Perturbation of matrices with large rank

Set up

Let A be a fixed $n \times n$ full rank hermitian matrix and let Z be GOE/GUE. We are mainly interested in answering the following question:

How do the eigenvalues of $A + Z$ differ from the ones of A?

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Let A be a fixed $n \times n$ full rank hermitian matrix and let Z be GOE/GUE. We are mainly interested in answering the following question:

How do the eigenvalues of $A + Z$ differ from the ones of A?

Let $\lambda_i(X)$ be the i^{th} largest eigenvalue of X and denote by $v_i(X)$ its corresponding eigenvector. Also define $\delta_i := \min(\lambda_{i-1} - \lambda_i, \lambda_i - \lambda_{i+1})$ and $\delta := \min_i \delta_i$.

Main Theorems

In these slides we let C, C' and C'' be constants (they might denote different constants from one line to the other).

Main Theorems

Theorem (Main Theorem 1)

Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ which satisfy $\lambda_1 - \lambda_i \geq (i-1)\log^3 i$ for any $i > 1$. Let $\Delta > 10$, then the following holds with probability at least $\left(1 - \frac{100}{\Delta^{\log \Delta}}\right)$:

$$\|A + Z\| \leq \|A\| + \Delta,$$

where Z is a GUE.

Matin Theorem 1

Observation

Note that the Main Theorem 1 is optimal up to the logarithmic factor. To see this, let $C > 0$ be a constant, $Z = (\xi_{ij})_{i,j \le n}$ be a GUE and let $\epsilon > 1/(2C)$ be also fixed.

$$A = \begin{bmatrix} \epsilon n & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then the following holds:

$$\|A + Z\| \ge |(A+Z)e_1|_2 = |(\epsilon n + \xi_{11}, \xi_{12}, \dots, \xi_{1n})|_2 \approx \epsilon n + \frac{1}{2\epsilon} > \epsilon n + C.$$

Main Theorem 2

Theorem (Main Theorem 2)

Let A be a Hermitian matrix with distinct eigenvalues and C a big constant. Define

$$c = \min_{i \neq j} \frac{\lambda_i - \lambda_j}{C \cdot (i - j) \log^3(n)}.$$

Then, for any $\epsilon \leq c$ the following holds with probability $1 - \frac{1}{n^{10}}$. For all $1 \leq i \leq n$

$$\lambda_i(A + \epsilon Z) = \lambda_i(A) + \epsilon\gamma + O(\epsilon / \log n),$$

where γ is $\mathcal{N}(0,1)$ and Z is GUE.

Corollary

Let C be a big constant and A be a Hermitian such that

$$\lambda_i(A) - \lambda_j(A) \geq C(j - i)\log^3 n.$$

Then for any $1 \leq i \leq n$ the following is true with probability $1 - \frac{1}{n^{10}}$:

$$\lambda_i(A_Z) = \lambda_i(A) + \gamma + O(1/\log n),$$

where γ is $\mathcal{N}(0,1)$ and Z is GUE.

Corollary

Let C be a big constant and A be a Hermitian such that

$$\lambda_i(A) - \lambda_j(A) \geq C(j-i)\log^3 n.$$

Then for any $1 \leq i \leq n$ *the following is true with probability* $1 - \frac{1}{n^{10}}$*:*

$$\lambda_i(A_Z) = \lambda_i(A) + \gamma + O(1/\log n),$$

where γ *is* $\mathcal{N}(0,1)$ *and Z is GUE.*

Observation

Corollary suggests that if A is diagonal, adding Z to it has the same effect with adding only the diagonal elements of Z.

Proof of Main Theorem 2

We prove Main Theorem 2 in several steps. First step is Theorem 1.1 below which is a weaker version of Main Theorem 1.

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Theorem (Theorem 1.1)

Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ which satisfy $\lambda_1 - \lambda_i \geq (i-1)\log^3(n)$ for any $i > 1$. Let $\Delta > \log n$, then the following holds with probability at least $\left(1 - e^{-50\Delta}\right) \cdot \left(1 - \frac{C}{n^{50}}\right)$:

$$\|A + Z\| \leq \|A\| + \Delta,$$

where Z is a GUE.

Proof of Main Theorem 2

Since Z is GUE, we can assume, without loos of generality that

$$A := \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n).$$

Let $g_i := \lambda_1 - \lambda_i \geq (i-1)\log^3(i-1)$ for any i and $Z := \big(-\xi_{ij} \big)$.

Proof of Main Theorem 2

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$$A := \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n).$$

Let $g_i := \lambda_1 - \lambda_i \geq (i-1)\log^3(i-1)$ for any i and $Z := \big(- \xi_{ij} \big)$.

We want to prove that, with high probability (depending on Δ)

$$\sup_{|v|=1} v^t (A + Z) v \leq \lambda_1 + \Delta.$$

Proof of Main Theorem 2

This implies that $M := (\lambda_1 + \Delta)I - A - Z$ is whp positive definite.

$$M := \begin{bmatrix} \Delta + \xi_{11} & \xi_{12} & \xi_{13} & \cdots & \xi_{1n} \\ \xi_{21} & g_2 + \Delta + \xi_{22} & \xi_{23} & \cdots & \xi_{2n} \\ \xi_{31} & \xi_{32} & g_3 + \Delta + \xi_{33} & \cdots & \xi_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots & g_n + \Delta + \xi_{nn} \end{bmatrix}$$

Proof of Main Theorem 2

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Let M_k be the top left $k \times k$ minor. We want to prove that all M_k's have positive determinant, which will imply that M is positive definite.

Proof of Main Theorem 2

Lemma

Let $k \geq \Delta^{1/4}$. Assume M_k is positive definite and that $\lambda_k(M_k) > 0$. Define

$$S_k^{(i)} := \sum_{i=1}^{k} \frac{1}{\lambda_i^i(M_k)}.$$

Assume further that

$$S_k^{(1)}, S_k^{(2)} \leq C(k),$$

where $C(k) = 100 + \sum_{i=1}^{k-1} \frac{2}{i \cdot log^2(n)}$. Then, the following hold with probability at least $\left(1 - \frac{C}{n\sqrt{\log n}}\right)$:

$$\lambda_{k+1}(M_{k+1}) > 0,$$

$$S_{k+1}^{(1)}, S_{k+1}^{(2)} \leq C(k+1).$$

Proof of Lemma

Let U_k be the unitary matrix such that $U_k^T M_k U_k = \text{diagonal}(\sigma_1, \sigma_2, ..., \sigma_k)$. Let

$$M' = \begin{bmatrix} U_k^* & 0 \\ 0 & 1 \end{bmatrix} M_{k+1} \begin{bmatrix} U_k & 0 \\ 0 & 1 \end{bmatrix} := \begin{bmatrix} \sigma_1(M_k) & 0 & \dots & 0 & \xi_1 \\ 0 & \sigma_2(M_k) & \dots & 0 & \xi_2 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \sigma_k(M_k) & \xi_k \\ \xi_1 & \xi_2 & \dots & \xi_k & c \end{bmatrix},$$

where $c := g_{k+1} + \Delta + \xi_{k+1,k+1}$ and $\xi_1, \xi_2, ..., \xi_k$ are i.i.d. Gaussian.

Proof of Lemma

Let $P(x) := \det(A - xI)$ be the characteristic polynomial of M'. It follows that:

$$P(x) = (c - x)\prod_{i=1}^{k}(\sigma_i(M_k) - x) - \sum_{i=1}^{k} \xi_i^2 \prod_{j \neq i}(\sigma_j(M_k) - x).$$

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For $x \neq \sigma_1(M_k), \sigma_2(M_k), ., , , \sigma_k(M_k)$, define:

$$f(x) := \frac{P(x)}{\prod_i(\sigma_i(M_k) - x)} = c - x - \sum_{i=1}^{k}\frac{\xi_i^2}{\sigma_i(M_k) - x},$$

so x is a root for P which is not $\sigma_i(M_k)$ for some i, iff x is a root of f.

Proof of Lemma

Note that with probability $1 - C/n^{\sqrt{\log n}}$ we have that $|\xi_i| \leq C' \sqrt{\log n}$ for all $1 \leq i \leq k$. It follows that with probability $1 - C/n^{\sqrt{\log n}}$ we have that for any $x \geq 0$ (we write σ_i for $\sigma_i(M_k)$ when it is no confusion):

$$
\begin{aligned}
f(-x) &= \left(\prod_{i=1}^{k}(\sigma_i + x)\right)\left(c + x - C'\sum_{i=1}^{k}\frac{\xi_i^2}{\sigma_i + x}\right) \\
&\geq \left(\prod_{i=1}^{k}(\sigma_i + x)\right)\left(c - C'\log n \cdot S_k^{(1)}\right) \\
&\geq \left(\prod_{i=1}^{k}\sigma_i\right)(c - \log n \cdot C_1(k)) \\
&> 0.
\end{aligned}
$$

We conclude that with probability $1 - C/n^{\sqrt{\log n}}$, all the roots of P are strictly positive.

Proof of Lemma

Recall:

$$M' = \begin{bmatrix} \sigma_1 & 0 & 0 & \dots & 0 & \xi_1 \\ 0 & \sigma_2 & 0 & \dots & 0 & \xi_2 \\ 0 & 0 & \sigma_3 & \dots & 0 & \xi_3 \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \dots & \sigma_k & \xi_k \\ \xi_1 & \xi_2 & \xi_3 & \dots & \xi_k & c \end{bmatrix}$$

The idea is to compute the elements of M'^{-1} and use the Trace and the Frobenius norm formulas to bound $S_{k+1}^{(1)}$ and $S_{k+1}^{(2)}$.

Proof of Lemma

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$$M' = \begin{bmatrix} \sigma_1 & 0 & 0 & \dots & 0 & \xi_1 \\ 0 & \sigma_2 & 0 & \dots & 0 & \xi_2 \\ 0 & 0 & \sigma_3 & \dots & 0 & \xi_3 \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \dots & \sigma_k & \xi_k \\ \xi_1 & \xi_2 & \xi_3 & \dots & \xi_k & c \end{bmatrix}$$

The idea is to compute the elements of M'^{-1} and use the Trace and the Frobenius norm formulas to bound $S_{k+1}^{(1)}$ and $S_{k+1}^{(2)}$.

Recall that with probability $1 - C/n^{\sqrt{\log n}}$, we have $|\xi_i| \leq C'\sqrt{\log n}$, for all $i \leq k$. From now on, we condition on this event.

Proof of Lemma

Let

$$S_k^{(j)*} = \sum_{i=1}^{n} \frac{\xi_i^2}{\sigma_i^j}$$

and

$$S_k^{(j)**} = \sum_{i=1}^{n} \frac{\xi_i^4}{\sigma_i^j}.$$

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Note that:

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Note that:

- $C(k)$ is bounded,
- $S_k^{(1)*}$ and $S_k^{(2)*} \leq C(k)C' \log n$
- $S_k^{(3)*} = \sum_{i=1}^{k} \frac{\xi_i^2}{\sigma_i^3} \leq \left(\sum_{i=1}^{k} \frac{\xi_i^2}{\sigma_i^2}\right)\left(\sum_{i=1}^{k} \frac{1}{\sigma_i}\right) \leq C(k)^2 C'^2 \log n.$

Proof of Lemma

Now we are ready to compute the elements of M'^{-1} and estimate $S_{k+1}^{(1)}$ and $S_{k+1}^{(2)}$. Note that since M' has the *almost diagonal* form, we can compute specifically each entry of M^{-1}.

Proof of Lemma

Now we are ready to compute the elements of M'^{-1} and estimate $S_{k+1}^{(1)}$ and $S_{k+1}^{(2)}$. Note that since M' has the *almost diagonal* form, we can compute specifically each entry of M^{-1}.

$$\det(M') = \det(M_{k+1}) = c\det(M_k) - \sum_{i=1}^{k} \det(M_k)\frac{\xi_i^2}{\sigma_i} = \det(M_k)\,(c - S_k^*)$$

Proof of Lemma

Now we are ready to compute the elements of M'^{-1} and estimate $S^{(1)}_{k+1}$ and $S^{(2)}_{k+1}$. Note that since M' has the *almost diagonal* form, we can compute specifically each entry of M^{-1}.

$$\det(M') = \det(M_{k+1}) = c\det(M_k) - \sum_{i=1}^{k} \det(M_k)\frac{\xi_i^2}{\sigma_i} = \det(M_k)\,(c - S_k^*)$$

We use the adjoint formula to find the elements of the inverse of M'.

$$M'^{-1}(k+1, k+1) = \frac{\det M_k}{\det M_{k+1}}$$

Proof of Lemma

$$M'^{-1}(i,i) = \frac{\det M_k}{\det M_{k+1}} \left(\frac{c}{\sigma_i} - \frac{1}{\sigma_i} \sum_{j \neq i} \frac{\xi_j^2}{\sigma_j} \right)$$
$$= \frac{\det M_k}{\det M_{k+1}} \left(\frac{c}{\sigma_i} - \frac{S_k^{(1)*}}{\sigma_i} + \frac{\xi_i^2}{\sigma_i^2} \right) \text{ for } i \neq k+1$$

Proof of Lemma

$$
\begin{aligned}
M'^{-1}(i,i) &= \frac{\det M_k}{\det M_{k+1}}\left(\frac{c}{\sigma_i} - \frac{1}{\sigma_i}\sum_{j\neq i}\frac{\xi_j^2}{\sigma_j}\right) \\
&= \frac{\det M_k}{\det M_{k+1}}\left(\frac{c}{\sigma_i} - \frac{S_k^{(1)*}}{\sigma_i} + \frac{\xi_i^2}{\sigma_i^2}\right) \text{ for } i \neq k+1
\end{aligned}
$$

$$
M'^{-1}(i,j) = \frac{(-1)^{i+j}\det M_k}{\det M_{k+1}}\left(\frac{\xi_i\xi_j}{\sigma_i\sigma_j}\right) \text{ for } i \neq j \neq k+1.
$$

Proof of Lemma

$$M'^{-1}(i,i) = \frac{\det M_k}{\det M_{k+1}} \left(\frac{c}{\sigma_i} - \frac{1}{\sigma_i} \sum_{j \neq i} \frac{\xi_j^2}{\sigma_j} \right)$$

$$= \frac{\det M_k}{\det M_{k+1}} \left(\frac{c}{\sigma_i} - \frac{S_k^{(1)*}}{\sigma_i} + \frac{\xi_i^2}{\sigma_i^2} \right) \text{ for } i \neq k+1$$

$$M'^{-1}(i,j) = \frac{(-1)^{i+j} \det M_k}{\det M_{k+1}} \left(\frac{\xi_i \xi_j}{\sigma_i \sigma_j} \right) \text{ for } i \neq j \neq k+1.$$

$$M'(k+1,i) = (-1)^{k+1+i} \frac{\det M_k}{\det M_{k+1}} \frac{\xi_i}{\sigma_i}$$

Proof of Lemma

It follows that:

$$\begin{aligned} S_{k+1}^{(1)} = \mathrm{Trace}(M'^{-1}) &= \frac{\det M_k}{\det M_{k+1}} \left(1 + cS_k^{(1)} - S_k^{(1)} S_k^{(1)*} + S_k^{(2)*}\right) \\ &= \frac{1 + cS_k^{(1)} - S_k^{(1)} S_k^{(1)*} + S_k^{(2)*}}{c - S_k^{(1)*}} \\ &= S_k^{(1)} + \frac{S_k^{(2)*} + 1}{c - S_k^{(1)*}} \\ &\leq S_k^{(1)} + \frac{2}{k \cdot \log^2(n)} \text{ whp} \\ &\leq C_1(k) + \frac{2}{k \cdot \log^2(n)} = C_1(k+1). \end{aligned}$$

Proof of Lemma

Similarly, but more messy:

$$
\begin{aligned}
S_{k+1}^{(2)} &= \sum_{i,j} M_{k+1}^{-1}(i,j)^2 \\
&= \sum_{i} M_{k+1}^{-1}(i,i)^2 + \sum_{i\neq j\neq k+1} M_{k+1}^{-1}(i,j)^2 + 2\sum_{i\neq k+1} M_{k+1}^{-1}(i,k+1)^2 \\
&= \left(\frac{\det M_k}{\det M_{k+1}}\right)^2 \left(\sum_i \left(\frac{c}{\sigma_i} - \frac{S_k^{(1)*}}{\sigma_i} + \frac{\xi_i^2}{\sigma_i^2}\right)^2 + \sum_{i\neq j} \frac{\xi_i^2\xi_j^2}{\sigma_i^2\sigma_j^2} + 2\sum_i \frac{\xi_i^2}{\sigma_i^2}\right) \\
&= \left(\frac{\det M_k}{\det M_{k+1}}\right)^2 \Big(c^2 S_k^{(2)} + (S_k^{(1)*})^2 S_k^{(2)} + \\
&\quad + S_k^{(4)**} - 2cS_k^{(1)*}S_k^{(2)} + 2cS_k^{(3)*} - 2S_k^{(1)*}S_k^{(3)*} + (S_k^{(2)*})^2 - S_k^{(4)**} + 2S_k^{(2)*}\Big)
\end{aligned}
$$

Proof of Lemma

$$= \left(\frac{\det M_k}{\det M_{k+1}}\right)^2 \left((c^2 + (S_k^{(1)*})^2 - 2cS_k^{(1)*})S_k^{(2)} + \right.$$

$$\left. + (S_k^{(2)*})^2 + 2S_k^{(2)*} + S_k^{(3)*}(2c - 2S_k^{(1)*}) \right)$$

$$= \frac{(c - S_k^{(1)*})^2 S_k^{(2)} + (S_k^{(2)*})^2 + 2S_k^{(2)*} + S_k^{(3)*}(2c - 2S_k^{(1)*})}{(c - S_k^{(1)*})^2}$$

$$= S_k^{(2)} + \frac{(S_k^{(2)*})^2 + 2S_k^{(2)*} + S_k^{(3)*}(2c - 2S_k^{(1)*})}{(c - S_k^{(1)*})^2}$$

$$\leq S_k^{(2)} + \frac{1}{k \cdot \log^2(n)}$$

$$\leq C_2(k) + \frac{1}{k \cdot \log^2(n)} = C_2(k+1).$$

Proof of Theorem 1

Now, that we have completed the proof of Lemma 1, we are ready to complete the proof of Theorem 1. The base case of the induction follows trivially by noting that

$$M_{\Delta^{1/4}} = \Delta \cdot I + (M_{\Delta^{1/4}} - \Delta \cdot I),$$

But,

$$\|M_{\Delta^{1/4}} - \Delta \cdot I\|_{Fr} \leq \Delta^{3/4} \text{ with probability } 1 - e^{-50\Delta}.$$

so $\sigma_{\min}(M_{\Delta^{1/4}}) \geq \Delta/2$. Let p_k be the probability that all the top-left minors, from 1 to k are positives and $S_k^{(1)}, S_k^{(2)} \leq C(k)$. Hence,

$$p_{\Delta^{1/4}} \geq 1 - e^{-50\Delta}$$

Proof of Theorem 1

By Lemma 1,

$$p_n \geq \left(1 - e^{-50\Delta}\right) \prod_{k=\Delta^{1/4}}^{n} \left(1 - \frac{C}{n^{\sqrt{\log n}}}\right)$$
$$\geq \left(1 - e^{-50\Delta}\right) \cdot \left(1 - \frac{C}{n^{50}}\right)$$

Proof of Theorem 1

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$$\geq \left(1 - e^{-50\Delta}\right) \cdot \left(1 - \frac{C}{n^{50}}\right)$$

The Proof of Main Theorem 2 follows by the Sylvester's criterion for positive definite matrices.

Theorem 1.2

The second step in our proof is to turn the upper bound of $\lambda_1(A)$ into a lower bound. Note that for the lower bound, we do not need any condition on the eigenvalues of A.

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Theorem (Theorem 1.2)

Suppose $\Delta \geq \log n$ *and* $\lambda_1 - \lambda_i \geq (i-1)\log^3(n)$, *then with probability at least* $1 - \frac{C}{n^{50}}$ *the following holds :*

$$\lambda_1(A+Z) \geq \lambda_1(A) - \Delta,$$

where Z *is GUE.*

Proof of Theorem 1.2.

Suppose $\lambda_1(A+Z) < \lambda_1(A) - \Delta$ and wlog assume $A =$ diagonal$(\lambda_i)_{i=1,..,n}$. Then the matrix $A + Z - \lambda_1 I + \Delta I$ has no positive eigenvalue, i.e.

$$M := \lambda_1 I - \Delta I - A - Z \text{ is positive definite.}$$

Proof of Theorem 1.2.

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$$M := \lambda_1 I - \Delta I - A - Z \text{ is positive definite.}$$

However, note that: $M(1,1) = \xi - \Delta$ where ξ is $\mathcal{N}(0,1)$ distributed. Since we have that with probability at least $1 - \frac{C}{n^{50}}$, $\det(M(1,1)) < 0$, by the Sylvester's criteria we have that M is not positive definite.

□

The third step, is to generalize Theorems 1 and 2 to other indices.

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Theorem (Theorem 1.3)

Let $i \geq 1$. Suppose $|\lambda_i - \lambda_j| \geq C|j-i| \log^3 n$ for any $j \neq i$. Then the following holds with probability at least $\left(1 - e^{-50\Delta}\right) \cdot \left(1 - \frac{C}{n^{50}}\right)$.

$$-\Delta \leq \lambda_i(A+Z) - \lambda_i(A) \leq \Delta.$$

Proof of Theorem 1.3.

Wlog assume that $A = \mathrm{diag}(\sigma_i)$. Note that:

$$\begin{aligned}\lambda_i(A+Z) &= \inf_{\dim(S)=n+1-i} \sup_{v,w\in S} w^T(A+Z)v \\ &\leq \sup_{v,w\in \mathrm{span}\, e_i,\ldots e_n} w^T(A+Z)v \\ &= \|A_i + Z\|,\end{aligned}$$

where

$$A_i := \begin{bmatrix} \sigma_i & 0 & 0 & \ldots & 0 \\ 0 & \sigma_{i+1} & 0 & \ldots & 0 \\ 0 & 0 & \sigma_{i+2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \sigma_n \end{bmatrix}.$$

The upper bound follows by applying Theorem 1.1 to A_i. □

Proof.

For the lower bound note that:

$$
\begin{aligned}
\lambda_i(A+Z) &= \sup_{\dim(S)=i} \inf_{v,w\in S} w^T(A+Z)v \\
&\geq \inf_{v,w\in \text{span } e_1,\dots e_i} w^T(A+Z)v \\
&= \lambda_{\min}(A_i+Z) \\
&= 1-\lambda_{\max}(I-(A_1+Z)),
\end{aligned}
$$

$$
A_i := \begin{bmatrix} \sigma_1 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & 0 \\ 0 & 0 & \sigma_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_i \end{bmatrix}.
$$

The lower bound follows by applying Theorem 1.1 to $I - A_i$. □

Corollary (Corollary 1)

Let A be a Hermitian matrix such that $\lambda_i(A) - \lambda_j(A) \geq C \cdot (j-i)\log^3(n)$ for any $j > i$. Then, with probability at least $1 - C/n^{10}$, we have that for any $i > 1$

$$|\lambda_i(A+Z) - \lambda_1(A+Z)| \geq \frac{(i-1)\cdot \log^2(n)}{2}.$$

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$$|\lambda_i(A+Z) - \lambda_1(A+Z)| \geq \frac{(i-1)\cdot \log^2(n)}{2}.$$

Proof.

Apply Theorem 1.3 for $i = 1, 2, ...n$ and $\Delta = 10\log(n)$. □

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$$|\lambda_i(A+Z) - \lambda_1(A+Z)| \geq \frac{(i-1)\cdot \log^2(n)}{2}.$$

Proof.

Apply Theorem 1.3 for $i = 1, 2, \ldots n$ and $\Delta = 10\log(n)$. □

Observation

Note that Theorems 1.1, 1.2, 1.3 and Corollary 1 holds even if we replace Z with εZ, where $\varepsilon \in [0,1]$.

Dyson Brownian motion

Recall from Dyson Brownian Motion that:

$$\lambda_1(A+Z) - \lambda_1(A) = B_1 + \int_0^1 \sum_{i=2}^{n} \frac{dt}{\lambda_1(A+Z_t) - \lambda_i(A+Z_t)} dt + o(1),$$

where B_1 is $\mathcal{N}(0,1)$ and Z_t is GUE with variance t.

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where B_1 is $\mathcal{N}(0,1)$ and Z_t is GUE with variance t. From Corollary 1, we have that, for fixed $t \in [0,1]$:

$$\sum_{i=2}^{n} \frac{dt}{\lambda_1(A+Z_t) - \lambda_i(A+Z_t)} \leq 2 \sum_{i=2}^{n} \frac{dt}{i \cdot \log^2(n)} \leq \frac{2dt}{\log(n)}$$

with probability at least $1 - C/n^{10}$.

By Theorem 1.3 and a union argument we have that with probability $\left(1 - C/n^{10}\right)^n$

$$|\lambda_i(A + Z_k) - \lambda_i(A)| \leq C \cdot \log(n),$$

for every $k = i/n$ and $i = 1, 2, ..., n$.

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Conditioned on the event that for avery i we have

$$|\lambda_i(A + Z_k) - \lambda_i(A)| \leq C \cdot \log(n),$$

we deduce that for any $j > i$

$$\lambda_i(A + Z_k) - \lambda_j(A + Z_k) \geq C'(j - i) \log^3(n).$$

Let $0 < \varepsilon < 1/n$, then

$$\lambda_i(A + Z_{k+\varepsilon}) - \lambda_j(A + Z_{k+\varepsilon}) \geq C'(j - i)\log^3(n) - 2\|Z_{k+\varepsilon} - Z_k\|,$$

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Let $0 < \varepsilon < 1/n$, then

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For the simplicity of the argument assume $C' = 1$ so we do not have to worry about the constants.

Suppose there exits $\varepsilon \in (0, 1/n)$ such that:

$$\|Z_{k+\varepsilon} - Z_k\| \geq \log n^3 \text{ and } \left\|Z_{k+1/n} - Z_k\right\| \leq \log n.$$

Call this event **E**.

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Let $t = \min \varepsilon'$ such that $\|Z_{k+\varepsilon'} - Z_k\| \geq \log n^3$, so $t \leq \varepsilon$ on **E**.

$$\begin{aligned}
\mathbf{P}(\mathbf{E}) &\leq \int_0^\varepsilon \mathbf{P}(t = x) \cdot \mathbf{P}(\|Z_{k+1/n} - Z_{k+t}\| \geq \log^2 n) dx \\
&\leq \int_0^\varepsilon \mathbf{P}(t = x) \cdot e^{-100n} dx \\
&\leq e^{-100n}.
\end{aligned}$$

Proof of Theorem 1

It follows that:

$$\mathbf{P}\left(\exists t \text{ such that} \sum_{i=2}^{n} \frac{1}{\lambda_1(A+Z_t)-\lambda_i(A+Z_t)} \geq 2\log(n)\right)$$
$$\leq n\left(\mathbf{P}(\mathbf{E})+\mathbf{P}(\|Z_{1/n}-Z_0\| \geq \log n)\right)$$

$$\leq 2ne^{-100n}$$

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$$\leq 2ne^{-100n}$$

This implies that with probability $(1-1/n^{10})(1-2ne^{-100n})$

$$\int_0^1 \sum_{i=2}^{n} \frac{dt}{\lambda_1(A+Z_t)-\lambda_i(A+Z_t)} dt = O(1/\log n)$$

which translates as:

$$\lambda_1(A+Z)-\lambda_1(A) = B_1 + o(1).$$

Proof of Theorem 1

Let A be a Hermitian matrix and let

$$\epsilon = \min_{j : j > 1} \frac{\lambda_1 - \lambda_j}{C \cdot (j-1) \log^3(n)}.$$

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Note that $\frac{1}{\epsilon}A$ satisfies the conditions from Corollary 1, hence:

$$\lambda_1\left(\frac{1}{\epsilon}A + Z\right) - \lambda_1\left(\frac{1}{\epsilon}A\right) = B_1 + o(1),$$

which implies Theorem 1 for $i = 1$. For general i, the proof is identical.

Observation

- *The power of the logarithm in Theorem 1 is not optimal. A straight-forward analysis of our proof revels that* $2 + \epsilon$ *is enough for ant* $\varepsilon > 0$.

Questions?