Special topics for classical damped wave models

PDE Course for Master and PhD students
at the Vietnam Institute for Advanced Study in Mathematics

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Schedule

1. 22.11.2017, 09.00-11.30: **Phase space analysis for classical damped wave models**
Content: First we derive representations of solutions by using Fourier multipliers. This allows to discuss decay behavior and decay rate of the wave type energy. Moreover, we explain the influence of additional regularity in the data. One of the most important properties is described by the so-called diffusion phenomenon for damped wave models. We explain this phenomenon. Some conclusions complete the first part of lectures.

2. 24.11.2017, 09.00-11.30: **Semilinear classical damped wave models**
Content: We begin with explanations of the Fujita exponent as the critical exponent in semilinear heat models. Then we show that this exponent is critical in semilinear classical damped wave models with power nonlinearity, too. Stability of the zero solution is proved by using decay estimates for solutions to parameter-dependent Cauchy problems, Duhamel's principle and a fixed point argument. The optimality of the critical exponent is proved by applying the test function method. This implies blow-up (in finite time) of weak solutions even for small data.

3. 30.11.2017, 09.00-11.30: **Fujita via Strauss - a never ending story**
Content: First we recall the story of the Strauss exponent appearing as the critical exponent in semilinear wave models. We introduce Kato's lemma as an important tool to prove blow-up results. Finally, we introduce recent results for some semilinear wave models with power nonlinearity and with scale invariant mass and dissipation. These models are taken from a grey zone, where an interaction of Fujita and Strauss critical exponents appears.

1 The classical damped wave model

First of all we mention that solutions to classical damped wave equations have qualitative properties as existence of a forward wave front, finite propagation speed of perturbations or existence of a domain of dependence.

1.1 Representation of solutions by using Fourier multipliers

Let us turn to the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x).$$

Step 1 *Transformation of the dissipation term into a mass term*
We introduce a new function $w = w(t,x)$ by $w(t,x) := e^{\frac{1}{2}t}u(t,x)$. Then w satisfies the Cauchy problem

$$w_{tt} - \Delta w - \frac{1}{4}w = 0, \quad w(0,x) = \varphi(x), \quad w_t(0,x) = \frac{1}{2}\varphi(x) + \psi(x).$$

In opposite to the Klein-Gordon equation now appears a *negative mass term*. This negative mass needs some special considerations.
Step 2 *Application of partial Fourier transformation*
The application of partial Fourier transformation gives the following ordinary differential equation for $v = v(t,\xi) = F_{x\to\xi}(w(t,x))(t,\xi)$:

$$v_{tt} + \Big(|\xi|^2 - \frac{1}{4}\Big)v = 0, \quad v(0,\xi) = v_0(\xi) := F(\varphi)(\xi),$$
$$v_t(0,\xi) = v_1(\xi) := \frac{1}{2}\, F(\varphi)(\xi) + F(\psi)(\xi).$$

We make a distinction of cases for $\{\xi \in \mathbb{R}^n : |\xi| > \frac{1}{2}\}$ (the coefficient $|\xi|^2 - \frac{1}{4}$ is positive) and for $\{\xi \in \mathbb{R}^n : |\xi| < \frac{1}{2}\}$ (the coefficient $|\xi|^2 - \frac{1}{4}$ is negative).
Case 1 $\{\xi : |\xi| > \frac{1}{2}\}$
Using $|\xi|^2 > \frac{1}{4}$ we define a new positive variable $|\eta|$ by $|\eta|^2 := |\xi|^2 - \frac{1}{4} > 0$. So we get the ordinary differential equation $v_{tt} + |\eta|^2 v = 0$. We obtain immediately the following representation of solution $v(t,\xi)$:

$$v(t,\xi) = \cos\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\; t\Big)\, v_0(\xi) + \frac{\sin\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\; t\Big)}{\sqrt{|\xi|^2 - \frac{1}{4}}}\, v_1(\xi).$$

Case 2 $\{\xi : |\xi| < \frac{1}{2}\}$

The solution to the transformed differential equation is

$$\begin{aligned} v(t,\xi) &= \Big(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}\,t} \\ &+\Big(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{\frac{1}{2}\sqrt{1-4|\xi|^2}\,t} \\ &= v_0(\xi)\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big) + \frac{2v_1(\xi)}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big). \end{aligned}$$

If we consider the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

with data $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, then we conclude from the above representations of solutions the next result after taking into consideration that only the *behavior for large frequencies is important for the regularity of solutions.*

Theorem 1.1. *Let the data $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, $s \in \mathbb{R}^1$, $n \geq 1$ be given in the Cauchy problem*

$$u_{tt} - \Delta u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

Then there exists for all $T > 0$ a uniquely determined (in general) distributional solution

$$u \in C\big([0,T], H^s(\mathbb{R}^n)\big) \cap C^1\big([0,T], H^{s-1}(\mathbb{R}^n)\big).$$

We have the a priori estimate

$$\|u(t,\cdot)\|_{H^s} + \|u_t(t,\cdot)\|_{H^{s-1}} \leq C(T)\big(\|\varphi\|_{H^s} + \|\psi\|_{H^{s-1}}\big).$$

Finally, the solution depends continuously on the data.

Let us discuss how to prove this theorem.

Remark 1.1. *The statements of Theorem 1.1 are true for the solutions to the Cauchy problem*

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

Consequently, the dissipation term has no influence on the regularity of solutions. Dissipation terms have an essential influence on energy estimates, they produce a decay of the energy. This will be explained in the next section.

1.2 Decay behavior and decay rate of the wave energy

We know that the wave energy

$$E_W(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2\Big)\,dx$$

of Sobolev solutions to the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

is a decreasing function if $E_W(u)(0)$ is finite. This follows from differentiation of the energy $E_W(u)(t)$ with respect to t and integration by parts. We assume that all these steps can be carried out, that is, the data are supposed to be smooth enough (it is sufficient to assume for our purpose that the data (φ,ψ) belong to the so-called energy space $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$). Then, we derive

$$\begin{aligned} E_W'(u)(t) &= \frac{1}{2}\int_{\mathbb{R}^n} \big(2u_t u_{tt} + 2\nabla u\cdot\nabla u_t\big)\,dx \\ &= \int_{\mathbb{R}^n} \big(u_t(\Delta u - u_t) + \nabla u\cdot\nabla u_t\big)\,dx = \int_{\mathbb{R}^n} -u_t(t,x)^2 dx \leq 0. \end{aligned}$$

Thus, the energy is decreasing for increasing t. We can not expect energy conservation. This seems to be no surprise because of the damping term. It arises the question for the behavior of the energy for $t\to\infty$. Of special interest is the question whether the energy $E_W(u)(t)$ tends to 0 for $t\to\infty$. Such a behavior is called *decay.*
Applying phase space analysis allows to verify that the energy $E_W(u)(t)$ is even decaying for $t\to\infty$. We are able to derive for $E_W(u)(t)$ an *optimal decay behavior* with an optimal *decay rate.*

Theorem 1.2. *The solution to the Cauchy problem*

$$u_{tt} - \Delta u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

with data $\varphi\in H^1(\mathbb{R}^n)$ and $\psi\in L^2(\mathbb{R}^n)$ satisfies the following estimates for $t\geq 0$:

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\leq C\big(\|\varphi\|_{L^2} + \|\psi\|_{H^{-1}}\big), \\ \|\nabla u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{1}{2}}\big(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\big), \\ \|u_t(t,\cdot)\|_{L^2} &\leq C(1+t)^{-1}\big(\|\varphi\|_{H^1} + \|\psi\|_{L^2}\big). \end{aligned}$$

Consequently, the wave energy satisfies the estimate

$$E_W(u)(t) \leq C(1+t)^{-1}\big(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2\big).$$

Remark 1.2. *We see that the kinetic energy decays faster than the elastic energy. To get these estimates we suppose for the data* (φ, ψ) *the regularity* $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *which is stronger than the regularity* $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*. The last regularity guarantees for a Sobolev solution to become an energy solution. Try to understand what kind of estimates we would have in the case of data to belong to* $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*. In which step of the following proof do we use the assumption* $\varphi \in L^2(\mathbb{R}^n)$*?*

Proof. Step 1 *Transformation of the energy in the phase space*
Let $\hat{u}$ be the Fourier transform of u, that is, $\hat{u}(t,\xi) = F_{x\to\xi}(u(t,x))(t,\xi)$. We transfer the energy in the phase space as follows:

$$\begin{aligned} E_W(u)(t) &= \frac{1}{2}\Big(\|\nabla u(t,\cdot)\|_{L^2}^2 + \|u_t(t,\cdot)\|_{L^2}^2\Big) \\ &= \frac{1}{2}\Big(\||\xi|\hat{u}(t,\cdot)\|_{L^2}^2 + \|\hat{u}_t(t,\cdot)\|_{L^2}^2\Big). \end{aligned}$$

Here we applied the formula of Parseval-Plancherel. After introducing $u(t,x) = e^{-\frac{1}{2}t}w(t,x)$ and $v(t,\xi) = F_{x\to\xi}(w)(t,\xi)$ it follows $\hat{u}(t,\xi) = e^{-\frac{1}{2}t}v(t,\xi)$. For the *elastic energy* we will use

$$|\xi|\hat{u}(t,\xi) = e^{-\frac{1}{2}t}\,|\xi|v(t,\xi),$$

for the *kinetic energy* we will use

$$\hat{u}_t(t,\xi) = e^{-\frac{1}{2}t}\,\Big(v_t(t,\xi)\; - \frac{1}{2}\,v(t,\xi)\Big).$$

Step 2 *Estimate of the solution itself*
We will divide the phase space $\mathbb{R}^n_\xi$ into several regions.
Case 1 $\{\xi :\ |\xi| \geq 1\}$
The representation of solution yields

$$|\hat{u}(t,\xi)| \leq Ce^{-\frac{t}{2}}\Big(|v_0(\xi)| + \frac{|v_1(\xi)|}{|\xi|}\Big).$$

Case 2 $\{\xi :\ |\xi| \in (\frac{1}{4}, 1)\}$
The representation of solution yields

$$|\hat{u}(t,\xi)| \leq Ce^{-\delta t}\big(|v_0(\xi)| + |v_1(\xi)|\big)$$

with a suitable positive constant δ.
Case 3 $\{\xi :\ |\xi| \leq \frac{1}{4}\}$
The representation of solution yields

$$|\hat{u}(t,\xi)| \leq C\big(|v_0(\xi)| + |v_1(\xi)|\big).$$

Summarizing we conclude the first estimate.

Step 3 Estimate of the elastic energy

We will divide the phase space $\mathbb{R}^n_\xi$ into several regions already motivated in Section 1.1 (cf. with Cases 1 and 2 there). We shall use the notations from the previous section.

Case 1 $\{\xi : |\xi| > \frac{1}{2}\}$

First we notice

$$|\xi|\hat{u}(t,\xi) = e^{-\frac{1}{2}t}\Big(\cos\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\, t\Big)\, |\xi| v_0(\xi) + t\, \frac{\sin\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\, t\Big)}{\sqrt{|\xi|^2 - \frac{1}{4}}\, t}\, |\xi| v_1(\xi)\Big).$$

By using the formula of Parseval-Plancherel this helps us to estimate the elastic energy $\|\nabla u(t,\cdot)\|^2_{L^2}$. We have

$$\begin{aligned}
\||\xi|\hat{u}(t,\xi)\|^2_{L^2\{|\xi|>\frac{1}{2}\}} &= \int_{|\xi|>\frac{1}{2}} |\xi|^2|\hat{u}(t,\xi)|^2\, d\xi \le 2\Big(\int_{|\xi|>\frac{1}{2}} e^{-t}|\xi|^2|v_0(\xi)|^2\, d\xi \\
&\quad + \int_{\frac{1}{2}<|\xi|\le 1} \underbrace{\frac{\sin^2\big(\sqrt{|\xi|^2-\frac{1}{4}}\, t\big)}{\big(\sqrt{|\xi|^2-\frac{1}{4}}\, t\big)^2}}_{\frac{\sin^2\alpha}{\alpha^2}\le C} t^2 e^{-t}|\xi|^2|v_1(\xi)|^2\, d\xi \\
&\quad + \int_{|\xi|>1} \underbrace{\frac{1}{|\xi|^2-\frac{1}{4}}|\xi|^2}_{\le C} e^{-t}|v_1(\xi)|^2\, d\xi\Big) \\
&\le 2e^{-t}\int_{\mathbb{R}^n} |\xi|^2|v_0(\xi)|^2\, d\xi \ + \ Ct^2e^{-t}\int_{\mathbb{R}^n} |v_1(\xi)|^2\, d\xi \ + \ Ce^{-t}\int_{\mathbb{R}^n} |v_1(\xi)|^2\, d\xi.
\end{aligned}$$

Summarizing *we obtain an exponential decay for large frequencies.* It holds

$$\int_{|\xi|>\frac{1}{2}} |\xi|^2|\hat{u}(t,\xi)|^2\, d\xi \le Ct^2e^{-t}\int_{\mathbb{R}^n} \big(|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2\big)\, d\xi.$$

We need the regularity $\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ for the data (φ,ψ).

Case 2 $\{\xi : |\xi| < \frac{1}{2}\}$

To estimate the elastic energy we use

$$\begin{aligned}|\xi|\hat{u}(t,\xi) &= |\xi|e^{-\frac{1}{2}t}\Big(\Big(\frac{v_0(\xi)}{2}-\frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}\\ &\quad+\Big(\frac{v_0(\xi)}{2}+\frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}\Big)\\ &= v_0(\xi)|\xi|\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)e^{-\frac{1}{2}t}\\ &\quad+\frac{2v_1(\xi)|\xi|}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)e^{-\frac{1}{2}t}.\end{aligned}$$

We divide the interval $(0,\frac{1}{2})$ for $|\xi|$ in two subintervals.
Case 2a $\{\xi:|\xi|\in[\frac{1}{4},\frac{1}{2})\}$:
Here we estimate as follows:

$$\begin{aligned}|\xi||\hat{u}(t,\xi)| &= \Big|\, v_0(\xi)|\xi|\underbrace{\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)}_{\le\cosh(\frac{\sqrt{3}}{4}\,t)}e^{-\frac{1}{2}t}\\ &\quad+\underbrace{\frac{\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)}{\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}t}_{\le Ct\,\cosh(\frac{\sqrt{3}}{4}\,t)}\;v_1(\xi)|\xi|e^{-\frac{1}{2}t}\,\Big|\\ &\le \Big|\, v_0(\xi)|\xi|\underbrace{\cosh\Big(\frac{\sqrt{3}}{4}\,t\Big)e^{-\frac{1}{2}t}}_{\le e^{-\delta t},\,\delta>0}+C\underbrace{v_1(\xi)|\xi|}_{\le|v_1(\xi)|}\underbrace{\cosh\Big(\frac{\sqrt{3}}{4}\,t\Big)\,te^{-\frac{1}{2}t}}_{\le e^{-\delta t},\,\delta>0}\,\Big|,\end{aligned}$$

and obtain with a suitable positive constant δ the estimate

$$\int_{\frac{1}{4}\le|\xi|<\frac{1}{2}}|\xi|^2|\hat{u}(t,\xi)|^2\,d\xi\le Ce^{-\delta t}\int_{\mathbb{R}^n}\big(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2\big)\,d\xi.$$

Here we get an exponential decay and use again the regularity $\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ of the data (φ,ψ).
Case 2b $\{\xi:|\xi|\in(0,\frac{1}{4})\}$:
By using the property

$$\sqrt{x+y}\le\sqrt{x}+\frac{y}{2\sqrt{x}}\quad\text{for any}\;\; x>0\quad\text{and}\;\; y\ge -x$$

it follows the inequality

$$-4|\xi|^2\le -1+\sqrt{1-4|\xi|^2}\le -2|\xi|^2\;\;\text{for}\;\;|\xi|<\frac{1}{2}.$$

With this inequality we proceed as follows:

$$\begin{aligned}
&\int_{|\xi|<\frac{1}{4}} |\xi|^2 |\hat{u}(t,\xi)|^2 \, d\xi \\
&\quad \le \int_{|\xi|<\frac{1}{4}} \big(|v_0(\xi)|^2|\xi|^2 + |v_1(\xi)|^2|\xi|^2\big)\Big(\underbrace{e^{-t-\sqrt{1-4|\xi|^2}\,t}}_{\le e^{-t}} + \underbrace{e^{-t+\sqrt{1-4|\xi|^2}\,t}}_{\le e^{-2|\xi|^2 t}}\Big)\, d\xi \\
&\quad \le Ce^{-t} \int_{|\xi|<\frac{1}{4}} \big(|v_0(\xi)|^2|\xi|^2 + |v_1(\xi)|^2|\xi|^2\big)\, d\xi \\
&\quad + C \int_{|\xi|<\frac{1}{4}} \big(|v_0(\xi)|^2 + |v_1(\xi)|^2\big)|\xi|^2 e^{-2|\xi|^2 t}\, d\xi.
\end{aligned}$$

For $t \ge 1$, we may estimate the second term on the right-hand side of the last inequality by

$$\begin{aligned}
&C \int_{|\xi|<\frac{1}{4}} \big(|v_0(\xi)|^2 + |v_1(\xi)|^2\big)|\xi|^2 e^{-2|\xi|^2 t}\, d\xi \\
&\quad \le C \sup_{|\xi|<\frac{1}{4},\, t\ge 1} \frac{t|\xi|^2}{t} e^{-2|\xi|^2 t} \int_{\mathbb{R}^n} \big(|v_0(\xi)|^2 + |v_1(\xi)|^2\big)\, d\xi \\
&\quad \le C\frac{1}{t} \underbrace{\sup_{|\xi|<\frac{1}{4},\, t\ge 1} t|\xi|^2 e^{-2|\xi|^2 t}}_{\le C} \int_{\mathbb{R}^n} \big(|v_0(\xi)|^2 + |v_1(\xi)|^2\big)\, d\xi.
\end{aligned}$$

For $t \in [0,1]$ we use $|\xi|^2 e^{-2|\xi|^2 t} \le C$. Summarizing we have shown for small frequencies

$$\int_{|\xi|<\frac{1}{4}} |\xi|^2 |\hat{u}(t,\xi)|^2 \, d\xi \;\le\; C(1+t)^{-1} \int_{\mathbb{R}^n} \big(|v_0(\xi)|^2 + |v_1(\xi)|^2\big)\, d\xi.$$

In this case we need the regularity $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ for the data (φ, ψ).
Summarizing all estimates of the Cases 1 to 2b we may conclude the desired decay estimate for the elastic energy.

Step 4 Estimate of the kinetic energy

We use the identity $\|u_t(t,\cdot)\|^2_{L^2(\mathbb{R}^n_x)} = \|\hat{u}_t(t,\cdot)\|^2_{L^2(\mathbb{R}^n_\xi)}$ with

$$\hat{u}_t(t,\xi) = e^{-\frac{1}{2}t}\Big(v_t(t,\xi) - \frac{1}{2}\, v(t,\xi)\Big).$$

Case 1 $\{\xi : |\xi| > \frac{1}{2}\}$

Using

$$v_t(t,\xi) = -\sqrt{|\xi|^2 - \frac{1}{4}}\sin\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\, t\Big)v_0(\xi) + \cos\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\, t\Big)v_1(\xi)$$

we obtain

$$\hat{u}_t(t,\xi) = e^{-\frac{1}{2}t}\Big(v_1(\xi)\Big(\cos\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big) \;-\; \frac{1}{2}\frac{\sin\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)}{\sqrt{|\xi|^2\;-\;\frac{1}{4}}}\Big)$$

$$- v_0(\xi)\Big(\frac{1}{2}\cos\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big) \;+\; \sqrt{|\xi|^2-\frac{1}{4}}\sin\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)\Big)\Big).$$

Repeating the approach to estimate the elastic energy gives

$$\|\hat{u}_t(t,\cdot)\|^2_{L^2\{|\xi|>\frac{1}{2}\}}$$

$$\le C\int_{|\xi|>\frac{1}{2}} e^{-t}|v_1(\xi)|^2\underbrace{\Big(\cos\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)-\frac{1}{2}\frac{\sin\big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\big)}{\sqrt{|\xi|^2\;-\;\frac{1}{4}}}\Big)^2}_{\le C\,(1+t)^2}d\xi$$

$$+ C\int_{|\xi|>\frac{1}{2}} e^{-t}|v_0(\xi)|^2\Big(\sqrt{|\xi|^2-\frac{1}{4}}\sin\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)$$

$$+ \underbrace{\frac{1}{2}\cos\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)}_{\le C}\Big)^2 d\xi.$$

The inequality $(|\xi|^2-\frac{1}{4})\sin^2\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big) \le |\xi|^2$ implies for $\{\xi : |\xi| > \frac{1}{2}\}$

$$\int_{|\xi|>\frac{1}{2}} |\hat{u}_t(t,\xi)|^2\,d\xi \le C(1+t)^2e^{-t}\int_{\mathbb{R}^n}\big(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2\big)\,d\xi.$$

We need the regularity $\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ for the data (φ,ψ) and obtain an exponential decay in time.

Case 2 $\{\xi : |\xi| < \frac{1}{2}\}$

We get immediately

$$\hat{u}_t(t,\xi)$$

$$= \frac{1}{2}e^{-\frac{1}{2}t}\Big(\sqrt{1-4|\xi|^2}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big) - \cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)\Big)v_0(\xi)$$

$$+e^{-\frac{1}{2}t}\Big(\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big) - \frac{1}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)\Big)v_1(\xi).$$

Again we divide the interval $[0,\frac{1}{2})$ into two subintervals.

Case 2a $\{\xi : |\xi| \in [\frac{1}{4},\frac{1}{2})\}$:

Here we show the exponential decay of the kinetic energy. On the one hand we use

$$\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)+\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)\le 2\cosh\Big(\frac{\sqrt{3}}{4}t\Big),$$

on the other hand we use

$$\Big|\frac{1}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big)\Big|\le C_\varepsilon t \;\text{ for }\; \frac{1}{2}\sqrt{1-4|\xi|^2}\,t\le\varepsilon.$$

Both estimates lead to

$$\|\hat{u}_t(t,\cdot)\|^2_{L^2\{|\xi|\in[\frac{1}{4},\frac{1}{2})\}}\;\le\; Ce^{-\delta t}\int_{\mathbb{R}^n}\big(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2\big)\,d\xi$$

with a suitable positive δ. Here we need the regularity $\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ for the data (φ,ψ). Moreover, we derived an exponential decay in time.

Case 2b $\{\xi : |\xi|<\frac{1}{4}\}$:

In this case we obtain

$$\begin{aligned}\hat{u}_t(t,\xi)=&\Big(\frac{v_0(\xi)}{4}+\frac{v_1(\xi)}{2\sqrt{1-4|\xi|^2}}\Big)\big(\sqrt{1-4|\xi|^2}-1\big)e^{-\frac{1}{2}t+\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}\\ &-\Big(\frac{v_0(\xi)}{4}-\frac{v_1(\xi)}{2\sqrt{1-4|\xi|^2}}\Big)\big(\sqrt{1-4|\xi|^2}+1\big)e^{-\frac{1}{2}t-\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}.\end{aligned}$$

Hence, we can estimate as follows:

$$|\hat{u}_t(t,\xi)|\le\Big|\Big(\frac{v_1(\xi)}{2\sqrt{1-4|\xi|^2}}+\frac{v_0(\xi)}{4}\Big)\Big(\underbrace{\sqrt{1-4|\xi|^2}-1}_{\le -2|\xi|^2}\Big)\underbrace{e^{-\frac{1}{2}t+\frac{1}{2}\sqrt{1-4|\xi|^2}t}}_{\le e^{-|\xi|^2t},\ |\xi|<\frac{1}{2}}\Big|.$$

Recalling the estimates for the elastic energy a similar approach leads to

$$\begin{aligned}
\|\hat{u}_t(t,\cdot)\|^2_{L^2\{|\xi|<\frac14\}} &\le C\int_{|\xi|<\frac14}\big(|v_1(\xi)|^2+|v_0(\xi)|^2\big)|\xi|^4\big(e^{-t}+e^{-2|\xi|^2t}\big)\,d\xi\\
&\le Ce^{-t}\int_{|\xi|<\frac14}\big(|v_1(\xi)|^2+|v_0(\xi)|^2\big)\,d\xi\\
&\quad + C\int_{|\xi|<\frac14}\big(|v_1(\xi)|^2+|v_0(\xi)|^2\big)|\xi|^4e^{-2|\xi|^2t}\,d\xi\\
&\le Ce^{-t}\int_{|\xi|<\frac14}\big(|v_1(\xi)|^2+|v_0(\xi)|^2\big)\,d\xi\\
&\quad + C\frac{1}{t^2}\underbrace{\sup_{|\xi|<\frac14,\,t\ge1} t^2|\xi|^4e^{-2|\xi|^2t}}_{\le c}\int_{|\xi|<\frac14}\big(|v_1(\xi)|^2+|v_0(\xi)|^2\big)\,d\xi\\
&\le Ce^{-t}\int_{|\xi|<\frac14}\big(|v_1(\xi)|^2+|v_0(\xi)|^2\big)\,d\xi\\
&\quad +\frac{C}{(1+t)^2}\int_{|\xi|<\frac14}\big(|v_1(\xi)|^2+|v_0(\xi)|^2\big)\,d\xi\\
&\le \frac{C}{(1+t)^2}\int_{\mathbb{R}^n}\big(|v_1(\xi)|^2+|v_0(\xi)|^2\big)\,d\xi.
\end{aligned}$$

Here we need the regularity $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ for the data (φ,ψ). Summarizing all the estimates from the Cases 1 to 2b we have proved the third inequality for the kinetic energy. Thus all statements from the theorem are proved. □

Which part of the phase space does the decay behavior of the energy influence?

The decay behavior is influenced by the small frequencies. But, which properties of solutions do the large frequencies influence? The large frequencies influence the necessary regularity of the data.

The reader can find a detailed discussion on the classical damped wave model in [34].

1.3 The diffusion phenomenon for damped wave models

At a first glance the properties of solutions to heat or wave models are completely different. One will not expect any relation between heat and wave models. In general, this is true. But, already the following exercise hints to something.

Exercise Let us consider the mixed problem

$$\begin{aligned}
&\varepsilon^2u_{tt}-u_{xx}+u_t=0,\quad u(0,x,\varepsilon)=\varphi(x),\quad u_t(0,x,\varepsilon)=\psi(x),\quad x\in(0,L),\\
&\qquad u(t,0,\varepsilon)=u(t,L,\varepsilon)=0\ \text{ for }\ t>0,
\end{aligned}$$

with sufficiently smooth data φ and ψ. We assume that the compatibility conditions are satisfied. Let $u = u(t,x,\varepsilon)$ be the unique (distributional) solution of this mixed problem (without explaining the precise regularity). Prove, that for every fixed (t,x) the following relation holds: $\lim_{\varepsilon\to 0} u(t,x,\varepsilon) = w(t,x)$, where $w = w(t,x)$ solves the mixed problem

$$w_t - w_{xx} = 0, \quad w(0,x) = \varphi(x), \ \ x \in (0,L),$$
$$w(t,0) = w(t,L) = 0 \ \text{ for } \ t > 0.$$

To what does this exercise hint?

From time to time mathematicians use instead of the heat equation

$$w_t - w_{xx} = 0$$

which solutions possess an *infinite speed of propagation* the damped wave equation

$$\varepsilon^2 u_{tt} - u_{xx} + u_t = 0, \ \varepsilon^2 > 0 \quad \text{is small.}$$

Now solutions have a *finite speed of propagation*. The speed depends on ε. One can prove the relation $\lim_{\varepsilon\to 0} u(t,x,\varepsilon) = w(t,x)$.

The main result of this section is a relation between solutions of the heat and of the classical damped wave equation. On the one hand we have the a priori estimate

$$\|w(t,\cdot)\|_{L^2} \leq C\|\varphi\|_{L^2} \ \text{ for solutions to } \ w_t - \Delta w = 0, \ w(0,x) = \varphi(x).$$

On the other hand we have from Theorem 1.2 the a priori estimate

$$\|u(t,\cdot)\|_{L^2} \leq C\big(\|\varphi\|_{L^2} + \|\psi\|_{H^{-1}}\big) \ \text{ for solutions to}$$
$$u_{tt} - \Delta u + u_t = 0, \ u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x).$$

Problem: Let (φ,ψ) be given data in the Cauchy problem for the classical damped wave equation. Can we find a data $\tilde{\varphi}$ in the Cauchy problem for the heat equation such that the difference of the corresponding solutions $u(t,\cdot) - w(t,\cdot)$ decays for $t \to \infty$ in the L^2-norm? Take into consideration that the estimates for $\|u(t,\cdot)\|_{L^2}$ and $\|w(t,\cdot)\|_{L^2}$ are optimal, thus we can not expect any decay for $t \to \infty$. If we show that the difference decays in the L^2-norm, then it is said that the *asymptotic behavior* of both solutions coincide for $t \to \infty$.

In the following we will give a positive answer to the last question. This effect is called *diffusion phenomenon* which was originally observed in [16] and was, for example, studied among other things in the papers [60], [37], or for an abstract model in [17].

To apply the above a priori estimates for solutions to the Cauchy problem for the heat and for the wave equation as well we shall assume $\varphi, \psi \in L^2(\mathbb{R}^n)$. Then let us turn to the Cauchy problems

$$\begin{aligned} &u_{tt} - \Delta u + u_t = 0 \\ &u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} &w_t - \Delta w = 0 \\ &w(0,x) = \varphi(x) + \psi(x). \end{aligned}$$

We introduce a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(s) = 1$ for $|s| \leq \frac{\varepsilon}{2} \ll 1$ and $\chi(s) = 0$ for $|s| \geq \varepsilon$ which localizes to small frequencies. Then we have the following remarkable result:

Theorem 1.3. *The difference of solutions to the above Cauchy problems satisfies the following estimate:*

$$\Big\| F^{-1}_{\xi\to x}\Big(\chi(\xi)F_{x\to\xi}\big(u(t,x)-w(t,x)\big)\Big)\Big\|_{L^2} \le C(1+t)^{-1}\|(\varphi,\psi)\|_{L^2}.$$

Proof. We use for small frequencies $|\xi| < \frac{1}{2}$ the following representation for the solution $u = u(t,x)$ from Section 1.1:

$$\begin{aligned}
&F_{x\to\xi}(u)(t,\xi)\\
&= e^{-\frac{1}{2}t}\Big(\Big(\frac{1}{2}F(\varphi)(\xi) - \frac{\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}\\
&\quad +\Big(\frac{1}{2}F(\varphi)(\xi) + \frac{\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}\Big).
\end{aligned}$$

We have for $w = w(t,x)$ the representation of solution

$$F_{x\to\xi}(w)(t,\xi) = e^{-|\xi|^2 t}\big(F(\varphi)(\xi)+F(\psi)(\xi)\big).$$

Taking into consideration the relations

$$\begin{aligned}
&\sqrt{1+s} = 1+\frac{s}{2}-\frac{s^2}{8}+O(s^3)\\
&\text{and } \frac{1}{\sqrt{1+s}} = 1-\frac{s}{2}+O(s^2) \ \text{ for } \ s\to+0
\end{aligned}$$

we get

$$\begin{aligned}
&\sqrt{1-4|\xi|^2} = 1-2|\xi|^2-2|\xi|^4+O(|\xi|^6)\\
&\text{and } \frac{1}{\sqrt{1-4|\xi|^2}} = 1+2|\xi|^2+O(|\xi|^4) \ \text{ for } \ |\xi|\to+0.
\end{aligned}$$

These relations allow to conclude

$$\begin{aligned}
&\Big\| F^{-1}_{\xi\to x}\Big(\chi(\xi)F_{x\to\xi}\big(u(t,x)-w(t,x)\big)\Big)\Big\|_{L^2} = \Big\|\chi(\xi)F_{x\to\xi}\big(u(t,x)-w(t,x)\big)\Big\|_{L^2}\\
&= \Big\|\chi(\xi)\Big(\Big(\frac{1}{2}F(\varphi)(\xi) - \Big(\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)\Big)\\
&\quad +\Big(\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)\Big)O(|\xi|^2)\Big)e^{-\frac{1}{2}t+O(|\xi|^2)t}e^{-\frac{1}{2}t}\\
&\quad +\Big(\frac{1}{2}F(\varphi)(\xi) + \Big(\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)\Big) + \Big(F(\varphi)(\xi)+2F(\psi)(\xi)\Big)|\xi|^2\\
&\quad +\Big(\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)\Big)O(|\xi|^4)\Big)e^{\frac{1}{2}t-|\xi|^2t-|\xi|^4t+O(|\xi|^6)t}e^{-\frac{1}{2}t}\\
&\quad -e^{-|\xi|^2 t}\big(F(\varphi)(\xi)+F(\psi)(\xi)\big)\Big)\Big\|_{L^2}.
\end{aligned}$$

On the one hand we have

$$\Big\|\chi(\xi)\Big(-F(\psi)(\xi)+\Big(\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)\Big)O(|\xi|^2)\Big)e^{(-1+O(|\xi|^2))t}\Big\|_{L^2}$$
$$\leq Ce^{-ct}\|(\varphi,\psi)\|_{L^2}$$

with a positive constant $c<1$ depending on the support of χ. On the other hand we have

$$\begin{aligned}
&\Big\|\chi(\xi)\Big(\Big(F(\varphi)(\xi)+F(\psi)(\xi)+\big(F(\varphi)(\xi)+2F(\psi)(\xi)\big)|\xi|^2\\
&\quad+\Big(\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)\Big)O(|\xi|^4)\Big)\\
&\quad\times e^{-|\xi|^2t-|\xi|^4t+O(|\xi|^6)t}-e^{-|\xi|^2t}\big(F(\varphi)(\xi)+F(\psi)(\xi)\big)\Big)\Big\|_{L^2}\\
&\leq\Big\|\chi(\xi)\big(F(\varphi)(\xi)+F(\psi)(\xi)\big)\Big(e^{-|\xi|^2t-|\xi|^4t+O(|\xi|^6)t}-e^{-|\xi|^2t}\Big)\Big\|_{L^2}\\
&\quad+\Big\|\chi(\xi)\Big(\big(F(\varphi)(\xi)+2F(\psi)(\xi)\big)|\xi|^2\\
&\quad+\Big(\frac{1}{2}F(\varphi)(\xi)+F(\psi)(\xi)\Big)O(|\xi|^4)\Big)e^{-|\xi|^2t+O(|\xi|^4)t}\Big\|_{L^2}.
\end{aligned}$$

We denote the two terms on the right-hand side of the last inequality by J_1 and by J_2. Let us assume $t\geq 1$. So, we obtain the estimates

$$\begin{aligned}
J_1&=\Big\|\chi(\xi)\big(F(\varphi)(\xi)+F(\psi)(\xi)\big)(-|\xi|^4t+O(|\xi|^6)t)e^{-|\xi|^2t}\\
&\quad\times\underbrace{\int_0^1 e^{(-|\xi|^4t+O(|\xi|^6)t)s}ds}_{\leq 1}\Big\|_{L^2}\\
&\leq C\Big\|\chi(\xi)\big(F(\varphi)(\xi)+F(\psi)(\xi)\big)\frac{|\xi|^4t^2}{t}\,e^{-|\xi|^2t}\Big\|_{L^2}\leq C\,t^{-1}\|(\varphi,\psi)\|_{L^2}
\end{aligned}$$

and

$$J_2\leq C\Big\|\chi(\xi)\big(|F(\varphi)(\xi)|+|F(\psi)(\xi)|\big)\frac{|\xi|^2t}{t}\,e^{-c|\xi|^2t}\Big\|_{L^2}\leq C\,t^{-1}\|(\varphi,\psi)\|_{L^2}.$$

For $t\in(0,1]$ and for $k=1,2$ we have

$$J_k\leq C\,\|(\varphi,\psi)\|_{L^2}.$$

Summarizing all derived estimates gives

$$J_k\leq C(1+t)^{-1}\|(\varphi,\psi)\|_{L^2}\ \text{ for }\ k=1,2.$$

The proof is complete. □

The diffusion phenomenon contains the information that the solution to the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

has asymptotically a *parabolic structure* (compare with the behavior of solutions to the Cauchy problem for the heat equation) from the point of view of L^2-estimates for the solution itself.

Why do we consider the diffusion phenomenon only for small frequencies?

Following the approach in Section 1.2 we conclude for large frequencies and for large times t the estimates

$$\|F^{-1}_{\xi\to x}\big((1-\chi(\xi))\hat{w}(t,\cdot)\big)\|_{L^2} \le C_0 e^{-C_2 t}\|(\varphi,\psi)\|_{L^2}$$

and

$$\|F^{-1}_{\xi\to x}\big((1-\chi(\xi))\hat{u}(t,\cdot)\big)\|_{L^2} \le C_0 e^{-C_1 t}\|(\varphi,\psi)\|_{L^2}$$

with some positive constants C_1 and C_2.

Thus we already have an exponential decay. This is optimal. There is no any reason to study in detail for the difference of Fourier transforms localized to large frequencies a better decay than the exponential one.

1.4 Decay behavior under additional regularity of data

We learned in Theorem 1.2 that the energy of solutions to classical damped wave models decays. This decay becomes faster under additional regularity of the data (φ,ψ). Let us turn again to the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

under the additional regularity assumption $(\varphi,\psi) \in L^m(\mathbb{R}^n) \times L^m(\mathbb{R}^n)$, $m \in [1,2)$. In the following we restrict ourselves to explain modifications in the treatment, in particular, how to use this additional regularity. For large frequencies we do not change our approach because under regularity $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ of the data we have an exponential decay. But for small frequencies the additional regularity $L^m(\mathbb{R}^n) \times L^m(\mathbb{R}^n)$ leads to better decay estimates.

Setting

$$\frac{1}{2} = \frac{1}{r} + \frac{1}{m'}$$

and after using Hölder's inequality we get

$$\begin{aligned}
\||\xi|\hat{u}(t,\xi)\|^2_{L^2\{|\xi|<\frac{1}{4}\}} &\le C\int_{|\xi|<\frac{1}{4}} |\xi|^2 e^{-|\xi|^2 t}\big(|v_0(\xi)|^2 + |v_1(\xi)|^2\big)\,d\xi \\
&\le C\big(\|v_0\|^2_{L^{m'}} + \|v_1\|^2_{L^{m'}}\big)\Big(\int_{|\xi|<\frac{1}{4}} \big(|\xi|^2 e^{-|\xi|^2 t}\big)^{\frac{r}{2}}\,d\xi\Big)^{\frac{2}{r}} \\
&\le C\big(\|\varphi\|^2_{L^m} + \|\psi\|^2_{L^m}\big)\Big(\int_{|\xi|<\frac{1}{4}} \big(|\xi|^2 e^{-|\xi|^2 t}\big)^{\frac{m}{2-m}}\,d\xi\Big)^{\frac{2-m}{m}}.
\end{aligned}$$

Here $m' \in (2, \infty]$ is the conjugate exponent to $m \in [1, 2)$. Let us only estimate the integral on the right-hand side. By using polar coordinates we obtain for large t the estimate

$$\int_{|\xi|<\frac{1}{4}} |\xi|^{\frac{2m}{2-m}} e^{-|\xi|^2 \frac{tm}{2-m}} \, d\xi = C \int_0^{\frac{1}{4}} r^{\frac{2m}{2-m}} e^{-r^2 \frac{tm}{2-m}} r^{n-1} \, dr$$
$$\le C \Big(\frac{2-m}{tm} \Big)^{\frac{n}{2} + \frac{m}{2-m}} \int_0^\infty s^{n-1+\frac{2m}{2-m}} e^{-s^2} \, ds \le C \Big(\frac{1+tm}{2-m} \Big)^{-\frac{n}{2} - \frac{m}{2-m}}.$$

Summarizing implies

$$\| |\xi| \hat{u}(t,\xi) \|^2_{L^2\{|\xi|<\frac{1}{4}\}} \le C \Big(\frac{1+tm}{2-m} \Big)^{-\frac{n(2-m)}{2m} - 1} \big(\|\varphi\|^2_{L^m} + \|\psi\|^2_{L^m} \big)$$
$$\le C_m (1+t)^{-\frac{n(2-m)}{2m} - 1} \big(\|\varphi\|^2_{L^m} + \|\psi\|^2_{L^m} \big).$$

Mapping properties of the Fourier transformation explain why to suppose additional regularity $L^m(\mathbb{R}^n)$ for $m \in [1, 2)$, only.
Similar estimates can be derived for $\|\partial_t^j \hat{u}(t,\xi)\|^2_{L^2\{|\xi|<\frac{1}{4}\}}$ with $j = 0, 1$.
All these estimates together imply the following result.

Theorem 1.4. *The solution to the Cauchy problem*

$$u_{tt} - \Delta u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

satisfies the following estimates for $t \ge 0$:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\le C_m (1+t)^{-\frac{n(2-m)}{4m}} \big(\|\varphi\|_{H^1 \cap L^m} + \|\psi\|_{L^2 \cap L^m} \big), \\
\|\nabla u(t,\cdot)\|_{L^2} &\le C_m (1+t)^{-\frac{1}{2} - \frac{n(2-m)}{4m}} \big(\|\varphi\|_{H^1 \cap L^m} + \|\psi\|_{L^2 \cap L^m} \big), \\
\|u_t(t,\cdot)\|_{L^2} &\le C_m (1+t)^{-1 - \frac{n(2-m)}{4m}} \big(\|\varphi\|_{H^1 \cap L^m} + \|\psi\|_{L^2 \cap L^m} \big).
\end{aligned}$$

Consequently, the energy satisfies the estimate

$$E_W(u)(t) \le C_m (1+t)^{-1 - \frac{n(2-m)}{2m}} \big(\|\varphi\|^2_{H^1 \cap L^m} + \|\psi\|^2_{L^2 \cap L^m} \big).$$

Remark 1.3. *The statement of the last theorem coincides for $m = 2$ (we suppose no additional regularity for the data) with the statement of Theorem 1.2.*

2 Semilinear classical damped wave models with source nonlinearity

Let us consider the Cauchy problem for semilinear wave models with time-dependent speed of propagation

$$u_{tt} - a(t)^2 \Delta u = f(u), \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

We will explain the definition of energies. The term u_{tt} yields the kinetic energy $\frac{1}{2}\|u_t(t,\cdot)\|^2_{L^2}$, the term $a(t)^2\Delta u$ implies the elastic type energy (we should take account of the time-dependent coefficient) $\frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2_{L^2}$. But how does the nonlinear term $f(u)$ influence the definition of the energy? Let us define the primitive $F(u) = \int_0^u f(s)ds$. It could be an idea to include this term into the energy, so we could propose as a suitable energy

$$E_W(u)(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2_{L^2} + \frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2_{L^2} + \int_{\mathbb{R}^n} -F(u)dx.$$

Is this a good idea? It depends heavily on $-F(u)$. The energy is supposed to be nonnegative. So, we expect nonnegativity of $-F(u)$ for all u. But this is not always satisfied. For this reason we distinguish between

1. *an absorbing nonlinearity* $f(u)$: $-F(u) \geq 0$ appears (is absorbed) in the definition of the energy. A typical example is $f(u) = -|u|^{p-1}u$, $p > 1$,

2. *a source nonlinearity* $f(u)$: $-F(u)$ is not nonnegative, thus, it does not appear in the definition of the energy. It should be treated as a source. Typical examples are $f(u) = |u|^{p-1}u$, $p > 1$, or $f(u) = |u|^p$, $p > 1$.

Let us turn to the Cauchy problem

$$u_{tt} - \Delta u + u_t = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

We learned that the nonlinear term $|u|^p$ is a source nonlinearity. For this reason the global existence (in time) of small data solutions is of interest. This means, that we try to prove that the steady-state solution $u \equiv 0$ of the Cauchy problem with homogeneous data is stable in a suitable evolution space. Small perturbations of the data in suitable Banach spaces preserve the property of the Cauchy problem to have globally (in time) solutions. It turns out that there exists a critical exponent p_{crit}, a threshold between global and non-global existence of small data solutions. For the above semilinear damped wave model this critical exponent is actually the *Fujita exponent* $p_{Fuj}(n) = 1 + \frac{2}{n}$. We shall discuss this issue in the next two sections.
A suitable energy of solutions is the wave energy $E_W(u)(t)$.

2.1 Fujita discovered the critical exponent

In his pioneering paper (see [10]) Fujita proved the following two results.

Theorem 2.1. *Let us consider the Cauchy problem*

$$u_t - \Delta u = u^p, \;\; u(0,x) = \varphi(x).$$

The data φ does not vanish identically and is supposed to be nonnegative and to belong to the function space $B^2(\mathbb{R}^n)$. Let $p \in (1, 1+\frac{2}{n})$. Then there is no global (in time) classical solution satisfying for any $T>0$ the estimate

$$|u(t,x)| \le M_T \exp(|x|^\beta) \;\; \textit{for all} \;\; (t,x) \in [0,T] \times \mathbb{R}^n, \, \beta \in (0,2).$$

Here $B^2(\mathbb{R}^n)$ denotes the space of functions with continuous and bounded derivatives up to order 2.

Theorem 2.2. *Let us consider the Cauchy problem*

$$u_t - \Delta u = u^p, \;\; u(0,x) = \varphi(x),$$

where the data φ is supposed to be nonnegative and to belong to the function space $B^2(\mathbb{R}^n)$. Let $p \in (1+\frac{2}{n}, \infty)$. Take any positive number γ. Then there exists a positive number δ with the following property:
If $\varphi(x) \le \delta G_n(\gamma, x)$, then there is a global (in time) classical solution satisfying the estimate

$$0 \le u(t,x) \le M G_n(t+\gamma, x) \;\; \textit{for all} \;\; (t,x) \in [0,T] \times \mathbb{R}^n.$$

Here

$$G_n(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\Big(- \frac{|x|^2}{4t} \Big)$$

is the Gauss kernel.
Both Theorems 2.1 and 2.2 imply that $p_{Fuj}(n)$ is really the critical exponent. The case $p = p_{Fuj}(n)$ remained open. Later a blow up result for $p = p_{Fuj}(n)$ has been proved in [15] or in [25].

2.2 Global existence of small data solutions

2.2.1 Main result

To formulate the following theorem we need the abbreviation $p_{GN}(n) = \frac{n}{n-2}$ for $n \ge 3$. This number is connected with the Gagliardo-Nirenberg inequality. The space for the data (φ, ψ) is defined as follows:

$$\mathcal{A}_{1,1} := \big(H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\big) \times \big(L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\big).$$

Theorem 2.3. *Let $n \leq 4$ and let*

$$\begin{cases} p > p_{Fuj}(n) & \text{if } n = 1, 2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if } n = 3, \\ p = 2 = p_{GN}(4) & \text{if } n = 4. \end{cases}$$

Let $(\varphi, \psi) \in \mathcal{A}_{1,1}$. Then the following statement holds with a suitable constant $\varepsilon_0 > 0$: if

$$\|(\varphi, \psi)\|_{\mathcal{A}_{1,1}} \leq \varepsilon_0,$$

then there exists a unique globally (in time) energy solution u belonging to the function space

$$C\big([0, \infty), H^1(\mathbb{R}^n)\big) \cap C^1\big([0, \infty), L^2(\mathbb{R}^n)\big).$$

Moreover, there exists a constant $C > 0$ such that the solution and its energy terms satisfy the decay estimates

$$\begin{aligned} \|u(t, \cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{4}} \|(\varphi, \psi)\|_{\mathcal{A}_{1,1}}, \\ \|\nabla u(t, \cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{4}-\frac{1}{2}} \|(\varphi, \psi)\|_{\mathcal{A}_{1,1}}, \\ \|u_t(t, \cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{4}-1} \|(\varphi, \psi)\|_{\mathcal{A}_{1,1}}. \end{aligned}$$

Remark 2.1. *We obtain the global (in time) existence of energy solutions only in low dimensions $n \leq 4$. This depends on the weak assumptions for the data. They are chosen from the energy space only with an additional regularity L^1. More restrictions of the data space or using estimates on L^p basis with $p \in [1, 2)$ allow, in general, to prove the global existence of small data solutions in higher dimensions for $p > p_{Fuj}(n)$, too (see, for example, [36], [56], [19], [18]).*

2.2.2 Main steps of our approach

We explain the main steps of our approach to prove Theorem 2.3. This approach can be used to study large classes of semilinear models.
Linear Cauchy problem: Let us consider the corresponding linear Cauchy problem

$$w_{tt} - \Delta w + w_t = 0, \;\; w(0, x) = \varphi(x), \;\; w_t(0, x) = \psi(x).$$

Then the solution $w = w(t, x)$ can be written in the following form

$$w(t, x) = K_0(t, 0, x) *_{(x)} \varphi(x) + K_1(t, 0, x) *_{(x)} \psi(x).$$

Here $K_0(t, 0, x) *_{(x)} \varphi(x)$ is the solution of the above Cauchy problem with second Cauchy data $\psi \equiv 0$. On the contrary, $K_1(t, 0, x) *_{(x)} \psi(x)$ is the solution of the above Cauchy problem with first Cauchy data $\varphi \equiv 0$. Now let us turn to the following classical damped wave model with source:

$$v_{tt} - \Delta v + v_t = f(t,x), \;\; v(0,x) = 0, \;\; v_t(0,x) = 0.$$

Using Duhamel's principle we get the solution

$$v(t,x) = \int_0^t K_1(t,s,x) *_{(x)} f(s,x)\, ds.$$

The family of terms $\{K_1(t,s,x) *_{(x)} f(s,x)\}_{s\geq 0}$ is the solution of the family of parameter-dependent Cauchy problems

$$w_{tt} - \Delta w + w_t = 0, \;\; w(s,x) = 0, \;\; w_t(s,x) = f(s,x).$$

So, Duhamel's principle explains that we have to take account of solutions to a family of parameter-dependent Cauchy problems, where the parameter appears in the description of the hyperplane $\{(t,x) \in \mathbb{R}^{n+1} : t = s\}$, where Cauchy data are posed. The classical damped wave equation has constant coefficients. Using the change of variables $t \to t - s$ in the last Cauchy problem implies the relation $K_l(t,s,x) = K_l(t-s,0,x)$ for $l = 0,1$.

Choice of spaces for solutions and data: This is a very important step. The choice of the space for the data is, in general, connected with the choice of the space for solutions. On the one hand the choice of data (the choice of l in the function spaces below) may cause some additional difficulties in the treatment. On the other hand the space of data may influence qualitative properties of solutions (e.g. compact support for all times or decay behavior for all times). We propose as space for solutions the *evolution space*

$$X(t) := C\big([0,t], H^l_m(\mathbb{R}^n)\big) \cap C^1\big([0,t], H^{l-1}_m(\mathbb{R}^n)\big)$$

for $m \in (1,2]$, $l \in \mathbb{N}$, $l \geq 1$ and for all $t > 0$. The data are taken from the function space

$$(H^l_m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (H^{l-1}_m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)).$$

So, we assume an additional regularity L^1 for the data (φ, ψ).

Estimates for solutions and some of its partial derivatives: To fix a norm in $X(t)$ we need so-called $(L^m \cap L^1) \to L^m$ estimates for solutions and some of their partial derivatives

$$\begin{aligned}
&\big\|\partial_x^\alpha u(t,\cdot)\big\|_{L^m} \leq C f_{|\alpha|}(t) \|(\varphi,\psi)\|_{(H^l_m \cap L^1)\times(H^{l-1}_m \cap L^1)} \text{ for } |\alpha| \leq l,\\
&\big\|\partial_x^\alpha u_t(t,\cdot)\big\|_{L^m} \leq C g_{|\alpha|}(t) \|(\varphi,\psi)\|_{(H^l_m \cap L^1)\times(H^{l-1}_m \cap L^1)} \text{ for } |\alpha| \leq l-1.
\end{aligned}$$

Then we introduce in $X(t)$ the norm

$$\begin{aligned}
&\|u\|_{X(t)}\\
&:= \sup_{0\leq\tau\leq t} \Big(\sum_{|\alpha|\leq l} f_{|\alpha|}(\tau)^{-1} \|\partial_x^\alpha u(\tau,\cdot)\|_{L^m} + \sum_{|\alpha|\leq l-1} g_{|\alpha|}(\tau)^{-1} \|u_\tau(\tau,\cdot)\|_{L^m} \Big).
\end{aligned}$$

Fixed point formulation: We introduce for arbitrarily given data

$$(\varphi, \psi) \in (H^l_m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (H^{l-1}_m(\mathbb{R}^n) \cap L^1)(\mathbb{R}^n)$$

the operator

$$\begin{aligned} N : u \in X(t) \to Nu := K_0(t, 0, x) *_{(x)} \varphi(x) + K_1(t, 0, x) *_{(x)} \psi(x) \\ + \int_0^t K_1(t-s, 0, x) *_{(x)} |u(s, x)|^p \, ds. \end{aligned}$$

Then we show that the following estimates are satisfied:

$$\begin{aligned} &\|Nu\|_{X(t)} \leq C_0 \, \|(\varphi, \psi)\|_{(H^l_m \cap L^1) \times (H^{l-1}_m \cap L^1)} + C_1(t) \|u\|^p_{X_0(t)}, \\ &\|Nu - Nv\|_{X(t)} \leq C_2(t) \|u - v\|_{X_0(t)} \big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)} \big) \end{aligned}$$

for $t \in [0, \infty)$ with nonnegative constants C_0, $C_1(t)$ and $C_2(t)$. Here we used the evolution space $X_0(t) := C\big([0, t], H^l_m\big)$ with the norm

$$\|u\|_{X_0(t)} := \sup_{0 \leq \tau \leq t} \Big(\sum_{|\alpha| \leq l} f_{|\alpha|}(\tau)^{-1} \|\partial^\alpha_x u(\tau, \cdot)\|_{L^m} \Big).$$

Application of Banach's fixed point theorem: The estimates for the image Nu of the last step allow to apply Banach's fixed point theorem. In this way we get simultaneously a unique solution to $Nu = u$ locally in time for large data and globally in time for small data. To prove the local (in time) existence we use $C_1(t)$, $C_2(t)$ tend to 0 for t tends to 0, while to to prove the global (in time) existence we use $C_1(t) \leq C_3$ and $C_2(t) \leq C_3$ for all $t \in [0, \infty)$ with a suitable nonnegative constant C_3.
Let us only verify how to prove the global existence in time.
In fact, taking the recurrence sequence $u_{-1} := 0$, $u_k := N(u_{k-1})$ for $k = 0, 1, 2, \cdots$ into account we apply the estimate for $\|Nu\|_{X(t)}$ with small norm

$$\|(\varphi, \psi)\|_{(H^l_m \cap L^1) \times (H^{l-1}_m \cap L^1)} = \varepsilon.$$

Then we arrive at $\|u_k\|_{X(t)} \leq 2C_3\varepsilon$ for any $\varepsilon \in [0, \varepsilon_0]$ with $\varepsilon_0 = \varepsilon_0(2C_3)$ sufficiently small. Once this uniform estimate is established we use the estimate for $\|Nu - Nv\|_{X(t)}$ and find

$$\|u_{k+1} - u_k\|_{X(t)} \leq C_3 \varepsilon^{k-1}, \; \|u_{k+1} - u_k\|_{X(t)} \leq 2^{-1} \|u_k - u_{k-1}\|_{X(t)}$$

for $\varepsilon \leq \varepsilon_0$ sufficiently small. We get inductively $\|u_k - u_{k-1}\|_{X(t)} \leq C_3 2^{-k}$ so that $\{u_k\}_k$ is a Cauchy sequence in the Banach space $X(t)$ converging to the unique solution of $Nu = u$ for all $t > 0$. Here we used that the constant C_3 appearing in the last estimates is independent of $t \in [0, \infty)$.

2.2.3 Proof of the main result

Proof. Now let us prove Theorem 2.3 by following all the steps of the approach of the previous section.
The space for the data is $\mathcal{A}_{1,1} := (H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$. The space of energy solutions is $X(t) = C([0,t], H^1(\mathbb{R}^n)) \cap C^1([0,t], L^2(\mathbb{R}^n))$. Taking into consideration the estimates of Theorem 1.4 we choose

$$f_{|\alpha|}(t) = (1+t)^{-\frac{n+2|\alpha|}{4}} \text{ for } |\alpha| \le 1, \;\; g_0(t) = (1+t)^{-\frac{n+4}{4}}.$$

So we introduce in $X(t)$ the norm

$$\begin{aligned}\|u\|_{X(t)} := \sup_{0\le\tau\le t} \big(&(1+\tau)^{\frac{n}{4}} \|u(\tau,\cdot)\|_{L^2} + (1+\tau)^{\frac{n+2}{4}} \|\nabla u(\tau,\cdot)\|_{L^2} \\ &+ (1+\tau)^{\frac{n+4}{4}} \|u_\tau(\tau,\cdot)\|_{L^2} \big).\end{aligned}$$

Moreover, we define the evolution space $X_0(t) = C([0,t], H^1(\mathbb{R}^n))$ with the norm

$$\|u\|_{X_0(t)} := \sup_{0\le\tau\le t} \big((1+\tau)^{\frac{n}{4}} \|u(\tau,\cdot)\|_{L^2} + (1+\tau)^{\frac{n+2}{4}} \|\nabla u(\tau,\cdot)\|_{L^2} \big).$$

It remains to show the estimates

$$\begin{aligned}&\|Nu\|_{X(t)} \le C_0 \,\|(\varphi,\psi)\|_{(H^1\cap L^1)\times(L^2\cap L^1)} + C_1(t)\|u\|^p_{X_0(t)}, \\ &\|Nu - Nv\|_{X(t)} \le C_2(t)\|u-v\|_{X_0(t)} \big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)} \big)\end{aligned}$$

for the operator N of the previous section. These estimates will follow from the next proposition in which the restriction on the power p and on the dimension n of Theorem 2.3 will appear.

Proposition 2.1. *Let u and v be elements of $X(t)$. Then under the assumptions of Theorem 2.3 the following estimates hold for $j + l = 0, 1$:*

$$\begin{aligned}&(1+t)^l (1+t)^{\frac{n}{4}+\frac{j}{2}} \|\nabla^j \partial_t^l Nu(t,\cdot)\|_{L^2} \le C \,\|(\varphi,\psi)\|_{\mathcal{A}_{1,1}} + C\|u\|^p_{X_0(t)}, \\ &(1+t)^l (1+t)^{\frac{n}{4}+\frac{j}{2}} \|\nabla^j \partial_t^l \big(Nu(t,\cdot) - Nv(t,\cdot)\big)\|_{L^2} \\ &\qquad\qquad \le C\|u-v\|_{X_0(t)} \big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)} \big).\end{aligned}$$

Here the nonnegative constant C is independent of $t \in [0,\infty)$.

Proof. We have

$$\begin{aligned}\nabla^j \partial_t^l Nu(t,\cdot) &= \nabla^j \partial_t^l K_0(t,0,x) *_{(x)} \varphi(x) + \nabla^j \partial_t^l K_1(t,0,x) *_{(x)} \psi(x) \\ &\quad + \nabla^j \partial_t^l \int_0^t K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds.\end{aligned}$$

The estimates of Theorem 1.4 imply immediately

$$\begin{aligned}
&\|\nabla^j \partial_t^l K_0(t,0,x) *_{(x)} \varphi(x) + \nabla^j \partial_t^l K_1(t,0,x) *_{(x)} \psi(x)\|_{L^2} \\
&\quad \le C\,(1+t)^{-l}(1+t)^{-\frac{n}{4}-\frac{j}{2}}\|(\varphi,\psi)\|_{\mathcal{A}_{1,1}}
\end{aligned}$$

for the admissible range of j and l. So, we restrict ourselves to the integral term in the representation of $\nabla^j \partial_t^l Nu(t,\cdot)$. Using $K_1(0,0,x)=0$ it follows

$$\begin{aligned}
&\nabla^j \partial_t^l \int_0^t K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds \\
&\quad = \int_0^t \nabla^j \partial_t^l K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds.
\end{aligned}$$

What we shall do is to use different estimates of solutions to the family of parameter-dependent Cauchy problems

$$w_{tt} - \Delta w + w_t = 0, \;\; w(s,x) = 0, \;\; w_t(s,x) = |u(s,x)|^p.$$

On the interval $[0,\frac{t}{2}]$: Here we use the $L^2 \cap L^1 \to L^2$ estimates of Theorem 1.4. So, additional regularity of the data is required.
On the interval $[\frac{t}{2},t]$: Here we use the $L^2 \to L^2$ estimates of Theorem 1.2. So, no additional regularity of the data is required.

Following this strategy we get

$$\begin{aligned}
&\Big\| \int_0^t \nabla^j \partial_t^l K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds \Big\|_{L^2} \\
&\quad \le C \int_0^{\frac{t}{2}} (1+t-s)^{-(\frac{n}{4}+\frac{j}{2}+l)} \big\| |u(s,x)|^p \big\|_{L^2 \cap L^1} \, ds \\
&\qquad + C \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{j}{2}-l} \big\| |u(s,x)|^p \big\|_{L^2} \, ds.
\end{aligned}$$

We use

$$\begin{aligned}
\big\| |u(s,x)|^p \big\|_{L^1 \cap L^2} &\le C \|u(s,\cdot)\|_{L^p}^p + \|u(s,\cdot)\|_{L^{2p}}^p, \\
\big\| |u(s,x)|^p \big\|_{L^2} &\le C \|u(s,\cdot)\|_{L^{2p}}^p.
\end{aligned}$$

Now Gagliardo-Nirenberg inequality comes into play. We may estimate

$$\begin{aligned}
\|u(s,\cdot)\|_{L^p}^p &\le C \|u(s,\cdot)\|_{L^2}^{p(1-\theta(p))} \, \|\nabla u(s,\cdot)\|_{L^2}^{p\,\theta(p)}, \\
\|u(s,\cdot)\|_{L^{2p}}^p &\le C \|u(s,\cdot)\|_{L^2}^{p(1-\theta(2p))} \, \|\nabla u(s,\cdot)\|_{L^2}^{p\,\theta(2p)},
\end{aligned}$$

where

$$\theta(p) = \frac{n(p-2)}{2p}, \qquad \theta(2p) = \frac{n(p-1)}{2p}.$$

We remark that the restriction $\theta(p) \geq 0$ implies that $p \geq 2$, whereas the restriction $\theta(2p) \leq 1$ implies that $p \leq p_{GN}(n)$ if $n \geq 3$. So, we use the estimates for $u(t,\cdot)$ and $\nabla u(t,\cdot)$ only. This is the main motivation for introducing the space $X_0(t)$. Taking into consideration $\theta(p) < \theta(2p)$ implies

$$\begin{aligned}
\big\||u(s,x)|^p\big\|_{L^2 \cap L^1} &\leq C\|u\|^p_{X_0(s)}(1+s)^{-p(\frac{n}{4}+\frac{\theta(p)}{2})} = \|u\|^p_{X_0(s)}(1+s)^{-\frac{(p-1)n}{2}}, \\
\big\||u(s,x)|^p\big\|_{L^2} &\leq C\|u\|^p_{X_0(s)}(1+s)^{-p(\frac{n}{4}+\frac{\theta(2p)}{2})} = \|u\|^p_{X_0(s)}(1+s)^{-\frac{(2p-1)n}{4}}.
\end{aligned}$$

After summarizing and using $\|u\|_{X_0(s)} \leq \|u\|_{X_0(t)}$ for $s \leq t$ we may conclude

$$\begin{aligned}
&\Big\| \int_0^t \nabla^j \partial_t^l K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds \Big\|_{L^2} \\
&\quad \leq C\|u\|^p_{X_0(t)} \int_0^{\frac{t}{2}} (1+t-s)^{-(\frac{n}{4}+\frac{j}{2}+l)}(1+s)^{-\frac{(p-1)n}{2}} ds \\
&\qquad + C\|u\|^p_{X_0(t)} \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{j}{2}-l}(1+s)^{-\frac{(2p-1)n}{4}} \, ds.
\end{aligned}$$

The first integral is estimated by $(1+t)^{-(\frac{n}{4}+\frac{j}{2}+l)}$. Indeed, since $p > p_{Fuj}(n)$, the function $(1+t)^{-\frac{(p-1)n}{2}}$ belongs to $L^1(\mathbb{R}^1_+)$. We treat the second integral as follows:

$$\begin{aligned}
&\int_{\frac{t}{2}}^t (1+t-s)^{-\frac{j}{2}-l}(1+s)^{-\frac{(2p-1)n}{4}} \, ds \\
&\quad \leq C(1+t)^{-\frac{(2p-1)n}{4}} \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{j}{2}-l} ds \\
&\quad \leq C(1+t)^{-\frac{(2p-1)n}{4}+1-\frac{j}{2}-l}((\log(1+t))^l \leq C(1+t)^{-(\frac{n}{4}+\frac{j}{2}+l)}
\end{aligned}$$

for $j+l = 0,1$. This completes the estimates for $\nabla^j \partial_t^l Nu(t,\cdot)$. Exactly in the same way we prove the desired estimates for $\nabla^j \partial_t^l \big(Nu(t,\cdot) - Nv(t,\cdot)\big)$. The considerations base on the following relations:

$$\begin{aligned}
&\nabla^j \partial_t^l \int_0^t K_1(t-s,0,x) *_{(x)} \big(|u(s,x)|^p - |v(s,x)|^p\big) \, ds \\
&\quad = \int_0^t \nabla^j \partial_t^l K_1(t-s,0,x) *_{(x)} \big(|u(s,x)|^p - |v(s,x)|^p\big) \, ds,
\end{aligned}$$

and

$$\begin{aligned}
&\big\| |u(s,x)|^p - |v(s,x)|^p \big\|_{L^1} \\
&\qquad \leq C\|u(s,\cdot)-v(s,\cdot)\|_{L^p} \big(\|u(s,\cdot)\|_{L^p}^{p-1} + \|v(s,\cdot)\|_{L^p}^{p-1} \big), \\
&\big\| |u(s,x)|^p - |v(s,x)|^p \big\|_{L^2} \\
&\qquad \leq C\|u(s,\cdot)-v(s,\cdot)\|_{L^{2p}} \big(\|u(s,\cdot)\|_{L^{2p}}^{p-1} + \|v(s,\cdot)\|_{L^{2p}}^{p-1} \big).
\end{aligned}$$

□

We conclude by Proposition 2.1 the statements of Theorem 2.3. □

2.3 Application of the test function method

In this section we shall show that the Fujita exponent $p_{Fuj}(n)$ is really the critical exponent. Here we apply the test function method which was introduced in the paper [62]. Our main concern is the following result.

Theorem 2.4. *Let us consider the Cauchy problem for the classical damped wave equation with power nonlinearity*

$$u_{tt} - \Delta u + u_t = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

in $[0,\infty)\times\mathbb{R}^n$ *with* $n \geq 1$ *and* $p \in (1, 1+\frac{2}{n}]$. *Let* $(\varphi,\psi) \in \mathcal{A}_{1,1}$ *satisfy the assumption*

$$\int_{\mathbb{R}^n} \big(\varphi(x) + \psi(x)\big)\, dx > 0.$$

Then there exists a locally (in time) defined energy solution

$$u \in C\big([0,T), H^1(\mathbb{R}^n)\big) \cap C^1\big([0,T), L^2(\mathbb{R}^n)\big).$$

This solution can not be continued to the interval $[0,\infty)$ *in time.*

Remark 2.2. *Following the proof to Theorem 2.3 we obtain a local (in time) energy solution*

$$u \in C\big([0,T), H^1(\mathbb{R}^n)\big) \cap C^1\big([0,T), L^2(\mathbb{R}^n)\big).$$

For this reason we restrict ourselves to prove that this solution does not exist globally on the interval $[0,\infty)$ *in time.*

Proof. We first introduce test functions $\eta = \eta(t)$ and $\phi = \phi(x)$ having the following properties:

1. $\eta \in C_0^\infty[0,\infty)$, $0 \le \eta(t) \le 1$,

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{2}, \\ 0 & \text{for } t \ge 1, \end{cases}$$

2. $\phi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \phi(x) \le 1$,

$$\phi(x) = \begin{cases} 1 & \text{for } |x| \le \frac{1}{2}, \\ 0 & \text{for } |x| \ge 1, \end{cases}$$

3. $\frac{\eta'(t)^2}{\eta(t)} \le C$ for $\frac{1}{2} < t < 1$, and $\frac{|\nabla\phi(x)|^2}{\phi(x)} \le C$ for $\frac{1}{2} < |x| < 1$.

Let $R \in [0,\infty)$ be a large parameter. We define the test function

$$\chi_R(t,x) := \eta_R(t)\phi_R(x) := \eta\Big(\frac{t}{R^2}\Big)\phi\Big(\frac{x}{R}\Big).$$

We put

$$Q_R := [0,R^2] \times B_R, \quad B_R := \{x \in \mathbb{R}^n : |x| \le R\}.$$

We note that the support of χ_R is contained in the set Q_R. Moreover, $\chi_R \equiv 1$ on $[0,\frac{R^2}{2}] \times B_{\frac{R}{2}}$. We suppose that the energy solution $u = u(t,x)$ exists globally in time. We define the functional

$$I_R := \int_{Q_R} |u(t,x)|^p \chi_R(t,x)^q \, d(x,t) = \int_{Q_R} \big(u_{tt} - \Delta u + u_t\big)\chi_R(t,x)^q \, d(x,t).$$

Here q is the Sobolev conjugate of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. After integration by parts we obtain

$$\begin{aligned} I_R &= -\int_{B_R} (\varphi + \psi)\phi_R^q \, dx + \int_{Q_R} u\partial_t^2(\chi_R^q) \, d(x,t) \\ &\quad - \int_{Q_R} u\partial_t(\chi_R^q) \, d(x,t) - \int_{Q_R} u\Delta(\chi_R^q) \, d(x,t) \\ &:= -\int_{B_R} (\varphi + \psi)\phi_R^q \, dx + J_1 + J_2 + J_3. \end{aligned}$$

By the assumption on the data (φ,ψ) it follows that $I_R < J_1 + J_2 + J_3$ for sufficiently large R. We shall estimate separately J_1, J_2 and J_3. Here we use the notations

$$\hat{Q}_{R,t} := \Big[\frac{R^2}{2}, R^2\Big] \times B_R, \quad \hat{Q}_{R,x} := [0,R^2] \times \big(B_R \setminus B_{\frac{R}{2}}\big).$$

We first estimate J_3. Noting

$$\Delta(\chi_R^q) = R^{-2}q(q-1)\eta_R^q(t)\phi_R^{q-2}(x)\Big|\nabla\phi\Big(\frac{x}{R}\Big)\Big|^2 + R^{-2}q\eta_R^q(t)\phi_R^{q-1}(x)(\Delta\phi)\Big(\frac{x}{R}\Big)$$

and the assumed properties for the test functions we may conclude

$$|J_3| \leq CR^{-2} \int_{\hat{Q}_{R,x}} |u| \chi_R^{q-1} \, d(x,t).$$

Application of Hölder's inequality implies

$$\begin{aligned}
|J_3| \quad & \leq CR^{-2} \Big(\int_{\hat{Q}_{R,x}} |u|^p \chi_R^q(t,x) \, d(x,t) \Big)^{1/p} \Big(\int_{\hat{Q}_{R,x}} 1 \, d(x,t) \Big)^{1/q} \\
& \leq CR^{-2} I_{R,x}^{\frac{1}{p}} \Big(\int_{\hat{Q}_{R,x}} 1 \, d(x,t) \Big)^{1/q} \leq C I_{R,x}^{\frac{1}{p}} R^{\frac{n+2}{q}-2},
\end{aligned}$$

where

$$I_{R,x} := \int_{\hat{Q}_{R,x}} |u|^p \chi_R^q(t,x) \, d(x,t).$$

Since $1 < p \leq 1 + 2/n$ the last inequality gives $|J_3| \leq C I_{R,x}^{\frac{1}{p}}$. Next, we estimate J_1. Noting

$$\partial_t^2(\chi_R^q) = \frac{1}{R^4} q(q-1) \phi_R^q(x) \eta_R^{q-2}(t) \Big(\eta' \Big(\frac{t}{R^2} \Big) \Big)^2 + \frac{1}{R^4} q \phi_R^q(x) \eta_R^{q-1}(t) \eta'' \Big(\frac{t}{R} \Big)$$

and using the properties of the test functions again we estimate J_1 as follows:

$$\begin{aligned}
|J_1| \leq & \, C \frac{1}{R^4} \Big(\int_{\hat{Q}_{R,t}} |u|^p \chi_R^q(t,x) \, d(x,t) \Big)^{1/p} \Big(\int_{\hat{Q}_{R,t}} 1 \, d(x,t) \Big)^{1/q} \\
= & \, I_{R,t}^{\frac{1}{p}} \frac{1}{R^4} \Big(\int_{\hat{Q}_{R,t}} 1 \, d(x,t) \Big)^{1/q} \leq C I_{R,t}^{\frac{1}{p}} R^{\frac{n+2}{q}-2} \leq C I_{R,t}^{\frac{1}{p}},
\end{aligned}$$

where

$$I_{R,t} := \int_{\hat{Q}_{R,t}} |u|^p \chi_R^q(t,x) \, d(x,t).$$

Finally, we estimate J_2. By

$$\partial_t(\chi_R^q) = \frac{1}{R^2} q \phi_R^q(x) \eta_R^{q-1}(t) \eta' \Big(\frac{t}{R^2} \Big),$$

we have

$$\begin{aligned}
|J_2| \leq & \, C \frac{1}{R^2} \int_{\hat{Q}_{R,t}} |u| \chi_R^{q-1} \, d(x,t) \\
& \leq C \frac{1}{R^2} \Big(\int_{\hat{Q}_{R,t}} |u|^p \chi_R^q \, d(x,t) \Big)^{1/p} \Big(\int_{\hat{Q}_{R,t}} 1 \, d(x,t) \Big)^{1/q} \\
& \leq C I_{R,t}^{\frac{1}{p}} \frac{1}{R^2} R^{\frac{n}{q}} \Big(\int_{\frac{R^2}{2}}^{R^2} 1 \, dt \Big)^{1/q} \leq C I_{R,t}^{\frac{1}{p}} R^{\frac{n+2}{q}-2} \leq C I_{R,t}^{\frac{1}{p}}.
\end{aligned}$$

Putting the estimates of J_1, J_2 and J_3 together we obtain

$$I_R \leq C(I_{R,t}^{1/p} + I_{R,x}^{1/p})$$

for large R. It is obvious that $I_{R,t}, I_{R,x} \leq I_R$. Hence, we have

$$I_R \leq C I_R^{1/p}.$$

This means $I_R \leq C$, that is, I_R is uniformly bounded for all R. By letting $R \to +\infty$ one may conclude

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p \, dx \, dt = \lim_{R\to\infty} I_R < \infty.$$

Now we recall the inequality

$$I_R \leq C(I_{R,t}^{\frac{1}{p}} + I_{R,x}^{\frac{1}{p}}).$$

By the integrability of $|u|^p$ and noting the shape of the region $\hat{Q}_{R,t}$ and $\hat{Q}_{R,x}$ we conclude

$$\lim_{R\to\infty} \left(I_{R,t}^{1/p} + I_{R,x}^{1/p}\right) = 0.$$

This implies

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p \, d(x,t) = \lim_{R\to\infty} I_R = 0.$$

Hence, $u \equiv 0$. But this is a contradiction to our assumptions for the data. □

Remark 2.3. *The test function method bases on a contradiction argument. Under suitable assumptions for the data no global in time solution does exist. Here one has to explain what kind of solutions do we have in mind. We formulated Theorem 2.4 in correspondence with Theorem 2.3. Both results are related to energy solutions. Following the proof to Theorem 2.4 we see that the same statement holds for Sobolev solutions as well. We may also exclude global in time Sobolev solutions under suitable assumptions for the data. But we do not get any information about blow up time or life span estimates or about blow up mechanisms.*
The test function method was originally developed for proving sharpness of the Fujita exponent as the critical exponent for semilinear parabolic equations. Later it was recognized that this method can also be applied for classical damped wave models. These are models with a "parabolic like behavior" from the point of view of decay estimates. Attempts to apply this method to classical wave models fail in the sense, that sharpness of critical exponents can not be expected in general.

3 Fujita via Strauss - a never ending story

3.1 Semilinear classical wave models with source nonlinearity

The application of estimates for solutions to linear equations is a very useful tool to study the global (in time) existence of small data solutions for semilinear equations. A lot of activities have been devoted to the Cauchy problem for wave equations with power nonlinearity

$$u_{tt} - \Delta u = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

For $1 < p < p_K(n) = \frac{n+1}{n-1}$ ($p_K(n)$ denotes the Kato exponent) the nonexistence of global (in time) generalized solutions for data with compact support was proved in [22].
On the other hand, in [21] it was shown that $p_{crit} = 1 + \sqrt{2}$ is the critical exponent for the global existence of classical small data solutions when $n = 3$. Here classical solution means $u \in C^2\big([0,\infty) \times \mathbb{R}^n\big)$. A bit later, it was conjectured in [50] that the critical exponent $p_{crit}(n)$ ($p_{crit}(3) = 1 + \sqrt{2}$) is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

This critical exponent is called Strauss exponent. In the further considerations we use the notation $p_0(n)$ for the Strauss exponent. This conjecture was verified in [12] and [13] for classical solutions when $n = 2$. For $n > 3$, the paper [47] proved the nonexistence of global (in time) solutions in $C\big([0,\infty), L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)\big)$ for suitable small data and for $1 < p < p_0(n)$. Later the supercritical case $p > p_0(n)$ was treated in [31]. There the authors proved the existence of global weak solutions belonging to $L^\infty\big([0,\infty), L^q(\mathbb{R}^n, d\mu)\big)$ with a weighted measure $d\mu$ up to $n \leq 8$ and for all n in the case of radial initial data. In [11] the authors removed the assumption of spherical symmetry. The global existence also breaks down at the critical exponent $p = p_0(n)$ as it was shown in [46] for $n = 2, 3$ and in [61] and, independently, in [66] for $n \geq 4$.
Some results verifying *Strauss' conjecture* are summarized in the following table. The table is taken from the paper [53].

	$p < p_0(n)$	$p = p_0(n)$	$p_0(n) < p < p_{conf}(n)$
$n = 2$	Glassey [12]	Schaeffer [46]	Glassey [13]
$n = 3$	John [21]	Schaeffer [46]	John [21]
$n \geq 4$	Sideris [47]	Yordanov-Zhang [61], Zhou Yi [66]	Georgiev-Lindblad-Sogge [11]

Remark 3.1. *The power $p_{conf}(n) = \frac{n+3}{n-1}$ is well-known as "conformal power" and one can obtain (see [32]) the global (in time) existence of small data solutions when $p \geq p_{conf}(n)$, too, under suitable regularity assumptions for the data.*

In the following two sections we are going to derive a local (in time) existence result (Section 3.2) and to show a blow up result in the special case $n \leq 3$ (Section 3.3). We skip to describe methods how to prove the global (in time) existence of small data solutions for $p > p_0(n)$. The reason is, that due to the lack of $L^1 - L^q$ estimates the necessary tools are more complicated than the ones for the classical damped waves.

3.2 Local existence (in time) of Sobolev solutions

According to Duhamel's principle, the Sobolev solution u of the Cauchy problem

$$u_{tt} - \Delta u = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

satisfies

$$u(t,x) = u_0(t,x) + \int_0^t K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds,$$

where

$$u_0(t,x) = K_0(t,0,x) *_{(x)} \varphi(x) + K_1(t,0,x) *_{(x)} \psi(x),$$
$$\widehat{K}_0(t,0,|\xi|) = \cos(t|\xi|) \;\text{ and }\; \widehat{K}_1(t,0,|\xi|) = \frac{\sin(t|\xi|)}{|\xi|}.$$

To derive a local (in time) existence result we are going to use the $L^r - L^q$ estimates for solutions to the Cauchy problem

$$u_{tt} - \Delta u = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

3.2.1 $L^r - L^q$ estimates

Let us consider the Cauchy problem for the free wave equation

$$u_{tt} - \triangle u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

By taking $\varphi \equiv 0$ and without asking for additional regularity of the data, one still may expect some singular $L^r - L^q$ estimates. One may conclude the following $L^r - L^q$ estimates on the conjugate line:

$$\|u(t,\cdot)\|_{L^q} \leq C \, t^{1-\frac{n}{r}+\frac{n}{q}} \|\psi\|_{L^r}$$

uniformly for any $t > 0$ and for

$$\frac{n+1}{2}\Big(\frac{1}{r} - \frac{1}{q}\Big) \leq 1 \leq n\Big(\frac{1}{r} - \frac{1}{q}\Big).$$

In particular, it is true for $(\frac{1}{r}, \frac{1}{q}) = P_1 := \big(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} - \frac{1}{n+1}\big)$. Since these estimates are true for $(\frac{1}{r}, \frac{1}{q}) = P_0 := \big(\frac{1}{2}, \frac{1}{2}\big)$, by interpolation we conclude it on the line with end points P_0 and

P_1.
More in general, the estimates in [43] and [51] imply that the solution to the above Cauchy problem satisfies $L^r - L^q$ estimates if, and only if, the point $(\frac{1}{r}, \frac{1}{q})$ belongs to the closed triangle with vertices

$$P_1 = \left(\tfrac{1}{2} + \tfrac{1}{n+1}, \tfrac{1}{2} - \tfrac{1}{n+1}\right),\ P_2 = \left(\tfrac{1}{2} - \tfrac{1}{n-1}, \tfrac{1}{2} - \tfrac{1}{n-1}\right),$$
$$\text{and } P_3 = \left(\tfrac{1}{2} + \tfrac{1}{n-1}, \tfrac{1}{2} + \tfrac{1}{n-1}\right).$$

In the case $n = 1$ or $n = 2$ we define $P_2 = (0, 0)$ and $P_3 = (1, 1)$. Moreover, the asymptotic behavior in t follows by homogeneity, namely, there exists a positive constant C such that the $L^r - L^q$ estimates

$$\|u(t, \cdot)\|_{L^q} \leq C\ t^{1-\frac{n}{r}+\frac{n}{q}} \|\psi\|_{L^r}$$

hold uniformly for any $t > 0$.
If we ask for additional regularity of the data, besides to avoid singular estimates at $t = 0$ we can also enlarge the admissible range for r, q in the $L^r - L^q$ estimates. For instance, combining results from [48] and [35], the estimates

$$\|u(t, \cdot)\|_{L^q} \leq C(1 + t)^{(n-1)|\frac{1}{q}-\frac{1}{2}|} \|\varphi\|_{H^s_q} + t(1 + t)^{\max\{(n-1)|\frac{1}{q}-\frac{1}{2}|-1,0\}} \|\psi\|_{H^r_q}$$

hold for $q \in (1, \infty)$ if, and only if,

$$(n - 1)\Big|\frac{1}{q} - \frac{1}{2}\Big| \leq s \quad \text{and} \quad (n - 1)\Big|\frac{1}{q} - \frac{1}{2}\Big| \leq r + 1,\ r \geq 0.$$

Therefore, apart from the case $q = 2$, in general, one can not expect $L^q - L^q$ estimates for the solutions of the free wave equation.
It is interesting to compare $L^r - L^q$ estimates for the solution to the Cauchy problem for the free wave equation with the ones for the Klein-Gordon equation

$$v_{tt} - \triangle v + v = 0,\ \ v(0, x) = \varphi(x),\ \ v_t(0, x) = \psi(x).$$

In [33] the authors proved that for every $t > 0$ the operator $T_t : (\varphi, \psi) \to v(t, \cdot)$ $(\varphi \equiv 0)$ is bounded from $L^r(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if, and only if, the point $(\frac{1}{r}, \frac{1}{q})$ belongs to the same closed triangle $P_1P_2P_3$.

3.2.2 Main result and its proof

So, in the next two results we fix $P_1 = (\frac{1}{r}, \frac{1}{q})$ with

$$q = \frac{2(n + 1)}{n - 1} \quad \text{and} \quad r = \frac{2(n + 1)}{n + 3}.$$

Lemma 3.1. *If $(\varphi, \psi) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, $n \geq 2$, with supp $\varphi, \psi \subset \{|x| \leq R\}$, then $u_0 \in C\big([0, T], L^q(\mathbb{R}^n)\big)$ for all $T > 0$ with supp $u_0(t, \cdot) \subset \{|x| \leq t + R\}$.*

Proof. The statements of Theorem 1.1 and Remark 1.1 imply $u_0 \in C\big([0,T], H^1(\mathbb{R}^n)\big)$ for all $n \geq 1$ and $T > 0$. The conclusion of the lemma follows by using the well-known domain of dependence property of solutions to the wave equation and that $H^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ thanks to Sobolev's embedding theorem. $\square$

Now we shall prove the existence of a uniquely determined local (in time) Sobolev solution for compactly supported data.

Theorem 3.1. *Let $(\varphi, \psi) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, $n \geq 2$ with supp $\varphi, \psi \subset \{|x| \leq R\}$. If $1 \leq p \leq \frac{n+3}{n-1}$, then there exists a positive T and a uniquely determined local (in time) Sobolev solution*

$$u \in C\big([0,T], L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)\big) \quad \textit{with supp } u(t,\cdot) \subset \{|x| \leq t+R\}.$$

Proof. With $q = \frac{2(n+1)}{n-1}$ we define the space

$$X(T) := \Big\{u \in C\big([0,T], L^q(\mathbb{R}^n)\big) : \text{supp } u(t,\cdot) \subset \{|x| \leq t+R\} \text{ for } t \in [0,T]\Big\}.$$

This is a Banach space with the norm $\|u\|_{X(T)} := \max_{t\in[0,T]} \|u(t,\cdot)\|_{L^q}$. We introduce the operator

$$N : u \in X(T) \to Nu := u_0 + \int_0^t K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds \text{ for } t \in (0,T].$$

As in Section 2.2.2 our goal is to show that for some $T = T(\varphi,\psi)$ the operator N maps $X(T)$ into itself and is Lipschitz continuous for all $(u,v) \in X(T) \times X(T)$. In other words, we are going to prove the estimates

$$\begin{aligned}
&\|Nu\|_{X(T)} \leq C_0(\varphi,\psi) + C_1(\varphi,\psi) T^{\frac{2}{n+1}} \|u\|^p_{X(T)}, \\
&\|Nu - Nv\|_{X(T)} \leq C_2(\varphi,\psi) T^{\frac{2}{n+1}} \|u-v\|_{X(T)} \big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).
\end{aligned}$$

Lemma 3.1 shows that $u_0 \in X(T)$. Now we use the $L^r - L^q$ estimate, away of the conjugate line, of Section 3.2.1 for solutions to the free wave equation. Here we take account of the fact that $(\frac{1}{r}, \frac{1}{q})$ coincides with the point P_1 from Section 3.2.1. Then we have

$$\|K_1(t,0,x) *_{(x)} \psi(x)\|_{L^q} \leq C\, t^{1-\frac{n}{r}+\frac{n}{q}} \|\psi\|_{L^r} \text{ for } \ t > 0.$$

Hence, the integral term can be estimated as follows:

$$\begin{aligned}
&\Big\| \int_0^t K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds \Big\|_{L^q} \\
&\quad \leq \int_0^t \big\| K_1(t-s,0,\cdot) *_{(x)} |u(s,\cdot)|^p \big\|_{L^q} \, ds \\
&\quad \leq C \int_0^t (t-s)^{1-\frac{n}{r}+\frac{n}{q}} \| |u(s,\cdot)|^p \|_{L^r} \, ds \\
&\quad = C \int_0^t (t-s)^{1-\frac{n}{r}+\frac{n}{q}} \| |u(s,\cdot)| \|^p_{L^{rp}} \, ds \\
&\quad \leq C \|u\|^p_{X(T)} \int_0^t (t-s)^{1-\frac{n}{r}+\frac{n}{q}} \, ds \leq C t^{\frac{2}{n+1}} \|u\|^p_X,
\end{aligned}$$

because of $1 - \frac{n}{r} + \frac{n}{q} > -1$ for $n \geq 2$. Here we use on the one hand the compact support property of $u(t,\cdot)$ and suppose on the other hand that $rp \leq q$. The last inequality implies the condition $1 \leq p \leq \frac{n+3}{n-1}$. This leads to $Nu \in L^\infty\big((0,T), L^q(\mathbb{R}^n)\big)$. Moreover, using similar arguments as in the proof to Lemma 3.1 we conclude that N maps $X(T)$ into itself.
Thanks to Young's inequality we have

$$||u|^p - |v|^p| \leq C|u-v|(|u|+|v|)^{p-1}.$$

Using Hölder's inequality we conclude

$$\| |u|^p - |v|^p \|_{L^r} \leq C \|u-v\|_{L^{rp}} \big(\|u\|^{p-1}_{L^{rp}} + \|v\|^{p-1}_{L^{rp}} \big).$$

Therefore,

$$\|Nu - Nv\|_{X(T)} \leq C T^{\frac{2}{n+1}} \|u-v\|_{X(T)} \big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)} \big)$$

for any $u, v \in X(T)$. The term $T^{\frac{2}{n+1}}$ implies that N is a contraction mapping on $X(T)$ if T is sufficiently small. This completes the proof of the existence of a uniquely determined local (in time) Sobolev solution after applying a contraction argument for a, in general, small T. $\square$

3.3 Nonexistence of global (in time) classical solutions

Now our aim is to introduce tools to show that the Strauss' conjecture is really true. We restrict our attention to lower space dimensions only. For higher dimensions, besides the result in [47], we also refer to [20] for a more elementary treatment.
The main idea is to consider the functional

$$F(t) = \int_{\mathbb{R}^n} u(t,x) \, dx,$$

and to verify that this functional satisfies a nonlinear ordinary differential inequality and, additionally, admits a lower bound in order to apply a version of Kato's lemma taken from [47]. We apply this proposition to prove that a solution can not exist beyond a certain time.

Theorem 3.2. *Let $n \leq 3$ and $u \in C^2([0,T) \times \mathbb{R}^n)$ be a classical solution of*

$$u_{tt} - \Delta u = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

Assume that $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, supp$(\varphi,\psi) \subset \{|x| \leq R\}$, where

$$C_\varphi = \int_{\mathbb{R}^n} \varphi(x)\; dx > 0 \;\; \text{and} \;\; C_\psi = \int_{\mathbb{R}^n} \psi(x)\; dx > 0.$$

If $1 < p < p_0(n)$ ($p > 1$ for $n = 1$), then T is necessarily finite.

Remark 3.2. *Under the assumptions of Theorem 3.2 and the hypothesis $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ it is well-known that there exists a unique local (in time) classical solution to the semilinear wave equation with power nonlinearity for all $p > 1$. For simplicity, we prove the nonexistence of global (in time) classical solutions, but the argument does not use at all the smoothness of the solution, for more details see [47].*

Proof. We will not explicitly consider the case $n = 1$. The proof we will give can be easily extended to the case $n = 1$. If $n = 1$, then no critical value of p appears because the solution does not decay uniformly to zero as $t \to \infty$. So, one can expect blow up for all $p > 1$.
Let $n = 3$. By Theorem 3.1, we have supp $u(t,\cdot) \subset \{|x| \leq t + R\}$. Hence, after integration with respect to the spatial variables (boundary integrals vanish), we obtain

$$F''(t) = d_t^2 \int_{\mathbb{R}^n} u(t,x)\,dx = \int_{\mathbb{R}^n} \partial_t^2 u(t,x)\,dx = \int_{\mathbb{R}^n} |u(t,x)|^p\,dx,$$

thanks to the divergence theorem. Using the compact support property of $u(t,\cdot)$ and Hölder's inequality with $q = \frac{p}{p-1}$ we get

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} u(t,x)\,dx\Big|^p &= \Big|\int_{|x|\leq t+R} u(t,x)\,dx\Big|^p \\ &\leq \Big(\int_{|x|\leq t+R} 1\,dx\Big)^{\frac{p}{q}}\Big(\int_{\mathbb{R}^n} |u(t,x)|^p\,dx\Big) \leq C(t+R)^{n(p-1)}F''(t). \end{aligned}$$

Thus, we have obtained the following differential inequality:

$$F''(t) \geq C(t+R)^{-n(p-1)}|F(t)|^p \quad \text{for all } \; 0 \leq t < T.$$

If $u_0 = u_0(t,x)$ is a classical solution to the Cauchy problem for the free wave equation $u_{tt} - \Delta u = 0$ with data φ and ψ, then the divergence theorem implies

$$\int_{\mathbb{R}^n} u_0(t,x)\,dx = C_\psi t + C_\varphi.$$

In three or lower dimensions the Riemann function $K(t-s,0,\cdot)$ is non-negative. So, we may conclude $u(t,x) \geq u_0(t,x)$. Moreover, in three dimensions the Huygens' principle states that

$$\operatorname{supp} u_0(t,\cdot) \subset \Omega = \big\{x \in \mathbb{R}^3 : t-R < |x| < t+R\big\} \ \text{ for } \ t > R.$$

Therefore, using Hölder's inequality we have

$$\begin{aligned} C_\psi t + C_\varphi &= \int_{\mathbb{R}^n} u_0(t,x)dx = \int_\Omega u_0(t,x)dx \\ &\leq \int_\Omega u(t,x)dx \leq (\operatorname{vol}\Omega)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^3} |u(t,x)|^p dx\Big)^{1/p} \\ &\leq C(t+R)^{\frac{2(p-1)}{p}} \Big(\int_{\mathbb{R}^3} |u(t,x)|^p dx\Big)^{\frac{1}{p}} = C(t+R)^{\frac{2(p-1)}{p}} (F''(t))^{\frac{1}{p}}. \end{aligned}$$

By our hypothesis, we have $C_\psi > 0$, thus, we may conclude

$$F''(t) \geq Ct^{2-p} \ \text{ for large } \ t.$$

Integrating twice gives under the assumption $p < 3$ the estimate

$$F(t) \geq Ct^{4-p} \ \text{ for large } \ t.$$

Proposition 4.9 implies that $T < \infty$, provided that $1 < p < 1+\sqrt{2}$. Indeed, for $n = 3$, the functional $F(t)$ satisfies a nonlinear ordinary differential inequality and, additionally, admits a lower bound as in Proposition 4.9, where we choose $q = 3(p-1)$ and $r = 4-p$. Moreover, $(p-1)r > q-2$ if, and only if, $p < 1+\sqrt{2}$. Hence, the application of Proposition 4.9 implies $T < \infty$.
By the lack of Huygens' principle, the proof for $n = 2$ is more delicate and for more details we refer to Glassey [12]. □

3.4 Some remarks - life span estimates

Remark 3.3. *Let us explain the solvability behavior of the Cauchy problem*

$$u_{tt} - \Delta u = |u|^p, \ \ u(0,x) = \varphi(x), \ \ u_t(0,x) = \psi(x)$$

in the 3d case. Due to Theorem 3.1, we know that for $p \in [1,3]$ *we have a local (in time) solution belonging to the evolution space* $C\big([0,T], L^4(\mathbb{R}^3)\big)$. *Theorem 3.2 yields, that this solution has, in general, a blow up behavior for* $p \in (1, 1+\sqrt{2})$. *It is known by [46] that for* $p = 1+\sqrt{2}$ *the solution may blow up in finite time for suitable small data. The existence of global (in time) classical solutions for sufficiently smooth initial data with compact support is proved in [21] for* $p \in (1+\sqrt{2}, 3]$. *In [11] the authors proved the existence of global (in time) Sobolev solutions in the space* $L^{p+1}(\mathbb{R}^{3+1}, d\mu)$ *with a weighted measure* $d\mu$ *for* $p \in (1+\sqrt{2}, 3]$

and for initial data in $C_0^\infty(\mathbb{R}^3)$. More recently, under the assumption of radial initial data $\psi \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ (for simplicity take $\varphi \equiv 0$), in [8] the authors obtained the existence of global Sobolev solutions $u \in C\big([0,\infty), L^3(\mathbb{R}^3)\big)$, without any assumption on the support of the data.

Remark 3.4. *In Remark 2.3, we verified that the test function method bases on a contradiction argument. So this method does not give any information about blow up time or life span estimate or about the blow up mechanism.*
The application of Kato type lemmas (see Proposition 4.9) gives the information that the functional

$$F(t) := \int_{\mathbb{R}^n} u(t,x)\,dx$$

may blow up in finite time. This describes a blow up mechanism. We can expect also estimates for the life span time $T(\varepsilon)$ (see Remark 3.5).

Remark 3.5. *The conclusion of Theorem 3.2 is true even by assuming small data, let us say, $u(0,x) = \varepsilon\varphi(x)$ and $u_t(0,x) = \varepsilon\psi(x)$ with small ε. An important topic of recent research is to determine the lifespan $T = T(\varepsilon)$ of solutions. Here, we define $T(\varepsilon) = \sup\{t_0 > 0\}$, where the solution exists on the time interval $[0,t_0]$ for arbitrarily fixed (φ,ψ). One should pay attention in which sense solutions do exist, as classical ones, energy solutions, Sobolev solutions or distributional solutions. In order to have a good overview about results on lower and upper bounds for the lifespan, we refer to the paper [52].*
The following estimates for the lifespan $T(\varepsilon)$ were conjectured for $1 < p < p_0(n)$ $(n \geq 3)$ or $2 < p < p_0(2)$ $(n = 2)$ in [52]:

$$c\,\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n)}} \leq T(\varepsilon) \leq C\,\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n)}}, \quad \gamma(p,n) = 2 + (n+1)p - (n-1)p^2,$$

where the positive constants c, C are independent of ε. Results verifying this conjecture are summarized in the following table from [52]:

	lower bounds for $T(\varepsilon)$	upper bounds for $T(\varepsilon)$
$n = 2$	Zhou [63]	Zhou [63]
$n = 3$	Lindblad [30]	Lindblad [30]
$n \geq 4$	Lai-Zhou [27]	(rescaling argument of Sideris [47])

In [52] the author presents a simpler proof for upper bounds for $T(\varepsilon)$ by using an improved Kato type lemma without any rescaling argument.
If $p = p_0(n)$, then it was conjectured that

$$\exp\big(c\,\varepsilon^{-p(p-1)}\big) \leq T(\varepsilon) \leq \exp\big(C\,\varepsilon^{-p(p-1)}\big),$$

where the positive constants c, C are independent of ε. Results verifying this conjecture are summarized in the following table from [52]:

	lower bounds for $T(\varepsilon)$	upper bounds for $T(\varepsilon)$
$n=2$	Zhou [63]	Zhou [63]
$n=3$	Zhou [64]	Zhou [64]
$n \geq 4$	Lindblad - Sogge [31] (for $n \leq 8$ or radially symmetric solutions)	Takamura - Wakasa [54]

3.5 Strauss exponent versus Fujita exponent

3.5.1 Shift of Strauss

In Section 3.1 we introduced the Strauss exponent $p_0(n)$ as critical exponent for the Cauchy problem for the wave equation with source power nonlinearity

$$u_{tt} - \Delta u = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

A dissipation term may have an improving influence on the critical exponent. If we are interested in the Cauchy problem for the classical damped wave equation

$$u_{tt} - \Delta u + u_t = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x),$$

then it is shown in Sections 2.2 and 2.3 that the critical exponent is the Fujita exponent $p_{Fuj}(n)$. It holds $p_{Fuj}(n) < p_0(n)$. In this way we may understand the improving influence of the classical dissipation term u_t. There exists a class of damped wave models for which the critical exponent depends somehow on the Fujita exponent and the Strauss exponent as well. This class is described by scale-invariant linear damped wave operators and reads as follows:

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x),$$

where $\mu > 0$ is a real parameter. It was recently shown in [3] and [4] that $p_{Fuj}(n)$ is still the critical exponent when $\mu \geq \frac{5}{3}$ if $n = 1$, $\mu \geq 3$ if $n = 2$ and $\mu \geq n+2$ if $n \geq 3$.
It seems to be a challenge to determined the critical exponent in the case $\mu \in (0, n+2)$. In particular, it seems to be interesting to understand the transfer of $p_{Fuj}(n)$ to $p_0(n)$. The interested reader can find a first result in [7]. The authors consider the above model for $\mu = 2$, that is, the Cauchy problem

$$u_{tt} - \Delta u + \frac{2}{1+t} u_t = |u|^p, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

In this special case a change of variables transforms this Cauchy problem to the Cauchy problem

$$v_{tt} - \Delta v = (1+t)^{-(p-1)} |v|^p, \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x).$$

So, one can apply tools for wave models with power nonlinearity and a time-dependent coefficient. The authors prove the conjecture $p_{crit}(n) = p_0(n+2)$, so we have a shift of the

Strauss exponent by 2, in dimensions $n = 2, 3$. Later the first two authors of [7] proved this conjecture for all odd dimensions [5]. We still feel an improving influence of the dissipation term because of $p_{Fuj}(2) = p_0(4) = 2 < p_0(2)$ for $n = 2$ and $p_{Fuj}(3) < p_0(5) < p_0(3)$ for $n = 3$.
To prove the conjecture for $n = 2, 3$ the authors use the following tools:

1. the blow up technique of Glassey (see [12]), in particular, a Kato type lemma (see Proposition 4.9) and the considerations in Section 3.3,
2. for $n = 2$ Klainerman's vector fields are used to derive a suitable energy estimate in Klainerman-Sobolev spaces (see [24] and [65]),
3. for $n = 3$ radial data are supposed and the existence of small data radial solutions is proved by the aid of pointwise estimates (see [1] and [26]).

3.5.2 Interplay between Strauss and Fujita

The goal of the last section is to explain a non-existence result for global (in time) solutions of the Cauchy problem for semi-linear wave equation with scale-invariant dissipation and mass and power non-linearity, i.e., for solutions to the following model:

$$\begin{cases} v_{\tau\tau} - \Delta_y v + \dfrac{\mu_1}{1+\tau} v_\tau + \dfrac{\mu_2^2}{(1+\tau)^2} v = |v|^p, & \tau > 0,\ y \in \mathbb{R}^n, \\ v(0, y) = v_0(y), \quad y \in \mathbb{R}^n, \\ v_\tau(0, y) = v_1(y), \quad y \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

assuming in some sense that the damping and the mass terms make the equation *hyperbolic-like* from the point of view of the critical exponent diving the set of admissible exponents into one set which allows to prove a blow-up behavior for global (in time) solutions and a second set which allows to prove a (global) in time result of at least small data Sobolev solutions. The previous model is called *scale-invariant*, since the corresponding linear model is invariant under the so-called *hyperbolic scaling*

$$\widetilde{v}(\tau, y) = v(\lambda(1+\tau) - 1, \lambda y), \quad \lambda > 0.$$

Let us formulate analytically, in terms of μ_1 and μ_2^2, the assumption that we require for our model in this paper.
If we define

$$\delta := (\mu_1 - 1)^2 - 4\mu_2^2,$$

then our assumption for these coefficients is

$$\delta \in (0, 1]. \tag{3.2}$$

The quantity δ describes in some sense the interplay between the damping and the mass term in (3.1) and in the corresponding linear problem. Considering the transformation

$$u(t,x) = (1+\tau)^{\frac{\mu_1-1}{2}+\frac{\sqrt{\delta}}{2}} v(\tau,y) \qquad \tau = (1+t)^{\ell+1}-1, \;\; y = (1+\ell)x, \tag{3.3}$$

where

$$\ell = \tfrac{1-\sqrt{\delta}}{\sqrt{\delta}}, \quad k = \tfrac{1-\mu_1-\sqrt{\delta}}{2\sqrt{\delta}}(p-1) + \tfrac{2(1-\sqrt{\delta})}{\sqrt{\delta}}, \tag{3.4}$$

then we find that u solves the following Cauchy problem:

$$\begin{cases} u_{tt} - (1+t)^{2\ell}\Delta_x u = (\ell+1)^2(1+t)^k|u|^p, \quad t>0, \;\; x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x\in\mathbb{R}^n, \\ u_t(0,x) = u_1(x), \quad x\in\mathbb{R}^n, \end{cases} \tag{3.5}$$

for suitable u_0, u_1. In particular, we see that condition (3.2) allows the choice of a nonnegative ℓ. Therefore, we consider the Cauchy problem (3.5) for general $\ell \geq 0$, $k > -2$ and nonnegative compactly supported data u_0 and u_1. Since, we can derive a non-existence result for global (in time) solutions to this last Cauchy problem, then using the inverse transformation in (3.3) we obtain a blow-up result for (3.1) provided that (3.2) is satisfied.
Let us sketch the historical background of blow-up results for solutions to the Cauchy problem

$$\begin{cases} w_{tt} - \Delta_x w + b(t)w_t + m^2(t)w = |w|^p, \qquad t>0, \;\; x\in\mathbb{R}^n, \\ w(0,x) = w_0(x), \quad x\in\mathbb{R}^n, \\ w_t(0,x) = w_1(x), \quad x\in\mathbb{R}^n, \end{cases} \tag{3.6}$$

that are related somehow to our scale-invariant model (3.1). In [56] the authors have proved the blow-up of solutions in the case of sub-Fujita exponents (that is, for $1 < p < p_{Fuj}(n) := 1 + \frac{2}{n}$) by using a blow-up result for ordinary differential inequalities (cf. [56, Proposition 3.1]). On the other hand, we have drastically less blow-up results concerning classical Klein-Gordon equations with power nonlinearity on the right-hand side. In [23], for example, a blow-up result has been proved in space dimensions $n = 1, 2, 3$ and for sub-Fujita exponents.
Let us now recall some results to semi-linear wave models (3.6) with time-dependent dissipation $b(t)w_t$ and without any mass term, where $b(t) = \mu_1(1+t)^{-\beta}$ with $\beta \in (-1,1]$ and $\mu_1 > 0$. A blow-up result is proved in [29] by using the test function method, if $\beta \in (-1,1)$ and provided that $1 < p \leq p_{Fuj}(n)$. Later in [6] the authors generalized this blow-up result to more general damping terms $b(t)w_t$ by using a *modified test function method* (cf. [4]). More precisely, the dissipation $b(t)w_t$, that is considered in [6], is *effective* according to the classification given in [59].
Afterwards the case $\beta = 1$ was considered in [58]. In this paper the author proves two blow-up results for the scale-invariant case, for $1 < p \leq p_{Fuj}(n)$ if $\mu_1 > 1$ and for $1 < p \leq p_{Fuj}(n + \mu_1 - 1)$ if $0 < \mu_1 \leq 1$, assuming a suitable integral sign condition for the Cauchy

data. Also in this case the test function method is used in order to prove these results. In particular, for $\mu_1 > 1$ the same result has been substantially already proved with the modified test function method in [4].
Recently, in [28] the authors took into consideration the scale-invariant wave equation with damping in the case in which, in some sense, we call the model hyperbolic-like. They have shown a nonexistence result for global (in time) solutions for

$$p_{Fuj}(n) \leq p < p_0(n+2\mu_1) \quad \text{and} \quad 0 < \mu_1 < \frac{n^2+n+2}{2(n+2)},$$

where the upper bound for μ_1 guarantees the non-emptiness of the range for p, by using an improved version of Kato's lemma, which allows to control the life-span of the solution from above (see [52]).
Finally, let us mention blow-up results which are known for the scale-invariant case when also the mass term is present. In [39, 40] it is proved that the solution blows up for

$$1 < p \leq p_{Fuj}\Big(n + \frac{\mu_1 - 1 - \sqrt{\delta}}{2}\Big)$$

assuming $\delta \geq 0$ and suitable sign conditions for the initial data (moreover, in [40], also the compactness of the supports of data is required). Although the range of p, for which the solution is not globally in time defined, is the same in both results, a different approach is used in the corresponding proofs. While in [39] the test function method is considered, in [40] it is employed a proper modification of the blow-up result for ordinary differential inequalities introduced first in [56] for the constant coefficients case and adapted then in [38] for coefficients $b(t) = \mu_1(1+t)^{-\beta}$, where $\beta \in [0,1)$.
Furthermore, in [39] a further nonexistence result is shown in the case in which the coefficients of the damping and mass term satisfy $\delta = 1$. In more detail, it is proved that the solution blows up in finite time (using Kato's lemma) provided that

$$1 < p \leq \max\Big\{p_0(n+\mu_1), p_{Fuj}\Big(n + \frac{\mu_1}{2} - 1\Big)\Big\}$$

and the data are nonnegative and compactly supported. In particular the case $\mu_1 = 2, \mu_2^2 = 0$ (already considered in [7]) is included as a special case there.
For sake of brevity, we put

$$\phi(\tau) := \tfrac{\tau^{\ell+1}}{\ell+1} \qquad \text{for } \tau \geq 0.$$

Moreover, if $a(t) := (1+t)^\ell$ is the time-dependent speed of propagation for the transformed Cauchy problem (3.5), then we denote by $A(t)$ the primitive of a that vanishes for $t = 0$, namely

$$A(t) := \int_0^t a(s)ds = \tfrac{1}{\ell+1}\big((1+t)^{\ell+1} - 1\big) = \phi(1+t) - \phi(1).$$

Then we are able to prove in [42] the following blow-up result.

Theorem 3.3. *Assume that* $u \in \mathcal{C}^2([0,T) \times \mathbb{R}^n)$ *is a classical solution to* (3.5) *with* $\ell \geq 0, k > -2$ *and nonnegative, compactly supported initial data* $(u_0, u_1) \in \mathcal{C}^2(\mathbb{R}^n) \times \mathcal{C}^1(\mathbb{R}^n)$ *such that* u_0 *is not identically* 0. *If the exponent* $p > 1$ *satisfies one of the following conditions:*

$$p < p_{NE}(n;\ell,k) := \max\{p_0(n;\ell,k), p_1(n;\ell,k)\}, \tag{3.7}$$

$$p = p_{NE}(n;\ell,k) = p_1(n;\ell,k) \qquad \text{if } n = 1, \tag{3.8}$$

$$p = p_{NE}(n;\ell,k) = p_0(n;\ell,k) \qquad \text{if } n \geq 2, \tag{3.9}$$

where

$$p_1(n;\ell,k) := \frac{(\ell+1)n + k + 1}{(\ell+1)n - 1}$$

and $p_0(n;\ell,k)$ *is the positive root of the quadratic equation*

$$((\ell+1)n - 1)p^2 - ((\ell+1)n + 2k + 1 - 2\ell)p - 2(\ell+1) = 0, \tag{3.10}$$

then u *blows up in finite time, that is,* $T < \infty$.

Remark 3.6. *Using the same notations of the previous statement, we find in particular that for* $\ell = k = 0$ *the exponent* $p_0(n;\ell,k)$ *coincides with the Strauss exponent* $p_0(n)$. *Moreover, since* $p_1(n;0,0) = \frac{n+1}{n-1} < p_0(n)$ *for any* $n \geq 2$, *we find the well-known blow-up result for the free wave equation with power non-linearity in the special case* $\ell = k = 0$.

4 Background material-Useful inequalities

First we remember a corollary of the Riesz-Thorin interpolation theorem (see [55]) for linear continuous operators

$$T \in L\big(L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)\big)$$

mapping $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. The main concern of the Riesz-Thorin interpolation theorem is to explain that if a linear operator T is defined on both $L^{p_0}(\mathbb{R}^n)$ and $L^{p_1}(\mathbb{R}^n)$ and maps boundedly into $L^{q_0}(\mathbb{R}^n)$ and $L^{q_1}(\mathbb{R}^n)$, respectively, then the operator can be interpolated to yield a bounded operator on $L^{p_\theta}(\mathbb{R}^n)$ into $L^{q_\theta}(\mathbb{R}^n)$, where p_θ and q_θ are appropriately defined intermediate exponents.

Proposition 4.1. *Let $1 \le p_0, p_1, q_0, q_1 \le \infty$. If T is a linear continuous operator from*

$$L\big(L^{p_0}(\mathbb{R}^n) \to L^{q_0}(\mathbb{R}^n)\big) \cap L\big(L^{p_1}(\mathbb{R}^n) \to L^{q_1}(\mathbb{R}^n)\big),$$

then T belongs to

$$L\big(L^{p_\theta}(\mathbb{R}^n) \to L^{q_\theta}(\mathbb{R}^n)\big) \quad \textit{for each } \theta \in (0,1),$$

too, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Moreover, the following norm estimates are true:

$$\|T\|_{L(L^{p_\theta}(\mathbb{R}^n) \to L^{q_\theta}(\mathbb{R}^n))} \le \|T\|^{1-\theta}_{L(L^{p_0}(\mathbb{R}^n) \to L^{q_0}(\mathbb{R}^n))} \|T\|^{\theta}_{L(L^{p_1}(\mathbb{R}^n) \to L^{q_1}(\mathbb{R}^n))}.$$

One application of this proposition is in proving Young's inequality.

Proposition 4.2. *(Young's inequality)*
Let $f \in L^r(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ be two given functions. Then the following estimates hold for the convolution $u := f * g$:

$$\|u\|_{L^q} \le \|f\|_{L^r} \|g\|_{L^p} \;\; \textit{for all } 1 \le p \le q \le \infty \;\; \textit{and } 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.$$

Proof. First, we use

$$\|u\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1} \;\text{ and }\; \|u\|_{L^\infty} \le \|f\|_{L^1} \|g\|_{L^\infty}.$$

The Proposition 4.1 implies

$$\|u\|_{L^q} \le \|f\|_{L^1} \|g\|_{L^q} \;\text{ for all }\; q \in [1, \infty].$$

Finally, taking account of Hölder's inequality

$$\|u\|_{L^\infty} \le \|f\|_{L^p} \|g\|_{L^q},\; \frac{1}{q} + \frac{1}{p} = 1,$$

and again of Proposition 4.1 leads to the desired statement. □

Sometimes one needs interpolation between Sobolev spaces. Here we refer to the following interpolation result from [44], Theorem A.10.

Proposition 4.3. *Let the linear operator T satisfy*

$$T\,:\,W_1^n(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n),\ \ \textit{bounded with norm}\ M_0,$$
$$T\,:\,L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),\ \ \textit{bounded with norm}\ M_1.$$

Then there exist constants $C_1 = C_1(q,n)$ and $C_2 = C_2(q,n)$ such that the operator T satisfies the following mapping properties, too:

$$T\,:\,W_p^{N_p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n),\ \ \textit{bounded with norm}\ M_q \le C_1 M_0^{1-\theta} M_1^\theta,$$
$$T\,:\,H_p^{N_p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n),\ \ \textit{bounded with norm}\ M_q \le C_2 M_0^{1-\theta} M_1^\theta$$

with $p \in (1,2)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\theta = \frac{2}{q}$ and $N_p > n(\frac{1}{p} - \frac{1}{q})$.

The following inequality can be found in [9], Part 1, Theorem 9.3.

Proposition 4.4. *(classical Gagliardo-Nirenberg inequality)*
Let $j, m \in \mathbb{N}$ with $j < m$, and let $u \in C_0^m(\mathbb{R}^n)$, i.e. $u \in C^m(\mathbb{R}^n)$ with compact support. Let $\theta \in [\frac{j}{m}, 1]$, and let p, q, r in $[1,\infty]$ be such that

$$j - \frac{n}{q} = \Big(m - \frac{n}{r}\Big)\theta - \frac{n}{p}(1-\theta).$$

Then

$$\|D^j u\|_{L^q} \le C_{n,m,j,p,r,\theta} \|D^m u\|_{L^r}^\theta\ \|u\|_{L^p}^{1-\theta}$$

provided that

$$\Big(m - \frac{n}{r}\Big) - j \notin \mathbb{N},\ \ \textit{that is,}\ \ \frac{n}{r} > m - j\ \ \textit{or}\ \ \frac{n}{r} \notin \mathbb{N}.$$

If

$$\Big(m - \frac{n}{r}\Big) - j \in \mathbb{N},$$

then Gagliardo-Nirenberg inequality holds provided that $\theta \in [\frac{j}{m}, 1)$.

Remark 4.1. *Let us give some explanations. If $j = 0$, $m = 1$ and $r = p = 2$, then the Gagliardo-Nirenberg inequality reduces to the special Gagliardo-Nirenberg inequality*

$$\|u\|_{L^q} \le C \|\nabla u\|_{L^2}^{\theta(q)}\ \|u\|_{L^2}^{1-\theta(q)},$$

where $\theta(q)$ is given from the equation

$$-\frac{n}{q} = \Big(1 - \frac{n}{2}\Big)\theta(q) - \frac{n}{2}\big(1-\theta(q)\big) = \theta(q) - \frac{n}{2}.$$

It is clear that $\theta(q) \geq 0$ if, and only if, $q \geq 2$. Analogously $\theta(q) \leq 1$ if, and only if, either $n = 1, 2$ or $q \leq \frac{2n}{n-2}$. Applying a density argument the above inequality holds for any $u \in H^1(\mathbb{R}^n)$. Assuming $q < \infty$, then the special Gagliardo-Nirenberg inequality holds for any finite $q \geq 2$ if $n = 1, 2$ and for any $q \in [2, \frac{2n}{n-2}]$ if $n \geq 3$.

There exist numerous generalizations of the classical Gagliardo-Nirenberg inequality. As an example we present the following fractional Gagliardo-Nirenberg type inequality from [14].

Proposition 4.5. *The generalized Gagliardo-Nirenberg inequality*

$$\|u\|_{\dot{B}^s_{p,q}} \leq C\|u\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot{B}^{s_1}_{p_1,\infty}}$$

holds for all $u \in \dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^n) \cap \dot{B}^{s_1}_{p_1,\infty}(\mathbb{R}^n)$ if, and only if,

$$\frac{n}{p} - s = (1-\theta)\Big(\frac{n}{p_0} - s_0\Big) + \theta\Big(\frac{n}{p_1} - s_1\Big), \quad \frac{n}{p_0} - s_0 \neq \frac{n}{p_1} - s_1,$$
$$s \leq (1-\theta)s_0 + \theta s_1, \quad \text{and } p_0 = p_1 \text{ if } s = (1-\theta)s_0 + \theta s_1,$$

where $0 < q < \infty$, $0 < p, p_0, p_1 \leq \infty$, $s, s_0, s_1 \in \mathbb{R}^1$, $\theta \in (0,1)$.

We use the following corollary from Proposition 4.5.

Corollary 4.1. *Let $a \in (0, \sigma)$. Then, we have the following inequality for $m \in (1, \infty)$:*

$$\||D|^a u\|_{L^q} \leq C\||D|^\sigma u\|^{\theta_{a,\sigma}(q,m)}_{L^m} \|u\|^{1-\theta_{a,\sigma}(q,m)}_{L^m} \quad \text{for all } u \in H^\sigma_m(\mathbb{R}^n),$$

where

$$\frac{a}{\sigma} \leq \theta_{a,\sigma}(q,m) < 1 \quad \text{and} \quad \theta_{a,\sigma}(q,m) = \frac{n}{\sigma}\Big(\frac{1}{m} - \frac{1}{q} + \frac{a}{n}\Big),$$
$$\text{hence,} \quad m \leq q < \frac{mn}{[n + m(a-\sigma)]^+}.$$

Proof. We use the notations from the monograph [45]. The operator $|D|^a$ generates an isomorphism from $L^p(\mathbb{R}^n)$ onto $\dot{H}^{-a}_p(\mathbb{R}^n)$ for $p \in (1, \infty)$ and $a \in \mathbb{R}^1$ (see [2] or [57]). The space $\dot{H}^{-a}_p(\mathbb{R}^n)$ coincides with $\dot{F}^{-a}_{p,2}(\mathbb{R}^n)$ for $p \in (1, \infty)$ (see [49]). The continuous embedding

$$\dot{B}^s_{p,\min\{p,2\}}(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p,2}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p,\infty}(\mathbb{R}^n)$$

(see [57]) implies the inequality

$$\||D|^a u\|_{L^q} \leq C\|u\|_{\dot{B}^a_{q,\min\{q,2\}}}.$$

Now we apply the Gagliardo-Nirenberg inequality from Proposition 4.5 in the form

$$\|u\|_{\dot{B}^a_{q,\min\{q,2\}}} \leq C\|u\|^{\theta_{a,\sigma}(q,m)}_{\dot{B}^\sigma_{m,\infty}} \|u\|^{1-\theta_{a,\sigma}(q,m)}_{\dot{B}^0_{m,\infty}},$$

where all assumptions for its application are satisfied. Finally, the desired inequality follows by the chain of inequalities

$$\begin{aligned}\||D|^a u\|_{L^q} &\le C\|u\|_{\dot{B}^{\sigma}_{m,\infty}}^{\theta_{a,\sigma}(q,m)}\|u\|_{\dot{B}^{0}_{m,\infty}}^{1-\theta_{a,\sigma}(q,m)} \le C\||D|^\sigma u\|_{\dot{F}^{0}_{m,2}}^{\theta_{a,\sigma}(q,m)}\|u\|_{\dot{F}^{0}_{m,2}}^{1-\theta_{a,\sigma}(q,m)} \\ &\le C\||D|^\sigma u\|_{L^m}^{\theta_{a,\sigma}(q,m)}\|u\|_{L^m}^{1-\theta_{a,\sigma}(q,m)}.\end{aligned}$$

This completes the proof. □

Remark 4.2. *The statement of Corollary 4.1 remains true for*

$$\frac{a}{\sigma} \le \theta_{a,\sigma}(q,m) \le 1, \quad \text{hence,} \;\; m \le q \le \frac{mn}{n+m(a-\sigma)}$$

(see [41]).

Sometimes the following result from [45] for fractional powers is very helpful.

Proposition 4.6. *Let $p > 1$ and $v \in H^s_m(\mathbb{R}^n)$, where $s \in \Big(\frac{n}{m}, p\Big)$. Then the following estimates hold:*

$$\begin{aligned}\||v|^p\|_{H^s_m} &\le C\|v\|_{H^s_m}\|v\|_{L^\infty}^{p-1}, \\ \|v|v|^{p-1}\|_{H^s_m} &\le C\|v\|_{H^s_m}\|v\|_{L^\infty}^{p-1}.\end{aligned}$$

We derive the following corollary from Proposition 4.6.

Corollary 4.2. *Under the assumptions of Proposition 4.6 the following estimates hold:*

$$\begin{aligned}\||v|^p\|_{\dot{H}^s_m} &\le C\|v\|_{\dot{H}^s_m}\|v\|_{L^\infty}^{p-1}, \\ \|v|v|^{p-1}\|_{\dot{H}^s_m} &\le C\|v\|_{\dot{H}^s_m}\|v\|_{L^\infty}^{p-1}.\end{aligned}$$

Proof. We only prove the first inequality. For this reason we write the estimate from Proposition 4.6 in the form

$$\||v|^p\|_{\dot{H}^s_m} + \||v|^p\|_{L^m} \le C\big(\|v\|_{\dot{H}^s_m} + \|v\|_{L^m}\big)\|v\|_{L^\infty}^{p-1}.$$

Using instead of v the dilation $v_\lambda(\cdot) := v(\lambda\cdot)$ in the last inequality we obtain with

$$\|u_\lambda\|_{\dot{H}^s_m} = \lambda^{s-\frac{n}{m}}\|u\|_{\dot{H}^s_m} \;\text{ and }\; \|u_\lambda\|_{L^m} = \lambda^{-\frac{n}{m}}\|u\|_{L^m}$$

and with λ to infinity the desired inequality. The other inequality can be proved in the same way. □

The Littlewood-Paley decomposition is a localization procedure in the frequency space for tempered distributions. One of the main motivations for introducing such a localization when dealing with nonlinear partial differential equations is that the derivatives act almost as homotheties on distributions with Fourier transform supported in a ball or an annulus. More precisely, we have the following proposition.

Proposition 4.7. *(Bernstein's inequalities)*
Let D be an annulus and B a ball. Then there exists a constant C such that for any nonnegative integer k, any couple of real (p,q) so that $q \geq p \geq 1$ and for any function $u \in L^p(\mathbb{R}^n)$ with supp $(F(u)) \subset \lambda B$ for some $\lambda > 0$, we have

$$\sup_{|\alpha|=k} \|\partial_x^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}.$$

On the other hand, if supp $(F(u)) \subset \lambda D$ for some $\lambda > 0$, then

$$C^{-k-1}\lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial_x^\alpha u\|_{L^p} \leq C^{k+1}\lambda^k \|u\|_{L^p}.$$

The proof of decay estimates or blow-up behavior of solutions to nonlinear Cauchy problems often relies on ordinary differential inequalities.

Proposition 4.8. *Let $y = y(t)$ be a bounded nonnegative function on the interval $[0,T)$, $T > 0$, satisfying the integral inequality*

$$y(t) \leq k_0(1+t)^{-\alpha} + k_1 \int_0^t (1+t-s)^{-\beta}(1+s)^{-\gamma} y(s)^\mu \, ds$$

for some constants $k_0, k_1 > 0$, $\alpha, \beta, \gamma \geq 0$ and $0 \leq \mu < 1$. Then we have the estimate

$$y(t) \leq C(1+t)^{-\theta}$$

for some constant $C > 0$ and

$$\theta = \min\Big\{\alpha; \beta; \frac{\gamma}{1-\mu}; \frac{\beta+\gamma-1}{1-\mu}\Big\}$$

with an exception given in the case $\alpha \geq \tilde{\theta}$ and

$$\tilde{\theta} := \min\Big\{\beta; \frac{\gamma}{1-\mu}\Big\} = \frac{\beta+\gamma-1}{1-\mu} \leq 1,$$

whereas

$$y(t) \leq C(1+t)^{-\tilde{\theta}} (\log(2+t))^{\frac{1}{1-\mu}}.$$

Remark 4.3. *The conclusion of Proposition 4.8 is also true for the case $\mu = 1$. In particular, if $\gamma > 0$ and $\beta + \gamma - 1 > 0$, we may take $\theta = \min\{\alpha; \beta\}$.*

Proof. First we consider the case $\mu = 0$. Let us divide the interval $[0,t]$ into two subintervals $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$. It holds

$$\frac{1}{2}(1+t) \leq (1+t-s) \leq 1+t \text{ for any } s \in \Big[0, \frac{t}{2}\Big],$$
$$\frac{1}{2}(1+t) \leq (1+s) \leq 1+t \text{ for any } s \in \Big[\frac{t}{2}, t\Big].$$

Hence, using the change of variables $\tau = t - s$ if needed, we get

$$\begin{aligned}
I(t) &:= \int_0^t (1+t-s)^{-\beta}\,(1+s)^{-\gamma}\,ds \\
&\le (1+t)^{-\beta}\int_0^{\frac{t}{2}}(1+s)^{-\gamma}\,ds + (1+t)^{-\gamma}\int_{\frac{t}{2}}^{t}(1+t-s)^{-\beta}\,ds \\
&= (1+t)^{-\beta}\int_0^{\frac{t}{2}}(1+s)^{-\gamma}\,ds + (1+t)^{-\gamma}\int_0^{\frac{t}{2}}(1+\tau)^{-\beta}\,d\tau \\
&\approx (1+t)^{-\min\{\beta;\gamma\}}\int_0^{\frac{t}{2}}(1+s)^{-\max\{\beta;\gamma\}}\,ds.
\end{aligned}$$

Therefore

$$I(t) \le C\begin{cases}(1+t)^{-\min\{\beta;\gamma\}} & \text{if } \max\{\beta;\gamma\} > 1,\\ (1+t)^{-\min\{\beta;\gamma\}}\log(2+t) & \text{if } \max\{\beta;\gamma\} = 1,\\ (1+t)^{1-\beta-\gamma} & \text{if } \max\{\beta;\gamma\} < 1.\end{cases}$$

The proof of the desired estimate follows immediately for $\mu = 0$. If $0 < \mu < 1$, then we define

$$M(t) := \sup_{0\le s\le t}(1+s)^{\theta}y(s).$$

So we may write

$$y(t) \le k_0(1+t)^{-\alpha} + k_1\int_0^t (1+t-s)^{-\beta}(1+s)^{-\gamma-\mu\theta}\,ds\; M(t)^{\mu}.$$

If $\max\{\beta;\gamma+\mu\theta\} \ne 1$, following the ideas to estimate $I(t)$, we get

$$y(t) \le k_0(1+t)^{-\alpha} + C(1+t)^{-\theta^{\sharp}}M(t)^{\mu},$$

with $\theta^{\sharp} = \min\{\beta;\gamma+\mu\theta;\beta+\gamma+\mu\theta-1\}$. One may verify that $\min\{\alpha;\theta^{\sharp}\} = \theta$. Hence,

$$(1+t)^{\theta}y(t) \le k_0 + CM(t)^{\mu}.$$

Thanks to $0 < \mu < 1$, this inequality implies $M(t) \le C$ and the proof is concluded. The exceptional case $\max\{\beta;\gamma+\mu\theta\} = 1$ can be treated in a similar way. $\square$

In Section 3.3 we apply the following version of Kato's lemma to prove a blow-up behavior of solutions to the Cauchy problem for semilinear wave equations.

Proposition 4.9. *Suppose $F \in C^2[a,b)$ and assume that for $a \le t < b$ we have*

$$F(t) \ge C_0(k+t)^r, \quad F''(t) \ge C_1(k+t)^{-q}F(t)^p,$$

for some positive constants C_0, C_1 and k. If $p > 1$, $r \ge 1$ and $(p-1)r > q-2$, then b must be finite.

Proof. By the hypotheses of the lemma we get

$$F''(t) \geq C_1(k+t)^{-q} C_0^p (k+t)^{pr} \geq C(k+t)^{pr-q}.$$

After integration one has

$$F'(t) - F'(a) \geq C \int_a^t (k+s)^{pr-q}\, ds.$$

Taking into consideration $pr - q \geq -1$ the last inequality implies that unless b is finite $F'(t)$ must be positive for t sufficiently large. Thus, one may assume that there exists an a_0 such that $a < a_0 < b$ and

$$F'(t) > 0 \text{ for all } t \in [a_0, b).$$

It follows from the assumptions on p, q and r that

$$\frac{1}{p} < 1 - \frac{q-2}{pr}.$$

Hence, there is a $\theta \in (0,1)$ such that

$$\frac{1}{p} < \theta < 1 - \frac{q-2}{pr}.$$

By interpolating between the assumed inequalities, one has

$$F''(t) \geq C_1(k+t)^{-q}\, F(t)^{\theta p + (1-\theta)p} \geq C(k+t)^{rp(1-\theta)-q}\, F(t)^{\theta p}.$$

Our choice of θ implies $\alpha = \theta p > 1$ and $\beta = q - rp(1-\theta) < 2$. Without loss of generality one can set $\beta \geq 0$. This leads to

$$F''(t)\, F'(t) \geq C(k+t)^{-\beta}\, F(t)^{\alpha}\, F'(t).$$

Integration of the last inequality yields

$$\begin{aligned}\frac{1}{2}\big(F'(t)^2 - F'(a_0)^2\big) &\geq C \int_{a_0}^t (k+s)^{-\beta} F(s)^{\alpha} F'(s)\, ds \\ &\geq C_2(k+t)^{-\beta}\big(F(t)^{1+\alpha} - F(a_0)^{1+\alpha}\big).\end{aligned}$$

Note that we can choose the constant C_2 so small that

$$F'(a_0)^2 \geq 2C_2(k+a_0)^{-\beta} F(a_0)^{1+\alpha}.$$

Here we take account of $F'(a_0) > 0$. It follows that

$$F'(t)^2 \geq 2C_2(k+t)^{-\beta} F(t)^{1+\alpha},$$

and, therefore,

$$F(t)^{-\frac{1+\alpha}{2}} F'(t) \geq C(k+t)^{-\frac{\beta}{2}}$$

for all $a_0 < t < b$. One final integration yields ($\alpha > 1$)

$$F(a_0)^{\frac{1-\alpha}{2}} - F(t)^{\frac{1-\alpha}{2}} \geq C\big((k+t)^{1-\frac{\beta}{2}} - (k+a_0)^{1-\frac{\beta}{2}}\big).$$

Since $\beta < 2$, it is clear that the time variable t can not be arbitrarily large. $\square$

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