Metric subregularity of multifunctions and applications *

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Abstract. The metric subregularity of multifunctions is a key notion in Variational Analysis and Optimization. In this paper, we establish firstly a cretirion for metric subregularity of multifunctions between metric spaces, by using the strong slope. Next, we use a combination of abstract coderivatives and contingent derivatives to derive verifiable first order conditions ensuring the metric subregularity of multifunctions between Banach spaces. By using second order approximations of convex multifunctions, we establish a second order condition for the metric subregularity of mixed smooth-convex constraint systems, which generalizes a result established recently by Gfrerer in [7].

Key Words. Error bound Metric subregularity Coderivative Contingent derivative .

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1 Introduction

Let $F : X \rightrightarrows Y$ be a multifunction between metric spaces X and Y, which are endowed with metrics both denoted by $d(\cdot,\cdot)$. Recall that the mapping F is said to be *metrically regular* at some $\bar{x} \in X$ with respect to $\bar{y} \in F(\bar{x})$ if there exist $\tau > 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that

$$d(x, F^{-1}(y)) \le \tau d(y, F(x)) \quad \text{for all} \quad (x, y) \in U \times V. \tag{1}$$

Where, we use the standard notation $d(x, C) = \inf_{z\in C} d(x, z)$, with the convention that $d(x, C) = +\infty$ whenever C is empty, and $B(x, r)$ stands for the ball centered at x with radius r.

According to the long history of metric regularity there is an abundant literature on conditions ensuring this property. This concept goes back to the surjectivity of a linear continuous mapping in the Banach Open Mapping Theorem and to its extension to nonlinear operators known as the Lyusternik & Graves Theorem ([21], [14], see also [8] and [11]).

A weaker poperty of metric regularity, called *metric subregularity,* where the inequality (1) should hold only for fixed $\bar{y}$, i.e., the set-valued mapping F is said to be metrically subregular at $\bar{x}$ with respect to $\bar{y} \in F(\bar{x})$ (or, shortly at $(\bar{x}, \bar{y}) \in \text{gph } F$) if there exist $\tau > 0$ and a neighborhood U of $\bar{x}$ such that

$$d(x, F^{-1}(\bar{y})) \le \tau d(\bar{y}, F(x)) \quad \text{for all} \quad x \in U. \tag{2}$$

When F is metrically regular at $(\bar{x}, \bar{y})$ and $F^{-1}(\bar{y}) = \{\bar{x}\}$, we say that F is *strongly metrically regular* at $(\bar{x}, \bar{y})$. The metric subregularity of F at $(\bar{x}, \bar{y})$ is equivalent to the *calmness* of the inverse mapping $F^{-1} : Y \rightrightarrows X$ at $(\bar{y}, \bar{x})$ in the following sense: There exist $\tau > 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that

$$e(F^{-1}(y) \cap U, F^{-1}(\bar{y})) \le \tau d(y, \bar{y}),$$

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where $e(Q, P)$ denotes the *excess* of Q to P,

$$e(P, Q) := \inf\{\varepsilon > 0 : \quad Q \subseteq B(P, \varepsilon)\}, \quad B(P, \varepsilon) := \{x \in X : \ d(x, P) < \varepsilon\}.$$

This metric subregularity, which is closely related to the error bound property, plays a crucial role to study constrained optimization problems for which the constraints can be formulated in an inclusion associated to a set-valued mapping F : $0 \in F(x)$. Recently, This property 'is founded'? a large range of applications in different areas of Variational Analysis as well as Optimization, such as for example, the theory of optimality conditions; the subdifferential theory, penalty methods in mathematical progamming; the convergence analysis of algorithms for solving equations or inclusions (see, e.g., [6], [16], [17], [19], [24], [32]). Recently, in [7], Gfrerer has established a first order *point-based* criteria for the metric subregularity of set-valued mapping. This characterization is based on the so-called *limit set critical for metric subregularity*. Then, the author has given a second order characterization for smooth constraint systems of the form:

$$0 \in g(x) - C, \tag{3}$$

where, $g : X \to Y$ is a C^1- mapping and $C \subseteq Y$ is a closed convex set.

By using the *strong slope*, the first purpose is to establish a criterion for the metric subregularity of multifunctions acting on metric spaces. Based on this characterization, we then give a point-based first order sufficient condition based on the abstract coderivative for metric subregularity, that is sharper than the one established in [7]. We also show that, for mixed smooth-convex constraint systems:

$$0 \in g(x) - F(x), \tag{4}$$

where, $g : X \to Y$ is a C^1-mapping and $F : X \rightrightarrows Y$ is a closed convex multifunction, this sufficient condition is actually a necessary condition. Secondly, we give a second order condition for metric subregularity of mixed smooth-convex system (4). This second order condition is based on second order approximations of convex multifunctions, and it generalizes the one in [7].

2 Metric subregularity on metric spaces

Let X, Y be metric spaces and let $F : X \rightrightarrows Y$ be a closed multifunction between X and Y. For a given $(\bar{x}, \bar{y}) \in \text{gph } F$, consider the following inclusion :

$$\text{Find } x \in X \text{ such that } \quad \bar{y} \in F(x). \tag{5}$$

Denote by S the solution set of inclusion (5), i.e.,

$$S := F^{-1}(\bar{y}) = \{x \in X : \quad \bar{y} \in F(x)\}. \tag{6}$$

In general, the distance function from $\bar{y}$ to $F(x)$ does not lower semicontinuous. Instead of it, we use its lower semicontinuous envelope function $\varphi : X \to X$ defined by

$$\varphi(x) := \liminf_{u \to x} d(\bar{y}, F(u)), \quad x \in X. \tag{7}$$

Obviuously, one has

$$S = \{x \in X : \ \varphi(x) = 0\},$$

and the metric subregularity of F at $(\bar{x}, \bar{y})$ is equivalent to the *error bound property* of the function φ at $\bar{x}$, that is, there exist $\tau > 0$ and $\delta > 0$ such that

$$d(x, S) \leq \tau \varphi(x) \quad \text{for all } \ x \in B(\bar{x}, \delta). \tag{8}$$

Recall from De Giorgi, Marino & Tosques [9], that the strong slope $|\nabla|\varphi(x)$ of the function φ at $x \in \text{Dom}\,\varphi$ is the quantity defined by $|\nabla|\varphi|(x) = 0$ if x is a local minimum of φ, and

$$|\nabla|\varphi(x) = \limsup_{y\to x, y\neq x} \frac{\varphi(x)-\varphi(y)}{d(x,y)},$$

otherwise. For $x \notin \text{Dom}\,\varphi$, we set $|\nabla|\varphi(x) = +\infty$.

For a locally Lipschitz mapping $h : X \to Y$ at $\bar{x} \in X$, we use the notation $\text{lip}\,(h,\bar{x})$ to denote the Lipschitz modulus of h at $\bar{x}$, that is,

$$\text{lip}\,(h,\bar{x}) := \limsup_{x,z\to\bar{x}} \frac{d(h(x),h(z))}{d(x,z)}.$$

Theorem 1 *Let X be a completed metric space and let Y be a metric space. Let $F : X \rightrightarrows Y$ be a closed multifunction and let be given $(\bar{x},\bar{y}) \in \text{gph}\,F$.*

(i). For any $x \notin S = F^{-1}(y)$, one has

$$m(x)d(x,S) \le \varphi(x). \tag{9}$$

Here the quantity $m(x)$ is defined by

$$m(x) := \liminf_{\varepsilon\to 0} \left\{ |\nabla\varphi|(z) : \begin{array}{l} d(x,z) \le (1-\varepsilon)d(x,S), \\ \varphi(z) \le \varphi(x),\ \frac{\varphi(z)}{d(z,S)} \le \frac{\varphi(x)}{(1-\varepsilon)(x,S)} \end{array} \right\}. \tag{10}$$

As a result, if

$$\liminf_{\substack{x\to\bar{x},\ x\notin S \\ \frac{\varphi(x)}{d(x,\bar{x})}\to 0}} |\nabla|\varphi(x) := m > 0, \tag{11}$$

then there exist $\tau > 0$ and $\delta > 0$ such that

$$d(x,S) \le \tau\varphi(x) \quad \textit{for all}\ x \in B(\bar{x},\delta).$$

That is, F is metrically subregular at $(\bar{x},\bar{y})$.

(ii). Assume that Y is a Banach space. If

$$\liminf_{\substack{x\to\bar{x},\ x\notin S \\ \frac{\varphi(x)}{d(x,\bar{x})}\to 0}} |\nabla|\varphi(x) = 0, \tag{12}$$

then there exists a locally Lipschitz mapping $h : X \to Y$ with $h(\bar{x}) = 0$; $lip(h,\bar{x}) = 0$ such that $F + h$ fails to be metrically subregular at $(\bar{x},\bar{y})$.

Proof. (i). Given $x \notin S$, since φ is nonnegative on the whole space, then

$$\varphi(x) \le \inf_{u\in X} \varphi(u) + \varphi(x).$$

By the Ekeland variational principle, for any $\varepsilon \in (0,1)$, we can find $z \in X$ such that

$$\begin{array}{l} d(x,z) \le (1-\varepsilon)d(x,S),\quad \varphi(z) \le \varphi(x); \\ \varphi(z) \le \varphi(u) + \frac{\varphi(x)}{(1-\varepsilon)d(x,S)}d(z,u) \quad \forall u \in X. \end{array}$$

The later relation follows that

$$\frac{\varphi(z)}{d(z,S)} \le \frac{\varphi(x)}{(1-\varepsilon)(x,S)},$$

and

$$(1-\varepsilon)d(x,S)\limsup_{u\to z} \frac{\varphi(z)-\varphi(u)}{d(z,u)} \le \varphi(x).$$

Therefore, one obtains inequality (9), by the definition of $m(x)$.

For the second part, suppose that (11) is satisfied. For a given $\kappa \in (m^{-1}, +\infty)$, there exists $\delta > 0$ such that

$$|\nabla|\varphi(z) > \tau^{-1} \quad \text{for all} \;\; z \in B(\bar{x}, \delta) \setminus S; \; \frac{\varphi(z)}{d(z,\bar{x})} < \delta. \tag{13}$$

Let $x \in B(\bar{x}, \delta/2)$ be given. If $\varphi(z)/d(z,\bar{x}) \geq \delta$, then

$$d(x,S) \leq d(x,\bar{x}) \leq \varphi(z)/\delta.$$

Suppose that $\varphi(z)/d(z,\bar{x}) < \delta$. For $\varepsilon > 0$ sufficiently small, and for all $z \in X$ satisfying

$$d(z,S) \leq (1-\varepsilon)d(x,S); \;\; \frac{\varphi(z)}{d(z,S)} \leq \frac{\varphi(x)}{(1-\varepsilon)(x,S)},$$

one has

$$z \in B(\bar{x}, \delta) \setminus S \quad \text{and} \quad \frac{\varphi(z)}{d(z,\bar{x})} < \delta.$$

By the definition of $m(x)$, one has $m(x) \geq m$. Hence , by setting $\tau := \max\{\kappa, \delta^{-1}\}$, one obtains

$$d(x,S) \leq \tau\varphi(x) \quad \text{for all} \quad x \in B(\bar{x}, \delta/2),$$

which completes the proof of the first part .

(ii). Suppose now (12) is verified. We can find a sequence $\{x_n\} \subseteq X$ satisfying

$$x_n \to \bar{x}; \;\; \bar{y} \notin F(x_n); \;\; \varphi(x_n)/\|x_n - \bar{x}\| < n^{-2}; \;\; \text{and} \;\; |\nabla|\varphi(x_n) < n^{-1}/4 \; \forall n.$$

Set $t_n := d(x_n, \bar{x})$, and by pick subsequences if necessary, we can assume that $t_{n+1} < \frac{t_n}{4}$ for all n. For each n, there exists ρ_n with $\rho_n \in (0, \frac{t_n}{3n})$ and $\rho_n < n\varphi(x_n)/2$ such that

$$\varphi(x_n) - \varphi(x) < \frac{d(x,x_n)}{4n} \quad \forall x \in B(x_n, \rho_n). \tag{14}$$

For each index n, by the definition of the semicontinuous envelope φ, we can find $u_n \in B(x_n, \rho_n/2)$ and $v_n \in F(u_n)$ such that

$$|\|\bar{y} - v_n\| - \varphi(x_n)| < \frac{\rho_n}{2n}. \tag{15}$$

Then by (14), one has

$$d(\bar{y}, F(x)) \geq \|\bar{y} - v_n\| - \frac{\rho_n}{4n} - \frac{d(x,x_n)}{4n} \quad \text{for all} \;\; x \in B(x_n, \rho_n). \tag{16}$$

Let $h : X \to Y$ be a mapping defined by

$$h(x) = \sum_{n=1}^{\infty} - \max\{1 - 2t_n^{-1}d(x,u_n), 0\}^2 \left(v_n - \bar{y} - \frac{\rho_n(v_n - \bar{y})}{n\|v_n - \bar{y}\|} \right).$$

Since Y is assumed to be a Banach space, and

$$\begin{array}{l} \sum_{n=1}^{\infty} \max\{1 - 2t_n^{-1}d(x,u_n), 0\}^2 \left\| v_n - \bar{y} - \frac{\rho_n(v_n - \bar{y})}{n\|v_n - \bar{y}\|} \right\| \\ \sum_{n=1}^{\infty}(\|v_n - \bar{y}\| - \rho_n/2n) \leq \sum_{n=1}^{\infty} t_n n^{-2} < \infty, \end{array} \tag{17}$$

then h is well defined on the whole space X.

For each n, for all $x \in B(u_n, \rho_n/2)$, and for any index $k \neq n$, if $k > n$ then

$$d(x,u_k) \geq d(\bar{x},x_n) - d(\bar{x},x_k) - d(x,x_n) - d(x_k,u_k) > t_n - t_k - \rho_n - \rho_k/2 > t_k/2;$$

otherwise, similarly one also has

$$d(x,u_k) \geq t_k - t_n - \rho_n - \rho_k/2 > t_k/2.$$

Consequently,

$$2t_k^{-1}d(x,u_k) \geq 1 \quad \text{for all } k \neq n; \ \text{ all } x \in B(u_k,\rho_k/2).$$

Hence,

$$h(x) = -\max\{1-2t_n^{-1}d(x,u_n),0\}^2\left(v_n - \bar{y} - \frac{\rho_n(v_n-\bar{y})}{n\|v_n-\bar{y}\|}\right) \text{ for all } x \in B(u_n,\rho_n/2);$$

moreover, $h(\bar{x}) = 0$, by the definition of (t_n). It implies that

$$d(\bar{y},F(u_n)+h(u_n)) \leq \|\bar{y} - v_n - h(u_n)\| = \rho_n/n. \tag{18}$$

On the other hand, for any $x \in B(u_n,\delta_n/2)$, by relations (15) and (16), one has

$$d(\bar{y},F(x)+h(x)) \quad \geq d(\bar{y},F(x)) - \|h(x)\| \geq \rho_n/n - \rho_n/4n - d(x,x_n)/4n \geq \rho_n/2n > 0.$$

This shows that $x \notin F^{-1}(\bar{y})$ for all $x \in B(u_n,\rho_n/2)$, that is, $d(u_n,F^{-1}(\bar{y})) \geq \rho_n/2$. This together with relation (18) shows that $F+h$ is not metrically regular at $(\bar{x},\bar{y})$.

To complete the proof, we demonstrate that h is locally Lipschitz at $\bar{x}$ with $\operatorname{lip}(h,\bar{x}) = 0$. Let $\varepsilon > 0$ be arbitrarily given. Since $\sum_{n=1}^{\infty} t_n^{-1}\|v_n - \bar{y}\| \leq \sum_{n=1}^{\infty} n^{-2} < \infty$, then there exists an index N such that $\sum_{n=N+1}^{\infty} t_n^{-1}\|v_n\| < \varepsilon/4$. By noting he functions $2t_n^{-1}d(\cdot,u_n)$ are Lipschitz with modulus $2t_n^{-1}$, respectively, and $2t_n^{-1}d(\bar{x},u_n) \geq 1$, for all n, we can find $\delta > 0$ such that for all $n = 1,...,N$, all $x,y \in B(\bar{x},\delta)$, one has

$$|\max\{1-2t_n^{-1}d(x,u_n),0\}^2 - \max\{1-2t_n^{-1}d(y,u_n),0\}^2| \leq \frac{\varepsilon}{2N}d(x,y).$$

Therefore, we have the follwing estimates

$$\begin{aligned}\|h(x)-h(y)\| &\leq \textstyle\sum_{n=1}^{N} |\max\{1-2t_n^{-1}d(x,u_n),0\}^2 - \max\{1-2t_n^{-1}d(y,u_n),0\}^2\|v_n-\bar{y}\|+ \\ &+ \textstyle\sum_{n=N+1}^{\infty} |\max\{1-2t_n^{-1}d(x,u_n),0\}^2 - \max\{1-2t_n^{-1}d(y,u_n),0\}^2|\|v_n-\bar{y}\| \\ &\leq \varepsilon d(x,y)/2 + 2d(x,y)\textstyle\sum_{n=N+1}^{\infty} t_n^{-1}\|v_n-\bar{y}\| < \varepsilon d(x,y).\end{aligned}$$

As ε is arbitrarily small, we conclude $\operatorname{lip}(h,\bar{x}) = 0$. The proof is completed. □

3 First order characterizations of the metric subregularity

Let X be a Banach space. We use the symbol ∂ to denote any *abstract subdifferentials*, that is any set-valued mapping which associates to every function defined on X and every $x \in X$ the set $\partial f(x) \subset X^\star$ (possibly empty), in such a way that

(C1) If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a l.s.c convex function, then ∂f coincides with the Fenchel-Moreau-Rockafellar subdifferential:

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \quad \forall y \in X\};$$

(C2) $\partial f(x) = \partial g(x)$ if $f(y) = g(y)$ for all y in a neighborhood of x.

(C3) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c function and $g : X \to \mathbb{R}$ be convex and Lipschitz. If $f + g$ attains a local minimum at x_0, then for any $\varepsilon > 0$, there exist $x_1, x_2 \in x_0 + \varepsilon B_X$, $x_1^* \in \partial f(x_1)$, $x_2^* \in \partial g(x_2)$, such that $|f(x_1) - f(x_0)| < \varepsilon$ and $\|x_1^* + x_2^*\| < \varepsilon$.

It is well known that the class of abstract subdifferentials includes Fréchet subdifferentials in Asplund spaces, viscosity subdifferentials in smooth Banach spaces as well as the Ioffe and the Clarke-Rockafellar subdifferentials in Banach spaces. For a closed subset C of X, the normal cone to C with respect to a subdifferential operator ∂ at $x \in C$ is defined by $N_\partial(C,x) = \partial\delta_C(x)$, where δ_C is the indicator function of C given by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise and we assume here that $\partial\delta_C(x)$ is *a cone* for any closed subset C of X.

Let X, Y be Banach spaces, and let ∂ be a subdifferential on $X \times Y$. Let $F : X \rightrightarrows Y$ be a closed multifunction (graph-closed) and let $(\bar{x}, \bar{y}) \in \mathrm{gph}F$. The multifunction $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x},\bar{y})(y^*) = \{x^* \in X^* : (x^*, -y^*) \in N_\partial(\mathrm{gph}F, (\bar{x},\bar{y}))\}$$

is called *the $\partial-$coderivative* of F at $(\bar{x}, \bar{y})$.

Let $F : X \rightrightarrows Y$ be a closed multifunction and as above, for given $(\bar{x}, \bar{y}) \in$ gph F, denote by $\varphi(x)$, $x \in X$ the lower semicontinuous envelope of the function $x \mapsto d(\bar{y}, F(x))$, and set

$$m := \liminf_{\substack{x\to\bar{x},\ x\notin S \\ \frac{\varphi(x)}{d(x,\bar{x})}\to 0}} |\nabla|\varphi(x), \tag{19}$$

We next establish a characterization of the metric subregularity by using abstract coderivatives of multifunctions.

Theorem 2 *Let X, Y be Banach spaces and let ∂ is a subdifferential operator on $X \times Y$. Suppose that $F : X \rightrightarrows Y$ be a closed multifunction between X and Y. For given $(\bar{x}, \bar{y}) \in$ gph F, if*

$$\liminf_{\substack{t\downarrow 0^+,\eta\downarrow 0^+ \\ v\to 0}} \inf\left\{ \|x^*\| : \begin{array}{l} (\bar{x}+tu, \bar{y}+tv) \in gphF,\ x^* \in D^*F(\bar{x}+tu,\bar{y}+tv)(y^*),\ \|y^*\| = 1, \\ \|u\| = 1,\ x+tu \notin F^{-1}(\bar{y}),\ t\|v\| \le (1+\eta)d(y, F(\bar{x}+tu)), \\ |\langle y^*, v\rangle - \|v\|| < \eta\|v\| \end{array} \right\} > 0, \tag{20}$$

then F is metrically subregular at $(\bar{x}, \bar{y})$.

This theorem follows directly from Theorem 1 and the following lemma, which gives an estimation for the quatity m by using the abstract subdifferential operator on $X \times Y$.

Lemma 3 *Let ∂ be a subdifferential on $X \times Y$. Then one has*

$$m \ge \liminf_{\substack{t\downarrow 0^+,\eta\downarrow 0^+ \\ v\to 0}} \inf\left\{ \|x^*\| : \begin{array}{l} (\bar{x}+tu, \bar{y}+tv) \in gphF,\ x^* \in D^*F(\bar{x}+tu,\bar{y}+tv)(y^*),\ \|y^*\| = 1, \\ \|u\| = 1,\ x+tu \notin F^{-1}(\bar{y}),\ t\|v\| \le (1+\eta)d(y, F(\bar{x}+tu)), \\ |\langle y^*, v\rangle - \|v\|| < \eta\|v\| \end{array} \right\}. \tag{21}$$

Proof. Let $\{x_n\} \subseteq X$ be such that

$$x_n \to \bar{x},\ x_n \notin F^{-1}(\bar{y}),\quad \lim_{n\to\infty} \frac{\varphi(x_n)}{\|x_n - \bar{x}\|} = 0,\quad \text{and} \quad \lim_{n\to\infty} |\nabla|\varphi(\cdot, y)(x_n) := m.$$

Take a sequence of positives (ε_n) such that $\varepsilon_n \in (0, \varphi(x_n))$ and $\varepsilon_n/\varphi(x_n) \to 0$. Then for each n, there is $\eta_n \in (0, \varepsilon_n)$ with $2\eta_n + \varepsilon_n < \varphi(x_n))$ and $1 - (m + \varepsilon_n + 2)\eta_n > 0$ such that $d(\bar{y}, F(z)) \ge \varphi(x_n)(1 - \varepsilon_n)$, $\forall z \in B(x_n, 4\eta_n)$ and

$$m + \varepsilon_n \ge \frac{\varphi(x_n) - \varphi(z)}{\|x_n - z\|} \quad \text{for all } z \in \bar{B}(x_n, \eta_n).$$

Equivalently,

$$\varphi(x_n) \le \varphi(z) + (m + \varepsilon_n)\|z - x_n\| \quad \text{for all } z \in \bar{B}(x_n, \eta_n).$$

Take $z_n \in B(x_n, \eta_n^2/4)$, $w_n \in F(z_n)$ such that $\|\bar{y} - w_n\| \leq \varphi(x_n) + \eta_n^2/4$. Then,

$$\|\bar{y} - w_n\| \leq \varphi(z) + (m + \varepsilon_n)\|z - x_n\| + \eta^2/4 \,\forall z \in \bar{B}(x_n, \eta_n).$$

Therefore,

$$\|\bar{y}-w_n\| \leq \|\bar{y}-w\| + \delta_{\mathrm{gph}F}(z, w) + (m+\varepsilon_n)\|z-z_n\| + (m+\varepsilon_n+1)\eta_n^2/4 \quad \forall (z, w) \in \bar{B}(x_n, \eta_n)\times Y.$$

By applying the Ekeland variational principle to the function

$$(z, w) \mapsto \|\bar{y} - w\| + \delta_{\mathrm{gph}F}(z, w) + (m + \varepsilon)\|z - z_n\|$$

on $\bar{B}(x_n, \eta_n) \times Y$, we can select $(z_n^1, w_n^1) \in (z_n, w_n) + \frac{\eta_n}{4} B_{X\times Y}$ with $(z_n^1, w_n^1) \in \mathrm{gph}F$ such that

$$\|y - w_n^1\| \leq \|y - w_n\| (\leq \varphi(x_n) + \eta^2/4); \tag{22}$$

and that the function

$$(z, w) \mapsto \|\bar{y} - w\| + \delta_{\mathrm{gph}F}(z, w) + (m + \varepsilon)\|z - u\| + (m + \varepsilon + 1)\eta\|(z, w) - (z_n^1, w_n^1)\|$$

attains a minimum on $\bar{B}(x_n, \eta_n) \times Y$ at (z_n^1, w_n^1). Hence, by (C3), we can find

$$w_n^2 \in B_Y(w_n^1, \eta_n);\ (z_n^3, w_n^3) \in B_{X\times Y}((z_n^1, w_n^1), \eta_n) \cap \mathrm{gph}F;$$

$$w_n^{2*} \in \partial\|\bar{y} - \cdot\|(w_n^2);\ (z_n^{3*}, -w_n^{3*}) \in N(\mathrm{gph}F, (z_n^3, w_n^3))$$

satisfying

$$\|w_n^{2*} - w_n^{3*}\| < (m + \varepsilon_n + 2)\eta_n \quad \text{and} \quad \|z_n^{3*}\| \leq m + \varepsilon + (m + \varepsilon_n + 2)\eta_n. \tag{23}$$

Since $w_n^{2*} \in \partial\|\bar{y} - \cdot\|(w_n^2)$ (note that $\|\bar{y} - w_n^2\| \geq \|\bar{y} - w_n\| - \|w_n^2 - w_n\| \geq \varphi(x_n) - \varepsilon_n - 2\eta_n > 0$), then $\|w_n^{2*}\| = 1$ and $\langle w_n^{2*}, w_n^2 - \bar{y}\rangle = \|\bar{y} - v_2\|$. Thus, from the first relation in (23), it follows that

$$\|w_n^{3*}\| \geq \|w_n^{2*}\| - (m + \varepsilon_n + 2)\eta_n = 1 - (m + \varepsilon_n + 2)\eta_n;$$

$$\|w_n^{3*}\| \leq \|w_n^{2*}\| + (m + \varepsilon_n + 2)\eta_n = 1 + (m + \varepsilon_n + 2)\eta_n.$$

Set

$$t_n = \|z_n^3 - \bar{x}\|;\ \ u_n = (z_n^3 - \bar{x})/t_n;\ \ v_n = (w_n^3 - \bar{y})/t_n,$$

and

$$y_n^* = w_n^{3*}/\|w_n^{3*}\|;\ \ x_n^* = z_n^{3*}/\|w_n^{3*}\|.$$

Since

$$\varphi(x_n) - \varepsilon_n \leq d(\bar{y}, F(\bar{y} + t_n u_n)) \leq t_n\|v_n\| \leq \|\bar{y} - w_n^1\| + \eta_n \leq \varphi(x_n) + \eta_n^2/4 + \eta_n;$$

and

$$t_n = \|z_n^3 - \bar{x}\| \geq \|x_n - \bar{x}\| - \|w_n^3 - x_n\| \geq \|x_n - \bar{x}\| - \eta_n^2/4 - 3\eta_n/4,$$

then

$$\|v_n\| \leq \frac{\varphi(x_n) + \eta_n^2/4 + \eta_n}{\|x_n - \bar{x}\| - \eta_n^2/4 - 3\eta_n/4}.$$

As $\varphi(x_n)/\|x_n - \bar{x}\| \to 0$ as well as $\eta_n/\varphi(x_n) \to 0$, one obtains

$$\lim_{n\to\infty} v_n = 0. \tag{24}$$

Next one has $x_n^* \in D^*F(\bar{x} + t_n u_n, \bar{y} + t_n v_n)(y_n^*)$ with $\|y_n^*\| = 1$ and by the second relation of (23), one derives that

$$\|x_n^*\| = \|z_n^{3*}\|/\|w_n^{3*}\| \leq \frac{m + \varepsilon_n + (m + \varepsilon_n + 2)\eta_n}{1 - (m + \varepsilon_n + 2)\eta_n}. \tag{25}$$

On the other hand,

$$
\begin{aligned}
t_n\langle y_n^*, v_n\rangle &= \tfrac{\langle w_n^{2*}, w_n^2-\bar{y}\rangle+\langle w_n^{2*}, w_n^3-w_n^2\rangle+\langle w_n^{3*}-w_n^{2*}, w_n^3-\bar{y}\rangle}{\|w^{3*}\|} \\
&\geq \tfrac{\|\bar{y}-w_n^2\|-2\eta_n-(m+\varepsilon_n+2)\eta_n\|\bar{y}-w_n^3\|}{1+(m+\varepsilon_n+2)\eta_n} \geq \tfrac{(1-(m+\varepsilon+2)\eta_n)t_n\|v_n\|-3\eta_n}{1-(m+\varepsilon+2)\eta_n}.
\end{aligned}
$$

Hence,

$$
|\langle y_n^*, v_n\rangle - \|v_n\|| \leq \frac{2(m+\varepsilon_n+2)\eta_n\|v_n\|+3\eta_n}{(1+(m+\varepsilon_n+2)\eta_n)t_n}. \tag{26}
$$

Since $\varepsilon_n/\varphi(x_n) \to 0$ and $\eta_n \in (0, \varepsilon_n)$, and

$$
\frac{\eta_n}{t_n v_n} \leq \frac{\eta_n}{\varphi(x_n) - \eta_n^2/4 - 2\eta_n} \to 0 \ \text{ as } \ n \to \infty,
$$

by combining relations (24), (25), (26), we complete the proof. $\square$

Let us recall from [7] the notion of *limit set critical* for metric subregularity of a set-valued mapping. The definition in [7] is stated for the Frechét subdifferential, however it is also valid for any subdifferential operator.

Definition 4 *Let X, Y be Banach spaces and let ∂ is a subdifferential operator on $X \times Y$. Let $F : X \rightrightarrows Y$ be a closed multifunction. For a given $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, the limit set critical for metric subregularity of F at $(\bar{x}, \bar{y})$ denote by $CrF(\bar{x}, \bar{y})$ is defined as the set of all $(v, x^*) \in Y \times X^*$ such that there exist sequences $(t_n) \downarrow 0$, $(v_n, x_n^*) \to (v, x^*)$, $(u_n, y_n^*) \in \mathcal{S}_X \times \mathcal{S}_{Y^*}$ with $x_n^* \in D^*F(\bar{x} + t_n u_n, \bar{y} + t_n v_n)(y_n^*)$. Where, $\mathcal{S}_X$ stands for the unit sphere in X.*

In [7], the author has established that $(0, 0) \notin \mathrm{Cr}F(\bar{x}, \bar{y})$ is a sufficient condition for the metric subregularity of F at $(\bar{x}, \bar{y})$. Theorem 2 permit us to obtain a sufficient condition which is weaker the one mentioned in [7] . We present a strict version of Definition 4.

Definition 5 *For a closed multifunction $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, the strict limit set of critical for the metric subregularity of F at $(\bar{x}, \bar{y})$ denote by $SCrF(\bar{x}, \bar{y})$ is defined as the set of all $(v, x^*) \in Y \times X^*$ such that there exist sequences $(t_n) \downarrow 0$, $(v_n, x_n^*) \to (v, x^*)$, $(u_n, y_n^*) \in \mathcal{S}_X \times \mathcal{S}_{Y^*}$ with $x_n^* \in D^*F(\bar{x} + t_n u_n, \bar{y} + t_n v_n)(y_n^*)$, $\bar{y} \notin F(\bar{x} + t_n u_n)$ $(\forall n)$, and $\frac{\langle y_n^*, v_n\rangle}{\|v_n\|} \to 1$.*

Obviously, $\mathrm{SCr}F(\bar{x}, \bar{y}) \subseteq \mathrm{Cr}F(\bar{x}, \bar{y})$. Theorem 2 yields immediately the following corollary.

Corollary 6 *Let X, Y be Banach spaces and let ∂ be a subdifferential operator on $X \times Y$. Let $F : X \rightrightarrows Y$ be a closed multifunction between X and Y and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. If $(0, 0) \notin SCrF(\bar{x}, \bar{y})$ then F is metrically subregular at $(\bar{x}, \bar{y})$.*

Let us consider the mixed smooth- convex inclusion of the form:

$$
0 \in g(x) - F(x) := G(x), \tag{27}
$$

where $g : X \to Y$ is a mapping of C^1 class around $\bar{x} \in G^{-1}(0)$; $F : X \rightrightarrows Y$ is a closed convex multifunction. In this case, the condition $(0, 0) \notin \mathrm{SCr}G(\bar{x}, 0)$ is also a necessary condition for the metric subregularity, as showed in the following proposition.

Proposition 7 *With the assumptions as above, the multifunction $G := g - F$ is metrically regular at $(\bar{x}, 0)$ if and only if $(0, 0) \notin SCrG(\bar{x}, 0)$.*

Proof. It suffices to prove the necessary part. Suppose that G is metrically subregular at $(\bar{x}, 0)$. There are $\tau > 0$, $\delta > 0$ such that

$$d(x, G^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x)) \quad \forall x \in B(\bar{x}, \delta) \tag{28}$$

Take $\varepsilon \in (0, \tau^{-1})$, since g is of C^1 class around $\bar{x}$, there is, say the same δ as above, such that

$$\|g(x_1) - g(x_2) - Dg(x_1)(x_1 - x_2)\| \leq \varepsilon \|x_1 - x_2\| \ \ \forall x_1, \, x_2 \in B(\bar{x}, \delta). \tag{29}$$

Let sequences (t_n), (u_n), (v_n), (x_n^*), (y_n^*) such that $(t_n) \downarrow 0$; $(u_n, y_n^*) \in \mathcal{S}_X \times \mathcal{S}_{Y^*}$; $x_n^* \in D^*G(\bar{x} + t_n u_n, \bar{y} + t_n v_n)(y_n^*)$; $\bar{y} \notin G(\bar{x} + t_n u_n)$ $(\forall n)$; $(v_n) \to 0$ and $\frac{\langle y_n^*, v_n \rangle}{\|v_n\|} \to 1$. We will prove that (x_n^*) does not converge to 0. Indeed, pick a sequence $(\varepsilon_n) \downarrow 0$ with $\tau_n(1 + \varepsilon_n) < \delta/2$, by assuming without loss of generality $t_n \in (0, \delta/2)$, for each n, there exists $z_n \in G^{-1}(\bar{y})$ such that

$$\|z_n - \bar{x} - t_n u_n\| \leq \tau(1 + \varepsilon_n) d(\bar{y}, G(\bar{x} + t_n u_n)) \leq \tau(1 + \varepsilon_n) t_n \|v_n\|. \tag{30}$$

Consequently, $z_n \in B(\bar{x}, \delta)$, for all n. By the standard sum rule for the coderivative (see, e.g., [22]),

$$x_n^* = Dg(\bar{x} + t_n u_n)^*(y_n^*) + z_n^*, \ \text{ with } \ z_n^* \in D^*F(\bar{x} + t_n u_n, g(\bar{x} + t_n u_n) - t_n v_n)(y_n^*).$$

Since F is a convex multifunction, then

$$\langle z_n^*, z - \bar{x} - t_n u_n \rangle + \langle y_n^*, w - g(\bar{x} + t_n u_n) + t_n v_n \rangle \leq 0 \ \ \forall (z, w) \in \text{gph } F.$$

By taking $(z, w) := (z_n, g(z_n))$ into account , one has

$$\langle x_n^*, \bar{x} + t_n u_n - z_n \rangle \geq -\langle y_n^*, g(\bar{x} + t_n u_n) - g(z) - Dg(\bar{x} + t_n u_n)(\bar{x} + t_n u_n - z_n) \rangle + t_n \langle y_n^*, v_n \rangle.$$

Therefore, from relations (29), (30), one obtains

$$\tau(1 + \varepsilon_n) t_n \|v_n\| \|x_n^*\| \geq \langle x_n^*, z_n - \bar{x} - t_n u_n \rangle \geq t_n \langle y_n^*, v_n \rangle - \varepsilon \tau (1 + \varepsilon_n) t_n \|v_n\|.$$

This follows that $\liminf_{n \to \infty} \|x_n^*\| \geq (1 - \tau\varepsilon) > 0$, which ends the proof. □

For this smooth-convex inclusion, as showed in the following proposition, which generalizes Proposition 3.9 in [7], the condition $(0, 0) \notin \text{Cr}G(\bar{x}, 0)$ is also actually a sufficient and necessary condition for either the strong metric subregularity or the the metric regularity of G at $(\bar{x}, 0)$.

Proposition 8 *For the mixed smooth-convex inclusion (27) and for given $\bar{x} \in G^{-1}(0)$, $(0, 0) \notin CrG(\bar{x}, 0)$ if and only if G is either strongly metrically subregular or metrically regular at $(\bar{x}, 0)$.*

Proof. The necessary part was proved in ([7], Proposition 3.8), it is valid for any closed multifunction G. For the sufficient part, suppose that $(0, 0) \notin \text{Cr}G(\bar{x}, 0)$, and assume that G is not strongly metrically subregular at $(\bar{x}, 0)$. Then there exists a sequence $(z_n) \to \bar{x}$ with $z_n \in G^{-1}(0)$, $z_n \neq \bar{x}$. Assume to contrary that G is not metrically regular at $(\bar{x}, 0)$. By the coderivative characterization of the metric regularity (see, e.g., [4]), we can find sequences $(x_n, y_n) \to (\bar{x}, 0)$ with $(x_n, y_n) \in \text{gph } G$; $x_n^* \in D^*G(x_n, y_n)(y_n^*)$ with $\|y_n^*\| = 1$ such that $\|x_n^*\| \to 0$. Then, there exists $z_n^* \in D^*F(x_n, g(x_n) - y_n)(-y_n^*)$ such that

$$x_n^* = Dg(x_n)^*(y_n^*) + z_n^*.$$

By taking a subsequences if necessary, we can assume that $z_n \neq x_n$ for all n, and by setting $r_n := \|z_n - x_n\| > 0$, and that

$$\|x_n^*\| < 1/(3n^2); \ \ \|y_n\| < r_n/(3n^2); \ \ \|g(z_n) - g(x_n) - Dg(\bar{x})(z_n - x_n)\| < r_n/(3n^2).$$

Since $z_n^* \in D^*F(x_n, g(x_n) - y_n)(-y_n^*)$, then

$$\langle z_n^*, z_n - x_n\rangle + \langle y_n^*, w - g(x_n) + y_n\rangle \leq 0 \;\; \forall (z,w) \in \text{gph } F. \tag{31}$$

Therefore,

$$\begin{array}{ll} \langle z_n^*, z_n\rangle + \langle y_n^*, g(z_n)\rangle & = \langle z_n^*, x_n\rangle + \langle y_n^*, g(x_n) - y_n\rangle \\ & +\langle x_n^*, z_n - x_n\rangle + \langle y_n^*, g(z_n) - g(x_n) - Dg(x_n)(z_n - x_n)\rangle + \langle y_n^*, y_n\rangle \\ & > \langle z_n^*, x_n\rangle + \langle y_n^*, g(x_n) - y_n\rangle - r_n/n^2. \end{array} .$$

The later relations imply that

$$\langle z_n^*, -z_n\rangle + \langle y_n^*, -g(z_n)\rangle < \inf_{(z,w)\in \text{gph } F} \{\langle z_n^*, -z\rangle + \langle y_n^*, -w\rangle\} + r_n/n^2.$$

By the Ekeland variational principle, we can find $(\bar{x}_n, g(\bar{x}_n) - \bar{y}_n) \in B((z_n, g(z_n)), r_n/n) \cap$ gph F such that

$$\langle z_n^*, z - \bar{x}_n\rangle + \langle y_n^*, w - g(\bar{x}_n) - \bar{y}_n \leq (\|z - \bar{x}_n\| + \|w - g(\bar{x}_n\|) + \bar{y}_n\|)/n$$

for all $(z,w) \in$ gph F. That is,

$$(z_n^*, y_n^*) \in N(\text{gph } F, (\bar{x}_n, g(\bar{x}_n)) - \bar{y}_n) + \frac{1}{n} B_{X^*\times Y^*}.$$

Therefore, there exists $(\bar{z}_n^*, \bar{y}_n^*)$ with $\bar{z}_n^* \in D^*F(\bar{x}_n, g(\bar{x}_n) - \bar{y}_n)(\bar{y}_n^*)$ such that

$$\|(z_n^*, y_n^*) - (\bar{z}_n^*, \bar{y}_n^*)\| \leq 1/n.$$

Set

$$w_n^* := \bar{y}_n^*/\|\bar{y}_n^*\|; \bar{x}_n^* := Dg(\bar{x}_n)^*(w_n^*) + \bar{z}_n^*/\|\bar{y}_n^*\| \in D^*G(\bar{x}_n, \bar{y}_n)(w_n^*);$$

$$\bar{x}_n := \bar{x} + t_n u_n; \quad \bar{y}_n := t_n v_n.$$

Then, it is easy to check from the above relations that

$$\bar{x}_n^* \to 0; \quad t_n \to 0 \quad \text{and} \;\; v_n \to 0,$$

which follow that $(0,0) \in \text{Cr}G(\bar{x}, 0)$. This completes the proof. □

4 Second order characterizations of the metric subregularity

Let X be a normed space, $S \subset X$ and $\bar{x} \in S$. The *tangent cone* $T(S,\bar{x})$ of S at $\bar{x}$ is defined by

$$T(S,\bar{x}) := \{v \in X : \;\; \exists (t_n) \downarrow 0, \;\; \exists (x_n) \subseteq S, \; x_n \to \bar{x}, \;\; v = \lim (x_n - \bar{x})/t_n\}.$$

We say that S is *first order tangentiable* at $\bar{x}$ if for every $\varepsilon > 0$, there is a neighborhood U of the origin such that

$$(S - \bar{x}) \cap U \subset [T(S;\bar{x})]_\varepsilon$$

where $[T(S;\bar{x})]_\varepsilon := \{x \in X \;:\; d(x/\|x\|, T(S;\bar{x})) < \varepsilon\} \cup \{0\}$ is the ε-conic neighborhood of $T(S;\bar{x})$.

We note that in a finitely dimensional space, every nonempty set is tangentiable at any point (see [12]).

We also recall that the contingent derivative of a multifunction $F : X \rightrightarrows Y$ at $(x,y) \in$ gph F , denoted by $\text{C}F(x,y)$, is a set valued map from X to Y defined by

$$\text{C}F(x,y)(u) := \{v \in Y \;:\; (u,v) \in T(\text{gph } F, (x,y))\}.$$

Lemma 9 *Let $S \subset X$, $\{x_n\} \subset S \setminus \{\bar{x}\}$ and $\bar{x} \in S$. Assume that S is tangentiable at $\bar{x}$, $T(S,\bar{x})$ is locally compact at the origin and $\{x_n\}$ converges to $\bar{x}$. Then the sequence $\{\frac{x_n-\bar{x}}{\|x_n-\bar{x}\|}\}$ has a convergent subsequence.*

Proof. Since S is tangentiable at $\bar{x}$, by passing to a subsequence if necessary, we may assume that

$$d(\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}, T(S;\bar{x})) < \frac{1}{n},\ \forall n.$$

Then for every n, there exists $u_n \in T(S,\bar{x})$ such that

$$\left\| \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} - u_n \right\| < \frac{1}{n}.$$

Since $T(S,\bar{x})$ is a cone and locally compact at 0, there exists a subsequence $\{u_{k_n}\} \subset \{u_n\}$ converges to some point u. Then $\{\frac{x_{k_n}-\bar{x}}{\|x_{k_n}-\bar{x}\|}\}$ is also converges to u. □

We note that in an infinite dimensional space there exist first order tangentiable sets with their tangent sets are locally compact at the origin such that they are not contained in any finitely dimensional subspace. For instant let H be a infinite dimensional Hilbert space with a countable base $\{e_1, e_2, \dots, e_n, \dots\}$ such that

$$\langle e_i, e_j \rangle = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Denote

$$S := \{e_n : \ n = 2, 3, ...\} \cup \{te_1 \ : \ t \geq 0\}.$$

Then $T(S;0) = \{te_1 \ : \ t \geq 0\}$. One can see that S is first order tangentiable at 0, $T(S;0)$ is locally compact at 0 and no finitely dimensional subspace of H contains S.

Now let $G : X \rightrightarrows Y$ be a multifunction. The following two propositions, giving the metric subregularity of the contingent derivative of a metrically regular multifunction, are generalizations of Proposition 2.1 in [7]. The first is for a general multifunctions. Here, instead of the finite dimensional assumption on X, we assume that $G^{-1}(\bar{y})$ is tangentiable at $\bar{x}$ and $T(G^{-1}(\bar{y}),\bar{x})$ is locally compact at the origin.

Proposition 10 *Let $G : X \rightrightarrows Y$ be a set-valued map from X to another normed space Y. Assume that G is metrically subregular at $(\bar{x},\bar{y}) \in \text{gph } G$ with some modulus κ. If $G^{-1}(\bar{y})$ is tangentiable at $\bar{x}$ and $T(G^{-1}(\bar{y}),\bar{x})$ is locally compact at the origin then the contingent derivative $CG(\bar{x},\bar{y})$ is metrically subregular at $(0,0)$ with modulus κ.*

Proof. The proof of this proposition is very direct from definition, which is similar to that of Proposition 2.1. in [7] except some changes concern to infinitely dimensional property of the space X. Let $u \in X$ and $\epsilon > 0$ be arbitrary. Choose $v \in CG(\bar{x},\bar{y})(u)$ such that $\|v\| < d(0, CG(\bar{x},\bar{y})(u)) + \epsilon$. Since $(u,v) \in T(\text{gph } G, (\bar{x},\bar{y}))$ there are sequences $t_n \downarrow 0$ and $(u_n, v_n) \to (u,v)$ such that $(\bar{x} + t_n u_n, \bar{y} + t_n v_n) \in \text{gph } G$. We have

$$d(\bar{x} + t_n u_n, G^{-1}(\bar{y})) \leq \kappa d(\bar{y}, G(\bar{x} + t_n u_n)) \leq \kappa t_n \|v_n\| \leq \kappa t_n [d(0, CG(\bar{x},\bar{y})(u)) + \epsilon]$$

for k sufficiently large. Then we can find $x_n \in G^{-1}(\bar{y})$ such that $x_n - \bar{x} - t_n u_n \in \kappa t_n [d(0, CG(\bar{x},\bar{y})(u)) + 2\epsilon]\mathcal{B}_X$. This implies the boundedness of the sequence $\{\frac{\|x_n-\bar{x}\|}{t_n}\}$. Hence we may assume that $\{\frac{\|x_n-\bar{x}\|}{t_n}\}$ converges to some α. On otherhand, by Lemma 9, we also may assume that the sequence $\{\frac{x_n-\bar{x}}{\|x_n-\bar{x}\|}\}$ converges to some point a. Set $\bar{u}_n := \frac{x_n-\bar{x}}{t_n}$. Then $\bar{u}_n \in u_n + \kappa[d(0, CG(\bar{x},\bar{y})(u)) + 2\epsilon]\mathcal{B}_X$ and $\bar{u}_n \to \bar{u} := \alpha a$. Therefore $\bar{u} \in u + \kappa[d(0, CG(\bar{x},\bar{y})(u)) + 2\epsilon]\mathcal{B}_X$. Since $\bar{y} \in G(\bar{x} + t_n \bar{u}_n)$ we have $0 \in CG(\bar{x},\bar{y})(\bar{u})$. Thus

$$d(u, CG(\bar{x},\bar{y})^{-1}(0)) \leq \|u - \bar{u}\| \leq \kappa[d(0, CG(\bar{x},\bar{y})(u)) + 2\epsilon].$$

Taking $\epsilon \to 0$ we complete the proof. $\square$

Now consider again the following mixed constraint system:

$$0 \in g(x) - F(x), \tag{32}$$

where, as the preceding section, $F : X \rightrightarrows Y$ is a closed and convex set-valued map and $g : X \to Y$ is assumed to be continuously differentiable in a neighbourhood of a point $\bar{x} \in (g-F)^{-1}(0)$. Set $G(x) := g(x) - F(x)$ and $\mathcal{C} := CG(\bar{x},0)^{-1}(0) = \{u \in X \ : \ Dg(\bar{x})(u) \in CF(\bar{x}, g(\bar{x}))(u)\}$. Denote by B_X the closed unit ball in X.

Proposition 11 *For the mixed smooth-convex constraint system (32), and for a given $\bar{x} \in G^{-1}(0) := (g-F)^{-1}(0)$, if G is metrically subregular at $(\bar{x},0)$ and assume that X is reflexive, then $CG(\bar{x},0)$ is metrically subregular at $(0,0)$ with the same modulus as G.*

Proof. Suppose that G is metrically regular at $(\bar{x},0)$ with modulus κ. Firstly, note that for the mixed smooth-convex constraint system (32), one has

$$\mathrm{C}G(\bar{x},0)(u) = Dg(\bar{x})(u) - \mathrm{C}F(\bar{x}, g(\bar{x}))(u), \ \ u \in X.$$

As in the proof of Proposition 10, for given $u \in X$, $\varepsilon > 0$, take $v \in CG(\bar{x},0)(u)$ such that $\|v\| < d(0, CG(\bar{x},0)(u)) + \epsilon$, and we can find sequences $t_n \downarrow 0$ and $(u_n, v_n) \to (u,v)$ such that $(\bar{x} + t_n u_n, t_n v_n) \in \text{gph } G$. There exist then $x_n \in G^{-1}(\bar{y})$ such that

$$\|x_n - \bar{x} - t_n u_n\| \leq \kappa t_n [d(0, CG(\bar{x},0)(u)) + 2\epsilon].$$

By setting $\bar{u}_n := \frac{x_n - \bar{x}}{t_n}$, since (u_n) is bounded and X is reflexive, then by passing to a subsequence if necessary, we can assume that (u_n) weakly converges to some $\bar{u} \in X$. Therefore, $\left(\frac{g(\bar{x}+t_n u_n) - g(\bar{x})}{t_n}\right)$ also weakly converges to $Dg(\bar{x})(\bar{u})$. Since gph F is convex, then

$$\begin{aligned}(\bar{u}, Dg(\bar{x})(\bar{u})) \in \mathrm{cl}_w\mathrm{cone}(\text{gph } F - (\bar{x}, g(\bar{x}))) \quad &= \mathrm{clcone}(\text{gph } F - (\bar{x}, g(\bar{x}))) \\ &= T(\text{gph } F, (\bar{x}, g(\bar{x}))).\end{aligned}$$

Consequently, $\bar{u} \in \mathrm{C}G(\bar{x},0)^{-1}(0)$, and one has

$$d(u, \mathrm{C}G(\bar{x},0)^{-1}(0)) \leq \|u - \bar{u}\| \leq \kappa[d(0, CG(\bar{x},0)(u)) + 2\epsilon].$$

As $\varepsilon > 0$ is arbitrary, this completes the proof. $\square$

In what follows, we make use of the following two assumptions.

Assumption 1. *There exist $\eta, R > 0$ such that for every $x, x' \in B(\bar{x}, R)$ the following inequality holds*

$$\|g(x) - g(x') - Dg(\bar{x})(x - x')\| \leq \eta max\{\|x - \bar{x}\|, \|x' - \bar{x}\|\}\|x - x'\|.$$

Assumption 2. *The second order directional derivative*

$$g''(\bar{x}; u) := \lim_{t \to 0^+} \frac{g(\bar{x} + tu) - g(\bar{x}) - tDg(\bar{x})(u)}{t^2/2}$$

exists for every $u \in \mathcal{C}$ and convergence is uniform with respect to u in bounded subsets of $\mathcal{C}$.

An consequence of Assumption 1,2 is

$$\|g''(\bar{x}, u)\| \leq 2\eta, \forall u \in \mathcal{C} \cap S_X. \tag{33}$$

Definition 12 *([7]) Let S be a closed convex subset of a Banach space Z, $A : X \to Z$ be a continuous linear map and $s \in S, u \in A^{-1}(T(S;s))$. Let ξ be a nonnegative real number. A set $\mathcal{I} \subset Z$ is called an inner second order approximation set for S at s with respect to A, u and ξ if*

$$\lim_{t\to 0^+} t^{-2} d(s + tAu + \frac{t^2}{2} w, S + t^2 \xi A \mathcal{B}_X) = 0 \tag{34}$$

holds for all $w \in \mathcal{I}$. A set-valued map $\mathcal{A}_{S,s,A,\xi} : A^{-1}(T(S;s)) \cap \mathcal{S}_X \rightrightarrows Z$ is an inner second order approximation mapping for S at s with respect to A, ξ if for each $u \in A^{-1}(T(S;s)) \cap \mathcal{S}_X$ the set $\mathcal{A}_{S,s,A,\xi}(u)$ is an inner second order approximation set with respect to A, u and ξ and the limit (34) holds uniformly for all $u \in A^{-1}(T(S;s)) \cap \mathcal{S}_X$ and all $w \in \mathcal{A}_{S,s,A,\xi}(u)$.

Denote by I_X is the identify map on X. It can see that

$$\mathcal{C} = (I_X, Dg(\bar{x}))^{-1}(T(gphF, (\bar{x}, g(\bar{x})))).$$

As usual, the support function of a set $C \subseteq X$ is denoted by $\sigma_C : X^* \to \mathbb{R} \cup \{+\infty\}$, and is defined by

$$\sigma_C(x^*) := \sup_{x\in C} \langle x^*, x\rangle, \quad x^* \in X^*.$$

Theorem 13 *Suppose that Assumptions 1,2 are fulfilled.*

1. If the contingent derivative $CG(\bar{x}, 0)$ is metrically subregular at $(0,0)$ and there are real $\xi \geq 0$ and an inner second order approximation map $\mathcal{A}$ for $\mathrm{gph}\, F$ at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and ξ such that for each sequence $\{({x_n}^*, {y_n}^*)\} \subset X^* \times \mathcal{S}_{Y^*}$ satisfying

$$\lim_{n\to\infty} [\langle ({x_n}^*, {y_n}^*), (\bar{x}, g(\bar{x}))\rangle - \sigma_{\mathrm{gph}\ F}({x_n}^*, {y_n}^*)] = \lim_{n\to\infty} \|Dg(\bar{x})^* {y_n}^* + {x_n}^*\| = 0$$

one has

$$\liminf_{n\to\infty} \sup_{u\in\mathcal{C}\cap\mathcal{S}_X} \{\langle {y_n}^*, g''(\bar{x}, u)\rangle - \sigma_{\mathcal{A}(u)}({x_n}^*, {y_n}^*)\} < 0, \tag{35}$$

then G is metrically subregular at $(\bar{x}, 0)$.

2. Conversely, if G is metrically subregular at $(\bar{x}, 0)$ and

$$\limsup_{t\to 0^+} \frac{d(g(\bar{x}) + tDg(\bar{x})(u), F(\bar{x} + tu))}{t^2} \tag{36}$$

is bounded on $\mathcal{C} \cap \mathcal{S}_X$ and convergence is uniform for all $u \in \mathcal{C} \cap \mathcal{S}_X$, then there are real $\xi \geq 0$ and an inner second order approximation map $\mathcal{A}$ $\mathrm{gph}\, F$ at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and ξ such that for each sequence $\{({x_n}^*, {y_n}^*)\} \subset X^* \times \mathcal{S}_{Y^*}$ satisfying

$$\lim_{n\to\infty} [\langle ({x_n}^*, {y_n}^*), (\bar{x}, g(\bar{x}))\rangle - \sigma_{\mathrm{gph}\ F}({x_n}^*, {y_n}^*)] = \lim_{n\to\infty} \|Dg(\bar{x})^* {y_n}^* + {x_n}^*\| = 0,$$

one has

$$\liminf_{n\to\infty} \sup_{u\in\mathcal{C}\cap\mathcal{S}_X} \{\langle {y_n}^*, g''(\bar{x}, u)\rangle - \sigma_{\mathcal{A}(u)}({x_n}^*, {y_n}^*)\} \leq 0. \tag{37}$$

Moreover, if $G^{-1}(0)$ is tangentiable at $\bar{x}$ and the tangent cone $T(G^{-1}(0), \bar{x})$ is locally compact at the origin then the contingent derivative $CG(\bar{x}, 0)$ is metrically subregular at $(0,0)$.

To prove this theorem we need the following lemma.

Lemma 14 *Suppose that Assumption 1, 2 are fulfilled and G is metrically subregular at $(\bar{x}, 0)$. If (36) is bounded on $\mathcal{C} \cap \mathcal{S}_X$ and convergence is uniform for all $u \in \mathcal{C} \cap \mathcal{S}_X$, then the mapping $\mathcal{A}(u) := \{(0, g''(\bar{x}, u))\}, u \in \mathcal{C} \cap \mathcal{S}_X$, is an inner second order approximation mapping for $\mathrm{gph}\, F$ at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and some $\xi > 0$.*

Proof. Let a sequence $t_n \to 0^+,\ u \in \mathcal{C}\cap\mathcal{S}_X$. Since G is metrically subregular at $(\bar{x}, 0)$ there exists some $\kappa > 0$ such that for n sufficiently large one has

$$d(\bar{x} + t_n u, G^{-1}(0)) \le \kappa d(0, G(\bar{x} + t_n u)) = \kappa d(g(\bar{x} + t_n u), F(\bar{x} + t_n u)).$$

Then for n sufficiently large, there exists $u_n \in X$ such that $g(\bar{x}+t_n u_n) \in F(\bar{x}+t_n u_n)$ and

$$t_n\|u - u_n\| \le \kappa d(g(\bar{x} + t_n u), F(\bar{x} + t_n u)) + {t_n}^2. \tag{38}$$

By combining Assumption 2 , (38), (33) together hypothesis, there exist some $\xi > 0, N \in \mathbb{N}$ not depend on u such that

$$t_n\|u - u_n\| \le \xi {t_n}^2, \forall n \ge N.$$

By this and by Assumption 1, for every $n \ge N$, we have

$$\begin{aligned}
&\frac{2}{{t_n}^2} d((\bar{x}, g(\bar{x})) + t_n(u, Dg(\bar{x})(u)) + \frac{{t_n}^2}{2}(0, g''(\bar{x}, u)), \mathrm{gph}\ F + {t_n}^2\xi(I_X, Dg(\bar{x}))\mathcal{B}_X) \\
&\le \frac{2}{{t_n}^2} d((\bar{x} + t_n u, g(\bar{x}) + t_n Dg(\bar{x})(u) + \frac{{t_n}^2}{2} g''(\bar{x}, u)), \mathrm{gph}\ F - t_n(I_X, Dg(\bar{x}))(u_n - u)) \\
&= \frac{2}{{t_n}^2} d((\bar{x} + t_n u_n, g(\bar{x}) + t_n Dg(\bar{x})(u_n) + \frac{{t_n}^2}{2} g''(\bar{x}, u)), \mathrm{gph}\ F) \\
&\le \frac{2}{{t_n}^2} d((\bar{x} + t_n u_n, g(\bar{x}) + t_n Dg(\bar{x})(u_n) + \frac{{t_n}^2}{2} g''(\bar{x}, u)), \{\bar{x} + t_n u_n\} \times F(\bar{x} + t_n u_n)) \\
&= \frac{2}{{t_n}^2} d(g(\bar{x}) + t_n Dg(\bar{x})(u_n) + \frac{{t_n}^2}{2} g''(\bar{x}, u)), F(\bar{x} + t_n u_n)) \\
&\le \frac{2}{{t_n}^2} \|g(\bar{x} + t_n u_n) - g(\bar{x}) - t_n Dg(\bar{x})(u_n) - \frac{{t_n}^2}{2} g''(\bar{x}, u)\| \\
&\le \frac{2}{{t_n}^2} \|g(\bar{x} + t_n u_n) - g(\bar{x} + t_n u) - t_n Dg(\bar{x})(u_n - u)\| + \\
&+ \frac{2}{{t_n}^2} \|g(\bar{x} + t_n u) - g(\bar{x}) - t_n Dg(\bar{x})(u) - \frac{{t_n}^2}{2} g''(\bar{x}, u)\| \\
&\le 2\xi\eta(1 + \xi t_n)t_n + \|\frac{2}{{t_n}^2}[g(\bar{x} + t_n u) - g(\bar{x}) - t_n Dg(\bar{x})(u)] - g''(\bar{x}, u)\|.
\end{aligned}$$

By Assumption 2, the last right hand part of inequalities above converges to 0 as $n \to \infty$ uniformly for all $u \in \mathcal{C} \cap \mathcal{S}_X$. Therefore $\mathcal{A}(u) := \{(0, g''(\bar{x}, u))\}, \forall u \in \mathcal{C} \cap \mathcal{S}_X$ is an inner second order approximation map for gph F at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and ξ . □

Proof of Theorem 13. 1. Suppose in the contrary that G is not metrically subregular at $(\bar{x}, 0)$. Then by Theorem 2, there exist sequences $x_n \to \bar{x},\ \epsilon_n \to 0,\ y_n \in F(x_n),\ {y_n}^* \in \mathcal{S}_{Y^*},\ {x_n}^* \in D^*F(x_n, y_n)(-{y_n}^*)$ such that

$$g(x_n) \notin F(x_n),\ \frac{d(0, g(x_n) - F(x_n))}{\|x_n - \bar{x}\|} \to 0,\ \|Dg(\bar{x})^*{y_n}^* + {x_n}^*\| \to 0. \tag{39}$$

$$\|g(x_n) - y_n\| \le (1 + \epsilon_n)d(0, g(x_n) - F(x_n)) \tag{40}$$

$$|\langle {y_n}^*, g(x_n) - y_n\rangle - \|g(x_n) - y_n\|| \le \epsilon_n\|g(x_n) - y_n\|. \tag{41}$$

Immediately, from definitions of ${x_n}^*, {y_n}^*$ and from (40) we have

$$\lim_{n\to\infty}[\langle({x_n}^*, {y_n}^*), (\bar{x}, g(\bar{x}))\rangle - \sigma_{\mathrm{gph}\ F}({x_n}^*, {y_n}^*)] = \lim_{n\to\infty}\langle({x_n}^*, {y_n}^*), (\bar{x}, g(\bar{x})) - (x_n, y_n)\rangle = 0. \tag{42}$$

Since $CG(\bar{x}, 0)$ is metrically subregular at $(0, 0)$ there exist $\kappa, \delta > 0$ such that for every $u \in B(0, \delta)$ one has $d(u, CG(\bar{x}, 0)^{-1}(0)) \le \kappa d(0, CG(\bar{x}, 0)(u))$,or equivelantly,

$$d(u, \mathcal{C}) \le \kappa d(Dg(\bar{x})(u), CF(\bar{x}, g(\bar{x}))(u)).$$

Hence for each n sufficiently large, there exist $u_n \in \mathcal{C} \cap \mathcal{S}_X, t_n \geq 0$ such that

$$\begin{aligned}\|x_n - \bar{x} - t_n u_n\| &\leq \kappa d(Dg(\bar{x})(x_n - \bar{x}), CF(\bar{x}, g(\bar{x}))(x_n - \bar{x}))\\ &\leq \kappa d(Dg(\bar{x})(x_n - \bar{x}), F(x_n) - g(\bar{x}))(\text{since } F(x_n) - g(\bar{x}) \subset CF(\bar{x}, g(\bar{x}))(x_n - \bar{x}))\\ &= \kappa d(g(\bar{x}) + Dg(\bar{x})(x_n - \bar{x}), F(x_n)).\end{aligned}$$

Then by Assumption 1 one has

$$\|x_n - \bar{x} - t_n u_n\| \leq \kappa[d(g(x_n), F(x_n)) + \eta\|x_n - \bar{x}\|^2] \tag{43}$$

which together (39) give

$$\|\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} - \frac{t_n}{\|x_n - \bar{x}\|}u_n\| \leq \kappa[\frac{d(g(x_n), F(x_n))}{\|x_n - \bar{x}\|} + \eta\|x_n - \bar{x}\|] \to 0(n \to \infty)$$

Hence

$$\frac{t_n}{\|x_n - \bar{x}\|} \to 1.$$

We have

$$\begin{aligned}&\langle({x_n}^*, {y_n}^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{gphF}({x_n}^*, {y_n}^*) =\\ &= \langle {x_n}^*, \bar{x} + t_n u_n\rangle + \langle {y_n}^*, g(\bar{x} + t_n u_n)\rangle - \langle({x_n}^*, {y_n}^*), (x_n, y_n)\rangle(\text{since } {x_n}^* \in D^*F(x_n, y_n)(-{y_n}^*))\\ &= \langle {x_n}^*, \bar{x} + t_n u_n - x_n\rangle + \langle {y_n}^*, g(\bar{x} + t_n u_n) - y_n\rangle\\ &= \langle {x_n}^*, \bar{x} + t_n u_n - x_n\rangle + \langle {y_n}^*, g(\bar{x} + t_n u_n) - g(x_n)\rangle + \langle {y_n}^*, g(x_n) - y_n\rangle\\ &\geq \langle {x_n}^*, \bar{x} + t_n u_n - x_n\rangle + \langle {y_n}^* \circ Dg(\bar{x}), \bar{x} + t_n u_n - x_n\rangle - \eta\max\{\|t_n u_n\|, \|x_n - \bar{x}\|\}\|\bar{x} + t_n u_n - x_n\| +\\ &+ (1 - \epsilon)\|g(x_n) - y_n\| \text{ (by Assumption 1 and (41))}\\ &\geq (1 - \epsilon_n)d(g(x_n), F(x_n)) - \|{x_n}^* + {y_n}^* \circ Dg(\bar{x})\|.\|\bar{x} + t_n u_n - x_n\| - \eta_n\|\bar{x} + t_n u_n - x_n\|\\ &(\eta_n := -\eta\max\{\|t_n u_n\|, \|x_n - \bar{x}\|\} \to 0)\\ &= (1 - \epsilon_n)d(g(x_n), F(x_n)) - \delta_n\|\bar{x} + t_n u_n - x_n\|(\delta_n := \|{x_n}^* + {y_n}^* \circ Dg(\bar{x})\| + \eta_n \to 0)\\ &\geq (1 - \epsilon_n - \kappa\delta_n)d(g(x_n), F(x_n)) - \eta\kappa\delta_n\|\bar{x} - x_n\|^2 \text{ (by 43)}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{{t_n}^2}[\langle({x_n}^*, {y_n}^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{gphF}({x_n}^*, {y_n}^*)] \geq\\ &\geq \frac{(1 - \epsilon_n - \kappa\delta_n)}{{t_n}^2}d(g(x_n), F(x_n)) - \eta\kappa\delta_n(\frac{\|\bar{x} - x_n\|}{t_n})^2.\end{aligned}$$

Hence

$$\liminf_{n\to\infty}\frac{1}{{t_n}^2}[\langle({x_n}^*, {y_n}^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{gphF}({x_n}^*, {y_n}^*)] \geq 0. \tag{44}$$

Since

$$\begin{aligned}&\langle({x_n}^*, {y_n}^*), (\bar{x} + t_n u_n, g(\bar{x}) + t_n Dg(\bar{x})(u_n))\rangle + \frac{{t_n}^2}{2}\sigma_{\mathcal{A}(u_n)}({x_n}^*, {y_n}^*) =\\ &= \sup_{(w_1, w_2)\in\mathcal{A}(u_n)}\langle({x_n}^*, {y_n}^*), (\bar{x}, g(\bar{x})) + t_n(u_n, Dg(\bar{x})(u_n)) + \frac{{t_n}^2}{2}(w_1, w_2)\rangle\\ &\leq \sigma_{\text{gph }F}({x_n}^*, {y_n}^*) + {t_n}^2\xi\|{x_n}^* + Dg(\bar{x})^*{y_n}^*\| + \circ({t_n}^2)\\ &(\text{since } (\bar{x}, g(\bar{x})) + t_n(u_n, Dg(\bar{x})(u_n)) + \frac{{t_n}^2}{2}(w_1, w_2) \in gphF + {t_n}^2\xi(I_X, Dg(\bar{x}))B_X + \circ({t_n}^2)\mathcal{B}_{X\times Y})\end{aligned}$$

one has

$$\begin{aligned}&\langle({x_n}^*, {y_n}^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{gphF}({x_n}^*, {y_n}^*) \leq\\ &\leq \langle({x_n}^*, {y_n}^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \langle({x_n}^*, {y_n}^*), (\bar{x} + t_n u_n, g(\bar{x}) + t_n Dg(\bar{x})(u_n))\rangle\\ &- \frac{{t_n}^2}{2}\sigma_{\mathcal{A}(u_n)}({x_n}^*, {y_n}^*) + {t_n}^2\xi\|{x_n}^* + Dg(\bar{x})^*{y_n}^*\| + \circ({t_n}^2)\\ &= \langle {y_n}^*, g(\bar{x} + t_n u_n) - g(\bar{x}) - t_n Dg(\bar{x})(u_n))\rangle - \frac{{t_n}^2}{2}\sigma_{\mathcal{A}(u_n)}({x_n}^*, {y_n}^*)\\ &+ {t_n}^2\xi\|{x_n}^* + Dg(\bar{x})^*{y_n}^*\| + \circ({t_n}^2).\end{aligned}$$

Therefore

$$\frac{2}{{t_n}^2}[\langle ({x_n}^*, {y_n}^*), (\bar{x}+t_nu_n, g(\bar{x}+t_nu_n))\rangle - \sigma_{gphF}({x_n}^*, {y_n}^*)] \leq$$

$$\leq \langle {y_n}^*, \frac{2}{{t_n}^2}[g(\bar{x}+t_nu_n) - g(\bar{x}) - t_nDg(\bar{x})(u_n)]\rangle - \sigma_{\mathcal{A}(u_n)}({x_n}^*, {y_n}^*) +$$

$$+ 2\xi\|{x_n}^* + Dg(\bar{x})^*{y_n}^*\| + \frac{\circ({t_n}^2)}{{t_n}^2}$$

$$= \langle {y_n}^*, g''(\bar{x}, u_n)\rangle - \sigma_{\mathcal{A}(u_n)}({x_n}^*, {y_n}^*) + \langle {y_n}^*, \frac{2}{{t_n}^2}[g(\bar{x}+t_nu_n) - g(\bar{x}) - t_nDg(\bar{x})(u_n)] -$$

$$- g''(\bar{x}, u_n)\rangle + 2\xi\|{x_n}^* + Dg(\bar{x})^*{y_n}^*\| + \frac{\circ({t_n}^2)}{{t_n}^2}$$

which together (39), (42), (45) and Assumption 2 imply

$$\liminf_{n\to\infty} \frac{2}{{t_n}^2}[\langle ({x_n}^*, {y_n}^*), (\bar{x}+t_nu_n, g(\bar{x}+t_nu_n))\rangle - \sigma_{\text{gph } F}({x_n}^*, {y_n}^*)] < 0$$

which contradict to (44).

2. By Lemma 14 there exists $\xi > 0$ such that $\mathcal{A}(u) := \{(0, g''(\bar{x}, u))\}, u \in \mathcal{C} \cap S_X$ is an inner second order approximation map for $gphF$ at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x})), \xi$. Then (37) holds immediately. The last assertion of Theorem 13 is obvious from Proposition 10. The proof is complete. □

Remark 15 *Theorem 13 above is a generalized version of Theorem 5.4 in [7], in which the set valued map F is assumed a constant map.*

When Y is finite dimensional, one can simplify the second order condition in the preceding theorem as follows.

Corollary 16 *Let Y be finite dimensional and suppose that Assumptions 1,2 are fulfilled. If the contingent derivative $CG(\bar{x}, 0)$ is metrically subregular at $(0,0)$ and there are real $\xi \geq 0$ and an inner second order approximation map $\mathcal{A}$ for* gph F *at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and ξ such that for each $y^* \in \mathcal{S}_{Y^*}$ satisfying*

$$\langle (-Dg(\bar{x})^*y^*, y^*), (\bar{x}, g(\bar{x}))\rangle = \sigma_{\text{gph } F}(-Dg(\bar{x})^*y^*, y^*),$$

one has

$$\sup_{u\in\mathcal{C}\cap\mathcal{S}_X} \{\langle y^*, g''(\bar{x}, u)\rangle - \sigma_{\mathcal{A}(u)}(-Dg(\bar{x})^*y^*, y^*)\} < 0, \qquad (45)$$

then G is metrically subregular at $(\bar{x}, 0)$.

Proof. It follows directly from Theorem 13 by passing the limit. □

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