=1}^{n} \frac{|u_i|^2}{\lambda'_1 - \lambda_k} \longrightarrow_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\lambda'_1 - \lambda_k} = G_\mu(\lambda'_1).$$

Take $\epsilon \to 0$, it follows that

$$\theta G_\mu(\lambda') \longrightarrow 1 \Rightarrow \lambda'_1 \longrightarrow G_\mu^{-1}(1/\theta).$$

However, if $1/\theta > G(b)$, then $G_\mu^{-1}(1/\theta)$ does not exist so, there is no such eigenvalue.

<u>Proof from rank r</u>

Let z be an eigenvalue for $\tilde{X}$ which is not an eigenvalue for X, then $\det(zI - \tilde{X}) = 0$, which implies:

$$\det(I - (zI - X)^{-1} P) = 0.$$

As before, assume $X = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ and let $P = U \Lambda U^*$, then:

$$0 = \det(I - (zI - X)^{-1} P) = \det\left(I - U^*(zI - X)^{-1} U \Lambda\right).$$

Define:

$$M_z := I - U^*(zI - X)^{-1} U \Lambda.$$

Lemma 5. *The following things are true about M:*

- $x \in \ker(zI - \tilde{X}) \Rightarrow U^* x \in \ker(M_z)$ *and*

$$x = (zI - X)^{-1} U \Lambda U^* x.$$

- $M_z(i,j) = \mathbb{1}_{i=j} - \theta_j \sum_k \frac{U(j,k) U^*(k,i)}{z - \lambda_k}$
- $M_z(i,j) \longrightarrow 0$ *if* $i \neq j$ *and* $M_z(i,i) \longrightarrow 1 - \theta_i G_\mu(z)$ *if* $i = j$.

Proof. Fix $\eta > 0$. As long as $z > \lambda_1 + \eta$ then $c_k := 1/(z - \lambda_k)$ are bounded and independent of the entries of U, which implies that $\sum_k \frac{U(j,k) U^*(k,i)}{z - \lambda_k} \longrightarrow 0$ for $i \neq j$. Take $\eta \to 0$. □

Now it is easy to see that z is an eigenvalue of $\tilde{X}$ outside $[a, b]$ if and only if z is a diagonal element of M_Z and $G(a) \leq 1/\theta_i \leq G(b)$.

□

Proof of Theorem **??**. Let's assume that $\theta_1 > \theta_2$. Suppose $\tilde{\lambda}_1 > b + \eta$, for some $\eta > 0$, is an eigenvalue for $\tilde{X}$ which is not an eigenvalue for X and let $\tilde{u}_1$ be the corresponding eigenvector. By Lemma **??** we know that

$$M_{G_\mu^{-1}(1/\theta_1)} \longrightarrow \text{diag}\left(0, 1 - \frac{\theta_2}{\theta_1}, 1 - \frac{\theta_3}{\theta_1}, ..., 1 - \frac{\theta_r}{\theta_1}, 1, 1, ..., 1\right).$$

Now Lemma **??** also implies that:

$$U^* \tilde{u}_1 \in \ker M_{G_\mu^{-1}(1/\theta_1)},$$

which implies:

$$U^*\tilde{u}_1 = c_1(1,0,0,...,0),$$

for some $c_1 > 0$.

This implies that:

$$\left\langle \tilde{u}_1, \bigoplus_{j>1} \ker(\theta_j I - P) \right\rangle \longrightarrow 0. \tag{2}$$

By Lemma **??** we know that:

$$\begin{aligned}
\tilde{u}_1 &= (\tilde{\lambda}_1 I - X)^{-1} U\Lambda U^* \tilde{u}_1 \\
&= (\tilde{\lambda}_1 I - X)^{-1} \sum_{j=1}^{r} \theta_j (u_j^* \tilde{u}_1) u_j \\
&= (\tilde{\lambda}_1 I - X)^{-1} \theta_1 (u_1^* \tilde{u}_1) u_1 + (\tilde{\lambda}_1 I - X)^{-1} \sum_{j=2}^{r} \theta_j (u_j^* \tilde{u}_1) u_j
\end{aligned}$$

We know that $\tilde{\lambda}_1 > \eta + b$, which implies

$$\|(\tilde{\lambda}_1 I - X)^{-1}\| < \infty,$$

and so, by equation (**??**)

$$(\tilde{\lambda}_1 I - X)^{-1} \sum_{j=2}^{r} \theta_j (u_j^* \tilde{u}_1) u_j \longrightarrow 0.$$

We conclude,

$$\begin{aligned}
1 = \|\tilde{u}_1\|^2 &= \theta_1^2 |u_1^* \tilde{u}_1|^2 \|(\tilde{\lambda}_1 I - X)^{-1} u_1\|^2 \\
&= \theta_1^2 |u_1^* \tilde{u}_1|^2 \sum_{i=1}^{r} \frac{|u_1(i)|^2}{(\tilde{\lambda}_1 - \lambda_i)^2} \\
&\longrightarrow \theta_1^2 |u_1^* \tilde{u}_1|^2 \int \frac{d\mu}{(\tilde{\lambda}_1 - t)^2} \\
&\longrightarrow \theta_1^2 |u_1^* \tilde{u}_1|^2 \left(-G'_\mu(\tilde{\lambda}_1) \right).
\end{aligned}$$

It follows that:

$$|u_1^* \tilde{u}_1|^2 \longrightarrow \frac{1}{-\theta_1^2 G'_\mu(\tilde{\lambda}_1)}.$$

Take $\eta \to 0$ to conclude the proof.

□

Consider the setting from the example. We have the following results regarding perturbation of small rank matrices.

Theorem 6 (GUE + rank 1 perturbation)**.** *[Peche] Assume that $X = \frac{1}{\sqrt{n}} Z$, where Z is GUE, with entries having variance σ^2. The following holds:*

$$\sqrt{n}\left(\lambda_1(\tilde{X}) - \left(\theta + \frac{1}{\theta} \right) \right) \longrightarrow \mathcal{N}(0, \sigma_\theta^2),$$

where $\sigma_\theta := (\sigma/\theta) \cdot \sqrt{\theta^2 - \sigma^2}$.

Theorem 7 (GUE + constant rank perturbation)**.** *[Peche] Let* $X = \frac{1}{\sqrt{n}}Z$*, where* Z *is GUE. Let* $P = \text{diag}(\theta_1, ..., \theta_1, \theta_2, ..., \theta_r, 0, 0, ..., 0)$ *with* θ_1 *of multiplicity* k *such that* k*,* r*,* θ_1 *are given numbers independent of* n *(*$k, r \in \mathbf{Z}_+$*) and* σ_i*'s lie in a compact set of* $(-\infty, \theta_1)$ *independent of* n*. Then the following holds:*

- *If* $\theta_1 < 1$*, then*
$$\lim_{n\to\infty} \mathbf{P}\left(n^{2/3}(\lambda_1 - 2) \leq x\right) = F_2^{TW}(x).$$
- *If* $\theta_1 = 1$*, then*
$$\lim_{n\to\infty} \mathbf{P}\left(n^{2/3}(\lambda_1 - 2) \leq x\right) = F_{k+2}^{TW}(x).$$
- *If* $\theta_1 > 1$*, then*
$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\theta_1^2}{\theta_1^2 - 1} n^{1/2}\left(\lambda_1 - \left(\theta_1 + \frac{1}{\theta_1}\right)\right) \leq x\right) = F^k_{GUE,\theta_1}(x),$$

where F_2^{TW}*,* F_{k+2}^{TW} *and* F^k_{GUE,θ_1} *are deterministic functions. (*F^k_{GUE,θ_1} *is the probability distribution of the largest eigenvalue of the* $k \times k$ *GUE with parameter* σ^2*)*

Theorem 8 (GUE + low rank perturbation)**.** *[Peche] Let* $X = \frac{1}{\sqrt{n}}Z$*, where* Z *is GUE. Let* $P = \text{diag}(\theta_1, ..., \theta_1, \theta_2, ..., \theta_r, 0, 0, ..., 0)$ *with* θ_1 *of multiplicity* k *such that* k *and* r *satisfy* $\lim_{n\to\infty} k/n = 0$ *and* $\lim_{n\to\infty} k/n = 0$*. Suppose that* θ_1 *is independent of* n *and* σ_i*'s lie in a compact set of* $(-\infty, \theta_1)$ *also independent of* n*. Then there exists deterministic functions of* n, k, r *and* θ_1*,* G_1, G_2, G_3 *such that the following holds:*

- *If* $\theta_1 \leq 1$*, then*
$$\lim_{n\to\infty} \mathbf{P}\left(G_1(\lambda_1 - G_2) \leq x\right) = F_2^{TW}(x).$$
- *If* $\theta_1 > 1$*, then*
$$\lim_{n\to\infty} \mathbf{P}\left(G_3(\lambda_1 - G_2) \leq x\right) = F_2^{TW}(x).$$

2 Dyson Brownian Motion

Theorem 9 (Wigner + rank 1 perturbation)**.** *[Capitaine, Donati-Martin, Feral] Suppose* X_n *is Wigner matrix with entries i.i.d up to the symmetry constrained, with variance* σ^2 *and distribution* μ *which satisfies the following Poincare inequality. "There exist a positive constant* C *such that for any* $f\mathbb{R} \to \mathbb{C}$*,* $f \in \mathcal{C}^1$ *and* $f, f' \in L^2(\mu)$*, the following holds:*
$$\mathbf{E}(|f - \mathbf{E}(f)|^2) \leq C \int |f'|^2 d\mu."$$
Let $P_n := \text{diag}(\theta, 0, 0, ..., 0)$ *or* $P_n(i,j) = \theta/n$ *for all* $1 \leq i, j \leq n$*, where* $\theta > \sigma$*, then the following holds:*
$$\lambda_1(\tilde{X}) \longrightarrow \theta + \frac{1}{\theta}.$$

Definition 10 (GOE/GUE)**.**

- We say that Z is GOE if and only if $Z_{ij} = Z_{ji}$ and Z_{ij} is $\mathcal{N}(0,1)$ for $i \neq j$ and $\mathcal{N}(0,2)$ is $i = j$ independent of each other.

- We say that Z is GUE if and only if $Z_{ij} = Z_{ji}$ and $Z_{ij} = (B_{ij} + \mathbf{i}B'_{ij})/\sqrt{2}$ and $Z_{ii} = B_{ii}$, where $B_{i,j}$ are $\mathcal{N}(0,1)$ independent of each other.

Let Z be a GOE/GUE matrix. We can think about the matrix $A + Z$ as a process which starts at time $t = 0$ in A and it smoothly moves up to time $t = 1$ when it reaches $A + Z$. A good candidate for the model would be:

$$A(t) = A + tZ.$$

Consider the relation

$$A(t)v_i(t) = \lambda_i(t)v_i(t)$$

and differentiate it with respect to t. We get:

$$Zv_i(t) + A(t)v_i^{(1)}(t) = \lambda_i^{(1)}(t)v_i(t) + \lambda_i(t)v_i^{(1)}(t).$$

We can left multiply by $v_i^T(i)$ to get:

$$v_i^T(t)Zv_i(t) + 0 = \lambda_i^{(1)}(t) + 0, \tag{3}$$

as v_i and $v_i^{(1)}$ are orthogonal.

The equation (**??**) is promising as we have a formula for the derivative of λ_i. Hence

$$\lambda_i(A + Z) - \lambda_i(A) = \int_0^1 \lambda_i^{(1)}(t)dt.$$

The problem with this model is that $v_i(t)$ depends on the matrix Z, so we can not use the *randomness* of Z to bound $v_i^T(t)Zv_i(t)$ unless $t = 0$. So we need a model in which at time t, $v_i(t)$ is independent of the noise that is added at time t. Here is when Brownian Motion comes in play.

Definition 11. We call $x(t)$ a Wigner process, if

- $x(t)$ is $\mathcal{N}(0,t)$ distributed for any t.
- $x(t) - x(t')$ is $\mathcal{N}(0, t - t')$ distributed for any $0 \le t \le t' \le 1$
- $x(t_1) - x(t_2)$ is independent of $x(t_3) - x(t_4)$ for any $t_1 > t_2 \ge t_3 > t_4$

Basically $x(t)$ is a random process and has the property that at time t the way it moves is independent of the actual value at time t. We can generalize this concept to matrices.

Basically,

$$x(t) = \lim_{dt \to 0} x(0) + \sum_{i=1}^{t/dt} \gamma_i,$$

where γ_i is $\mathcal{N}(0, dt)$ distributed.

Definition 12. We call $Z(t)$ a Wigner symmetric process, i.e.

- $Z(t)$ is symmetric $\mathcal{N}(0,t)$ distributed
- $Z(t) - Z(t')$ is $\mathcal{N}(0, t - t')$ distributed for any $0 \le t \le t' \le 1$

- $Z(t_1) - Z(t_2)$ is independent of $Z(t_3) - Z(t_4)$ for any $t_1 > t_2 \geq t_3 > t_4$

Theorem 13 (Dyson)**.** *Let $Z(t)$ be a Wiener process and let $\lambda_i(t)$ be the i^{th} eigenvalue at time t. Then the following relations hold:*

- $\mathbf{P}\left(\forall t > 0, \lambda_1(t) > \lambda_2(t) > ... > \lambda_n(t)\right) = 1.$
- $d\lambda_i = \frac{\sqrt{2}}{\sqrt{\beta}} dB_i + \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)}$, *where B_i is $\mathcal{N}(0,1)$ distributed and $\beta = 1$ for GOE and $\beta = 2$ for GUE.*

Similarly, $Z(t)$ can be seen as:

$$Z(t) = \lim_{dt \to 0} Z(0) + \sum_{i=1}^{t/dt} Z_i,$$

where Z_i's are GOE with variance dt.

We can thing about the model as follows. Fix n and let dt to be very small, going to 0 (independent of n). Then, we can divide our process in $1/dt$ processes as:

$$A(t) = \begin{cases} A + \sum_{i=1}^{t/dt} Z_i \text{ if } t \text{ is a multiple of } dt \\ A + \sum_{i=1}^{\lfloor t/dt \rfloor} Z_i + (t - \lfloor t/dt \rfloor) Z_{i+1} \text{ otherwise} \end{cases}$$

Now, since dt is very small, then we can approximate

$$v_i(t) \approx v_i(\lfloor t/dt \rfloor dt),$$

and hence it will be independent of Z_{i+1} so we can follow the approach from equation (**??**).

Observation 14. *There is one issue with the above approach: $dt \cdot Z$ is GOE with variance $\mathcal{N}(0, dt^2)$ and not dt. So we need to take the gap of length $\sqrt{dt}$ to be under the condition of equation (**??**).*

This implies that:

$$\lambda_i\left((k+1)\sqrt{dt}\right) \approx \lambda_i\left(k\sqrt{dt}\right) + v_i(k\sqrt{dt})^T Z_k v_i(k\sqrt{dt}),$$

where $v_i(k\sqrt{dt})$ is independent of Z_k. Since Z_k is $\mathcal{N}(0, dt)$, we get that:

$$\lambda_i^{(1)}\left(k\sqrt{dt} + s\right) \approx v_i^T(k\sqrt{dt}) Z_{k+1} v_i^T(k\sqrt{dt}) := \gamma_k,$$

for any $0 \leq s < \sqrt{dt}$, where γ_k is $\mathcal{N}(0, 2dt)$ distributed which depends only on Z_{k+1}.

Similarly, we can compute first $v_i^{(1)}$ to get to:

$$\lambda_i^{(2)}\left(k\sqrt{dt} + s\right) \approx 2 \sum_{j \neq i} \frac{\left(v_i^T(k\sqrt{dt}) Z_{k+1} v_j^T(k\sqrt{dt})\right)^2}{\lambda_i(k\sqrt{dt}) - \lambda_j(k\sqrt{dt})} := 2 \sum_{j \neq i} \frac{Z_k(i,j)^2}{\lambda_i(k\sqrt{dt}) - \lambda_j(k\sqrt{dt})},$$

for any $0 \leq s < \sqrt{dt}$, where $Z_k(i,j)$ is $\mathcal{N}(0, dt)$ distributed and depends only on Z_{k+1}.

Finally, using Taylor we have

$$\lambda_i((k+1)\sqrt{dt}) = \lambda_i(k\sqrt{dt}) + \sqrt{dt}\lambda_i^{(1)}(k\sqrt{dt}) + \frac{dt}{2}\lambda_i^{(2)}(k\sqrt{dt}) + \mathcal{O}((dt)^{3/2})$$

Hence:

$$\begin{aligned}
\lambda_i(A+Z) - \lambda_i(A) &= \sum_{k=1}^{1/\sqrt{dt}} \lambda_i\left((k+1)\sqrt{dt}\right) - \lambda_i(k\sqrt{dt}) \\
&= \sqrt{dt}\sum_{k=1}^{1/\sqrt{dt}} \gamma_k + \sum_{k\neq i}\sum_{j\neq i} \frac{Z_k(i,j)^2}{\lambda_i(k\sqrt{dt}) - \lambda_j(k\sqrt{dt})} + O(dt^{3/2}) \\
&\longrightarrow_{dt\to 0} B_i + \int_0^1 \left(\sum_{k\neq i} \frac{1}{\lambda_i(A+Z(t)) - \lambda_k(A+Z(t))}\right) dt + o(1) \text{ whp}
\end{aligned}$$

where B_i is $\mathcal{N}(0,2)$ distributed.

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