ON THE PETERSON HIT PROBLEM

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Abstract. Let \( P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k] \) be the polynomial algebra over the prime field of two elements, \( \mathbb{F}_2 \), in \( k \) variables \( x_1, x_2, \ldots, x_k \), each of degree 1. We study the hit problem, set up by F. Peterson, of finding a minimal set of generators for \( P_k \) as a module over the mod-2 Steenrod algebra, \( A \). In this paper, we study a minimal set of generators for \( A \)-module \( P_k \) in some so-called generic degrees and apply these results to explicitly determine the hit problem for \( k = 4 \).

Dedicated to Prof. N. H. V. Hưng on the occasion of his sixtieth birthday

1. Introduction and Statement of Results

Let \( V_k \) be an elementary abelian 2-group of rank \( k \). Denote by \( BV_k \) the classifying space of \( V_k \). It may be thought of as the product of \( k \) copies of the real projective space \( \mathbb{R}P^\infty \). Then
\[ P_k := H^*(BV_k) \cong \mathbb{F}_2[x_1, x_2, \ldots, x_k], \]
a polynomial algebra in \( k \) variables \( x_1, x_2, \ldots, x_k \), each of degree 1. Here the cohomology is taken with coefficients in the prime field \( \mathbb{F}_2 \) of two elements.

Being the cohomology of a space, \( P_k \) is a module over the mod 2 Steenrod algebra \( A \). The action of \( A \) on \( P_k \) can explicitly be given by the formula
\[ Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & \text{otherwise}, \end{cases} \]
and subject to the Cartan formula
\[ Sq^n(fg) = \sum_{i=0}^n Sq^i(f)Sq^{n-i}(g), \]
for \( f, g \in P_k \) (see Steenrod and Epstein [29]).

A polynomial \( f \) in \( P_k \) is called hit if it can be written as a finite sum \( f = \sum_{i>0} Sq^i(f_i) \) for some polynomials \( f_i \). That means \( f \) belongs to \( A^+P_k \), where \( A^+ \) denotes the augmentation ideal in \( A \). We are interested in the hit problem, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra \( P_k \) as a module over the Steenrod algebra \( A \). In other words, we want to find a basis of the \( \mathbb{F}_2 \)-vector space \( QP_k := P_k/A^+P_k = \mathbb{F}_2 \otimes_A P_k \).

The hit problem was first studied by Peterson [21, 22], Wood [36], Singer [27], and Priddy [23], who showed its relationship to several classical problems respectively in cobordism theory, modular representation theory, Adams spectral sequence for

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the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups. The vector space $\mathcal{P}_k^n$ was explicitly calculated by Peterson [21] for $k = 1, 2$, by Kameko [14] for $k = 3$. The case $k = 4$ has been treated by Kameko [16] and by us [30].

Several aspects of the hit problem were then investigated by many authors. (See Boardman [1], Bruner, Hà and Hưng [2], Carlisle and Wood [3], Crabb and Hubbuck [4], Giambalvo and Peterson [5], Hà [6], Hưng [7], Hưng and Nam [8, 9], Hưng and Peterson [10, 11], Janfada and Wood [12, 13], Kameko [14, 15], Minami [17], Mothebe [18], Nam [19, 20], Repka and Selick [24], Singer [28], Silverman [25], Walker and Wood [33, 34, 35], Wood [37, 38] and others.)

The $\mu$-function is one of the numerical functions that have much been used in the context of the hit problem. For a positive integer $n$, by $\mu(n)$ one means the smallest number $r$ for which it is possible to write

$$n = \sum_{1 \leq i \leq r} (2d_i - 1),$$

where $d_i > 0$. A routine computation shows that $\mu(n) = s$ if and only if there exists uniquely a sequence of integers $d_1 > d_2 > \ldots > d_{s-1} \geq d_s > 0$ such that

$$n = 2d_1 + 2d_2 + \ldots + 2d_{s-1} + 2d_s - s. \quad (1.1)$$

From this it implies $n - s$ is even and $\mu(\frac{n - s}{2}) \leq s$.

Denote by $(\mathcal{P}_k^n)$ the subspace of $\mathcal{P}_k^n$ consisting of all the classes represented by homogeneous polynomials of degree $n$ in $P_k$.

Peterson [21] made the following conjecture, which was subsequently proved by Wood [36].

**Theorem 1.1** (Wood [36]). If $\mu(n) > k$, then $(\mathcal{P}_k^n) = 0$.

One of the main tools in the study of the hit problem is Kameko’s homomorphism $\widetilde{\mathcal{Q}}_*: \mathcal{P}_k \to \mathcal{P}_k$. This homomorphism is induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{\mathcal{Q}}_*: P_k \to P_k$, given by

$$\widetilde{\mathcal{Q}}_*(x) = \begin{cases} y, & \text{if } x = x_1x_2\ldots x_ky^2, \\ 0, & \text{otherwise}, \end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{\mathcal{Q}}_*$ is not an $A$-homomorphism. However, $\widetilde{\mathcal{Q}}_*^0 \mathcal{Q}_*^{2t} = \mathcal{Q}_*^{t} \widetilde{\mathcal{Q}}_*^0$, and $\widetilde{\mathcal{Q}}_*^0 \mathcal{Q}_*^{2t+1} = 0$ for any non-negative integer $t$.

**Theorem 1.2** (Kameko [14]). Let $m$ be a positive integer. If $\mu(2m + k) = k$, then $\widetilde{\mathcal{Q}}_*: (\mathcal{P}_k^{2m+k}) \to (\mathcal{P}_k^m)$ is an isomorphism of $GL_k$-modules.

Based on Theorems 1.1 and 1.2, the hit problem is reduced to the case of degree $n$ with $\mu(n) = s < k$.

The hit problem in the case of degree $n$ of the form (1.1) with $s = k - 1$, $d_i - 1 > d_{i-1} > d_i > 1$ for $2 \leq i < k$ and $d_{k-1} > 1$ was studied by Crabb and Hubbuck [4], Nam [19] and Repka and Selick [24].

In this paper, we explicitly determine the hit problem for the case $k = 4$. First, we study the hit problem for the cases of degree $n$ of the form (1.1) for $s = k - 1$. The following theorem gives an inductive formula for the dimension of $(\mathcal{P}_k^n)$ in this case.
Theorem 1.3. Let \( n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1) \) with \( d_i \) positive integers such that \( d_1 > d_2 > \ldots > d_{k-2} \geq d_{k-1} \), and let \( m = \sum_{1 \leq i \leq k-2} (2^{d_i} - d_{k-1}) - 1 \). If \( d_{k-1} \geq k-1 \geq 1 \), then
\[
\dim(QP_k)_n = (2^k - 1) \dim(QP_{k-1})_m.
\]

For \( d_{k-1} \geq k \), the theorem follows from the results in Nam [19] and the present author [32]. However, for \( d_{k-1} = k - 1 \), the theorem is new.

Based on Theorem 1.3, we explicitly compute \( QP_4 \).

Theorem 1.4. Let \( n \) be an arbitrary positive integer with \( \mu(n) < 4 \). The dimension of the \( \mathbb{F}_2 \)-vector space \( (QP_4)_n \) is given by the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( s = 1 )</th>
<th>( s = 2 )</th>
<th>( s = 3 )</th>
<th>( s = 4 )</th>
<th>( s \geq 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2s+1 - 3 )</td>
<td>4</td>
<td>15</td>
<td>35</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>( 2s+1 - 2 )</td>
<td>6</td>
<td>24</td>
<td>50</td>
<td>70</td>
<td>80</td>
</tr>
<tr>
<td>( 2s+1 - 1 )</td>
<td>14</td>
<td>35</td>
<td>75</td>
<td>89</td>
<td>85</td>
</tr>
<tr>
<td>( 2s+2 + 2s+1 - 3 )</td>
<td>46</td>
<td>94</td>
<td>105</td>
<td>105</td>
<td>105</td>
</tr>
<tr>
<td>( 2s+3 + 2s+1 - 3 )</td>
<td>87</td>
<td>135</td>
<td>150</td>
<td>150</td>
<td>150</td>
</tr>
<tr>
<td>( 2s+4 + 2s+1 - 3 )</td>
<td>136</td>
<td>180</td>
<td>195</td>
<td>195</td>
<td>195</td>
</tr>
<tr>
<td>( 2s+t+1 + 2s+1 - 3, t \geq 4 )</td>
<td>150</td>
<td>195</td>
<td>210</td>
<td>210</td>
<td>210</td>
</tr>
<tr>
<td>( 2s+1 + 2s - 2 )</td>
<td>21</td>
<td>70</td>
<td>116</td>
<td>164</td>
<td>175</td>
</tr>
<tr>
<td>( 2s+2 + 2s - 2 )</td>
<td>55</td>
<td>126</td>
<td>192</td>
<td>240</td>
<td>255</td>
</tr>
<tr>
<td>( 2s+3 + 2s - 2 )</td>
<td>73</td>
<td>165</td>
<td>241</td>
<td>285</td>
<td>300</td>
</tr>
<tr>
<td>( 2s+4 + 2s - 2 )</td>
<td>95</td>
<td>179</td>
<td>255</td>
<td>300</td>
<td>315</td>
</tr>
<tr>
<td>( 2s+5 + 2s - 2 )</td>
<td>115</td>
<td>175</td>
<td>255</td>
<td>300</td>
<td>315</td>
</tr>
<tr>
<td>( 2s+t + 2s - 2, t \geq 6 )</td>
<td>125</td>
<td>175</td>
<td>255</td>
<td>300</td>
<td>315</td>
</tr>
<tr>
<td>( 2s+2 + 2s+1 + 2s - 3 )</td>
<td>64</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
</tr>
<tr>
<td>( 2s+3 + 2s+2 + 2s - 3 )</td>
<td>155</td>
<td>210</td>
<td>210</td>
<td>210</td>
<td>210</td>
</tr>
<tr>
<td>( 2s+t+1 + 2s+t + 2s - 3, t \geq 3 )</td>
<td>140</td>
<td>210</td>
<td>210</td>
<td>210</td>
<td>210</td>
</tr>
<tr>
<td>( 2s+3 + 2s+1 + 2s - 3 )</td>
<td>140</td>
<td>225</td>
<td>225</td>
<td>225</td>
<td>225</td>
</tr>
<tr>
<td>( 2s+u+1 + 2s+1 + 2s - 3, u \geq 3 )</td>
<td>120</td>
<td>210</td>
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<td>210</td>
<td>210</td>
</tr>
<tr>
<td>( 2s+u+2 + 2s+2 + 2s - 3, u \geq 2 )</td>
<td>225</td>
<td>315</td>
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<td>315</td>
</tr>
<tr>
<td>( 2s+t+u + 2s+t + 2s - 3, u \geq 2, t \geq 3 )</td>
<td>210</td>
<td>315</td>
<td>315</td>
<td>315</td>
<td>315</td>
</tr>
</tbody>
</table>

The space \( QP_4 \) was also computed in Kameko [10] by using computer calculation. However the manuscript is unpublished at the time of the writing.

Carlisle and Wood showed in [8] that the dimension of the vector space \( (QP_k)_m \) is uniformly bounded by a number depended only on \( k \). In 1990, Kameko made the following conjecture in his Johns Hopkins University PhD thesis [14].

Conjecture 1.5 (Kameko [14]). For every nonnegative integer \( m \),
\[
\dim(QP_k)_m \leq \prod_{1 \leq i \leq k} (2^i - 1).
\]

The conjecture was shown by Kameko himself for \( k \leq 3 \) in [14]. From Theorem 1.4 we see that the conjecture is also true for \( k = 4 \).

By induction on \( k \), using Theorem 1.3 we obtain the following.
Corollary 1.6. Let \( n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1) \) with \( d_i \) positive integers. If \( d_1 - d_2 \geq 2, d_{i-1} - d_i \geq i - 1, 3 \leq i \leq k - 1, d_{k-1} \geq k - 1, \) then
\[
\dim(QP_k)_n = \prod_{1 \leq i \leq k} (2^i - 1).
\]

For the case \( d_{i-1} - d_i \geq i, 2 \leq i \leq k - 1, \) and \( d_{k-1} \geq k, \) this result is due to Nam [19]. This corollary also shows that Kameko's conjecture is true for the degree \( n \) as given in the corollary.

By induction on \( k, \) using Theorems 1.3, 1.4 and the fact that the dual of the Kameko squaring is an epimorphism, one gets the following.

Corollary 1.7. Let \( n = \sum_{1 \leq i \leq k-2} (2^{d_i} - 1) \) with \( d_i \) positive integers and let \( d_{k-1} = 1, n_r = \sum_{1 \leq i \leq r-2} (2^{d_i} - d_{i-1} - 1) - 1 \) with \( r = 5, 6, \ldots, k. \) If \( d_1 - d_2 \geq 4, d_{i-2} - d_{i-1} \geq i, \) for \( 4 \leq i \leq k \) and \( k \geq 5, \) then
\[
\dim(QP_k)_n = \prod_{1 \leq i \leq k} (2^i - 1) + \sum_{5 \leq r \leq k} \left( \prod_{r+1 \leq i \leq k} (2^i - 1) \right) \dim \ker(\tilde{Sq}_{n_r}^0),
\]
where \( (\tilde{Sq}_{n_r}^0 : (QP_r)_{2n_r+r} \to (QP_r)_{n_r} \) denotes the squaring operation \( \tilde{Sq}_{n_r}^0 \) in degree \( 2n_r + r. \) Here, by convention, \( \prod_{r+1 \leq i \leq k} (2^i - 1) = 1 \) for \( r = k. \)

This corollary has been proved in [32] for the case \( d_{i-2} - d_{i-1} > i + 1 \) with \( 3 \leq i \leq k. \)

Obviously \( 2n_r + r = \sum_{1 \leq i \leq r-2} (2^{e_i} - 1), \) where \( e_i = d_i - d_{i-1} + 1 \) for \( 1 \leq i \leq r - 2. \) So, in degree \( 2n_r + r \) of \( P_r, \) there is a so-called spike \( x = x_1^{2^{e_1}-1} x_2^{2^{e_2}-1} \ldots x_r^{2^{e_r-2}-1}, \) i.e. a monomial whose exponents are all of the form \( 2^e - 1 \) for some \( e. \) Since the class \([x] \) in \((QP_r)_{2n_r+r} \) represented by the spike \( x \) is nonzero and \( \tilde{Sq}_{n_r}^0([x]) = 0, \) we have \( \ker(\tilde{Sq}_{n_r}^0) \neq 0, \) for any \( 5 \leq r \leq k. \) Therefore, by Corollary 1.7, Kameko’s conjecture is not true in degree \( n = 2n_k + k \) for any \( k \geq 5, \) where \( n_k = 2^{d_1-1} + 2^{d_2-1} + \ldots + 2^{d_{k-2}-1} - k + 1. \)

This paper is organized as follows. In Section 2, we recall some needed information on the admissible monomials in \( P_k \) and Singer’s criterion on the hit monomials. We prove Theorem 1.3 in Section 3 by describing a basis of \((QP_k)_n \) in terms of a given basis of \((QP_{k-1})_n \). In Section 4, we recall the results on the hit problem for \( k \leq 3. \) Theorem 1.4 will be proved in Section 5 by explicitly determining all of the admissible monomials in \( P_4. \)

The first formulation of this paper was given in a 240-page preprint in 2007 [30], which was then publicized to a remarkable number of colleagues. One year latter, we found the negative answer to Kameko’s conjecture on the hit problem [31, 32]. Being led by the insight of this new study, we have remarkably reduced the length of the paper.

2. Preliminaries

In this section, we recall some results in Kameko [14] and Singer [28] which will be used in the next sections.
Notation 2.1. Throughout the paper, we use the following notations.

\[ \mathbb{N}_k = \{1, 2, \ldots, k\}, \]
\[ X_I = X_{i_1, i_2, \ldots, i_r} = x_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_r} \ldots x_k \]
\[ = \prod_{i \in \mathbb{N}_k \setminus I} x_i, \quad I = \{i_1, i_2, \ldots, i_r\} \subset \mathbb{N}_k, \]

In particular, we have
\[ X_{\emptyset} = 1, \]
\[ X_{\emptyset} = x_1 x_2 \ldots x_k, \]
\[ X_i = x_1 \ldots \hat{x}_i \ldots x_k, \quad 1 \leq i \leq k. \]

Let \( \alpha_i(a) \) denote the \( i \)-th coefficient in dyadic expansion of a nonnegative integer \( a \). That means \( a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots \), for \( \alpha_i(a) = 0 \) or \( 1 \) and \( i \geq 0 \).

Denote by \( \alpha(a) \) the number of one in dyadic expansion of \( a \).

Let \( x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \in P_k \). Denote by \( \nu_j(x) = a_j \), \( 1 \leq j \leq k \). Set
\[ I_i(x) = \{j \in \mathbb{N}_k : \alpha_i(\nu_j(x)) = 0\}, \]
for \( i \geq 0 \). Then we have
\[ x = \prod_{i \geq 0} X_{I_i(x)}, \]

For a polynomial \( f \) in \( P_k \), we denote by \([f]\) the class in \( QP_k \) represented by \( f \).

For a subset \( S \subset P_k \), we denote
\[ [S] = \{[f] : f \in S\} \subset QP_k. \]

Definition 2.2. For a monomial \( x \), define two sequences associated with \( x \) by
\[ \omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \]
\[ \sigma(x) = (a_1, a_2, \ldots, a_k), \]
where \( \omega_i(x) = \sum_{1 \leq j \leq k} \alpha_{i-1}(\nu_j(x)) = \deg X_{I_{i-1}(x)}, \quad i \geq 1. \)

The sequence \( \omega(x) \) is called the weight vector of \( x \) (see Wood [37]). The weight vectors and the sigma vectors can be ordered by the left lexicographical order.

Let \( \omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots) \) be a sequence of nonnegative integers such that \( \omega_i = 0 \) for \( i \leq 0 \). Define \( \deg \omega = \sum_{i > 0} 2^{i-1} \omega_i \). Denote by \( P_k(\omega) \) the subspace of \( P_k \) spanned by all monomials \( y \) such that \( \deg y = \deg \omega \), \( \omega(y) \leq \omega \) and \( P^k_\omega(\omega) \) the subspace of \( P_k \) spanned by all monomials \( y \in P_k(\omega) \) such that \( \omega(y) < \omega \). Denote by \( A^+_\omega \) the subspace of \( A \) spanned by all \( Sq^j \) with \( 1 \leq j < 2^\omega \).

Definition 2.3. Let \( \omega \) be a sequence of nonnegative integers and \( f, g \) two homogeneous polynomials of the same degree in \( P_k \).

i) \( f \equiv g \) if and only if \( f - g \in A^+_\omega P_k \).

ii) \( f \cong (s, \omega) g \) if and only if \( f - g \in A^+_\omega P_k + P^k_\omega(\omega) \).

Since \( A^+_\omega P_k = 0 \), \( f \cong (s, \omega) g \) if and only if \( f - g \in P^k_\omega(\omega) \). If \( x \) is a monomial in \( P_k \) and \( \omega = \omega(x) \), then we denote \( x \cong_s g \) if and only if \( x \cong (s, \omega(x)) g \).

Obviously, the relations \( \equiv \) and \( \cong (s, \omega(x)) \) are equivalence relations.

We recall some relations on the action of the Steenrod squares on \( P_k \).

Proposition 2.4. Let \( f \) be a homogeneous polynomial in \( P_k \).

i) If \( i > \deg f \), then \( Sq^i(f) = 0 \). If \( i = \deg f \), then \( Sq^i(f) = f^2 \).

ii) If \( i \) is not divisible by \( 2^s \), then \( Sq^i(f^{2^s}) = 0 \) while \( Sq^i(2^s) = (Sq^i(f))^{2^s} \).
Proposition 2.5. Let \( x, y \) be monomials and \( f, g \) homogeneous polynomials in \( P_k \) such that \( \deg x = \deg f, \deg y = \deg g. \)

i) If \( \omega_i(x) \leq 1 \) for \( i > s \) and \( x \simeq_s f \), then \( xy^{2^r} \simeq_s fy^{2^r} \).

ii) If \( \omega_i(x) = 0 \) for \( i > s \), \( x \simeq_s f \) and \( y \simeq_r g \), then \( xy^{2^r} \simeq_{s+r} fy^{2^r} \).

Proof. Suppose that

\[
x + f + \sum_{1 \leq i < 2^s} Sq^i(z_i) = h \in P_k^-(\omega(x))
\]

where \( z_i \in P_k \). From this and Proposition 2.4, we have \( Sq^i(z_1)y^{2^r} = Sq^i(z_1y^{2^r}) \).

Observe that \( \omega_i(xy^{2^r}) = \omega_i(x) \) for \( i = 1, 2, \ldots, s \). If \( z \) is a monomial and \( z \in P_k^-(\omega(x)) \), then there exists an index \( i \geq 1 \) such that \( \omega_j(z) = \omega_j(x), j = 1, 2, \ldots, i-1 \) and \( \omega_i(z) < \omega_i(x) \). If \( i > s \), then \( \omega_j(x) = 1, \omega_i(z) = 0 \). Then we have

\[
\alpha_{i-1} \left( \deg x - \sum_{j=1}^{i-1} 2^{j-1} \omega_j(x) \right) = \alpha_{i-1} \left( 2^{i-1} + \sum_{j=i+1}^{i-1} 2^{j-1} \omega_j(x) \right) = 1.
\]

On the other hand, since \( \deg x = \deg z \), \( \omega_i(z) = 0 \) and \( \omega_j(z) = \omega_j(x), j = 1, 2, \ldots, i-1 \), one gets

\[
\alpha_{i-1} \left( \deg x - \sum_{j=1}^{i-1} 2^{j-1} \omega_j(x) \right) = \alpha_{i-1} \left( \deg z - \sum_{j=1}^{i-1} 2^{j-1} \omega_j(z) \right)
= \alpha_{i-1} \left( 2^{i-1} \omega_j(z) \right) = 0.
\]

This is a contradiction. Hence \( 1 \leq i \leq s \).

From these about equalities and the fact that \( h \in P_k^-(\omega(x)) \), one gets

\[
xy^{2^r} + fy^{2^r} + \sum_{1 \leq i < 2^s} Sq^i(z_1y^{2^r}) = hy^{2^r} \in P_k^-(\omega(xy^{2^r})).
\]

The first part of the proposition is proved.

Suppose that \( y + g + \sum_{1 \leq i < 2^s} Sq^i(u_j) = h_1 \in P_k^-(\omega(y)) \), where \( u_j \in P_k \). Then

\[
xy^{2^r} = xy^{2^r} + xh_1^{2^r} + \sum_{1 \leq i < 2^s} xSq^{2^r}(u_j^{2^r}).
\]

Since \( \omega_i(x) = 0 \) for \( i > s \) and \( h_1 \in P_k^-(\omega(y)) \), we get \( xh_1^{2^r} \in P_k^-(\omega(xy^{2^r})) \). Using the Cartan formula and Proposition 2.4 we obtain

\[
xSq^{2^r}(u_j^{2^r}) = Sq^{2^r}(xu_j^{2^r}) + \sum_{0 \leq b < j} Sq^{2^r}(x)(Sq^{j-b}(u_j))^{2^r}.
\]

Since \( \omega_i(x) = 0 \) for \( i > s \), we have \( x = \prod_{0 \leq i < s} X_i^{2^r} \). Using the Cartan formula and Proposition 2.4 we see that \( Sq^{2^r}(x) \) is a sum of polynomials of the form

\[
\prod_{0 \leq i < s} (Sq^{b_i}(X_i^{2^r})))^{2^r},
\]

where \( \sum_{0 \leq i < s} b_i 2^i = b 2^s \) and \( 0 \leq b_i \leq \deg X_i^{2^r} \). Let \( \ell \) be the smallest index such that \( b_\ell > 0 \) with \( 0 \leq \ell < s \). Suppose that a monomial \( z \) appears as a term of the polynomial \( \left( \prod_{0 \leq i < s} (Sq^{b_i}(X_i^{2^r})))^{2^r} \right) (Sq^{j-b}(u_j))^{2^r} \). Then \( \omega_i(z) = \deg X_{i-1}^{2^r} = \deg X_i^{2^r} \).
\[ \omega_t(x) = \omega_t(xy^{2^r}) \] for \( t \leq \ell \), and \( \omega_{t+1}(z) = \deg X_{t+1}(z) - b_t < \deg X_{t+1}(z) = \omega_{t+1}(x) = \omega_{t+1}(xy^{2^r}) \). Hence
\[
\left( \prod_{0 \leq i < s} (Sq^h(X_{t+1}(z))^2) \right) (Sq^{i-b}(u_j))^2 \in P_k^-(\omega(xy^{2^r})).
\]
This implies \( Sq^{2^r}(x)(Sq^{i-b}(u_j))^2 \in P_k^-(\omega(xy^{2^r})) \) for \( 0 < b < j \). So one gets
\[
xy^{2^r} + xg^{2^r} + \sum_{1 \leq l < 2^r} Sq^l(xy^{2^r}) \in P_k^-(\omega(xy^{2^r})).
\]

Since \( h \in P_k^-(\omega(x)) \), we have \( hg^{2^r} \in P_k^-(\omega(xy^{2^r})) \). Using Proposition 2.4 and the Cartan formula, we get
\[
xy^{2^r} + xg^{2^r} + \sum_{1 \leq l < 2^r} Sq^l(xy^{2^r}) = hg^{2^r} \in P_k^-(\omega(xy^{2^r})).
\]

Note that \( 1 \leq j2^s < 2^{r+s} \) for \( 1 \leq j < 2^r \). Combining the above equalities gives
\[
xy^{2^r} - fg^{2^r} \in \mathcal{A}_{r+s}P_k + P_k^-(\omega(xy^{2^r})).
\]
This implies \( xy^{2^r} \sim_{r+s} xg^{2^r} \sim_{r+s} fg^{2^r} \).

The proposition is proved. \( \square \)

**Definition 2.6.** Let \( x, y \) be monomials of the same degree in \( P_k \). We say that \( x < y \) if and only if one of the following holds

i) \( \omega(x) < \omega(y) \);

ii) \( \omega(x) = \omega(y) \) and \( \sigma(x) < \sigma(y) \).

**Definition 2.7.** A monomial \( x \) is said to be inadmissible if there exist monomials \( y_1, y_2, \ldots, y_t \) such that \( y_j < x \) for \( j = 1, 2, \ldots, t \) and \( x - \sum_{j=1}^t y_j \in \mathcal{A}P_k \).

A monomial \( x \) is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree \( n \) in \( P_k \) is a minimal set of \( \mathcal{A} \)-generators for \( P_k \) in degree \( n \).

**Definition 2.8.** A monomial \( x \) is said to be strictly inadmissible if and only if there exist monomials \( y_1, y_2, \ldots, y_t \) such that \( y_j < x \), for \( j = 1, 2, \ldots, t \) and \( x - \sum_{j=1}^t y_j \in \mathcal{A}P_k \) with \( s = \max \{ i : \omega_i(x) > 0 \} \).

It is easy to see that if \( x \) is strictly inadmissible, then it is inadmissible. The following theorem is a modification of a result in [14].

**Theorem 2.9** (Kameko [14], Sum [32]). Let \( x, y, w \) be monomials in \( P_k \) such that \( \omega_i(x) = 0 \) for \( i > r > 0 \), \( \omega_s(w) \neq 0 \) and \( \omega_i(w) = 0 \) for \( i > s > 0 \).

i) If \( w \) is inadmissible, then \( xw^{2^r} \) is also inadmissible.

ii) If \( w \) is strictly inadmissible, then \( xw^{2^r}y^{2^{r+s}} \) is inadmissible.

**Proposition 2.10** ([32]). Let \( x \) be an admissible monomial in \( P_k \). Then we have

i) If there is an index \( i_0 \) such that \( \omega_{i_0}(x) = 0 \), then \( \omega_i(x) = 0 \) for all \( i > i_0 \).

ii) If there is an index \( i_0 \) such that \( \omega_{i_0}(x) < k \), then \( \omega_i(x) < k \) for all \( i > i_0 \).

Now, we recall a result of Singer [28] on the hit monomials in \( P_k \).

**Definition 2.11.** A monomial \( z \) in \( P_k \) is called a spike if \( \nu_j(z) = 2^{s_j} - 1 \) for \( s_j \) a nonnegative integer and \( j = 1, 2, \ldots, k \). If \( z \) is a spike with \( s_1 > s_2 > \ldots > s_{r-1} \geq s_r > 0 \) and \( s_j = 0 \) for \( j > r \), then it is called a minimal spike.

The following is a criterion for the hit monomials in \( P_k \).
Theorem 2.12 (Singer [23]). Suppose \( x \in P_k \) is a monomial of degree \( n \), where \( \mu(n) \leq k \). Let \( z \) be the minimal spike of degree \( n \). If \( \omega(x) < \omega(z) \), then \( x \) is hit.

From this theorem, we see that if \( z \) is a minimal spike, then \( P_k(\omega(z)) \subset A^+P_k \).

The following lemmas were proved in [22].

Lemma 2.13 ([23]). Let \( n = \sum_{1 \leq i \leq k-1} (2d_i - 1) \) with \( d_i \) positive integers such that \( d_1 > d_2 > \ldots > d_{k-2} \geq d_{k-1} > 0 \), and \( x \) a monomial of degree \( n \) in \( P_k \). If \( |x| \neq 0 \), then \( \omega_i(x) = k - 1 \) for \( 1 \leq i \leq d_{k-1} \).

Lemma 2.14 ([23]). Let \( n = \sum_{1 \leq i \leq k-1} (2d_i - 1) \) with \( d_i \) positive integers such that \( d_1 > d_2 > \ldots > d_{k-2} \geq d_{k-1} > 0 \), and \( x \) a monomial in \( P_k \) such that \( \omega_i(x) = k - 1 \) for \( i = 1, 2, \ldots, s \leq d_{k-1} \) and \( \omega_i(x) = 0 \) for \( i > s \). Suppose \( y, f \) and \( g \) are polynomials in \( P_k \) with \( \deg f = \deg x \) and \( \deg y = \deg g = (n - \deg x)/2^s = 2^{d_1-s} + \ldots + 2^{d_{i-1}-s} + 2^{d_{i-1}-s} - k + 1 \).

i) If \( x \preceq f \), then \( xy^{2^s} \equiv fg^{2^s} \).

ii) If \( y \equiv g \), then \( xy^{2^s} \equiv xg^{2^s} \).

For latter use, we set
\[
P_k^0 = \langle \{ x = x_1^{a_1}x_2^{a_2} \ldots x_k^{a_k} : a_1a_2 \ldots a_k = 0 \} \rangle,
\]
\[
P_k^+ = \langle \{ x = x_1^{a_1}x_2^{a_2} \ldots x_k^{a_k} : a_1a_2 \ldots a_k > 0 \} \rangle.
\]

It is easy to see that \( P_k^0 \) and \( P_k^+ \) are the \( A \)-submodules of \( P_k \). Furthermore, we have the following.

Proposition 2.15. We have a direct summand decomposition of the \( \mathbb{F}_2 \)-vector spaces
\[
QP_k = QP_k^0 \oplus QP_k^+.
\]
Here \( QP_k^0 = P_k^0/A^+P_k^0 \) and \( QP_k^+ = P_k^+/(A^+P_k^0) \).

3. Proof of Theorem [13]

We denote
\[
\mathcal{N}_k = \{(i; I) ; I = (i_1, i_2, \ldots, i_r)，1 \leq i < i_1 < \ldots < i_r \leq k, 0 \leq r < k\}.
\]
Let \( (i; I) \in \mathcal{N}_k \) and \( j \in \mathbb{N}_k \). Denote by \( r = \ell(I) \) the length of \( I \), and
\[
I \cup j = \begin{cases} I, & \text{if } j \not\in I, \\ (i_1, \ldots, i_{t-1}, j, i_t, \ldots, i_r), & \text{if } i_{t-1} < j < i_t, 1 \leq t \leq r + 1. \end{cases}
\]
Here \( i_0 = 0 \) and \( i_{r+1} = k + 1 \).

For \( 2 \leq h < k \), we set \( \mathcal{N}_{h-1} \cup h = \{(i; I \cup h); (i; I) \in \mathcal{N}_{h-1}\} \). Then we have
\[
\mathcal{N}_k = (\mathcal{N}_1 \cup 2) \cup \ldots \cup (\mathcal{N}_{k-1} \cup k) \cup \{(1; \emptyset), \ldots, (k; \emptyset)\}.
\]

(3.1)

For \( 1 \leq i \leq k \), define the homomorphism \( f_i = f_{k+1} : P_{k-1} \to P_k \) of algebras by substituting
\[
f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ x_{j+1}, & \text{if } i \leq j < k. \end{cases}
\]
Definition 3.1. Let \((i; I) \in \mathbb{N}_k\), let \(r = \ell(I)\), and let \(u\) be an integer with \(1 \leq u \leq r\). A monomial \(x \in P_{k-1}\) is said to be \(u\)-compatible with \((i; I)\) if all of the following hold:

i) \(\nu_{i_1}(x) = \nu_{i_2}(x) = \ldots = \nu_{(u-1)}(x) = 2^r - 1\),

ii) \(\nu_{i_u}(x) > 2^r - 1\),

iii) \(\alpha_{r-t}(\nu_{i_u}(x)) = 1\), \(\forall t, 1 \leq t < u\),

iv) \(\alpha_{r-t}(\nu_{i_{u-1}}(x)) = 1\), \(\forall t, u < t \leq r\).

Clearly, a monomial \(x\) can be \(u\)-compatible with a given \((i; I) \in \mathcal{N}_k, r = \ell(I) > 0\), for at most one value of \(u\). By convention, \(x\) is \(1\)-compatible with \((i; \emptyset)\).

Definition 3.2. Let \((i; I) \in \mathcal{N}_k\), \(x(I,u) = x_{i_u}^{2^r-1} + \ldots + x_{i_1}^{2^r-u} \prod_{u < t \leq r} x_t^{2^r-t}\) for \(1 \leq u \leq r = \ell(I)\), \(x(\emptyset,1) = 1\). For a monomial \(x\) in \(P_{k-1}\), we define the monomial \(\phi_{(i; I)}(x)\) in \(P_k\) by setting

\[
\phi_{(i; I)}(x) = \begin{cases} 
(x_{i_1}^{2^r-1} f_i(x))/x_{(I,u)}, & \text{if there exists } u \text{ such that } x \text{ is } u\text{-compatible with } (i, I), \\
0, & \text{otherwise.}
\end{cases}
\]

Then we have an \(\mathbb{F}_2\)-linear map \(\phi_{(i; I)} : P_{k-1} \to P_k\). In particular, \(\phi_{(i; \emptyset)} = f_i\).

Let \(x = X^{2^d-1} y^{2^d}\), with \(y\) a monomial in \(P_{k-1}\) and \(X = x_1 x_2 \ldots, x_{k-1} \in P_{k-1}\).

If \(r < d\), then \(x\) is \(1\)-compatible with \((i; I)\) and

\[
\phi_{(i; I)}(x) = \phi_{(i; I)}(X^{2^d-1}) f_i(y)^{2^d} = x_{i_1}^{2^r-1} \prod_{1 \leq t \leq r} x_t^{2^d-2^r-t-1} X^{2^d-1} f_i(y)^{2^d}. \quad (3.2)
\]

If \(d = r\), \(\nu_{i_1}(y) = 0, j = i_1, i_2, \ldots, i_{u-1}\) and \(\nu_{i_u}(y) > 0\), then \(x\) is \(u\)-compatible with \((i; I)\) and

\[
\phi_{(i; I)}(x) = \phi_{(i_u; J_u)}(X^{2^d-1}) f_i(y)^{2^d}, \quad (3.3)
\]

where \(J_u = (i_{u+1}, \ldots, i_r)\).

Let \(B\) be a finite subset of \(P_{k-1}\) consisting of some homogeneous polynomials in degree \(n\). We set

\[
\Phi^0(B) = \bigcup_{1 \leq i \leq k} \phi_{(i; \emptyset)}(B) = \bigcup_{1 \leq i \leq k} f_i(B).
\]

\[
\Phi^+(B) = \bigcup_{(i; I) \in \mathcal{N}_k, 0 < \ell(I) \leq k-1} \phi_{(i; I)}(B) \setminus P^0_k.
\]

\[
\Phi(B) = \Phi^0(B) \cup \Phi^+(B).
\]

It is easy to see that if \(B_{k-1}(n)\) is a minimal set of generators for \(P_{k-1}\) in degree \(n\), then \(\Phi^0(B_{k-1}(n))\) is a minimal set of generators for \(A\)-module \(P_{k-1}^0\) in degree \(n\) and \(\Phi^+(B_{k-1}(n)) \subset P_{k-1}^+\).

Proposition 3.3. Let \(n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)\) with \(d_i\) positive integers such that \(d_1 > d_2 > \ldots > d_{k-2} \geq d_{k-1} \geq k-1 \geq 1\). If \(B_{k-1}(n)\) is a minimal set of generators for \(A\)-module \(P_{k-1}\) in degree \(n\), then \(B_k(n) = \Phi(B_{k-1}(n))\) is also a minimal set of generators for \(A\)-module \(P_k\) in degree \(n\).

For \(d_{k-1} \geq k\), this proposition is a modification of a result in Nam [19]. For \(d_{k-2} = d_{k-1} > k\), it has been proved in [32].

We prepare some lemmas for the proof of this proposition.
Lemma 3.4. Let \( j_0, j_1, \ldots, j_{d-1} \in \mathbb{N}_k \). Then there is \((i; I) \in \mathcal{N}_k\) such that
\[
x = \prod_{0 \leq t < d} X_{j_t}^2 \simeq_{d-1} \phi_{(i; I)}(X^{2^d-1}),
\]
where \( i = \min \{j_0, j_1, \ldots, j_{d-1}\} \).

Lemma 3.5. Let \( n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1) \) with \( d_i \) positive integers such that \( d_1 > d_2 > \ldots > d_{k-2} \geq d_{k-1} > 0 \), and let \( y_0 \) be a monomial in \((P_k)_{m-1}\), \( y_i = y_0 x_i \) for \( 1 \leq i \leq k \), and \((i; I) \in \mathcal{N}_k\).

i) If \( 0 < r = \ell(I) < d = d_{k-1} \), then
\[
\phi_{(i; I)}(X^{2^d-1}) y_i^{2^d} = \sum_{1 \leq j < i} \phi_{(j; I)}(X^{2^d-1}) y_j^{2^d} + \sum_{i \leq j < k} \phi_{(i; I)}(X^{2^d-1}) y_j^{2^d},
\]
where \( i_j = \min \{j, I\}, I_j = I \) for \( j < \min I \), and \( I_j = (I \cup j) \setminus \{i_j\} \) for \( j \geq \min I \).

ii) If \( r + 1 < d \), then
\[
\phi_{(i; I)}(X^{2^d-1}) y_i^{2^d} = \sum_{1 \leq j < i} \phi_{(j; I \cup j)}(X^{2^d-1}) y_j^{2^d} + \sum_{i \leq j < k} \phi_{(i; I \cup j)}(X^{2^d-1}) y_j^{2^d}.
\]

Denote by \( I_t = (t + 1, t + 2, \ldots, k) \) for \( 1 \leq t \leq k \). Set
\[
Y_t = \sum_{r=t}^{k} \phi_{(i; I_t)}(X^{2^d-1}) x_r^{2^d}, \quad d > k + 1 - t.
\]

Lemma 3.6. For \( 1 < t \leq k \),
\[
Y_t \simeq_{(k, \omega)} \sum_{(j; I)} \phi_{(j; I)}(X^{2^d-1}) x_j^{2^d},
\]
where the sum runs over some \((j; J) \in \mathcal{N}_k\) with \( 1 \leq j < t \), \( J \subset I_{t-1}, J \neq I_{t-1} \) and \( \omega = \omega(X_1^{2^d-1} x_1^{2^d}) \).

We assume that all elements of \( B_{k-1}(n) \) are monomials. Denote by \( B = B_{k-1}(n) \).
We set
\[
\mathcal{C} = \{ z \in B : \nu_1(z) > 2^{k-1} - 1 \},
\]
\[
\mathcal{D} = \{ z \in B : \nu_1(z) = 2^{k-1} - 1, \nu_2(z) > 2^{k-1} - 1 \},
\]
\[
\mathcal{E} = \{ z \in B : \nu_1(z) = \nu_2(z) = 2^{k-1} - 1 \}.
\]

Since \( \omega_k(z) \geq k-3 \) for all \( z \in B \), we have \( B = \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \). If \( d = d_{k-1} > k - 1 \), then \( \mathcal{D} = \mathcal{E} = \emptyset \). If \( d_{k-2} > d_{k-1} = k - 1 \), then \( \mathcal{E} = \emptyset \). We set \( \bar{B} = \{ z ; X^{2^d-1} x_1^{2^d} \in B \} \). If either \( d \geq k \) or \( I \neq I_1 \), then \( \phi_{(i; I)}(z) = \phi_{(i; I_1)}(X^{2^d-1}) f_i(z)^{2^d} \). If \( d = d_{k-1} = k - 1 \), then
\[
\phi_{(i; I)}(z) = \begin{cases} 
\phi_{(2; I_2)}(X^{2^d-1}) f_1(z)^{2^d}, & \text{if } z \in \mathcal{C}, \\
\phi_{(3; I_3)}(X^{2^d-1}) f_2(z)^{2^d}, & \text{if } z \in \mathcal{D}, \\
\phi_{(4; I_4)}(X^{2^d-1}) f_3(z)^{2^d}, & \text{if } z \in \mathcal{E}. 
\end{cases}
\]

For any \((i; I) \in \mathcal{N}_k\), we define the homomorphism \( p_{(i; I)} : P_k \to P_{k-1}\) of algebras by substituting
\[
p_{(i; I)}(x_j) = \begin{cases} 
x_j, & \text{if } 1 \leq j < i, \\
\sum_{s \in I} x_{s-1}, & \text{if } j = i, \\
x_{j-1}, & \text{if } i < j \leq k.
\end{cases}
\]
Lemma 3.7. Let \( z \in B, (i; I), (j; J) \in \mathcal{N}_k \) and \( \ell(J) \leq \ell(I) \).

i) If either \( d \geq k \) or \( d = k - 1 \) and \( I \neq I_1 \), then

\[
p_{(j; J)}(\phi_{(i; I)}(z)) = \begin{cases} z, & \text{if } (j; J) = (i; I), \\ 0, & \text{if } (j; J) \neq (i; I). \\ \end{cases}
\]

ii) If \( z \in C \) and \( d = k - 1 \), then

\[
p_{(i; I)}(\phi_{(1; I_1)}(z)) = \begin{cases} z, & \text{if } (i; I) = (1; I_1), \\ 0 \mod(D \cup E), & \text{if } (i; I) = (2; I_2) \\ 0, & \text{otherwise}. \\ \end{cases}
\]

iii) If \( z \in D, \) then

\[
p_{(i; I)}(\phi_{(1; I_1)}(z)) = \begin{cases} z, & \text{if } (i; I) = (1; I_1), (1; I_2), (2; I_2), \\ 0 \mod(E), & \text{if } (i; I) = (3; I_3), \\ 0, & \text{otherwise}. \\ \end{cases}
\]

iv) If \( z \in E, \) then

\[
p_{(i; I)}(\phi_{(1; I_1)}(z)) = \begin{cases} z, & \text{if } I_3 \subset I, \\ 0, & \text{otherwise}. \\ \end{cases}
\]

The above lemmas will be proved in the end of the section.

We recall the following.

Lemma 3.8 (Nam [19]). Let \( x \) be a monomial in \( P_k \). Then \( x \equiv \sum \bar{x} \), where \( \bar{x} \) are monomials with \( \nu_i(\bar{x}) = 2^i - 1 \) and \( t = \alpha(\nu_1(\bar{x})) \).

Proof of Proposition 3.3. Denote by \( \mathcal{P}(n) \) the subspace of \( (P_k)_n \) spanned by all elements of the set \( B_k(n) \).

Let \( x \) be a monomial of degree \( n \) in \( P_k \) and \( [x] \neq 0 \). By Lemma 2.13, we have \( \omega_i(x) = k - 1 \) for \( 1 \leq i \leq d_{k-1} = d \). Hence we obtain \( x = (\prod_{0 \leq t < d} x_{j_t}^{2^t}) \bar{y}^{2d} \), for suitable monomial \( \bar{y} \in (P_k)_m \), with \( m = \sum_{1 \leq i \leq k-2} (2^{d_i} - d - 1) \).

According to Lemmas 3.4 and 2.14, there is \( (i; I) \in \mathcal{N}_k \) such that

\[
x = \left( \prod_{0 \leq t < d} x_{j_t}^{2^t} \right) \bar{y}^{2d} \equiv \phi_{(i; I)}(X^{2^{d-1}})\bar{y}^{2d}, \tag{3.5}
\]

where \( r = \ell(I) < d \).

Set \( h_u = 2^{d_1} - u + \ldots + 2^{d_{k-2}} - u + 2^{d_{k-1}} - u - k + 1 \), for \( 0 \leq u \leq d \). We have \( h_0 = n, h_d = m, 2h_d + k = h_u - 1 = h_{u-1} + \mu(2h_d + k + 1) = k - 1 \) for \( 1 \leq u \leq d \).

By Theorem 1.2, the squaring operation \( (\tilde{\mathcal{S}}q_s)_{h_u} : (QP_{k-1})_{h_{u-1}} \to (QP_{k-1})_{h_u} \) is an isomorphism of \( \mathbb{F}_2 \)-vector spaces. So the iterated squaring operation

\[
(\tilde{\mathcal{S}}q_s)^d = (\tilde{\mathcal{S}}q_s)^d \cdot \ldots \cdot (\tilde{\mathcal{S}}q_s)^d \in (QP_{k-1})_m \to (QP_{k-1})_m
\]

is also an isomorphism of \( \mathbb{F}_2 \)-vector spaces. Hence

\[
\widehat{B}_{k-1}(m) = (\tilde{\mathcal{S}}q_s)^d(B_{k-1}(n)) = \{ \bar{z} \in (P_{k-1})_m : X^{2^{d-1}}\bar{z}^{2d} \in B_{k-1}(n) \}
\]

is a minimal set of \( \mathcal{A} \)-generators for \( P_{k-1} \) in degree \( m \).
Now, we prove \([x] \in [P(n)]\). The proof is divided into many cases.

**Case 3.5.1.** \(\bar{y} = f_i(y)\) with \(y \in (P_{k-1})_m\).

Since \(y \in (P_{k-1})_m\), we have \(y \equiv \bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_s\) with \(\bar{z}_t\) monomials in \(\bar{B}_{k-1}(m)\). Using Lemma 2.14, we get
\[
x = \phi_{(i; J)}(X^{2^d-1})f_i(y)^{2^d} = \sum_{1 \leq t \leq s} \phi_{(i; J)}(X^{2^d-1})f_i(\bar{z}_t)^{2^d}.
\]

Since \(\phi_{(i; J)}(X^{2^d-1})f_i(\bar{z}_t)^{2^d} = \phi_{(i; J)}(X^{2^d-1} - \bar{z}_t^{2^d}) = X^{2^d-1} - \bar{z}_t^{2^d} \in B_{k-1}(n)\), we get \([x] \in [P(n)]\).

**Case 3.5.2.** \(d \geq k, \bar{y} = x^a_i f_i(y)\) with \(y \in (P_{k-1})_{m-a}\).

If \(i = 1\) and either \(I \neq I_1\) or \(d > k\), then \(d - r - 1 \geq 1\). Applying Lemma 3.5(ii) with \(y_0 = x^{a-1}_1 f_1(y)\), we get
\[
x = \sum_{2 \leq j \leq k} \phi_{(1; J)(j)}(X^{2^d-1})(x^{a-1}_1 f_1(x_{j-1} - y))^{2^d}.
\]
From this and the inductive hypothesis, we obtain \([x] \in [P(n)]\).

If \(I = I_1\) and \(d = k\), then \(r = d - 1\). Using Lemma 3.5(i) with \(y_0 = x^{a-1}_1 f_1(y)\) and Lemma 3.6 we get
\[
x = \sum_{J \neq I_1} k \phi_{(2; J_2)}(X^{2^d-1})(x^k_{J_2} y_0)^{2^d} = \sum_{J \neq I_1} \phi_{(1; J)}(X^{2^d-1})(x^a_1 f_1(y))^{2^d}.
\]
Since \(J \neq I_1\), one gets \([x] \in [P(n)]\).

Suppose \(i > 1\). Then \(r+1 < k \leq d\). Applying Lemma 3.5(ii) with \(y_0 = x^{a-1}_1 f_i(y)\), we obtain
\[
x = \sum_{1 \leq j < i} \phi_{(j; J)(j)}(X^{2^d-1})y_j^{2^d} + \sum_{i < j \leq k} \phi_{(i; J)(j)}(X^{2^d-1})y_j^{2^d},
\]
where \(y_j = x_j y_0 = x^{a-1}_1 f_i(x_{j-1} - y)\) for \(j > i\). Using the inductive hypothesis, we get \([x] \in [P(n)]\). So the proposition is proved for \(d \geq k\).

In the remaining part of the proof, we assume that \(d = k - 1\).

**Case 3.5.3.** \((i; I) = (2; I_2)\) and \(\bar{y} = f_1(y)\) with \(y \in (P_{k-1})_m\), \(\nu_1(y) > 0\).

Since \(y \in (P_{k-1})_m\), we have \(y \equiv \bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_s\) with \(\bar{z}_t\) monomials in \(\bar{B}_{k-1}(m)\). Using Lemma 2.14, we get
\[
x = \phi_{(2; I_2)}(X^{2^d-1})f_1(y)^{2^d} = \sum_{1 \leq t \leq s} \phi_{(2; I_2)}(X^{2^d-1})f_1(\bar{z}_t)^{2^d}.
\]
If \(\nu_1(\bar{z}_t) > 0\), then \(\phi_{(2; I_2)}(X^{2^d-1})f_1(\bar{z}_t)^{2^d} = \phi_{(1; I_1)}(X^{2^d-1} - \bar{z}_t^{2^d})\). If \(\nu_1(\bar{z}_t) = 0\), then \(f_1(\bar{z}_t) = f_2(\bar{z}_t)\) and \(\phi_{(2; I_2)}(X^{2^d-1})f_1(\bar{z}_t)^{2^d} = \phi_{(2; I_2)}(X^{2^d-1} - \bar{z}_t^{2^d})\). Hence \([x] \in [P(n)]\).

**Case 3.5.4.** \((i; I) = (3; I_3)\) and \(\bar{y} = f_2(y)\) with \(y \in (P_{k-1})_m\), \(\nu_1(y) = 0, \nu_2(y) > 0\).
Since \( y \in (P_{k-1})_m \) and \( \nu_1(y) = 0 \), we have \( y \equiv \bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_s \) with \( \bar{z}_i \) polynomials in \( B_{k-1}(m) \) and \( \nu_1(\bar{z}_i) = 0 \). Using Lemma 2.14 we get
\[
x \equiv \phi(3,t_1)(X^{2^{d-1}})f_2(y)^{2^d} \equiv \sum_{1 \leq t \leq s} \phi(3,t_1)(X^{2^{d-1}})f_2(\bar{z}_t)^{2^d}.
\]

If \( \nu_2(\bar{z}_t) > 0 \), then \( \phi(3,t_1)(X^{2^{d-1}})f_2(\bar{z}_t)^{2^d} = \phi(1,t_1)(X^{2^{d-1}})\bar{z}_t^{2^d} \). If \( \nu_2(\bar{z}_t) = 0 \), then \( f_2(\bar{z}_t) = f_3(\bar{z}_t) \) and \( \phi(3,t_2)(X^{2^{d-1}})f_2(\bar{z}_t)^{2^d} = \phi(3,t_2)(X^{2^{d-1}})\bar{z}_t^{2^d} \). Hence \( [x] \in [\mathcal{P}(n)] \).

**Case 3.5.5.** \((i; I) = (4;I_4)\) and \( \bar{y} = f_3(y) \) with \( y \in (P_{k-1})_m \), \( \nu_1(y) = \nu_2(y) = 0 \).

Since \( y \in (P_{k-1})_m \) and \( \nu_1(y) = \nu_2(y) = 0 \), we have \( y \equiv \bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_s \) with \( \bar{z}_i \) polynomials in \( B_{k-1}(m) \) and \( \nu_1(\bar{z}_i) = \nu_2(\bar{z}_i) = 0 \). Using Lemma 2.14 we get
\[
x \equiv \phi(4,t_1)(X^{2^{d-1}})(f_3(y))^{2^d} \equiv \sum_{1 \leq t \leq s} \phi(4,t_1)(X^{2^{d-1}})f_3(\bar{z}_t)^{2^d}.
\]

If \( \nu_3(\bar{z}_t) > 0 \), then \( \phi(4,t_1)(X^{2^{d-1}})f_3(\bar{z}_t)^{2^d} = \phi(1,t_1)(X^{2^{d-1}})\bar{z}_t^{2^d} \). If \( \nu_3(\bar{z}_t) = 0 \), then \( f_3(\bar{z}_t) = f_4(\bar{z}_t) \) and \( \phi(4,t_1)(X^{2^{d-1}})f_3(\bar{z}_t)^{2^d} = \phi(4,t_1)(X^{2^{d-1}})\bar{z}_t^{2^d} \). Hence \( [x] \in [\mathcal{P}(n)] \).

**Case 3.5.6.** \( \bar{y} = x_i^2f_1(y) \) with \( y \in (P_{k-1})_m \), \( i = 1 \) and \( \ell(I) < k - 2 \).

According to Lemma 3.8 \( x_i^2f_1(y)^{2^d} \equiv x_1f_1(y) \), for some polynomial \( g \). So we assume \( s = 0 \). Using Lemma 3.5 ii) with \( y_0 = f_1(y) \), we have
\[
x \equiv \sum_{r=2}^k \phi(1;u,r)(X^{2^{d-1}})(f_1(x_{r-1}y))^{2^d}.
\]
Hence by Case 3.5.1 \( [x] \in [\mathcal{P}(n)] \).

**Case 3.5.7.** \( \bar{y} = x_i^2f_2(y) \) with \( y \in (P_{k-1})_m \), \( \nu_1(y) = 0 \), \( i = 2 \) and \( \ell(I) < k - 3 \).

Using Lemma 3.8 we need only to prove \([x] \in [\mathcal{P}(n)] \) for \( s = 0 \). Using Lemma 3.5 ii) with \( y_0 = f_2(y) \), one gets
\[
x \equiv \phi(1;u,2)(X^{2^{d-1}})(x_1f_2(y))^{2^d} + \sum_{r=3}^k \phi(2;u,r)(X^{2^{d-1}})(f_2(x_{r-1}y))^{2^d}.
\]
Since \( \nu_1(y) = 0 \), \( f_2(y) = f_1(y) \), from this equalities, Cases 3.5.1 and 3.5.6 we get \( [x] \in [\mathcal{P}(n)] \).

**Case 3.5.8.** \( \bar{y} = x_3^{2^d}f_3(y) \), with \( y \in (P_{k-1})_m \), \( \nu_1(y) = \nu_2(y) = 0 \) and \( i = 3 \).

We need only to prove \([x] \in [\mathcal{P}(n)] \) for \( s = 0 \). Note that since \( \nu_1(y) = \nu_2(y) = 0 \), we have \( f_1(y) = f_2(y) = f_3(y) \). If \( I = I_3 \), then by Case 3.5.4 \( [x] \in [\mathcal{P}(n)] \). If \( \ell(I) < k - 4 \), then using Lemma 3.5 ii) with \( y_0 = f_3(y) \), we get
\[
x \equiv \phi(1;u,3)(X^{2^{d-1}})(x_1f_1(y))^{2^d} + \phi(2;u,3)(X^{2^{d-1}})(f_2(x_1y))^{2^d} + \sum_{r=4}^k \phi(3;u,r)(X^{2^{d-1}})(f_3(x_{r-1}y))^{2^d}.
\]
From this equalities and Cases 3.5.1 3.5.6 3.5.7 we get \( [x] \in [\mathcal{P}(n)] \).
If $d_{k-2} > d_{k-1}$ and $I \neq I_3$, then $u_k(x) = \omega_1(y) + 1 = k - 2$. Hence $a_0(\nu_j(y)) = 1$ for $j = 3, \ldots, k - 1$. Applying Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12 we get

$$ x \equiv \phi(1; I)(X^{2d-1}f_1(x)(y))^{2^d} + \phi(2; I)(X^{2d-1}f_2(x)(y))^{2^d}. $$

Hence by Cases 3.5.6 and 3.5.7 we get

$$ x \equiv \phi(1; I)(X^{2d-1}f_1(x)(y))^{2^d} + \phi(2; I)(X^{2d-1}f_2(x)(y))^{2^d}. $$

By Cases 3.5.5, 3.5.6 and 3.5.7 we get $[x] \in [P(n)]$.

Suppose $d_{k-2} = d_{k-1}$ and $I = k - 4$. Then $I = I_{3, u} = (4, \ldots, \hat{u}, \ldots, k)$ with $4 \leq u \leq k$. Since $u_k(x) = \omega_1(y) + 1 = k - 3$, we have $\omega_1(y) = k - 4$. Hence there exists uniquely $3 \leq t < k$ such that $a_0(\nu_j(y)) = 0$.

If $t = u - 1$, then using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12 we get

$$ x \equiv \phi(1; I)(X^{2d-1}f_1(x)(y))^{2^d} + \phi(2; I)(X^{2d-1}f_2(x)(y))^{2^d} + \phi(4; I_4)(X^{2d-1}f_3(x)(y))^{2^d}. $$

By Cases 3.5.5, 3.5.6 and 3.5.7 we get $[x] \in [P(n)]$.

If $u = 4 < t + 1$, then using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12 we get

$$ x \equiv \phi(1; I)(X^{2d-1}f_1(x)(y))^{2^d} + \phi(2; I)(X^{2d-1}f_2(x)(y))^{2^d} + \phi(4; I_4)(X^{2d-1}f_3(x)(y))^{2^d}. $$

Applying Lemma 3.5(i) with $y_0 = f_3(x_1y/x_4)$ and Theorem 2.12 we have

$$ \phi(5; I_5)(X^{2d-1}f_3(x)(y))^{2^d} = \sum_{1 \leq i \leq 3} \phi(i; I_5)(X^{2d-1}f_3(x_1y/x_4))^2. $$

Since $\ell(I_5) = k - 5 < k - 4$, using Cases 3.5.6, 3.5.7 and the above equalities, we get $[x] \in [P(n)]$.

Suppose that $4 < u \neq t + 1$. Using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12 we obtain

$$ x \equiv \phi(1; I)(X^{2d-1}f_1(x)(y))^{2^d} + \phi(2; I)(X^{2d-1}f_2(x)(y))^{2^d} + \phi(4; I_4)(X^{2d-1}f_3(x)(y))^{2^d}. $$

Applying Lemma 3.5(i) with $y_0 = f_3(x_1y/x_3)$ and Theorem 2.12 we have

$$ \phi(4; I_4)(X^{2d-1}f_3(x)(y))^{2^d} = \sum_{1 \leq i \leq 3} \phi(i; I_4)(X^{2d-1}f_3(x_1y/x_3))^2. $$

Since $\ell(I \setminus 4) = k - 5 < k - 4$, using Cases 3.5.6, 3.5.7 and the above equalities, we get $[x] \in [P(n)]$.

**Case 3.5.9.** $\bar{y} = x_3^4x_4f_4(y)$ for $y \in (P_{k-1})_{m-b-c}$ with $\nu_j(y) = 0$, $j = 1, 2, 3$ and $i = 4$.

Using Lemmas 3.8 and 2.14 we assume that $b = 2s - 1$. We prove $[x] \in [P(n)]$ by double induction on $(\ell(I), c)$. If $c = 0$, then by Case 3.5.1 $[x] \in [P(n)]$. If $I \neq I_4$, then applying Lemma 3.5(ii) with $y_0 = x_4^b x_3^{-c} f_4(y)$, we have

$$ x \equiv \phi(1; I_4)(X^{2d-1}f_1(x_2^b x_3^{-c-1} y))^{2^d} + \phi(2; I_4)(X^{2d-1}f_2(x_2^b x_3^{-c-1} y))^{2^d} + \phi(3; I_4)(X^{2d-1}f_3(x_3^{-c-1} y))^{2^d} + \sum_{r=5}^{k} \phi(4; I_4)(X^{2d-1}f_4(x_3^{-c-1} y))^{2^d}. $$
From these equalities, Cases 3.5.6, 3.5.7, and the inductive hypothesis, we get \(|x| \in \{\mathcal{P}(n)\}.

If \(I = I_4\), then applying Lemma 3.5(i) with \(y_0 = x_3^b x_4^{c-1} f_4(y)\), we obtain
\[
x = \phi_{(1;I_4)}(X^{2d-1})(x_1 f_1(x_2^{b} x_3^{c-1} y))^{2d} + \phi_{(2;I_4)}(X^{2d-1})(x_2 f_2(x_2^{b} x_3^{c-1} y))^{2d} + \phi_{(3;I_4)}(X^{2d-1})(x_3^{b-1} f_3(x_4^{c-1} y))^{2d} = Y_5 y_0^{2d}.
\]

By Lemma 3.6 and Lemma 2.14
\[
Y_5 y_0^{2d} = \sum \phi_{(j;I)}(X^{2d-1})y_j^{2d},
\]
where \(1 \leq j < 5\), \(I \subset I_4\) and \(J \neq I_4\). From the above equalities, Cases 3.5.6, 3.5.7, and the inductive hypothesis, we get \(|x| \in \{\mathcal{P}(n)\}.

Case 3.5.10. \(\hat{y} = x_3^{b} f_3(y)\) for \(y \in \{P_{k-1}\}_{m-b}\) with \(\nu_1(y) = \nu_2(y) = 0\) and \(i = 3\).

We prove \(|x| \in \{\mathcal{P}(n)\}\) by double induction on \((\ell(I), b)\). If \(b = 0\), then by Case 3.5.1, \(|x| \in \{\mathcal{P}(n)\}\). If \(I = \mathcal{I}_3\), then by Case 3.5.4 \(|x| \in \{\mathcal{P}(n)\}\).

Suppose \(b > 0\). If \(\ell(I) < k - 4\), then applying Lemma 3.5(ii) with \(y_0 = x_3^{b-1} f_3(y)\), we obtain
\[
x = \phi_{(1;\mathcal{I}_3)}(X^{2d-1})(x_1 f_1(x_2^{b-1} y))^{2d} + \phi_{(2;\mathcal{I}_3)}(X^{2d-1})(x_2 f_2(x_2^{b-1} y))^{2d} + \sum \phi_{(3;\mathcal{I}_r)}(X^{2d-1})(x_3^{b-1} f_3(x_{r-1} y))^{2d}.
\]

Using Cases 3.5.6, 3.5.7 and the inductive hypothesis, we obtain \(|x| \in \{\mathcal{P}(n)\}.

Suppose that \(\ell(I) = k - 4\), and \(I = \mathcal{I}_{3,u} = (4, \ldots, \hat{u}, \ldots, k)\), \(3 < u \leq k\). If \(d_{k-2} > d_{k-1}\), then \(\omega_k(x) = \omega_1(y) = 1 = k - 2\). Hence \(a_0(\nu_j(y)) = 1\) for \(j = 3, \ldots, k - 1\).

Applying Lemma 3.5(i) with \(y_0 = x_3^{b-1} f_3(y)\) and Theorem 2.12 we get
\[
x = \phi_{(1;I)}(X^{2d-1})(x_1 f_1(x_2^{b-1} y))^{2d} + \phi_{(2;I)}(X^{2d-1})(x_2 f_2(x_2^{b-1} y))^{2d}.
\]

Hence by Cases 3.5.6 and 3.5.7 we get \(|x| \in \{\mathcal{P}(n)\}.

Suppose \(d_{k-2} = d_{k-1}\). Since \(\omega_k(x) = \omega_1(y) = 1 = k - 3\), we have \(\omega_1(y) = k - 4\).

Hence there exists uniquely \(3 \leq t \leq k - 1\) such that \(a_0(\nu_t(y)) = 0\).

If \(t = u - 1\), then using Lemma 3.5(i) with \(y_0 = x_3^{b-1} f_3(y)\) and Theorem 2.12 we get
\[
x = \phi_{(1;I)}(X^{2d-1})(x_1 f_1(x_2^{b-1} y))^{2d} + \phi_{(2;I)}(X^{2d-1})(x_2 f_2(x_2^{b-1} y))^{2d} + \phi_{(3;I_t)}(X^{2d-1})(x_3^{b-1} f_3(x_t y))^{2d}.
\]

From this equalities, Cases 3.5.6, 3.5.7 and 3.5.9 we get \(|x| \in \{\mathcal{P}(n)\}.

If \(u = 4 < t + 1\), then using Lemma 3.5(i) with \(y_0 = f_3(y)\) and Theorem 2.12 we get
\[
x = \phi_{(1;I)}(X^{2d-1})(x_1 f_1(x_2^{b-1} y))^{2d} + \phi_{(2;I)}(X^{2d-1})(x_2 f_2(x_2^{b-1} y))^{2d} + \phi_{(3;I_t)}(X^{2d-1})(x_3^{b-1} f_3(x_t y))^{2d}.
\]

Applying Lemma 3.5(i) with \(y_0 = x_3^{b-1} f_3(x_t y/x_4)\) and Theorem 2.12 we have
\[
\phi_{(3;I_t)}(X^{2d-1})(x_3^{b-1} f_3(x_t y/x_4))^{2d} = \sum_{1 \leq i \leq 3} \phi_{(i;I_t)}(X^{2d-1})(x_3^{b-1} x_i f_3(x_t y/x_4))^{2d}.
\]
From the above equalities, Cases 3.5.6, 3.5.7 and the inductive hypothesis, we get $|x| \in |\mathcal{P}(n)|$.

Suppose that $4 < u \neq t + 1$. Using Lemma 3.5(i) with $y_0 = f_3(y)$ and Theorem 2.12, we obtain

$$x \equiv \phi_{(1;I)}(X^{2d-1})(x_1f_1(x_2^{b-1}y))^{2d} + \phi_{(2;I)}(X^{2d-1})(x_2f_2(x_2^{b-1}y))^{2d} + \phi_{(4;I\setminus 4)}(X^{2d-1})(x_3^{b-1}f_3(x_3y))^{2d}.$$

From the above equalities, Cases 3.5.6, 3.5.7 and 3.5.9, we get $|x| \in |\mathcal{P}(n)|$.

**Case 3.5.11.** $\bar{y} = x_2^{a/2}f_2(y)$ for $y \in (P_{k-1})_{m-a}$ with $\nu_1(y) = 0$ and $i = 2$.

It suffices to prove $|x| \in |\mathcal{P}(n)|$ for $s = 0$. If $t(I) < k - 3$, then $|x| \in |\mathcal{P}(n)|$ by Case 3.5.7. If $I = I_2$, then by Case 3.5.3, $|x| \in |\mathcal{P}(n)|$. Suppose $t(I) = k - 3$. Then $I = I_2, u = (3, \ldots, \hat{u}, \ldots, k)$. If $u = 3$, then using Lemma 3.5(i) with $y_0 = f_2(y)$, we get

$$x \equiv \phi_{(1;I_3)}(X^{2d-1})(x_1f_1(y))^{2d} + \phi_{(3;I_3)}(X^{2d-1})(f_2(x_2y))^{2d} + \sum_{i=4}^{k} \phi_{(4;I_4)}(X^{2d-1})(f_2(x_{r-1}y))^{2d}.$$

Using Cases 3.5.4, 3.5.6, 3.5.9 and the above equalities, we obtain $|x| \in |\mathcal{P}(n)|$.

If $u > 3$, then using Lemma 3.5(ii) with $y_0 = f_2(y)$, we get

$$x \equiv \phi_{(1;I)}(X^{2d-1})(x_1f_1(y))^{2d} + \phi_{(3;I)}(X^{2d-1})(f_2(x_{u-1}y))^{2d} + \sum_{4 \leq r \leq k, r \neq u} \phi_{(3;I \setminus 3)}(X^{2d-1})(f_2(x_{r-1}y))^{2d}.$$

Using Cases 3.5.4, 3.5.6, 3.5.9 and the above equalities, we obtain $|x| \in |\mathcal{P}(n)|$.

**Case 3.5.12.** $\bar{y} = x_2^{a/2}x_3^{b/2}f_3(y)$ for $y \in (P_{k-1})_{m-a-b}$ with $\nu_1(y) = \nu_2(y) = 0$ and $i = 3$.

According to Lemma 3.8, we assume $a = 2^{s-1} - 1$. We prove $|x| \in |\mathcal{P}(n)|$ by double induction on $(t(I), b)$.

If $b = 0$, then by Case 3.5.1, $|x| \in P[n]$. If $I \neq I_3$, then using Lemma 3.5(iii) with $y_0 = x_2^{a/2}x_3^{b-1}f_3(y)$, we get

$$x \equiv \phi_{(1;I\setminus 3)}(X^{2d-1})(x_1f_1(x_2^{b-1}y))^{2d} + \phi_{(2;I\setminus 3)}(X^{2d-1})(f_2(x_2^{b-1}y))^{2d} + \sum_{r=4}^{k} \phi_{(3;I \setminus r)}(X^{2d-1})(x_3^{b-1}f_3(x_{r-1}y))^{2d}.$$

From this, Cases 3.5.6, 3.5.7 and the inductive hypothesis we obtain $|x| \in |\mathcal{P}(n)|$.

If $I = I_3$, then using Lemma 3.5(i) with $y_0 = x_2^{a/2}x_3^{b-1}f_3(y)$, we get

$$x \equiv \phi_{(1;I_3)}(X^{2d-1})(x_1f_1(x_2^{a/2}x_3^{b-1}y))^{2d} + \phi_{(2;I_3)}(X^{2d-1})(f_2(x_2^{b-1}y))^{2d} + Y_{4b_0}^{2d}.$$

By Lemma 3.6 and Lemma 2.14, we have

$$Y_{4b_0}^{2d} \equiv \sum_{(j;J)} \phi_{(j;J)}(X^{2d-1})y_j^{2d},$$
where $1 \leq j < 4$ and $J \subset I_3$ and $J \neq I_3$. Using Cases 3.5.6 3.5.11 the above equalities and the induction hypothesis, we obtain $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.13.** $y = x_1^2 f_2(y)$ for $y \in (P_{k-1})_{m-a}$ with $v_1(y) = 0$ and $i = 2$.

We prove $[x] \in [\mathcal{P}(n)]$ by double induction on $(\ell(I), a)$. If $a = 0$, then by Case 3.5.1 $[x] \in [\mathcal{P}(n)]$. If $I = I_2$, then by Case 3.5.3 $[x] \in [\mathcal{P}(n)]$. Suppose $a > 0$ and $\ell(I) < k - 3$. Applying Lemma 3.5 ii with $y_0 = x_2^{a-1} f_2(y)$, we get

$$x \equiv \phi_{(1; I, 2)}(X^{d-1})(x_1 f_1(x_1^{a-1} y))^{2^d} + \sum_{r=3}^{k} \phi_{(2; I, r)}(X^{d-1})(x_2^{a-1} f_2(x_r y))^{2^d}.$$  

Using Case 3.5.6 and the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$.

Suppose that $I = I_{2, u} = (3, \ldots, \hat{u}, \ldots, k)$, $3 \leq u \leq k$.

If $u = 3$, then $I = I_3$. Applying Lemma 3.5 i with $y_0 = x_2^{a-1} f_2(y)$, we get

$$x \equiv \phi_{(1; I, 3)}(X^{d-1})(x_1 f_1(x_1^{a-1} y))^{2^d} + \phi_{(3; I, 3)}(X^{d-1})(x_2^{a-1} f_2(x_3 y))^{2^d} + \sum_{r=4}^{k} \phi_{(4; I, r)}(X^{d-1})(x_r x_2^{a-1} f_2(y))^{2^d}.$$  

Applying Lemma 3.6 and Lemma 2.14 one gets

$$\sum_{r=4}^{k} \phi_{(4; I, r)}(X^{d-1})(x_r x_2^{a-1} f_2(y))^{2^d} = Y_0 y_0^{d^d}$$  

$$\equiv \sum_{(j, J)} \phi_{(j; J)}(X^{d-1})(x_j x_2^{a-1} f_2(y))^{2^d},$$  

where the last sum runs over some $(j; J)$ with $1 \leq j < 4$, $J \subset I_3$ and $J \neq I_3$. Since $\ell(J) < \ell(I_3) = k - 3$, from the above equalities, Cases 3.5.4 3.5.6 3.5.12 and the inductive hypothesis, we get $[x] \in [\mathcal{P}(n)]$.

If $u > 3$, applying Lemma 3.5 i with $y_0 = x_2^{a-1} f_2(y)$, we get

$$x \equiv \phi_{(1; I)}(X^{d-1})(x_1 f_1(x_1^{a-1} y))^{2^d} + \sum_{r=3}^{k} \phi_{(3; I, r)}(X^{d-1})(x_2^{a-1} f_2(x_r y))^{2^d}. $$  

From the last equalities, Cases 3.5.6 and 3.5.12 we have $[x] \in [\mathcal{P}(n)]$.

**Case 3.5.14.** $y = x_1^2 f_1(y)$ with $y \in (P_{k-1})_{m-2}$ and $i = 1$.

By Lemma 3.8 we need only to prove $[x] \in [\mathcal{P}(n)]$ for $s = 0$. If $\ell(I) < k - 2$, then $[x] \in [\mathcal{P}(n)]$ by Case 3.5.6. Suppose $\ell(I) = k - 2$ and $I = I_{1, u} = (2, \ldots, \hat{u}, \ldots, k)$.

If $u = 2$, then $I = I_2$. Applying Lemma 3.5 i with $y_0 = f_1(y)$, one gets

$$x \equiv \phi_{(2; I, 2)}(X^{d-1})(f_1(x_1 y))^{2^d} + \sum_{r=3}^{k} \phi_{(3; I, r)}(X^{d-1})(f_1(x_r y))^{2^d}. $$  

From the last equalities and Cases 3.5.3 3.5.12 we have $[x] \in [\mathcal{P}(n)]$.

If $u > 2$, then applying Lemma 3.5 i with $y_0 = f_1(y)$, one obtain

$$x \equiv \phi_{(2; I, 2)}(X^{d-1})(f_1(x_u y))^{2^d} + \sum_{2 \leq r \leq k, r \neq u} \phi_{(2; I, r)}(X^{d-1})(f_1(x_r y))^{2^d}. $$  

From the above equalities and Cases 3.5.3 3.5.13 we have $[x] \in [\mathcal{P}(n)]$.
Case 3.5.15. $\bar{y} = x_1^a x_2^b f_2(y)$ for $y \in (P_{k-1})_{m-a-b}$ with $\nu_1(y) = 0$ and $i = 2$.

We prove $[x] \in [P(n)]$ by double induction on $(\ell(I), b)$. By Lemma 3.8 we assume that $a = 2^s - 1$.

If $b = 0$, then $[x] \in [P(n)]$ by Case 3.5.1. Suppose that $b > 0$.

If $I \neq I_2$, then applying Lemma 3.5(ii) with $y_0 = x_1^a x_2^{b-1} f_2(y)$, we get
\begin{align*}
x & \equiv \phi_{(1, l, 2)}(X^{2^{d-1}})(x_1^{2^r} x_2^{b-1} f_2(y))^{2^d} \\
& \quad + \sum_{3 \leq r \leq k} \phi_{(2, l, r)}(X^{2^{d-1}})(x_1^a x_2^{b-1} f_2(x_{r-1} y))^{2^d}.
\end{align*}
From the last equalities, Case 3.5.14 and the inductive hypothesis, we have $[x] \in [P(n)]$.

If $I = I_2$, then applying Lemma 3.5(i) with $y_0 = x_1^a x_2^{b-1} f_2(y)$, we get
\begin{align*}
x & \equiv \phi_{(1, I_2)}(X^{2^{d-1}})(x_1^a x_2^{b-1} f_2(y))^{2^d} + \sum_{3 \leq r \leq k} \phi_{(3, I_2)}(X^{2^{d-1}})(x_1^a x_2^{b-1} f_2(y))^{2^d}.
\end{align*}

By Lemma 3.6 and Lemma 2.14 we have
\begin{align*}
\sum_{3 \leq r \leq k} \phi_{(3, I_2)}(X^{2^{d-1}})(x_1^a x_2^{b-1} f_2(y))^{2^d} & = Y_3 y_0^{2^d} \\
& \equiv \sum_{(j; J)} \phi_{(j; J)}(X^{2^{d-1}})(x_j x_1^a x_2^{b-1} f_2(y))^{2^d},
\end{align*}
where the last sum runs over some $(j; J)$ with $j = 1, 2, J \subset I_2$ and $J \neq I_2$.

From the above equalities, Case 3.5.14 and the inductive hypothesis, we have $[x] \in [P(n)]$.

Case 3.5.16. $\bar{y} = x_1^a f_1(y)$ for $y \in (P_{k-1})_{m-a}$ and $i = 1$.

If $a = 0$, then by Case 3.5.1 $[x] \in [P(n)]$. Suppose that $a > 0$. If $\ell(I) < k - 2$, then applying Lemma 3.5(ii) with $y_0 = x_1^a x_2^{b-1} f_2(y)$, we get
\begin{align*}
x & \equiv \phi_{(1, l, r)}(X^{2^{d-1}})(x_1^{a-1} f_1(x_{r-1} y))^{2^d}.
\end{align*}
Hence by the inductive hypothesis, we have $[x] \in [P(n)]$.

Suppose that $\ell(I) = k - 2$. Then $I = I_1, u = (2, \ldots, \hat{u}, \ldots, k)$. If $u = 2$, then applying Lemma 3.5(i) with $y_0 = x_1^{a-1} f_1(y)$ and Lemma 2.14 we get
\begin{align*}
x & \equiv \phi_{(2, I_2)}(X^{2^{d-1}})(x_1^{a-1} f_1(x_1 y))^{2^d} + \sum_{r = 3}^{k} \phi_{(3, I_2)}(X^{2^{d-1}})(x_1 x_1^{a-1} f_1(y))^{2^d}.
\end{align*}
By Lemma 3.6 and Lemma 2.14 we have
\begin{align*}
\sum_{r = 3}^{k} \phi_{(3, I_2)}(X^{2^{d-1}})(x_1 x_1^{a-1} f_1(y))^{2^d} & = Y_3 y_0^{2^d} \\
& \equiv \sum_{(j; J)} \phi_{(j; J)}(X^{2^{d-1}})(x_j x_1^{a-1} f_1(y))^{2^d},
\end{align*}
where the last sum runs over some $(j; J)$ with $j = 1, 2, J \subset I_2$ and $J \neq I_2$.

From the above equalities, Case 3.5.15 and the inductive hypothesis, we have $[x] \in [P(n)]$. 
If \( u > 2 \), then applying Lemma 3.5(i) with \( y_0 = x_{i}^{a-1}f_{1}(y) \), we get
\[
x \equiv \phi(2; I; J)(X^{2^{e-1}})(x_{i}^{a-1}f_{1}(x_{j}y))^{2^{e}} + \sum_{3 \leq u \leq k, r \neq u} \phi(2; I; J)(X^{2^{e-1}})(x_{r}^{-1}f_{1}(x_{r}^{-1}y))^{2^{e}}.
\]

From the above equalities, Case 3.5.15 and the inductive hypothesis, we have \([x] \in \mathcal{P}(n)\).

**Case 3.5.17.** \( y = x_{i}^{a}f_{1}(y) \) for \( y \in (P_{k-1})_{m-a} \).

If \( a = 0 \), then by Case 3.5.1 \([x] \in \mathcal{P}(n)\). If \( a > 0 \) and \( i = 1, 2 \), then by Cases 3.5.15 and 3.5.16 \([x] \in \mathcal{P}(n)\). If \( a > 0 \) and \( i > 2 \), then applying Lemma 3.5(ii) with \( y_{0} = x_{i}^{a-1}f_{1}(y) \), we get
\[
x = \sum_{1 \leq j < i} \phi(i; J, \gamma) + \sum_{i < j \leq k} \phi(i; I, J)(X^{2^{e-1}})y_{j}^{2^{e}},
\]
where \( y_{j} = x_{i}^{-1}f_{1}(x_{j}^{-1}y) \) for \( j > i \). Hence using the inductive hypothesis, we get \([x] \in \mathcal{P}(n)\). So we have proved \([x] \in \mathcal{P}(n)\) for all \( x \in (P_{k})_{n} \).

Now we prove that \([B_{k}(n)]\) is linearly independent in \( QP_{k} \). Suppose that there is a linear relation
\[
S = \sum_{(i; I, z) \in N_{k} \times B_{k-1}(n)} \gamma(i; I, z) \phi(i; I)(z) \equiv 0,
\]
(3.6)
where \( \gamma(i; I, z) \in \mathbb{F}_{2} \).

If \( d \geq k \), then by induction on \( \ell(I) \), we can show that \( \gamma(i; I, z) = 0 \), for all \((i; I) \in N_{k} \) and \( z \in B_{k-1}(n) \) (see [32] for the case \( d > k \)).

Suppose that \( d = k - 1 \). By Lemma 3.7, the homomorphism \( p_{j} = p_{j}(\gamma) \) sends the relation (3.6) to \( \sum_{z \in B_{k-1}(n)} \gamma(j; \gamma, z)z \equiv 0 \). This relation implies \( \gamma(j; \gamma, z) = 0 \) for any \( 1 \leq j \leq k \) and \( z \in B_{k-1}(n) \).

Suppose \( 0 < \ell(J) < k - 3 \) and \( \gamma(i; I, z) = 0 \) for all \((i; I) \in N_{k} \) with \( \ell(I) < \ell(J), 1 \leq i \leq k \) and \( z \in B_{k-1}(n) \). Then using Lemma 3.7 and the relation (3.4), we see that the homomorphism \( p_{j}(\gamma) \) sends the relation (3.6) to \( \sum_{z \in B_{k-1}(n)} \gamma(j; J, z)z \equiv 0 \). Hence we get \( \gamma(j; J, z) = 0 \) for all \( z \in B_{k-1}(n) \).

Now, let \((j; J) \in N_{k} \) with \( \ell(J) = k - 3 \). If \( J \neq I_{3} \), then using Lemma 3.7 we have \( p_{j}(\gamma)(\phi(i; I, z))z \equiv 0 \) for all \( z \in B_{k-1}(n) \) and \((i; I) \in N_{k} \) with \( (i; I) \neq (j; J) \). So we get
\[
p_{j}(\gamma) = \sum_{z \in B_{k-1}(n)} \gamma(j; J, z)z \equiv 0.
\]
Hence \( \gamma(j; I, z) = 0 \), for all \( z \in B_{k-1}(n) \).

According to Lemma 3.7, \( p_{j}(\gamma)(\phi(1; I_{3}))(z) \equiv 0 \) for \( z \in \mathcal{C} \) and \( p_{j}(\gamma)(\phi(1; I_{3}))(z) \in \langle \mathcal{E} \rangle \) for \( z \in \mathcal{D} \cup \mathcal{E} \). Hence we obtain
\[
p_{j}(\gamma) = \sum_{z \in \mathcal{C} \cup \mathcal{D}} \gamma(j; J, z)z \equiv 0 \mod \langle \mathcal{E} \rangle.
\]

So we get \( \gamma(j; I_{3}, z) = 0 \) for all \( z \in \mathcal{C} \cup \mathcal{D} \).

Now, let \((j; J) \in N_{k} \) with \( \ell(J) = k - 2 \). Suppose that \( I_{3} \notin J \). Then using Lemma 3.7, we have \( p_{j}(\gamma)(\phi(1; I_{3}))(z) \equiv 0 \) for all \( z \in \mathcal{B} \). Hence we get
\[
p_{j}(\gamma) = \sum_{z \in \mathcal{B}} \gamma(j; J, z)z \equiv 0.
\]
From this, we obtain $\gamma_{(j; I), z} = 0$ for all $z \in B$.

Suppose that $I_3 \subset J$. Then either $J = I_2, j = 1, 2$ or $J = I_3 \cup 2, j = 1$. According to Lemma 3.7, $p_{(j; I_2)}(\phi_{(1; I_1)}(z)) \in \langle D \cup E \rangle$ for all $z \in B$, $p_{(1; I_3 \cup 2)}(\phi_{(1; I_1)}(z)) \equiv 0$ for $z \in C \cup D$ and $p_{(1; I_3 \cup 2)}(\phi_{(1; I_1)}(z)) \in \langle E \rangle$ for $z \in E$. Hence we obtain

$$p_{(j; I_2)}(S) \equiv \sum_{z \in C} \gamma_{(j; I_2), z} z \equiv 0 \mod \langle D \cup E \rangle,$$

$$p_{(1; I_3 \cup 2)}(S) \equiv \sum_{z \in C \cup D} \gamma_{(1; I_3 \cup 2), z} z \equiv 0 \mod \langle E \rangle.$$

So $\gamma_{(j; I_2), z} = 0$ for $z \in C$ and $\gamma_{(1; I_3 \cup 2), z} = 0$ for $z \in C \cup D$. Since $\gamma_{(i; I), z} = 0$, for all $z \in C$ and $I \neq I_1$, applying Lemma 3.7, we have

$$p_{(1; I_1)}(S) \equiv \sum_{z \in C} \gamma_{(1; I_1), z} z \equiv 0 \mod \langle D \cup E \rangle.$$

Hence $\gamma_{(1; I_1), z} = 0$ for all $z \in C$. So the relation (3.6) becomes

$$S = \sum_{1 \leq i \leq 3, z \in E} \gamma_{(i; I_1), z} \phi_{(i; I_2)}(z) + \sum_{z \in E} \gamma_{(1; I_3 \cup 2), z} \phi_{(1; I_1)}(z)$$

$$+ \sum_{1 \leq i \leq 2, z \in D \cup E} \gamma_{(i; I_3), z} \phi_{(i; I_2)}(z) + \sum_{z \in D \cup E} \gamma_{(1; I_1), z} \phi_{(1; I_1)}(z) \equiv 0. \quad (3.7)$$

Using the relation (3.7) and Lemma 3.7

$$p_{(i; I_2)}(S) \equiv \sum_{z \in D} (\gamma_{(i; I_2), z} + \gamma_{(i; I_1), z}) z \equiv 0 \mod \langle E \rangle, \quad i = 1, 2.$$
We prove the lemma by induction on $\ell(I)$. Let $i, j$ be positive integers such that $0 < i < j \leq k$, and $a, b > 0$ with $a + b = 2^d - 1$. Then
\[ X_i^a X_j^b \simeq_2 X_i^{2^d-2} X_j = \phi_{(i,j)}(X^{2^d-1}). \]

Proof. We prove the lemma by induction on $b$. If $b = 1$, then
\[ X_i^a X_j^1 = X_i^{2^d-2} X_j. \]
So the lemma holds. Suppose that \( b > 1 \). Note that \( X_i^a X_j^b = x_i^b x_j^a X_{j-i}^{d-1} \). If \( a_0(b) = 0 \), then
\[
x \simeq_0 S^1(x_i^{b-1} x_j^a X_{j-i}^{d-1}) + x_i^{b-1} x_j^{a+1} X_{j-i}^{d-1} \simeq_1 X_i^{a+1} X_j^{b-1} \simeq_2 X_i^{d-2} X_j.
\]
If \( a_0(b) = 1 \), then
\[
x \simeq_0 S^1(x_i^{b-2} x_j^{a+1} X_{j-i}^{d-1}) + S^2(x_i^{b-2} x_j^a X_{j-i}^{d-1}) + x_i^{b-1} x_j^{a+1} X_{j-i}^{d-1}
\]\
\[
\simeq_2 x_i^{b-1} x_j^{a+1} X_{j-i}^{d-1} = X_i^{a+1} X_j^{b-1} \simeq_2 X_i^{d-2} X_j.
\]

The lemma is proved. \( \square \)

**Proof of Lemma 3.10** We prove the lemma by induction on \( d \). Suppose \( d = 2 \). If \( j_0 = j_1 = i \), then \( x = \phi_{(i,0)}(X^2) \). If \( j_0 > j_1 = i \), then \( x = X_i^2 X_j = \phi_{(i,i)}(X^2) \). Suppose \( d > 2 \). By the inductive hypothesis, there is \( (i_1; I') \in N_k \) such that \( \prod_{0 \leq i < d-1} X_i^{r_i} \simeq_{d-2} \phi_{(i_1; I')}(X_i^{d-2}) \), where \( i_1 = \min\{j_0, j_1, \ldots, j_{d-2}\} \). If \( j_{d-1} = i \), then the lemma holds with \( (i; I) = (i_1; I') \). Suppose that \( j_{d-1} \neq i_1 \).

If \( I' = \emptyset \), then using Lemma 3.10, we have
\[
x \simeq_{d-2} X_i^{d-2} X_{j_{d-1}} \simeq_2 \phi_{(i,i)}(X_i^{d-2}),
\]
where \( i = \min\{i_1, j_{d-1}\} = \min\{j_0, j_1, \ldots, j_{d-1}\} \). The lemma holds. Suppose \( I' = (i_1', i_2', \ldots, i_{r'}), 0 < r < d-2 \) and \( I_2 = (i_2', \ldots, i_{r'}) \), then
\[
\phi_{(i_i, I')}(X_i^{d-2}) = \phi_{(i_i; I_2)}(X_i^{d-2}) X_i^{d-2}.\]

If \( i_1 < j_{d-1} \) and \( r = d-2 \), then \( X_i X_j \simeq_1 X_i X_{j_{d-1}} \). Hence using Proposition 2.5(ii), one gets
\[
x \simeq_{d-2} \phi_{(i_i, I_2)}(X_i^{d-2}) X_i^{d-2} \simeq_1 X_i X_{j_{d-1}} \simeq_2 X_i^{d-2} X_{j_{d-1}}.\]

If \( i_1 < j_{d-1} \) and \( r < d-2 \), then using Lemma 3.10, we have \( X_i^{d-2-r} X_{j_{d-1}} \simeq_2 X_i^{d-2} X_{j_{d-1}}. \) Hence by Proposition 2.5(ii),
\[
x \simeq_{d-2} \phi_{(i_i, I_2)}(X_i^{d-2}) X_i^{d-2} X_{j_{d-1}} \simeq r+2 \phi_{(i_i, I_2)}(X_i^{d-2}) X_i^{d-2} X_{j_{d-1}} \simeq_{d-1} \phi_{(i_i, I_2)}(X_i^{d-2}) X_i^{d-2} X_{j_{d-1}} \simeq_{d-1} \phi_{(i_i, I_2)}(X_i^{d-2}) X_i^{d-2} X_{j_{d-1}} \simeq (\phi_{(j; I)}(X_i^{d-2}))
\]

By the inductive hypothesis, there is \( (j; I) \in N_k \) such that
\[
\phi_{(j; I)}(X_i^{d-2}) \simeq (\phi_{(j; I)}(X_i^{d-2})),
\]
for $0 < r < d - 2$. So, from the above equalities and Proposition 2.5(ii), we get

$$x \simeq_d 2 \phi_{(i;I)}(X^{2d-1})X_{i_i}^{d-2} = \phi_{(i_1;I_1)}(X^{2d-1}).$$

The lemma holds.

If $i_1 > j_d - 1$ and $r = d - 2$, then

$$x \simeq_{d-2} \phi_{(i_1;I_1)}(X_i^2)X_{j_d-1}^{d-2} = \phi_{(j_1;I_1)}(X^{2d-1}).$$

If $i_1 > j_d - 1$ and $r < d - 2$, then using Lemma 3.10 we have $X_{i_1}^{d-2}X_{j_d-1}^{d-1} \simeq_2 2X_{j_d-2}^{d-2}X_{i_1}$. Hence by Proposition 2.5(ii),

$$x \simeq_{d-2} \phi_{(i_1;I_1)}(X_i^2)X_{j_d-1}^{d-2} X_{i_1}^{d-2} = \phi_{(j_1;I_1)}(X^{2d-1}).$$

Since $r + 2 < d$, the lemma is proved.

From the proof of Lemma 3.4 we easily obtain the following.

**Corollary 3.11.** Let $(i; I) \in N_k$, $j \in N_k$ and a polynomial $y$ in $(P_k)_m$. If $j > i$ and $d > r + 1$, then

i) $\phi_{(i;I)}(X^{2d-1})X_j^{d-2} = \simeq_{d-1} \phi_{(i;I)}(X^{2d-1}).$

ii) $X_j^{d-1}\phi_{(i;I)}(X^{2d-1})X_j^{d-1} \simeq_{d-1} \phi_{(i;I)}(X^{2d-1}).$

**Proof of Lemma 3.10** Applying the Cartan formula, we have

$$S_{q^1}(X^{2d-1}y_i^c) = \sum_{1 \leq j \leq k} X_j^{2d-1}y_j^c,$$

where $c$ is a positive integer. From this, we obtain

$$X_j^{2d-1}y_j^c = \sum_{1 \leq j < i} X_j^{2d-1}y_j^c + \sum_{i < j \leq k} X_j^{2d-1}y_j^c.$$

If $d > r$, then $\phi_{(i;I)}(X^{2d-1})y_i^c = \phi_{(i;I^+)}(X^{2d-1})X_j^{2d-1}y_j^c$, with $c = d - r$ and $I^+ = (i_1, i_2, \ldots, i_r)$. Hence using Lemma 2.14 we get

$$\phi_{(i;I)}(X^{2d-1})y_i^c \equiv \sum_{1 \leq j < i} \phi_{(i;I^+)}(X^{2d-1})X_j^{2d-1}y_j^c + \sum_{i < j \leq k} \phi_{(i;I^+)}(X^{2d-1})y_j^c.$$

Applying Corollary 3.11 and Lemma 2.14, we have

$$\phi_{(i;I^+)}(X^{2d-1})X_j^{2d-1}y_j^c = \phi_{(i;I^+)}(X^{2d-1})y_j^c, \text{ for } j < i,$$

$$\phi_{(i;I^+)}(X^{2d-1})X_j^{2d-1}y_j^c = \phi_{(i;I^+)}(X^{2d-1})y_j^c, \text{ for } j > i.$$

Hence the first part of the lemma follows.

If $d > r + 1$, then $\phi_{(i;I)}(X^{2d-1})y_i^c = \phi_{(i;I)}(X^{2d-1})y_i^c$, with $c = d - r - 1$. Hence using Lemma 2.14 we get

$$\phi_{(i;I)}(X^{2d-1})y_i^c \equiv \sum_{1 \leq j < i} \phi_{(i;I)}(X^{2d-1})X_j^{2d-1}y_j^c + \sum_{i < j \leq k} \phi_{(i;I)}(X^{2d-1})y_j^c.$$
According to Corollary 3.11 and Lemma 2.14

\[
\phi(i; j)(X^{2^{r+1}-1} x_i x_j^{2^{r+1}}) = \phi(i; j)(X^{2^{r+1}}) x_j^{2^{r+1}}, \text{ for } j < i, \\
\phi(i; j)(X^{2^{r+1}-1} x_i x_j^{2^{r+1}}) = \phi(i; j)(X^{2^{r+1}}) x_j^{2^{r+1}}, \text{ for } j > i.
\]

So the second part of the lemma is proved.

We need the following lemmas for the proof of Lemma 3.6.

**Lemma 3.12.** For any integer \(0 < \ell \leq k\),

\[
X_{\ell+1}^{2^{\ell-1} x_{\ell+1}^{2^{\ell}}} \approx \sum_{r=\ell}^{k} \sum_{(i; l) \in N_{r-1}} \phi(i; l)(X^{2^{l-1}}) x_r^{2^{l}} + \sum_{r=\ell+1}^{k} X_{r+1}^{2^{\ell-1} x_{r+1}^{2^{\ell}}}.
\]

**Proof.** We prove the lemma by induction on \(\ell\). For \(\ell = 1\), the lemma is trivial. Suppose that \(\ell \geq 1\) and the lemma is true for \(\ell\). Using the Cartan formula we have

\[
X_{\ell+1}^{2^{\ell+1-1} x_{\ell+1}^{2^{\ell+1}}} = \sum_{r=1}^{\ell} X_{r}^{2^{\ell+1-1} x_{r}^{2^{\ell+1}}} + \sum_{r=\ell+2}^{k} X_{r}^{2^{\ell+1-1} x_{r}^{2^{\ell+1}}} + \sum_{r=\ell+2} \phi(i; l)(X^{2^{l-1}}) x_r^{2^{l}} + \sum_{r=\ell+2}^{k} X_{r}^{2^{\ell+1-1} x_{r}^{2^{\ell+1}}}.
\]

Using the inductive hypothesis and Proposition 2.5, we have

\[
X_{r}^{2^{\ell+1-1} x_{r}^{2^{\ell+1}}} \approx \sum_{r=\ell+1}^{k} \sum_{(i; l) \in N_{r-1}} \phi(i; l)(X^{2^{l-1}}) x_r^{2^{l}}.
\]

According to Corollary 3.11

\[
X_{r}^{2^{\ell+1-1} x_{r}^{2^{\ell+1}}} \approx \sum_{(i; l) \in N_{r-1}} \phi(i; l)(X^{2^{l-1}}) x_r^{2^{l}}.
\]

From the above equalities, we get

\[
X_{r}^{2^{\ell+1-1} x_{r}^{2^{\ell+1}}} \approx \sum_{(i; l) \in N_{r-1}} \phi(i; l)(X^{2^{l-1}}) x_r^{2^{l}} + \sum_{m=r+1}^{k} \sum_{(i; l) \in N_{r-1}} \phi(i; l)(X^{2^{l-1}}) x_m^{2^{l}} + \phi(r)(X^{2^{l+1-1}}) x_m^{2^{l+1}}.
\]
By a direct computation from the above equalities, using the relation 3.10, we have
\[
\sum_{r=1}^{\ell} X_{r}^{2^{\ell+1}-1}x_{r}^{2^{\ell+1}} + \sum_{r=1}^{\ell} \sum_{r\in N_{r-1}} \phi_{(i; I;r)}(X_{r}^{2^{\ell+1}-1}x_{r}^{2^{\ell+1}}) + \sum_{m=2}^{\ell-1} \sum_{r\in N_{r-1}} \phi_{(i; I;m)}(X_{m}^{2^{\ell+1}-1}x_{m}^{2^{\ell+1}})
\]
\[
+ \sum_{m=1}^{\ell-1} \sum_{r\in N_{r-1}} \phi_{(i; I;m)}(X_{m}^{2^{\ell+1}-1}x_{m}^{2^{\ell+1}}) + \sum_{m=1}^{\ell-1} \sum_{r\in N_{r-1}} \phi_{(i; I;m)}(X_{m}^{2^{\ell+1}-1}x_{m}^{2^{\ell+1}})
\]
\[
= \sum_{r=1}^{\ell} \sum_{r\in N_{r-1}} \phi_{(i; I;r)}(X_{r}^{2^{\ell+1}-1}x_{r}^{2^{\ell+1}}) + \sum_{m=2}^{\ell} \sum_{r\in N_{r-1}} \phi_{(i; I;m)}(X_{m}^{2^{\ell+1}-1}x_{m}^{2^{\ell+1}})
\]
\[
+ \sum_{m=1}^{\ell} \sum_{r\in N_{r-1}} \phi_{(i; I;m)}(X_{m}^{2^{\ell+1}-1}x_{m}^{2^{\ell+1}})
\]
\[
= \sum_{m=1}^{\ell} \sum_{r\in N_{r-1}} \phi_{(i; I;m)}(X_{m}^{2^{\ell+1}-1}x_{m}^{2^{\ell+1}}).
\]
Combining the above equalities, we get
\[
X_{\ell+1}^{2^{\ell+1}-1}x_{\ell+1}^{2^{\ell+1}} \simeq_{\ell+1} \sum_{r=\ell+1}^{\ell} \sum_{r\in N_{r-1}} \phi_{(i; I;r)}(X_{r}^{2^{\ell+1}-1}x_{r}^{2^{\ell+1}}) + \sum_{r=\ell+2}^{\ell} \sum_{r\in N_{r-1}} \phi_{(i; I;r)}(X_{r}^{2^{\ell+1}-1}x_{r}^{2^{\ell+1}}).
\]
The lemma is proved.

From the proof of this lemma, we obtain

**Corollary 3.13.** For \(2 \leq d \leq k\), we have
\[
\sum_{r=1}^{d} X_{r}^{d-1}x_{r}^{d} \simeq_{d} \sum_{r=1}^{k} \sum_{r\in N_{r-1}} \phi_{(i; I;r)}(X_{r}^{d-1}x_{r}^{d}).
\]

**Lemma 3.14.** For any integer \(d > k\), \(0 \leq r \leq d - k\) and \(0 < m < h \leq k\),
\[
Z := \phi_{(m; I;m)}(X_{r}^{d-r-1})X_{r}^{d-r} \simeq_{k-m+1} \phi_{(m; I;m)}(X_{r}^{d-1}).
\]

**Proof.** We prove the lemma by double induction on \((m, r)\). If \(m = k - 1\), then \(h = k\). By Lemma 3.10, we have
\[
\phi_{(k-1;k)}(X_{r}^{d-r-1})X_{k}^{d-r} = X_{k-1}^{d-r-2}X_{k}^{d-r+1} \simeq_{2} \phi_{(k-1;k)}(X_{r}^{d-1}).
\]
So, the lemma holds. Suppose that \(0 < m < k - 1\). If \(h = m + 1\), we have
\[
Z = \phi_{(m+2; I;m+2)}(X_{r}^{d-m-1-1})X_{r}^{d-k+m+1-2}X_{m+1}^{d-k+m+1}X_{m+1}^{d-k+m+1} \simeq_{2} \phi_{(k-1;k)}(X_{r}^{d-1}).
\]
According to Lemma 3.10
\[
Z \simeq_{k-m+1} \phi_{(m+2; I;m+2)}(X_{r}^{d-m-1-1})X_{r}^{d-k+m+1}X_{m+1}^{d-k-m-1}X_{m+1}^{2}.
\]
Hence using Proposition 2.5 we obtain
\[
Z \simeq_{k-m+1} \phi_{(m+2; I;m+2)}(X_{r}^{d-m-1-1})X_{r}^{d-k+m+1}X_{m+1}^{2k-m-1} = \phi_{(m; I;m)}(X_{r}^{d-1}).
\]
The lemma holds. Suppose that \( h > m + 1 \) and \( r = 1 \). We have

\[
Z = \phi(m+1; m+1)(X^{2^{k-m}-1})(X_m^{2^{d-k+m+1}}X_{h}^{2^{d-k+m+1}})^{2^{k-m}}.
\]

Since \( X_m^{2^{d-k+m+1}}X_{h}^{2^{d-k+m+1}} \approx 1 \) and the inductive hypothesis, we have

\[
Z \approx_{k-m+1} \phi(m+1; m+1)(X^{2^{k-m}-1})(X_m^{2^{d-k+m+1}}X_{h}^{2^{d-k+m+1}})^{2^{k-m}}
\]

By Lemma 3.12, we get

\[
Z \approx_{k-m+1} \phi(m+1; m+1)(X^{2^{k-m}-1})(X_m^{2^{d-k+m+1}}X_{h}^{2^{d-k+m+1}})^{2^{k-m}}.
\]

Applying Proposition 2.5 and the inductive hypothesis, we have

\[
Z \approx_{k-m+1} \phi(m+1; m+1)(X^{2^{k-m}-1})(X_m^{2^{d-k+m+1}}X_{h}^{2^{d-k+m+1}})^{2^{k-m}}
\]

According to Lemma 3.10, we have

\[
X_m^{2^{d-k+m+1}}X_{h}^{2^{d-k+m+1}} \approx_2 X_m^{2^{d-k+m+1}}X_{h}^{2^{d-k+m+1}}.
\]

Hence using Proposition 2.5 one gets

\[
Z \approx_{k-m+1} \phi(m+1; m+1)(X^{2^{k-m}-1})(X_m^{2^{d-k+m+1}}X_{h}^{2^{d-k+m+1}})^{2^{k-m}}
\]

Now, suppose that \( h > m + 1 \) and \( r > 1 \). Applying Proposition 2.5 and the inductive hypothesis, one gets

\[
Z = \phi(m+1; m+1)(X^{2^{d-r}-1})X_m^{2^{d-r}}X_{h}^{2^{d-r+1}}
\]

The lemma is proved. \( \square \)

**Lemma 3.15.** For any integer \( d \geq k \),

\[
X_k^{2^{d-1}}x_k^{2^{d-1}} \approx_k \sum_{(i; I) \in \mathcal{N}_{k-1}} \phi(i; I; \emptyset)(X^{2^{d-1}})x_k^{2^{d-1}}.
\]

**Proof.** By Lemma 3.12 we have

\[
X_k^{2^{d-1}}x_k^{2^{d-1}} \approx_k \sum_{(i; I) \in \mathcal{N}_{k-1}} \phi(i; I; \emptyset)(X^{2^{d-1}})x_k^{2^{d}}.
\]

Hence using Proposition 2.5 we get

\[
X_k^{2^{d-1}}x_k^{2^{d-1}} = X_k^{2^{d-1}}x_k^{2^{d}}X_{h}^{2^{d-2}} \approx_k \sum_{(i; I) \in \mathcal{N}_{k-1}} \phi(i; I; \emptyset)(X^{2^{d-1}})X_k^{2^{d-2}}x_k^{2^{d}}.
\]

Let \( (i; I) \in \mathcal{N}_{k-1} \). If \( I = \emptyset \), then using Lemma 3.10 we have

\[
\phi(i; I; \emptyset)(X^{2^{d-1}})X_k^{2^{d-2}}x_k^{2^{d}} = \phi(i; k)(X^{2^{d-1}})X_k^{2^{d-2}}x_k^{2^{d}}
\]

\[
X_k^{2^{d}}x_k^{2^{d}} = \phi(i; k)(X^{2^{d-1}})X_k^{2^{d-2}}x_k^{2^{d}}
\]

\[
\approx_{2} X_k^{2^{d-2}}X_k^{2^{d}}
\]

\[
= \phi(i; k)(X^{2^{d}}x_k^{2^{d}}).
\]
If $I = (i_1, \ldots, i_r), r > 0$, then $s = k - l(I \cup k) > 0$. Hence

$$Y := \phi_{(i_1; I \cup k)}(X^{2k-1})X^{2d-k}_{x_k^d} = \phi_{(i_1; I \cup k)}(X^{2k-1}X_k^{2d-k}x_k^{d-k}).$$

By Lemma 3.10

$$X_i^{2r-1}X_k^{2d-k}x_k^{d-k} \simeq 2X_i^{2d-k}x_k^{d-k}.$$

If $(i; I \cup k) \neq (1; I_1)$, then $s \geq 2$. Using Proposition 2.5 and Lemma 3.4, one gets

$$Y \simeq \phi_{(i_1; I \cup k)}(X^{2k-1})X_k^{2d-k}x_k^{d-k} = \phi_{(i_1; I \cup k)}(X^{2k-2}X_k^{2d-2}x_k^{d-2}).$$

Suppose that $(i; I \cup k) = (1; I_1)$. Then using Lemma 3.14 and Proposition 2.5 we have

$$\phi_{(1; I_1)}(X^{2d-1})X_k^{2d-2}x_k^{d-k} \simeq \phi_{(1; I_1)}(X^{2d-2})x_k^{d-k}.$$

The lemma is proved.

**Lemma 3.16.** $Y_1 \simeq (k, \omega) 0$ with $\omega = \omega(X_1^{2d-1}x_1^{d-k})$. More precisely,

$$Y_1 = \sum_{0 \leq k < k} S_q^{2d}(y_i) + h,$$

with $y_i$ polynomials in $P_k$, and $h \in P_k^{-}(\omega)$.

**Proof.** First we prove the following by induction on $m$

$$Y_1 \simeq (k, \omega) Y_m + \sum_{r = m}^{k} \sum_{i \in N_{m-1}} \phi_{(i; I \cup m-1)}(X^{2d-1})x_r^{2d}. \quad (3.9)$$

Note that

$$\phi_{(m; I_m)}(X^{2d-1})x_m^{2d} = \phi_{(m+1; I_{m+1})}(X^{2k-m-1})(X_m^{2m-1}x_m^{2m}X_k^{2d-m-k-2m})^{2k-m}.$$  

Applying Lemma 3.12 and Proposition 2.5 we have

$$X_m^{2m-1}x_m^{2d-k+m-2m} \simeq m \sum_{r = m+1}^{k} X_r^{2d-k+m-1}x_r^{d-k+m}$$

$$+ \sum_{r = m}^{k} \sum_{i \in N_{m-1}} \phi_{(i; I \cup r)}(X^{2m-1})X_r^{2d-k-m-2m}x_r^{d-k-m}. \quad (3.9)$$

Using Lemma 3.14 and Proposition 2.5 we have

$$\phi_{(m+1; I_{m+1})}(X^{2k-m-1})x_r^{2d-k-m} \simeq \phi_{(m+1; I_{m+1})}(X^{2d-1})x_r^{2d}.$$

From the above equalities, Proposition 2.5 and Lemma 3.4, one gets

$$\phi_{(m; I_m)}(X^{2d-1})x_m^{2d} \simeq [Y_{m+1} + \sum_{r = m}^{k} \sum_{i \in N_{m-1}} \phi_{(i; I \cup m-1)}(X^{2d-1})x_r^{2d}].$$
If either $r > m$ or $I \neq (2, \ldots, m - 1)$, then $(i; I \cup I_m \cup r) \neq (1; I_1)$. From the proof of Lemma 3.14 we have

$$\phi_{(i; I \cup I_m \cup r)}(X^{2^d-1})X_{r}^{2^d-2^k}x_r^{2^d} \simeq_k \phi_{(i; I \cup I_m \cup r)}(X^{2^d-1})x_r^{2^d}.$$

If $r = m$ and $I = (2, \ldots, m - 1)$, then $(i; I \cup I_m \cup m) = (1; I_1)$. By Lemma 3.14 we have

$$\phi_{(1; I_1)}(X^{2^d-1})X_m^{2^d-2^k}x_m^{2^d} \simeq_k \phi_{(1; I_1)}(X^{2^d-1})x_m^{2^d}.$$

Combining the above equalities, we get

$$\phi_{(m; I_m)}(X_m^{2^d-1})x_m^{2^d} \simeq_k Y_{m+1} + \sum_{r = m+1}^{k} \sum_{(i; I) \in \mathcal{N}_{m-1}} \phi_{(i; I \cup I_m \cup r)}(X^{2^d-1})x_r^{2^d}.$$

Using the above equalities and the inductive hypothesis, we get

$$Y_1 \simeq_{(k, \omega)} Y_{m+1} + \sum_{r = m}^{k} \left( \sum_{(i; I) \in \mathcal{N}_{m-1}} \phi_{(i; I \cup I_m \cup r)}(X^{2^d-1})x_r^{2^d} \right)$$

$$+ \sum_{r = m+1}^{k} \phi_{(m; I_m)}(X_m^{2^d-1})x_m^{2^d} + \sum_{(i; I) \in \mathcal{N}_{m-1}} \phi_{(i; I \cup I_m \cup r)}(X^{2^d-1})x_r^{2^d}$$

$$= Y_{m+1} + \sum_{r = m+1}^{k} \left( \sum_{(i; I) \in \mathcal{N}_{m-1}} \phi_{(i; I \cup I_m \cup r)}(X^{2^d-1})x_r^{2^d} \right) + \sum_{r = m+1}^{k} \phi_{(m; I_m)}(X_m^{2^d-1})x_m^{2^d}$$

(since $m \cup I_m = I_{m-1}$ and $I_m \cup r = I_m$ for $r > m$)

$$= Y_{m+1} + \sum_{r = m+1}^{k} \left( \sum_{(i; I) \in \mathcal{N}_{m}} \phi_{(i; I \cup I_m \cup r)}(X^{2^d-1})x_r^{2^d} \right)$$

(since $\mathcal{N}_m = \mathcal{N}_{m-1} \cup \{ \mathcal{N}_{m-1} \cup m \} \cup \{(m; \emptyset)\}$).

The relation (3.9) is proved.

Since $Y_k = X_k^{2^d-1}x_k^{2^d}$, using the relation (3.9) with $m = k$ and Lemma 3.12 one gets

$$Y_1 \simeq_{(k, \omega)} X_k^{2^d-1}x_k^{2^d} + \sum_{(i; I) \in \mathcal{N}_{k-1}} \phi_{(i; I \cup k)}(X^{2^d-1})x_k^{2^d} \simeq_{(k, \omega)} 0.$$

The lemma is proved. \hfill \frown

**Proof of Lemma 3.6** We have $Y_m = Z^{2^d-1}Y_1(x_m, \ldots, x_k)$ with $Z = x_1x_2 \ldots x_{m-1}$.

By Lemma 3.16, $Y_m$ is a sum of polynomials of the form $f = Z^{2^d-1}(S_{2^d}^i(y) + h)$ with $0 \leq i \leq k - m$, $y$ a monomial in $P_k-m+1 = P_k-m+1(x_m, \ldots, x_k)$ and $h \in P_k-m+1(\omega^*)$, $\omega^* = \omega((x_m+1 \ldots x_k)^{2^d-1}x_m^d)$. Then $Z^{2^d-1}h \in P_k(\omega)$ with $\omega = \omega(X_k^{2^d-1}x_k^{2^d})$. Using the Cartan formula, we have

$$f \simeq_{(0, \omega)} S_{2^d}^i(Z^{2^d-1}y) + \sum_{1 \leq t \leq 2^d} S_{2^d}^i(Z^{2^d-1}S_{2^d}^{2^d-1}(y)).$$
By a direct computation using the Cartan formula, we can show that if $0 < t < 2^i$, then $\omega_u(Sq^t(Z^{d-1})Sq^{2^i-t}(y)) < k - 1$ for some $u \leq d$. Hence one gets

$$ f \simeq_{(k, \omega)} Sq^t(Z^{d-1})y \simeq_{(k, \omega)} \sum_{0 < j < m} Z^{d-1}x_j^t y. $$

Since $\omega_u(Z^{d-1}x_j^t) = m - 2$ for $i < u \leq k$, if $Z^{d-1}x_j^t y \notin P_k^{-}(\omega)$, then $\omega_u(y) = k - m$ for $i < u \leq k$. According to Lemma 3.4 there is $(j; J) \in \mathcal{N}_k$ such that $Z^{d-1}x_j^t y \cong_i \phi(j; J)(X^{2^d-1})x_j^{2^d}$. Here $J \subset I_{m-1}$. Since $0 \leq \ell(J) = i \leq k - m < \ell(I_{m-1}) = k - m + 1$, we have $J \neq I_{m-1}$. The lemma is proved. □

The following will be used in the proof of Lemma 3.7.

**Lemma 3.17.** Let $(j; J), (i; I) \in \mathcal{N}_k$ with $\ell(I) < d$. Then

$$ p_{(j; J)}\phi_{(i; I)}(X^{2^{d-1}}) = \begin{cases} X^{2^{d-1}}, & (i; I) \subset (j; J), \\ 0, & (i; I) \not\subset (j; J). \end{cases} $$

**Proof.** Suppose that $(i; I) \not\subset (j; J)$. If $i \not\in (j; J)$, then from (3.2), we see that $p_{(j; J)}(\phi_{(i; I)}(X^{2^{d-1}}))$ is a sum of monomials of the form

$$ w = x_j^{2^{d-1}-1}f_{k-1;i'}(z), $$

for suitable monomial $z$ in $P_{k-2}$. Here $i' = i$ if $j > i$ and $i' = i - 1$ if $j < i$. In this case, we have $\alpha_i(2^r - 1) = 0$ and $\omega_{r+1}(w) < k - 1$. Hence $w \in P_{k-1}(\omega(d))$, where $\omega(d) = \omega(X^{2^{d-1}})$. Suppose that $i \in (j; J)$. Since $(i; I) \not\subset (j; J)$, there is $1 \leq t \leq r$, such that $i_t \not\in (j; J)$, then from (3.2), we see that $p_{(j; J)}(\phi_{(i; I)}(X^{2^{d-1}}))$ is a sum of monomials of the form

$$ w = x_j^{2^{d-1}-2^{r-t}-1}f_{k-1;i_t-1}(z), $$

for some monomial $z$ in $P_{k-2}$. It is easy to see that $\alpha_{r-t}(2^r - 2^{r-t} - 1) = 0$ and $\omega_{r+1}(w) < k - 1$. Hence $w \in P_{k-1}(\omega(d))$.

Suppose that $(i; I) \subset (j; J)$. If $i = j$, then from (3.2), we see that the polynomial $p_{(j; J)}(\phi_{(i; I)}(X^{2^{d-1}}))$ is a sum of monomials of the form

$$ w = \left( \prod_{1 \leq t \leq r} x_{t-1}^{2^{d-1}-2^{r-t}-1+b_t} \right) \left( \prod_{j+1 \in J \setminus I} x_j^{2^{d-1}+c_j} \right) \left( \prod_{j+1 \notin J} x_j^{2^{d-1}} \right), $$

where $b_1 + b_2 + \ldots + b_r + \sum_{j+1 \in J \setminus I} c_j = 2^r - 1$. If $c_j > 0$, then $\alpha_{u_j}(2^d - 1 + c_j) = 0$ with $u_j$ the smallest index such that $\alpha_{u_j}(c_j) = 1$. Hence $w \in P_{k-1}(\omega(d))$. If $b_t = 0$ for suitable $1 \leq t \leq r$, then $\alpha_{r-t}(2^r - 2^{r-t} - 1) = 0$ and $\omega_{r+1}(w) < k - 1$. Hence $w \in P_{k-1}(\omega(d))$. Suppose that $b_t > 0$ for any $t$. Let $v_t$ be the smallest index such that $\alpha_{v_t}(b_t) = 1$. If $v_t \neq r - t$, then $\alpha_{v_t}(2^r - 2^{r-t} - 1 + b_t) = 0$ and $w \in P_{k-1}(\omega(d))$. So $u_t = r - t$ and $b_t = 2^{r-t} + b'_t$ with $b'_t \geq 0$. If $b'_t > 0$, then $\alpha_{v_t}(2^r - 2^{r-t} - 1 + b_t) = \alpha_{v_t}(2^r - 1 + b'_t) = 0$ with $v'_t$ the smallest index such that $\alpha_{v'_t}(b'_t) = 1$. Hence $w \in P_{k-1}(\omega(d))$. This implies $b'_t = 0$ for $1 \leq t \leq r$ and $w = g$.

If $i \in J$, then from (3.2) we see that the polynomial $p_{(j; J)}(\phi_{(i; I)}(X^{2^{d-1}}))$ is a sum of monomials of the form

$$ w = x_j^{2^{d-1}+b_t} \left( \prod_{1 \leq t \leq r} x_j^{2^{d-1}-2^{r-t}-1+b_t} \right) \left( \prod_{j+1 \in I \setminus (i; I)} x_j^{2^{d-1}+c_j} \right) \left( \prod_{j+1 \notin J} x_j^{2^{d-1}} \right). $$

ON THE PETERSON HIT PROBLEM 29
where \(b_0 + b_1 + b_2 + \ldots + b_r + \sum_{j+1 \in J_{(i;I)}} c_j = 2^d - 1\). By a same argument as above, we see that \(w \in P_{k-1}^{-1}(\omega(d))\) if either \(c_j > 0\) or \(b_i \neq 2^{r-t}\) for some \(j, t\) with \(t > 0\). Suppose \(c_j = 0\) and \(b_i = 2^{r-t}\) with all \(j\) and \(t > 0\). Then \(2^d - 1 = b_0 + b_1 + b_2 + \ldots + b_r + \sum_{j+1 \in J_{(i;I)}} c_j = b_0 + 2^r - 1\) and \(w = X^{2^d-1}\). The lemma is proved.

The following is easily be proved by a direct computation.

**Lemma 3.18.** The following diagram is commutative:

\[
\begin{array}{ccc}
P_{k-1} & \xrightarrow{f_i} & P_k \\
p_{(i;i_t)} & & \downarrow \\
P_{k-2} & \xrightarrow{f_i} & P_{k-1}.
\end{array}
\]

**Proof of Lemma 3.18.**

i) Suppose that either \(d \geq k\) or \(d = k - 1\) and \(I \neq I_1\), then \(\phi_{(i;I)}(z) = \phi_{(i;I)}(X^{2^d-1})f_i(z)^{2^d}\). Hence the first part of the lemma follows from Lemma 3.17.

ii) According to (3.4), \(\phi_{(1;I_1)}(z) = \phi_{(2;I_2)}(X^{2^d-1})f_1(z)^{2^d}\). Hence from Lemmas 3.17 and 3.18, we have

\[
p_{(i;I)}(\phi_{(1;I_1)}(z)) = p_{(i;I)}(\phi_{(2;I_2)}(X^{2^d-1}))p_{(i;I)}(f_1(z))^{2^d}
\]

\[
\begin{cases}
z & \text{if } (i; I) = (1; I_1), \\
X^{2^d-1}f_1p_{(1;I_1)}(z)^{2^d} & \text{if } (i; I) = (2; I_2), \\
0 & \text{otherwise.}
\end{cases}
\]

iii) Let \(z \in D\). Using the relation (3.4), Lemma 3.17 and Lemma 3.18, one has

\[
p_{(i;I)}(\phi_{(1;I_1)}(z)) = p_{(i;I)}(\phi_{(3;I_3)}(X^{2^d-1}))p_{(i;I)}(f_2(z))^{2^d}
\]

\[
\begin{cases}
z & \text{if } I_2 \subset I, \\
X^{2^d-1}f_2p_{(2;I_2)}(z)^{2^d} & \text{if } (i; I) = (3; I_3), \\
0 & \text{otherwise.}
\end{cases}
\]

iv) Let \(z \in E\). Using the relation (3.4), Lemma 3.17 and Lemma 3.18, one gets

\[
p_{(4;I_4)}(\phi_{(4;I_4)}(z)) = p_{(4;I_4)}(\phi_{(4;I_4)}(X^{2^d-1}))p_{(4;I_4)}(f_3(z))^{2^d}
\]

\[
X^{2^d-1}f_3p_{(3;I_3)}((z)^{2^d}).
\]

If a monomial \(y\) is a term of \(f_3p_{(3;I_3)}((z)^{2^d})\), then \(\omega_1(y) < k - 3\). According to Theorem 2.12, \(y \equiv 0\). Hence \(X^{2^d-1}f_3p_{(3;I_3)}(z)^{2^d} \equiv 0\). So using Lemma 3.18 one gets

\[
p_{(i;I)}(\phi_{(1;I_1)}(z)) = p_{(i;I)}(\phi_{(4;I_4)}(X^{2^d-1}))p_{(i;I)}(f_3(z))^{2^d}
\]

\[
\begin{cases}
z & \text{if } I_3 \subset I, \\
0 & \text{otherwise}.
\end{cases}
\]

The lemma is completely proved.
4. The cases $k \leq 3$

In this section and the next sections, we denote by $B_k(n)$ the set of all admissible monomials of degree $n$ in $P_k$, $B^0_k(n) = B_k(n) \cap P^0_k$, $B^+_k(n) = B_k(n) \cap P^+_k$. For an $\omega$-vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ of degree $n$, we set $B_k(\omega) = B_k(n) \cap P_k(\omega)$, $B^+_k(\omega) = B^+_k(n) \cap P_k(\omega)$. Then $[B_k(\omega)]$ and $[B^+_k(\omega)]$, respectively are the bases of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $Q\Sigma^+_k(\omega)$.

If there is $i_0 = 0, i_1, i_2, \ldots, i_r > 0$ such that $i_1 + i_2 + \ldots + i_r = m$ and $\omega_{i_1} + \ldots + \omega_{i_{r-1}+t} = a_s, 1 \leq i \leq r$, then we denote $\omega = (a_{i_1}^{(i)}, a_{i_2}^{(i)}, \ldots, a_{i_r}^{(i)})$. If $i_u = 1$, then we denote $a_u^{(i)} = a_u$.

Using Lemma [5.3.3(i)] in Subsection [5.3] and Theorem [2.9] we easily obtain the following.

**Proposition 4.1.** For any $s \geq 1$,

$$B_k(1^{(s)}) = \{x_{i_1}^{2^{i_1}}, \ldots, x_{i_m}^{2^{i_m}} : 1 \leq i_1 < \ldots < i_m \leq k, 1 \leq m \leq \min\{s, k\}\}.$$

It is well known that if $n \neq 2^u - 1$ then $B_1(n) = \emptyset$. If $n = 2^u - 1$ for $u \geq 1$, then $B_1(n) = B_1(1^{(u)}) = \{x^{2^{u-1}}\}$. It is easy to see that $\Phi(B_1(0)) = \{1\} = B_2(0)$, $\Phi(B_1(1)) = \{x_1, x_2\} = B_2(1)$. According to Proposition [3.3] for $u > 1$, we have

$$B_2(2^u - 1) = \Phi(B_1(2^u - 1)) = \{x_1^{2^{u-1}}, x_2^{2^{u-2}}, x_3^{2^{u-1}}\},$$

By Theorem [1.1], $B_2(n) = \emptyset$ if $n \neq 2^{t+u} + 2^t - 2$ for all nonnegative integers $t, u$. We define the $\mathbb{F}_2$-linear map $\psi : (P_k)_m \to (P_k)_{2m+k}$ by $\psi(y) = x_8y^2$ for any monomial $y \in (P_k)_m$. From Theorem [1.2] and Theorem [1.3] we have

**Theorem 4.2** (Peterson [21]). If $n = 2^{t+u} + 2^t - 2$, with $t, u$ positive integers, then

$$B_2(n) = \psi(\Phi(B_1(2^u - 1)))$$

$$= \begin{cases} \{x_1x_2\}^{2^t-1}, & u = 0, \\ \{x_1^{2^{t+1}-1}, x_2^{2^t-1}, x_3^{2^{t+1}-1}\}, & u = 1, \\ \{x_1^{2^{u+u}}, x_2^{2^{u+1}}, x_3^{2^{u+1}}\}, & u > 1. \end{cases}$$

By Theorems [1.1] and [1.2] for $k = 3$, we need only to consider the cases of degree $n = 2^s - 2$, $n = 2^s - 1$ and $n = 2^{t+u} + 2^s - 2$ with $s, u$ positive integers. A direct computation using Theorem [1.3] we have

**Theorem 4.3** (Kameko [14]).

i) If $n = 2^s - 2$, then $B_3(2^s - 2) = \Phi(B_2(2^s - 2))$.

ii) If $n = 2^s - 1$, then $B_3(2^s - 1) = B_3(1^{(s)}) \cup \psi(\Phi(B_2(2^{s-1} - 2)))$.

iii) If $n = 2^{t+u} + 2^s - 2$, then

$$B_3(n) = \begin{cases} \Phi(B_2(8)) \cup \{x_3^3x_2^3x_3\}, & if $s = 1, t = 2$, \\ \Phi(B_2(2^{t+u} + 2^s - 2)), & otherwise. \end{cases}$$

5. Proof of Theorem [1.4]

For $1 \leq i \leq k$, define $\varphi_i : QP_k \to QP_k$, the homomorphism induced by the $A$-homomorphism $\overline{\varphi}_i : P_k \to P_k$, which is determined by $\overline{\varphi}_1(x_1) = x_1 + x_2, \overline{\varphi}_1(x_j) = x_j$ for $j > 1$, and $\overline{\varphi}_i(x_i) = x_{i-1}, \overline{\varphi}_i(x_{i-1}) = x_i, \overline{\varphi}_i(x_j) = x_j$ for $j \neq i, i-1, 1 < i \leq k$. Note that the general linear group $GL_k$ is generated by $\overline{\varphi}_i, 0 < i \leq k$ and the symmetric group $\Sigma_k$ is generated by $\overline{\varphi}_i, 1 < i \leq k$. 

Let $B$ be a finite subset of $P_k$ consisting of some monomials of degree $n$. To prove the set $[B]$ is linearly independent in $Q P_k$, we order the set $B$ by the order as in Definition 2.6 and denote the elements of $B$ by $d_i = d_{n,i}$, $0 < i \leq b = |B|$ in such away that $d_{n,i} < d_{n,j}$ if and only if $i < j$. Suppose there is a linear relation

$$S = \sum_{1 \leq j \leq b} \gamma_j d_{n,j} \equiv 0,$$

with $\gamma_j \in \mathbb{F}_2$. For $(i,I) \in \mathcal{N}_k$, we explicitly compute $p_{(i;I)}(S)$ in terms of a minimal set of $\mathcal{A}$-generators in $P_{k-1}$. Computing from some relations $p_{(i;I)}(S) \equiv 0$ with $(i,I) \in \mathcal{N}_k$ and $\mathcal{P}_i(S) \equiv 0$, we will obtain $\gamma_j = 0$ for all $j$.

5.1. The case of degree $n = 2^{s+1} - 3$.

In this subsection, we prove the following.

**Proposition 5.1.1.** For any $s \geq 1$, $\Phi(B_3(n))$ is a minimal set of generators for $\mathcal{A}$-module $P_4$ in degree $n = 2^{s+1} - 3$.

We need the following lemma for the proof of the proposition.

**Lemma 5.1.2.** If $x$ is an admissible monomial of degree $2^{s+1} - 3$ in $P_4$, then $\omega(x) = (3^{(s-1)}, 1)$.

**Proof.** It is easy to see that the lemma holds for $s = 1$. Suppose $s \geq 2$. Obviously, $z = x_1^{2^{s-1}-1} x_2^{2^{s-1}-1} x_3^{2^{s-1}-1}$ is the minimal spike of degree $2^{s+1} - 3$ in $P_4$ and $\omega(z) = (3^{(s-1)}, 1)$. Since $2^{s+1} - 3$ is odd, we get either $\omega_1(x) = 1$ or $\omega_1(x) = 3$. If $\omega_1(x) = 1$, then $\omega(x) < \omega(z)$. By Theorem 2.12, $x$ is hit. This contradicts the fact that $x$ is admissible. Hence we have $\omega_1(x) = 3$. Using Proposition 2.10 and Theorem 2.12, we obtain $\omega_i(x) = 3$, $i = 1, 2, \ldots, s-1$. From this, it implies

$$2^{s+1} - 3 = \deg x = \sum_{i \geq 1} 2^{i-1} \omega_i(x) = 3(2^{s-1} - 1) + \sum_{i < s} 2^{i-1} \omega_i(x).$$

The last equality implies $\omega_s(x) = 1$ and $\omega_i(x) = 0$ for $i > s$. The lemma is proved.

From Lemma 3.10, we have the following.

**Lemma 5.1.3.** The following monomials are strictly inadmissible:

$$X_1 x_1^2, X_i X_j^2, 1 \leq i < j \leq 4.$$

**Proof of Proposition 5.1.1.** We have $n = 2^{s+1} - 3 = 2^s + 2^{s-1} + 2^{s-1} - 3$. Hence the proposition follows from Theorem 1.3 for $s \geq 4$. According to Kameko [14],

$$B_3(n) = \{v_1 = X_1^{2^{s-1}-1} x_3^{2^{s-1}}, v_2 = X_2^{2^{s-1}-1} x_3^{2^{s-1}}, v_3 = X_3^{2^{s-1}-1} x_4^{2^{s-1}}\},$$

where $X = x_1 x_2 x_3$.

It is easy to see that $\Phi(B_3(1)) = \{x_1, x_2, x_3, x_4\}$. Hence the proposition holds for $s = 1$. For $s = 2$, using Lemma 5.1.3, we see that

$$\Phi^+(B_3(5)) = \{x_1 x_2 x_3 x_4, x_1 x_2 x_3^2 x_4, x_1 x_2^2 x_3 x_4\}$$

is a minimal set of generators for $(P_4)^+$. A direct computation using Lemmas 5.1.2 and 5.1.3 shows that for $s = 3$, $\Phi^+(B_3(13))$ is the set of 23 following monomials:

$$X_i^2 X_j x_m^4, 1 \leq i < j \leq 4, m \neq i, X_i^2 X_j x_4^4, 2 \leq i < j \leq 4, X_3 x_3^4, X_4 x_4^4.$$
Using Lemmas [5.1.2, 5.1.3] and Theorem [2.12] we see that if \( x \) is an admissible monomial of degree 13 in \( P^+_4 \), then \( x \in \Phi^+(B_3(13)) \). Hence \((QP^+_4)_{13}\) is generated by \([\Phi^+(B_3(13))]\). Now we prove that the set \([\Phi^+(B_3(13))]\) is linearly independent.

Suppose there is a linear relation
\[
p\sum_{j=1}^{23} \gamma_j d_j = 0,
\]
where \( \gamma_j \in \mathbb{F}_2, 1 \leq j \leq 23 \).

Consider the homomorphisms \( p_{(1;i)} : P_4 \rightarrow P_3, i = 2, 3, 4 \). By a direct computation from (5.1.3.1), we have
\[
p_{(1;2)}(\mathcal{S}) = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_7 v_3 = 0,
p_{(1;3)}(\mathcal{S}) = \gamma_3 v_1 + (\gamma_5 + \gamma_6)v_2 + \gamma_8 v_3 = 0,
p_{(1;4)}(\mathcal{S}) = (\gamma_4 + \gamma_5)v_1 + \gamma_6 v_2 + \gamma_9 v_3 = 0.
\]

From the above equalities it implies
\[
\begin{align*}
\gamma_j = 0, & \quad j = 1, 2, 3, 6, 7, 8, 9, \\
\gamma_5 = \gamma_6, & \quad \gamma_4 = \gamma_16.
\end{align*}
\]

Substituting (5.1.3.2) into the relation (5.1.3.1), we have
\[
\mathcal{S} = \gamma_4 d_4 + \gamma_5 d_5 + \sum_{10 \leq j \leq 23} \gamma_j d_j = 0.
\]

Applying the homomorphisms \( p_{(2;3)}, p_{(2;4)}, p_{(3;4)} : P_4 \rightarrow P_3 \) to (5.1.3.3), we get
\[
p_{(2;3)}(\mathcal{S}) = \gamma_{10} v_1 + (\gamma_{12} + \gamma_{16} + \gamma_{18}) v_2 + \gamma_{21} v_3 = 0,
p_{(2;4)}(\mathcal{S}) = (\gamma_{11} + \gamma_{15} + \gamma_{19}) v_1 + \gamma_{13} v_2 + \gamma_{22} v_3 = 0,
p_{(3;4)}(\mathcal{S}) = (\gamma_{14} + \gamma_{15} + \gamma_{16} + \gamma_{17}) v_1 + \gamma_{20} v_2 + \gamma_{23} v_3 = 0.
\]

Hence we get
\[
\begin{align*}
\gamma_j = 0, & \quad j = 10, 13, 20, 21, 22, 23, \\
\gamma_{12} + \gamma_{16} + \gamma_{18} = \gamma_{11} + \gamma_{15} + \gamma_{19} = 0, \\
\gamma_{14} + \gamma_{15} + \gamma_{16} + \gamma_{17} = 0.
\end{align*}
\]

Substituting (5.1.3.4) into the relation (5.1.3.3), we get
\[
\mathcal{S} = \gamma_4 d_4 + \gamma_5 d_5 + \gamma_{11} d_{11} + \gamma_{12} d_{12} + \sum_{14 \leq j \leq 19} \gamma_j d_j = 0.
\]

The homomorphisms \( p_{(1;(2,3))}, p_{(1;(2,4))}, p_{(1;(3,4))} : P_4 \rightarrow P_3 \), send (5.1.3.5) respectively to
\[
p_{(1;(2,3))}(\mathcal{S}) = (\gamma_{5} + \gamma_{12} + \gamma_{16}) v_2 + \gamma_{18} v_3 = 0,
p_{(1;(2,4))}(\mathcal{S}) = (\gamma_{4} + \gamma_{11} + \gamma_{15}) v_1 + \gamma_{19} v_3 = 0
p_{(1;(3,4))}(\mathcal{S}) = (\gamma_{4} + \gamma_{14} + \gamma_{15}) v_1 + (\gamma_{5} + \gamma_{16} + \gamma_{17}) v_2 = 0.
\]

From this we obtain
\[
\begin{align*}
\gamma_{18} &= \gamma_{19} = \gamma_{5} + \gamma_{12} + \gamma_{16} = 0, \\
\gamma_{4} + \gamma_{11} + \gamma_{15} &= \gamma_{4} + \gamma_{14} + \gamma_{15} = \gamma_{5} + \gamma_{16} + \gamma_{17} = 0.
\end{align*}
\]
Combining (5.1.3.2), (5.1.3.4) and (5.1.3.6), we obtain $\gamma_j = 0$, $j = 1, 2, \ldots, 23$. The proposition is proved.

5.2. The case of degree $n = 2^{s+1} - 2$.

It is well-known that, Kameko’s homomorphism

$$\widetilde{S}_q^0 : (QP_k)_{2m+k} \to (QP_k)_m$$

is an epimorphism. Hence we have

$$(QP_k)_{2m+k} \cong (QP_k)_m \oplus (QP_k^0)_{2m+k} \oplus (\text{Ker} \widetilde{S}_q^0 \cap (QP_k^+))_{2m+k},$$

and $(QP_k)_m \cong \{ \psi(B_k(m)) \} \subset (QP_k)_{2m+k}$.

For $k = 4$, from Theorem 4.3 it is easy to see that

$$\Phi(B_4(2)) = \Phi^0(B_3(2)) = \{ x_i x_j | 1 \leq i < j \leq 4 \}.$$

For $m = 2^s - 3$, $s \geq 2$, we have

$$|\Phi^0(B_3(6))| = 18, \ |\Phi^0(B_3(2^{s+1} - 2))| = 22, \ for \ s > 3,$$

$$|\psi(B_4(1))| = 4, \ Ker \widetilde{S}_q^0 \cap [B_4^+(6)] = \{ [x_1 x_2^2 x_3 x_4^3], [x_1 x_2 x_3^2 x_4^2] \}.$$

Hence $\dim (QP_4)_2 = 6, \ dim (QP_4)_6 = 24$.

The main result of this subsection is:

**Proposition 5.2.1.** For any $s \geq 3$, $(QP^+_{1})_{2^{s+1} - 2} \cap \text{Ker} \widetilde{S}_q^0$ is an $\mathbb{F}_2$-vector space of dimension 13 with a basis consisting of all the classes represented by the following admissible monomials:

$$d_1 = x_1 x_2 x_3^2 x_4^{2s-2} x_5^{2s-2}, \quad d_2 = x_1 x_2^2 x_3 x_4^{2s-4} x_5^{2s-1}, \quad d_3 = x_1 x_2^3 x_3 x_4^{2s-3} x_5^{2s-2},$$

$$d_4 = x_1 x_2^3 x_3 x_4^{2s-4} x_5^{2s-1}, \quad d_5 = x_1 x_2^3 x_3 x_4^{2s-4} x_5^{2s-2}, \quad d_6 = x_1 x_2^3 x_3 x_4^{2s-3} x_5^{2s-4},$$

$$d_7 = x_1 x_2^2 x_3 x_4^{2s-2} x_5^{2s-1}, \quad d_8 = x_1 x_2^2 x_3 x_4^{2s-4} x_5^{2s-2}, \quad d_9 = x_1 x_2^2 x_3 x_4^{2s-3} x_5^{2s-4},$$

$$d_{10} = x_1^3 x_2^2 x_3 x_4^{2s-2} x_5^{2s-4}, \quad d_{11} = x_1^3 x_2^2 x_3 x_4^{2s-4} x_5^{2s-2}, \quad d_{12} = x_1^3 x_2^2 x_3 x_4^{2s-3} x_5^{2s-4},$$

The proof of this theorem is based on some lemmas.

**Lemma 5.2.2.** If $x$ is an admissible monomial of degree $2^{s+1} - 2$ in $P_4$ and $[x] \in \text{Ker} \widetilde{S}_q^0$, then $\omega(x) = (2^s)$.\[Proof.\] We prove the lemma by induction on $s$. Obviously, the lemma holds for $s = 1$. Observe that $z = (x_1 x_2)^{2^s-1}$ is the minimal spike of degree $2^{s+1} - 2$ in $P_4$ and $\omega(z) = (2^s)$. Since $2^{s+1} - 2$ is even, using Theorem 2.12 and the fact that $[x] \in \text{Ker} \widetilde{S}_q^0$, we obtain $\omega_1(x) = 2$. Hence $x = x_i x_j y^2$, where $y$ is a monomial of degree $2^{s} - 2$ and $1 \leq i < j \leq 4$. Since $x$ is admissible, by Theorem 2.9 $y$ is also admissible. Now, the lemma follows from the inductive hypothesis.\]

The following lemma is proved by a direct computation.

**Lemma 5.2.3.** The following monomials are strictly inadmissible:

i) $x_1^i x_i x_j y^j$, $x_1^i x_j x_k y^i$, $i < j, k \neq i, j$. $x_1^i x_j x_k x_l y^m$, $x_1^i x_j x_k x_l y^m x_1 x_2 x_3 x_4$.

ii) $x_1 x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$.

iii) $x_1 x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$.
Proof of Proposition 5.2.1. Let $x$ be an admissible monomial in $P_4$ and $[x] \in \text{Ker} \tilde{S}^0_q$. By Lemma 5.2.2, $\omega_i(x) = 2$, for $1 \leq i \leq s$. By induction on $s$, we see that if $x \neq d_i$, for $i = 1, 2, \ldots, 13$, then there is a monomial $y$, which is given in Lemma 5.2.3 such that $x = wy^2u$ for some monomial $y$ and positive integer $u$. By Theorem 2.9, $x$ is inadmissible. Hence $\text{Ker} \tilde{S}_q \cap (QP_4^+)$ is spanned by the classes $[d_i]$ with $i = 1, 2, \ldots, 13$. Now, we prove that the classes $[d_i]$ with $i = 1, 2, \ldots, 13$, are linearly independent.

Suppose there is a linear relation

$$\sum_{1 \leq i \leq 13} \gamma_i d_i = 0, \quad (5.2.3.1)$$

with $\gamma_i \in \mathbb{F}_2$.

According to Kameko [14], for $s \geq 3$, $B_3(n) \cap (P_3^+)_n$ is the set consisting of 4 monomials:

$$w_1 = x_1x_2^2x_3^2 - x_2x_3^2 - 1, \quad w_2 = x_1x_2^2x_3^2 - x_2x_3^2 - 2,$$
$$w_3 = x_1x_2^2x_3^2 - x_2x_3^2 - 2, \quad w_4 = x_1x_2^2x_3^2 - x_2x_3^2 - 2.$$

Apply the homomorphisms $p_{(1;2)}, p_{(3;4)} : P_4 \to P_3$ to the relation (5.2.3.1) and we obtain

$$\gamma_2 w_1 + \gamma_4 w_2 + \gamma_3 w_3 + \gamma_7 w_4 \equiv 0,$$
$$\gamma_7 w_1 + \gamma_8 w_2 + \gamma_12 w_3 + \gamma_13 w_4 \equiv 0.$$

From these relations, we get $\gamma_i = 0$, $i = 2, 3, 4, 7, 8, 12, 13$. Then the relation (5.2.3.1) becomes

$$\gamma_1 d_1 + \gamma_5 d_5 + \gamma_6 d_6 + \gamma_9 d_9 + \gamma_10 d_{10} + \gamma_{11} d_{11} \equiv 0. \quad (5.2.3.2)$$

Apply the homomorphisms $p_{(1;4)}, p_{(2;3)} : P_4 \to P_3$ to the relation (5.2.3.2) and we get

$$(\gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11}) w_1 + \gamma_6 w_3 \equiv 0,$$
$$(\gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11}) w_2 + \gamma_9 w_3 \equiv 0.$$

These equalities imply $\gamma_6 = \gamma_9 = \gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11} = 0$. Hence we obtain

$$\gamma_1 d_1 + \gamma_5 d_5 + \gamma_10 d_{10} + \gamma_{11} d_{11} \equiv 0. \quad (5.2.3.3)$$

For $s > 3$, apply the homomorphisms $p_{(1;3)}, p_{(2;4)} : P_4 \to P_3$ to (5.2.3.3), we get

$$\gamma_1 w_2 + \gamma_5 w_3 \equiv 0,$$
$$\gamma_1 w_1 + \gamma_{10} w_3 \equiv 0.$$

From the above equalities, we get $\gamma_i = 0$, $i = 1, 2, \ldots, 13$.

For $s = 3$, apply the homomorphisms $p_{(1;3)}, p_{(2;4)} : P_4 \to P_3$ to (5.2.3.3), we get

$$(\gamma_1 + \gamma_{11}) w_2 + \gamma_8 w_3 \equiv 0,$$
$$(\gamma_1 + \gamma_{11}) w_1 + \gamma_{10} w_3 \equiv 0.$$

From the above equalities, we get $\gamma_i = 0$, $i = 1, 2, \ldots, 10, 12, 13$ and $\gamma_1 = \gamma_{11}$. So the relation (5.2.3.3) becomes

$$\gamma_1 (d_1 + d_{11}) \equiv 0.$$
Now, we prove that \(|d_1 + d_{11}| \neq 0\). Suppose the contrary, that the polynomial 
\(d_1 + d_{11} = x_1 x_2 x_3^6 x_4^6 + x_3^3 x_2^2 x_4^4\) is hit. Then by the unstable property of the action 
of \(A\) on the polynomial algebra, we have 
\[x_1 x_2 x_3^6 x_4^6 + x_3^3 x_2^2 x_4^4 = Sq^1(A) + Sq^2(B) + Sq^4(C),\]
for some polynomials \(A \in (P_4^+)_1, B \in (P_4^+)_2, C \in (P_4^+)_10\). Let \((Sq^2)^3\) acts on the 
both sides of the above equality. Since \((Sq^2)^3 Sq^1 = 0\) and \((Sq^2)^3 Sq^2 = 0\), we get 
\[(Sq^2)^3(x_1 x_2 x_3^6 x_4^6 + x_3^3 x_2^2 x_4^4) = (Sq^2)^3 Sq^4(C).\]
On the other hand, by a direct computation, it is not difficult to check that 
\[(Sq^2)^3(x_1 x_2 x_3^6 x_4^6 + x_3^3 x_2^2 x_4^4) \neq (Sq^2)^3 Sq^4(C),\]
for all \(C \in (P_4^+)_10\). This is a contradiction. Hence \(|d_1 + d_{11}| \neq 0\) and \(\gamma_1 = \gamma_{11} = 0\). 
The proposition is proved. \(\square\)

### 5.3. The case of degree \(n = 2^{s+1} - 1\).

First, we determine the \(\omega\)-vector of an admissible monomial of degree \(2^{s+1} - 1\) in \(P_4\).

**Lemma 5.3.1.** If \(x\) is an admissible monomial of degree \(2^{s+1} - 1\) in \(P_4\) then either 
\(\omega(x) = (1^{(s+1)})\) or \(\omega(x) = (3, 2^{(s-1)})\) or \(\omega(x) = (1, 3)\) for \(s = 2\).

**Proof.** Obviously, the lemma holds for \(s = 1\). Suppose \(s \geq 2\). By a direct 
computation we see that if \(w\) is a monomial in \(P_4\) such that \(\omega(w) = (1, 3, 2)\) or 
\(\omega(w) = (1, 1, 3)\), then \(w\) is strictly inadmissible.

Since \(2^{s+1} - 1\) is odd, we have either \(\omega_1(x) = 1\) or \(\omega_1(x) = 3\). If \(\omega_1(x) = 1\), 
then \(x = x_1 y^2\), where \(y\) is a monomial of degree \(2^s - 1\). Hence either \(\omega_1(y) = 1\) or 
\(\omega_1(y) = 3\). So the lemma holds for \(s = 2\). Suppose that \(s \geq 3\). If \(\omega_1(y) = 3\), then 
\(y = X_i y_1^2\), where \(y_1\) is a monomial of degree \(2^{s-1} - 2\). Since \(y_1\) is admissible, using 
Proposition 2.10 one gets \(\omega_1(y_1) = 2\). Hence \(x\) is inadmissible. If \(\omega_1(y) = 1\), then 
\(y = x_1 y_1^2\), where \(y_1\) is an admissible monomial of degree \(2^{s-1} - 1\). By the inductive 
hypothesis \(\omega(y_1) = (1^{(s-1)})\). So we get \(\omega(x) = (1^{(s+1)})\).

Suppose that \(\omega_1(x) = 3\). Then \(x = X_i y^2\), where \(y\) is an admissible monomial of 
degree \(2^s - 2\). Since \(x\) is admissible, by Lemma 5.2.3 \(\omega(y) = (2^{(s-1)})\). The lemma 
is proved. \(\square\)

For \(s = 1\), we have \((QP_4)_3 = (QP_4^0)_3\). Hence \(B_4(3) = \Phi^0(B_3(3))\). Using 
Proposition 4.1 and Theorem 4.3 we have 
\(|\Phi^0(B_3(3))| = 14, |\Phi^0(B_3(7))| = 26, |\Phi^0(B_3(15))| = 38,\]
\(|\Phi^0(B_3(2^{s+1} - 1))| = 42\), for \(s \geq 4\).

For \(s = 2\), \(B_4(7) = B_4(1^{(3)}) \cup B_4(1, 3) \cup B_4(3, 2)\). By a direct computation, we have 
\(B_4(1, 3) = \{x_1 X_1^2\}, B_4(3, 2) = \Phi(B_3(7))\).

Recall that 
\(B_3(2^{s+1} - 1) = B_3(1^{(s+1)}) \cup \psi(\Phi(B_2(2^s - 2)))\),
where \(B_2(2^s - 2) = \{x_1^{2^s-1}, x_2^{2^s-1}\}\). Hence \(B_3(3, 2^{(s-1)}) = \psi(\Phi(B_2(2^s - 2)))\).
Proposition 5.3.2. For any \( s \geq 3, B_4(3, 2^{(s-1)}) = (\Phi(B_3(3, 2^{(s-1)})) \cup A(s), \) where \( A(s) \) is determined as follows:

\[
\begin{align*}
A(3) &= \{ x_1^3 x_2 x_3 x_4^7, x_1^3 x_2 x_3 x_4, x_1^3 x_2 x_3^4 x_4, x_1 x_2 x_3 x_4, x_1 x_2 x_3 x_4^5 \}, \\
A(4) &= \{ x_1^3 x_2 x_3^{11} x_4^{13}, x_1^3 x_2 x_3 x_4^{13}, x_1^7 x_2 x_3^{8} x_4, x_1^7 x_2 x_3^{9} x_4^8 \}, \\
A(s) &= \{ x_1^3 x_2 x_3^{2^s-5} x_4^{2^s-3} \}, s \geq 5.
\end{align*}
\]

Combining Lemma 5.3.3 and Propositions 4.1, 5.3.2 we have

\[ B_4(2^{s+1} - 1) = B_4(1^{(s+1)}) \cup \Phi(B_3(3, 2^{(s-1)})) \cup A(s). \]

The following can easily be proved by a direct computation.

Lemma 5.3.3. The following monomials are strictly inadmissible:

i) \( x_i^2 x_j, x_i^3 x_j^4, 1 \leq i < j \leq 4 \),

ii) \( x_2^2 x_4^2, x_1 x_2 x_3^2, i = 2, 3, 4 \),

iii) \( x_3 x_2 x_4 x_5, x_1 x_2 x_3 x_5, x_3 x_2 x_3 x_5, i < j < k, \ell \neq i, j, k \),

iv) \( x_1^3 x_2^3 x_3 x_4, x_1^3 x_2^3 x_3^4, x_1 x_2 x_3^4 x_4, x_1 x_2 x_3^8 x_4, x_1 x_2 x_3^8 x_4, x_1 x_2 x_3^8 x_4, x_1 x_2 x_3^8 x_4, x_1 x_2 x_3^8 x_4, x_1 x_2 x_3^8 x_4, i = 1, 2, j = 3, 4 \).

Proof of Proposition 5.3.3. By a direct computation using Lemma 5.3.3 and Theorem 2.9 we see that if \( x \) is a monomial of degree \( 2^{s+1} - 1 \) in \( P_4 \) and \( x \notin \Phi(B_3(3, 2^{(s-1)})) \cup A(s), \) then there is a monomial \( u \) which is given in Lemma 5.3.3, such that \( x = w y^{2u} \) for some monomial \( y \) and integer \( u \geq 1 \). Hence \( x \) is inadmissible.

Now we prove that the set \( B_4(3, 2^{(s-1)}) \) is linearly independent in \( QP_4^+ \). For \( s = 3 \), we have \( |B_4(3, 2, 2)| = 36 \). Suppose there is a linear relation

\[ \mathcal{S} = \sum_{1 \leq i \leq 36} \gamma_i d_i \equiv 0, \]  \hspace{1cm} (5.3.3.1)

with \( \gamma_i \in \mathbb{F}_2 \) and \( d_i = d_{15,i} \).

A simple computation, we see that \( B_3(3, 2, 2) = \psi(\Phi(B_2(6))) \) is the set consisting of 6 monomials:

\[ v_1 = x_1 x_2 x_3 x_4, v_2 = x_1 x_2 x_3 x_4, v_3 = x_1 x_2 x_3 x_4, v_4 = x_1 x_2 x_3 x_4, v_5 = x_1 x_2 x_3 x_4, v_6 = x_1 x_2 x_3 x_4. \]

By a direct computation, we have

\[
\begin{align*}
p(1;2)(\mathcal{S}) &\equiv \gamma_3 v_2 + \gamma_4 v_3 + (\gamma_9 + \gamma_22) v_4 + (\gamma_{10} + \gamma_{23}) v_5 + (\gamma_{11} + \gamma_{24}) v_6 \equiv 0, \\
p(1;3)(\mathcal{S}) &\equiv (\gamma_1 + \gamma_{10}) v_1 + \gamma_5 v_2 + (\gamma_7 + \gamma_{20}) v_3 + \gamma_{13} v_5 + (\gamma_{15} + \gamma_{30}) v_6 \equiv 0, \\
p(1;4)(\mathcal{S}) &\equiv (\gamma_2 + \gamma_{19}) v_1 + (\gamma_6 + \gamma_{21} + \gamma_{27}) v_2 + \gamma_{8} v_3 + (\gamma_{12} + \gamma_{29}) v_4 + \gamma_{14} v_5 \equiv 0, \\
p(2;3)(\mathcal{S}) &\equiv (\gamma_1 + \gamma_3 + \gamma_5 + \gamma_9) v_1 + (\gamma_{16} + \gamma_{22}) v_2 \\
&\quad + (\gamma_{18} + \gamma_{20} + \gamma_{23} + \gamma_{26}) v_3 + \gamma_{32} v_5 + (\gamma_{34} + \gamma_{36}) v_6 \equiv 0, \\
p(2;4)(\mathcal{S}) &\equiv (\gamma_2 + \gamma_4 + \gamma_8 + \gamma_{11}) v_1 \\
&\quad + (\gamma_{17} + \gamma_{21}) v_2 + (\gamma_{19} + \gamma_{24} v_3 + \gamma_{31} + \gamma_{35}) v_4 + \gamma_{33} v_5 \equiv 0, \\
p(3;4)(\mathcal{S}) &\equiv (\gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{15}) v_1 + (\gamma_{25} + \gamma_{26} + \gamma_{27} + \gamma_{28}) v_2 \\
&\quad + (\gamma_{29} + \gamma_{30}) v_3 + (\gamma_{31} + \gamma_{32} + \gamma_{34} + \gamma_{34}) v_4 + (\gamma_{35} + \gamma_{36}) v_5 \equiv 0.
\end{align*}
\]
From these equalities, we obtain

\[
\begin{align*}
&\gamma_j = 0, \ j = 3, 4, 5, 8, 13, 14, 32, 33, \\
&\gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22}, \ \gamma_7 = \gamma_{11} = \gamma_{19} = \gamma_{24}, \ \gamma_7 = \gamma_{20}, \\
&\gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22}, \ \gamma_{10} = \gamma_{23}, \ \gamma_{17} = \gamma_{21}, \\
&\gamma_{12} = \gamma_{15} = \gamma_{29} = \gamma_{30}, \ \gamma_{31} = \gamma_{34} = \gamma_{35} = \gamma_{36}, \\
&\gamma_6 + \gamma_{21} + \gamma_{27} = \gamma_7 + \gamma_{10} + \gamma_{18} + \gamma_{26} = \gamma_{25} + \gamma_{26} + \gamma_{27} + \gamma_{28} = 0.
\end{align*}
\tag{5.3.3.2}
\]

A direct computation using 5.3.3.2 and Theorem 2.12, we get

\[
\begin{align*}
p_{1(2,3)}(S) &\equiv \gamma_{18}w_3 + \gamma_{26}w_5 + \gamma_{28}w_6 \equiv 0, \\
p_{1(2,4)}(S) &\equiv (\gamma_6 + \gamma_{10} + \gamma_{27})w_2 + \gamma_{25}w_4 + \gamma_{27}w_5 \equiv 0, \\
p_{1(3,4)}(S) &\equiv (\gamma_{17} + \gamma_{18})w_1 \\
&+ (\gamma_6 + \gamma_7 + \gamma_{17} + \gamma_{25} + \gamma_{26} + \gamma_{27})w_2 + (\gamma_{17} + \gamma_{28})w_3 \equiv 0.
\end{align*}
\]

Combining the above equalities and \tag{5.3.3.2}, one gets \(\gamma_j = 0\) for \(j \neq 1, 2, 9, 11, 12, 15, 16, 19, 22, 24, 29, 30, 31\) and \(\gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22}, \ \gamma_7 = \gamma_{11} = \gamma_{19} = \gamma_{24}, \ \gamma_{12} = \gamma_{15} = \gamma_{29} = \gamma_{30}, \ \gamma_{31} = \gamma_{34} = \gamma_{35} = \gamma_{36}\). Hence the relation \tag{5.3.3.1} becomes

\[
\begin{align*}
\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_3 + \gamma_{31}\theta_4 &\equiv 0. \\
\tag{5.3.3.3}
\end{align*}
\]

where

\[
\begin{align*}
\theta_1 &= d_1 + d_9 + d_{16} + d_{22}, \ &\theta_2 &= d_2 + d_{11} + d_{19} + d_{24}, \\
&\theta_3 &= d_{12} + d_{15} + d_{29} + d_{30}, \ &\theta_4 &= d_{31} + d_{34} + d_{35} + d_{36}.
\end{align*}
\]

Now, we prove that \(\gamma_1 = \gamma_2 = \gamma_{12} = \gamma_{31} = 0\).

The proof is divided into 4 steps.

**Step 1.** Under the homomorphism \(\varphi_1\), the image of \tag{5.3.3.3} is

\[
\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_3 + \gamma_{31}(\theta_4 + \theta_3) \equiv 0. \tag{5.3.3.4}
\]

Combining \tag{5.3.3.3} and \tag{5.3.3.4}, we get

\[
\gamma_{31}\theta_3 \equiv 0. \tag{5.3.3.5}
\]

If the polynomial \(\theta_3\) is hit, then we have

\[
\theta_3 = (S^q)^1(A) + (S^q)^2(B) + (S^q)^4(C),
\]

for some polynomials \(A \in (P^+_{4})_{14}, B \in (P^+_{4})_{13}, C \in (P^+_{4})_{11}\). Let \((S^q)^3\) act on the both sides of this equality. We get

\[
(S^q)^3(\theta_3) = (S^q)^3S^4(C),
\]

By a direct calculation, we see that the monomial \(x = x_1^8x_2^7x_3^4x_4^2\) is a term of \((S^q)^3(\theta_3)\). If this monomial is a term of \((S^q)^3S^4(y)\) for a monomial \(y \in (P^+_{4})_{11}\), then \(y = x_3^2f_2(z)\) with \(z \in P_3\) and \(\deg z = 4\). Using the Cartan formula, we see that \(x\) is a term of \(x_3^2(S^q)^3S^4(z) = x_3^2(S^q)^3(z^2) = 0\). Hence

\[
(S^q)^3(\theta_3) \neq (S^q)^3S^4(C),
\]

for all \(C \in (P^+_{4})_{11}\) and we have a contradiction. So \(\theta_3 \neq 0\) and \(\gamma_{31} = 0\).

**Step 2.** Since \(\gamma_{31} = 0\), the homomorphism \(\varphi_2\) sends \tag{5.3.3.3} to

\[
\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_4 \equiv 0. \tag{5.3.3.6}
\]

Using the relation \tag{5.3.3.6} and by the same argument as given in Step 1, we get \(\gamma_{12} = 0\).
Step 3. Since $\gamma_{31} = \gamma_{12} = 0$, the homomorphism $\varphi_3$ sends $5.3.3.3$ to
$$\gamma_1[\theta_1] + \gamma_2[\theta_3] = 0. \quad (5.3.3.7)$$
Using the relation $[5.3.3.7]$ and by the same argument as given in Step 2, we obtain
$\gamma_3 = 0$.

Step 4. Since $\gamma_{31} = \gamma_{12} = \gamma_2 = 0$, the homomorphism $\varphi_4$ sends $5.3.3.3$ to
$$\gamma_1\theta_2 = 0.$$ Using this relation and by the same argument as given in Step 3, we obtain $\gamma_1 = 0$.

For $s \geq 4$, $B_3(3, 2(s-1)) = \psi(\Phi(B_2(2^{s-1} - 2)))$ is the set consisting of 7 monomials:

$$v_1 = x_1 x_2^{s-1} x_3^{1-s} - x_3^{2-s}, \quad v_2 = x_1^2 x_2^{s-3} x_3^{2-s}, \quad v_3 = x_1^3 x_2^{s-1} x_3^{3-s}, \quad v_4 = x_1^4 x_2^{s-5} x_3^{4-s},$$
$$v_5 = x_1^{s-1} x_2 x_3^{3-s}, \quad v_6 = x_1^{s-1} x_2^2 x_3^{2-s}, \quad v_7 = x_1^{s-1} x_2^2 - x_3^{2-s} - x_3.$$

Suppose that $s = 4$. Then we have $|B_4((3, 2^{(3)}))| = 46$. Suppose there is a linear relation
$$S = \sum_{1 \leq j \leq 46} \gamma_j d_j = 0, \quad (5.3.3.8)$$
with $\gamma_j \in \mathbb{F}_2$ and $d_i = d_{31,i}$.

By a direct computation using Theorem $2.12$, we have
$$p_{(1;2)}(S) \equiv \gamma_3 w_2 + \gamma_4 w_3 + (\gamma_9 + \gamma_{25}) w_4 + \gamma_{12} w_5 + \gamma_{13} w_6 + \gamma_{14} w_7 \equiv 0,$$
$$p_{(1;3)}(S) \equiv (\gamma_1 + \gamma_{19}) w_1 + \gamma_5 w_2 + (\gamma_7 + \gamma_{23} + \gamma_{37} + \gamma_{39}) w_3$$
$$+ (\gamma_{10} + \gamma_{28}) w_4 + \gamma_{16} w_5 + \gamma_{18} w_7 \equiv 0,$$
$$p_{(1;4)}(S) \equiv (\gamma_2 + \gamma_{22}) w_1 + (\gamma_6 + \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{40}) w_2$$
$$+ \gamma_8 w_3 + (\gamma_{11} + \gamma_{34}) w_5 + \gamma_{17} w_6 \equiv 0.$$

From these equalities, we get
$$\begin{cases} 
\gamma_j = 0, & j = 3, 4, 5, 8, 11, 12, 13, 14, 16, 17, 18, \\
\gamma_9 = \gamma_{25}, \gamma_1 = \gamma_{19}, \gamma_7 = \gamma_{23} + \gamma_{37} + \gamma_{39} = 0, \gamma_{10} = \gamma_{28}, \\
\gamma_2 = \gamma_{22}, \gamma_6 = \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{40} = 0, \gamma_{15} + \gamma_{34} = 0. 
\end{cases} \quad (5.3.3.9)$$

Using the relations $5.3.3.9$, and Theorem $2.12$, we obtain
$$p_{(2;3)}(S) \equiv \gamma_1 w_1 + \gamma_3 w_2 + (\gamma_9 + \gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{26} + \gamma_{31} + \gamma_{39}) w_3$$
$$+ (\gamma_{35} + \gamma_{37}) w_4 + \gamma_{43} w_6 \equiv 0,$$
$$p_{(2;4)}(S) \equiv \gamma_2 w_1 + \gamma_{45} w_2 + (\gamma_{20} + \gamma_{24} + \gamma_{38} + \gamma_{40}) w_2 + \gamma_2 w_3 + \gamma_{36} w_4$$
$$+ (\gamma_{42} + \gamma_{46}) w_5 + \gamma_{44} w_6 \equiv 0,$$
$$p_{(3;4)}(S) \equiv \gamma_{15} w_1 + (\gamma_{30} + \gamma_{31} + \gamma_{32} + \gamma_{33}) w_2$$
$$+ \gamma_{15} w_3 + \gamma_{41} w_4 + (\gamma_{42} + \gamma_{43} + \gamma_{44} + \gamma_{45}) w_5 + \gamma_{42} w_6 \equiv 0.$$

From these equalities, we get
$$\begin{cases} 
\gamma_j = 0, & j = 1, 2, 15, 36, 41, 42, 43, 44, 45, 46, \\
\gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{26} + \gamma_{31} + \gamma_{39} = 0, \\
\gamma_{35} = \gamma_{37}, \gamma_{20} + \gamma_{24} + \gamma_{38} + \gamma_{40} = 0, \\
\gamma_{30} + \gamma_{31} + \gamma_{32} + \gamma_{33} = 0. 
\end{cases} \quad (5.3.3.10)$$
By a direct computation using (5.3.3.9), (5.3.3.10) and Theorem 2.12, we have

\[ P_{1;2}(S) \equiv (\gamma_7 + \gamma_{21} + \gamma_{23} + \gamma_{39})w_3 + \gamma_{26}w_4 + \gamma_{31}w_5 + \gamma_{33}w_7 \equiv 0, \]

\[ P_{1;2}(S) \equiv (\gamma_6 + \gamma_9 + \gamma_{20} + \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{38} + \gamma_{40})w_2 + \gamma_{27}w_4 + \gamma_{30}w_5 + \gamma_{32}w_6 \equiv 0, \]

\[ P_{1;3}(S) \equiv (\gamma_6 + \gamma_{10} + \gamma_{23} + \gamma_{24} + \gamma_{26} + \gamma_{27} + \gamma_{29} + \gamma_{30} + \gamma_{31} + \gamma_{32})w_2 + (\gamma_7 + \gamma_{23} + \gamma_{24} + \gamma_{33} + \gamma_{35} + \gamma_{38} + \gamma_{39} + \gamma_{40})w_3 + (\gamma_20 + \gamma_{21} + \gamma_{35})w_1 + \gamma_{29}w_4 \equiv 0, \]

\[ P_{2;3}(S) \equiv (\gamma_{10} + \gamma_{20} + \gamma_{23} + \gamma_{24} + \gamma_{26} + \gamma_{27} + \gamma_{29} + \gamma_{30} + \gamma_{35} + \gamma_{38} + \gamma_{39} + \gamma_{40})w_2 + (\gamma_9 + \gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{24} + \gamma_{26} + \gamma_{27} + \gamma_{29} + \gamma_{31} + \gamma_{32})w_3 + (\gamma_6 + \gamma_7 + \gamma_9 + \gamma_{10})w_1 + \gamma_{38}w_4 \equiv 0. \]

Combining the above equalities, (5.3.3.9) and (5.3.3.10), we get

\[
\begin{cases}
\gamma_j = 0, & j \neq 7, 10, 21, 23, 24, 28, 35, 37, 39, 40, \\
\gamma_7 = \gamma_{10} = \gamma_{28}, & \gamma_{21} = \gamma_{35} = \gamma_{37}, \\
\gamma_7 + \gamma_{21} + \gamma_{23} + \gamma_{39} = 0.
\end{cases}
\]

(5.3.3.11)

Hence we obtain

\[ \gamma_7 \theta_1 + \gamma_{21} \theta_2 + \gamma_{39} \theta_3 + \gamma_{24} \theta_4 \equiv 0, \]

(5.3.3.12)

where

\[ \theta_1 = d_{7} + d_{10} + d_{23} + d_{28}, \]

\[ \theta_2 = d_{21} + d_{23} + d_{35} + d_{37}, \]

\[ \theta_3 = d_{23} + d_{39}, \; \theta_4 = d_{24} + d_{40}. \]

Now, we prove \( \gamma_7 = \gamma_{21} = \gamma_{24} = \gamma_{39} = 0. \) The proof is divided into 4 steps.

**Step 1.** The homomorphism \( \varphi_1 \) sends (5.3.3.12) to

\[ \gamma_7 \theta_1 + \gamma_{21} (\theta_2 + \theta_1) + \gamma_{24} \theta_3 + \gamma_{39} \theta_4 \equiv 0. \]

(5.3.3.13)

Combining (5.3.3.12) and (5.3.3.13) gives

\[ \gamma_25 \theta_1 \equiv 0. \]

(5.3.3.14)

By an analogous argument as given in the proof of the proposition for the case \( s = 3, \; [\theta_1] \neq 0. \) So we get \( \gamma_{21} = 0. \)

**Step 2.** Applying the homomorphism \( \varphi_2 \) to (5.3.3.8), we obtain

\[ \gamma_7 \theta_2 + \gamma_{24} \theta_3 + \gamma_{39} \theta_4 = 0. \]

(5.3.3.15)

Using (5.3.3.15) and by a same argument as given in Step 1, we get \( \gamma_7 = 0. \)

**Step 3.** Under the homomorphism \( \varphi_3, \) the image of (5.3.3.8) is

\[ \gamma_{24}[\theta_2] + \gamma_{39}[\theta_4] = 0. \]

(5.3.3.16)

Using (5.3.3.16) and by a same argument as given in Step 3, we obtain \( \gamma_{24} = 0. \)

**Step 4.** Since \( \gamma_7 = \gamma_{22} = \gamma_{24} = 0, \) the homomorphism \( \varphi_3 \) sends (5.3.3.8) to

\[ \gamma_{39}[\theta_4] = 0. \]

From this equality and by a same argument as given in Step 3, we get \( \gamma_{39} = 0. \)
Lemma 5.4.1. Let Theorem 2.12 and Proposition 2.10, we get and of degree \( 2 \).

Observe that the monomial \( v_i \), \( i = 1, 2, \ldots, 7 \), we obtain \( \gamma_j = 0 \) for any \( j \). The proposition is proved. \( \square \)

5.4. The case of degree \( 2s+t+1 + 2s+1 - 3 \).

First of all, we determine the \( \omega \)-vector of an admissible monomial of degree \( n = 2s+t+1 + 2s+1 - 3 \) for any positive integers \( s, t \).

**Lemma 5.4.1.** Let \( x \) be a monomial of degree \( 2s+t+1 + 2s+1 - 3 \) in \( P_4 \) with \( s, t \) are the positive integers. If \( x \) is admissible, then either \( \omega(x) = (3^s, 1^{(t+1)}) \) or \( \omega(x) = (3^{(s+1)}, 2^{(t-1)}) \).

**Proof.** Observe that the monomial \( z = x_1^{2s+t+1-1} x_2^{2^{t-1}} x_3^{2^{t-1}} \) is the minimal spike of degree \( 2s+t+1 + 2s+1 - 3 \) in \( P_4 \) and \( \omega(z) = (3^s, 1^{(t+1)}) \). Since \( z \) is admissible and \( 2s+t+1 + 2s+1 - 3 \) is odd, using Theorem 2.12, we obtain \( \omega_1(x) = 3 \). Using Theorem 2.12 and Proposition 2.10, we get \( \omega_i(x) = 3 \) for \( i = 1, 2, \ldots, s \).

Let \( x' = \prod_{i \geq 1} x_{i,i+1}^{2^{t-1}} \). Then \( \omega_i(x') = \omega_i(x), i \geq 1 \) and \( \deg(x') = 2^{t+1} - 1 \). Since \( x \) is admissible, using Theorem 2.9, we see that \( x' \) is also admissible. By Lemmas 5.3.1, either \( \omega(x') = (1^{(t+1)}) \) or \( \omega(x') = (3, 2^{(t-1)}) \) or \( \omega(x') = (1, 3) \) for \( t = 2 \). By a direct computation we see that if \( \omega(x') = (1, 3) \), then \( x \) is inadmissible. So, the lemma is proved. \( \square \)

Using Theorem 1.3, we easily obtain the following.

**Proposition 5.4.2.** For any positive integers \( s, t \) with \( s \geq 3 \), \( \Phi(B_3(n)) \) is a minimal set of generators for \( A \)-module \( P_4 \) in degree \( n = 2s+t+1 + 2s+1 - 3 \).

Hence it suffices to consider the cases \( s = 1 \) and \( s = 2 \).

5.4.1. The subcase \( s = 1 \).

For \( s = 1 \), \( n = 2^{t+2} + 1 = (2^{t+2} - 1) + (2 - 1) + (2 - 1) \). Hence \( \mu(2^{t+2} + 1) = 3 \) and Kaneko’s homomorphism

\[
\widetilde{S}_0^0: (QP_3)_{2^{t+2}+1} \rightarrow (QP_3)_{2^{t+1}-1}
\]

is an isomorphism. So, we get

\[
B_3(n) = \psi(B_3(2^{t+1} - 1)) = \psi(B_3(1^{(t+1)})) \cup \psi(B_3(3, 2^{(t-1)})).
\]

**Proposition 5.4.3.** For any positive integer \( t \), \( \Phi(B_3(n)) \cup B(t) \) is the set of all the admissible monomials for \( A \)-module \( P_4 \) in degree \( n = 2^{t+2} + 1 \), where the set \( B(t) \) is determined as follows:

\[
B(1) = \{ x_1^3 x_2^2 x_3^4 x_4 \}, \quad B(2) = \{ x_1^3 x_2^5 x_3^8 x_4 \},
\]

\[
B(3) = \{ x_1^3 x_2^7 x_3^{11} x_4^{12}, x_1^3 x_2 x_3^{11} x_4^{12}, x_1^7 x_2 x_3^{11} x_4^{12}, x_1 x_2 x_3 x_4^{11}, x_1 x_2 x_3^{11} x_4 \},
\]

\[
B(t) = \{ x_1^3 x_2^7 x_3^{2t+1-5} x_4^{2^{t+1}-4}, x_1^7 x_2^{2^{t+1}-5} x_4^{2^{t+1}-4},
\]
\[
x_1^{2^{t+1}-5} x_3^{2^{t+1}-5} x_4^{2^{t+1}-5}, x_1 x_2 x_3^{2^{t+1}-8} x_4^{2^{t+1}-5} \}, \text{ for } t > 3.
\]
The following lemma is proved by a direct computation.

**Lemma 5.4.4.** The following monomials are strictly inadmissible:

i) \( X_2x_1^{12}, X_3^2x_1^4x_4, j = 1, 2, X_1x_2^2x_3^4, X_2^3x_2^2x^8x_1^4, j = 3, 4. \)

ii) \( X_3^2x_2^2x_3^4, X_3^2x_2^2x^8x_1^4, j = 3, 4. \)

iii) \( X_3^3x_2^3x_3^9x_1^{12}, X_4X_3^2x_1^2x_3^3x_2^4, \) \( X_3^3x_2^3x_3^9x_1^{12}x_4^2, \) \( j = 2, 4. \)

iv) \( X_3^3x_1^2x_3^{12}, 1 \leq i < j \leq 4, m \neq i, j. \)

v) \( X_4x_1^3x_3^2x_4^2x_1^2, j = 3, 4, X_3^3x_1^2x_3^2x_4, j = 2, 4. \)

vi) \( X_3^3x_1^2x_3^2x_4^2, X_3^3x_1^2x_3^2x_4^2x_4^2, X_3^3x_1^2x_3^2x_4^2x_4^2, \) \( X_4x_1^3x_3^2x_4^2, X_3^3x_1^2x_3^2x_4, i = 1, 2, j = 3, 4, X_3^3x_1^2x_3^2x_4. \)

**Proof of Proposition 5.4.4.** Let \( x \) be an admissible monomial of degree \( n = 2^{t+2} + 1. \) According to Lemma 5.4.1, \( x = X_4y^2 \) with \( y \) a monomial of degree \( 2t+1. \) Since \( x \) is admissible, by Theorem 2.12, \( y \) is admissible. By a direct computation, we see that if \( y \in B_4(2t+1) \) and \( X_4y^2 \notin \Phi(B_4(n)) \cup B(t), \) then there is a monomial \( w \) which is given in one of Lemma 5.1.3, \( \Phi(B_4) \cup B(t) \) such that \( X_4y^2 = wz^2u \) with some positive integer \( u \) and monomial \( z. \) By Theorem 2.9, \( x \) is inadmissible.

For \( t = 1, \) we have \( |C_4^+(9)| = 18. \) Suppose there is a linear relation

\[
S = \sum_{i=1}^{18} \gamma_id_i = 0, \tag{5.4.4.1}
\]

with \( \gamma_i \in \mathbb{Z}. \) A direct computation from the relations \( p_{(r;j)}(S) = 0, \) for \( 1 \leq r < j \leq 4, \) obtains \( \gamma_i = 0 \) for \( i \neq 1, 4, 9, 10, 11, 12 \) and \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_{11} = \gamma_{12}. \) Hence the relation \( \gamma_1 \theta = 0 \) for \( \theta = d_1 + d_4 + d_9 + d_{10} + d_{11} + d_{12}. \)

We prove \( \gamma_1 = 0. \) Suppose \( \theta \) is hit. Then we get

\[
\theta = Sq^1(A) + Sq^2(B) + Sq^4(C),
\]

for some polynomials \( A \in (P_4^+)_{3}, B \in (P_4^+)_{7}, C \in (P_4^+)_{5}. \) Let \( (Sq^2)^3 \) act on both sides of this equality. It is easy to check that \( (Sq^2)^3 Sq^4(C) = 0 \) for all \( C \in (P_4^+)_{5}. \) Since \( (Sq^2)^3 \) annihilates \( Sq^1 \) and \( Sq^2, \) the right hand side is sent to zero. On the other hand, a direct computation shows

\[
(Sq^2)^3(\theta) = (1, 2, 4, 8) + \text{symmetries} \neq 0.
\]

Hence we have a contradiction. So we obtain \( \gamma_1 = 0. \)

For \( t = 2, |B_4^+(17)| = 47. \) Suppose there is a linear relation

\[
S = \sum_{i=1}^{47} \gamma_id_i = 0, \tag{5.4.4.2}
\]

with \( \gamma_i \in \mathbb{Z} \) and \( d_i = d_{17}. \) A direct computation from the relations \( p_{(r;j)}(S) = 0, \) for \( (j; J) \in N_4, \) we obtain \( \gamma_i = 0 \) for \( i \neq 1, 4, 8, 9, 10, 11, 17, 18 \) and \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma_9 = \gamma_{10} = \gamma_{11} = \gamma_{12} = \gamma_9 = \gamma_{11} = \gamma_{17} = \gamma_{18}. \) Hence the relation \( \gamma_1 \theta = 0 \) where \( \theta = d_1 + d_4 + d_9 + d_{10} + d_{11} + d_{13} + d_{16} + d_{17} + d_{18}. \)

By a same argument as given in the proof of the proposition for \( t = 1, \) we see that \( \theta \neq 0. \) Hence \( \gamma_1 = 0. \)

For \( t = 3, \) we have \( |B_4^+(33)| = 84, \) and \( |B_4^+(2^{t+2} + 1)| = 94 \) for \( t \geq 4. \) Suppose there is a linear relation

\[
S = \sum_{i=1}^{84} \gamma_id_i = 0, \tag{5.4.4.3}
\]
with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{33,i}$. A direct computation from the relations $p_{(j; J)}(S) \equiv 0$, for $(j; J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i \not\in E$ with $E = \{1, 3, 8, 9, 13, 14, 17, 24, 25, 42, 43, 59, 60, 65, 66, 67\}$ and $\gamma_i = \gamma_1$ for all $i \in E$. Hence the relation (5.4.4.3) become $\gamma_1 \theta \equiv 0$ with $\theta = \sum_{i \in E} d_i$.

By a same argument as given in the proof of the proposition for $t = 1$, we see that $[\theta] \not\equiv 0$. Therefore $\gamma_1 = 0$.

Now, we prove the set $B_4^+(n)$ is linearly independent for $t > 3$. Suppose there is a linear relation

$$S = \sum_{i=1}^{94} \gamma_i d_i \equiv 0,$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{n,i}$. A direct computation from the relations $p_{(j; J)}(S) \equiv 0$, for $(j; J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i$. □

5.4.2. The subcase $s = 2$.

For $s = 2$, we have $n = 2^{t+3} + 5$. According to Theorem 1.2, the iterated Kameko homomorphism $$(\tilde{S}_q^0) : (QP_3)_{2^{t+3}+5} \to (QP_3)_{2^{t+1} - 1}$$ is an isomorphism. So we get

$$B_3(n) = \psi^2(B_3(2^{t+1} - 1)) = \psi^2(B_3(1^{t+1})) \cup \psi^2(\Phi(B_3(3, 2^{t-1}))).$$

**Proposition 5.4.5.**

i) $B_4(n) = \Phi(B_4(21)) \cup \{x_1^7 x_2^8 x_3^3 x_4^3, x_2^7 x_3^8 x_4^3, x_3^7 x_4^8 x_4^3, x_4^7 x_1^8 x_3^3, x_4^7 x_2^8 x_4^3\}$ is the set of all the admissible monomials for $A$-module $P_4$ in degree 21.

ii) For any integer $t > 1$, $\Phi(B_4(n))$ is the set of all the admissible monomials for $A$-module $P_4$ in degree $n = 2^{t+3} + 5$.

The following lemma is proved by a direct computation.

**Lemma 5.4.6.** The following monomials are strictly inadmissible:

i) $X_2^3 x_3^4, X_3^4 x_3^4, 1 \leq i < j \leq 4, X_3^4 x_4^4$.

ii) $X_2^3 x_3^4 x_4^4, X_3^4 x_3^4 x_4^4, X_2^3 x_4^4 x_4^4, X_4^4 x_4^4, i = 1, 2$.

iii) $x_2^3 x_3^4 x_4^4, x_2^3 x_3^4 x_4^4, x_3^4 x_3^4 x_4^4, x_3^4 x_3^4 x_4^4, j = 3, 4$

iv) $x_4^4 x_2^3 x_3^4 x_4^4, x_4^4 x_2^3 x_3^4 x_4^4, x_4^4 x_2^3 x_3^4 x_4^4, x_4^4 x_2^3 x_3^4 x_4^4$.

**Proof of Proposition 5.4.5.** Let $x$ be an admissible monomial of degree $n = 2^{t+3} + 5$. According to Lemma 5.4.1, $x = X_i y^2$ with $y$ a monomial of degree $2^{t+2} + 1$. Since $x$ is admissible, by Theorem 2.12, $y$ is admissible.

By a direct computation, we see that if $y \in B_4(2^{t+2} + 1)$ and $X_i y^2$ is not in the set given in Proposition 5.4.5, then there is a monomial $w$ which is given in one of Lemmas 5.1.3, 5.3.3, 5.4.6 such that $X_i y^2 = w z^2$ with some positive integer $u$ and monomial $z$.

By Theorem 2.9, $x$ is inadmissible. Hence $QP_4(n)$ is generated by the set given in the proposition.

For $t = 1$, we have $|B_4^+(21)| = 66$. Suppose there is a linear relation

$$S = \sum_{i=1}^{66} \gamma_i d_i \equiv 0,$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{21,i}$. 

By a simple computation, we see that $B_3(21)$ is the set consisting of 7 monomials:

$$
\begin{align*}
  v_1 &= x_1^3 x_2^3 x_4^{15}, \\
  v_2 &= x_1^3 x_2^7 x_3^{11}, \\
  v_3 &= x_1^3 x_2 x_3^3, \\
  v_4 &= x_1^3 x_2 x_3 x_3^{11}, \\
  v_5 &= x_1^3 x_2 x_2^2 x_3^{11}, \\
  v_6 &= x_1^{15} x_2^2 x_3^{11}, \\
  v_7 &= x_1^3 x_2 x_2^3.
\end{align*}
$$

A direct computation, we have

$$
\begin{align*}
  p(1;2)(S) &\equiv \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \gamma_4 v_4 + \gamma_5 v_5 + \gamma_6 v_6 + \gamma_7 \equiv 0, \\
  p(1;3)(S) &\equiv \gamma_4 v_4 + \gamma_6 v_6 + \gamma_8 + \gamma_9 + \gamma_4 v_4 + \gamma_12 v_4 \\
  &\quad + (\gamma_14 + \gamma_18 v_5 + \gamma_17) v_6 + \gamma_18 v_7 \equiv 0, \\
  p(1;4)(S) &\equiv (\gamma_5 + \gamma_26 + \gamma_7) v_1 + (\gamma_7 + \gamma_29 v_2 + \gamma_9) v_3 + (\gamma_13 + \gamma_37) v_4 \\
  &\quad + \gamma_15 v_5 + \gamma_18 v_6 + \gamma_59 v_7 \equiv 0, \\
  p(2;3)(S) &\equiv \gamma_19 v_1 + (\gamma_21 + \gamma_27 + \gamma_32 + \gamma_60) v_2 + (\gamma_23 + \gamma_30 + \gamma_34 + \gamma_38 + \gamma_40) v_3 \\
  &\quad + \gamma_43 v_4 + (\gamma_45 + \gamma_49 + \gamma_51) v_5 + \gamma_54 v_6 + \gamma_63 v_7 \equiv 0, \\
  p(2;4)(S) &\equiv (\gamma_20 + \gamma_26 + \gamma_33 + \gamma_37 + \gamma_41) v_1 + \gamma_22 + \gamma_29 + \gamma_35 + \gamma_61) v_2 \\
  &\quad + \gamma_24 v_3 + (\gamma_44 + \gamma_48 + \gamma_52) v_4 + \gamma_46 v_5 + \gamma_55 v_6 + \gamma_64 v_7 \equiv 0, \\
  p(3;4)(S) &\equiv (\gamma_25 + \gamma_26 + \gamma_27 + \gamma_28 + \gamma_29 + \gamma_30 + \gamma_31) v_1 \\
  &\quad + (\gamma_36 + \gamma_37 + \gamma_38 + \gamma_62) v_2 + \gamma_42 v_3 \\
  &\quad + (\gamma_47 + \gamma_48 + \gamma_49 + \gamma_50 + \gamma_65) v_4 + \gamma_53 v_5 + \gamma_56 v_6 + \gamma_66 v_7 \equiv 0.
\end{align*}
$$

From the above equalities, we get $\gamma_i = 0$, for $i = 1, 2, 3, 4, 9, 10, 11, 12, 15, 16, 17, 18, 19, 24, 42, 43, 46, 53, 54, 55, 56, 57, 58, 59, 63, 64, 66$ and $\gamma_6 = \gamma_27 - \gamma_8 + \gamma_30 + \gamma_49 = 0$, $\gamma_14 = \gamma_38$, $\gamma_7 + \gamma_26 + \gamma_48 = 0$, $\gamma_7 = \gamma_29$, $\gamma_13 = \gamma_37$, $\gamma_7 + \gamma_21 + \gamma_32 + \gamma_60 = 0$, $\gamma_14 + \gamma_23 + \gamma_30 + \gamma_44 + \gamma_49 + \gamma_51 = 0$, $\gamma_20 + \gamma_26 + \gamma_33 + \gamma_47 + \gamma_41 = 0$, $\gamma_7 + \gamma_22 + \gamma_35 + \gamma_61 = 0$, $\gamma_44 + \gamma_48 + \gamma_52 = 0$, $\gamma_6 + \gamma_7 + \gamma_25 + \gamma_26 + \gamma_28 + \gamma_30 + \gamma_31 = 0$, $\gamma_14 = \gamma_36 + \gamma_37 + \gamma_38 + \gamma_62 = 0$, $\gamma_47 + \gamma_48 + \gamma_49 + \gamma_50 + \gamma_65 = 0$.

With the aid of the above equalities have

$$
\begin{align*}
  p(1;2,3)(S) &\equiv \gamma_21 v_2 + (\gamma_8 + \gamma_23 + \gamma_30 + \gamma_45 + \gamma_49) v_3 + \gamma_32 v_4 \\
  &\quad + (\gamma_34 + \gamma_45 + \gamma_49 + \gamma_51) v_5 + (\gamma_40 + \gamma_51) v_6 + \gamma_60 v_7 \equiv 0, \\
  p(1;2,4)(S) &\equiv (\gamma_5 + \gamma_20 + \gamma_26 + \gamma_44 + \gamma_48) v_1 + \gamma_22 v_2 \\
  &\quad + (\gamma_33 + \gamma_44 + \gamma_48 + \gamma_52) v_4 + \gamma_35 v_5 + (\gamma_41 + \gamma_52) v_6 + \gamma_61 v_7 \equiv 0.
\end{align*}
$$

From this, we obtain $\gamma_i = 0$, for $i = 21, 22, 32, 35, 60, 61$ and $\gamma_8 + \gamma_23 + \gamma_30 + \gamma_45 + \gamma_49 = 0$, $\gamma_34 + \gamma_45 + \gamma_49 + \gamma_51 = 0$, $\gamma_40 = \gamma_51$, $\gamma_6 + \gamma_20 + \gamma_26 + \gamma_44 + \gamma_48 = 0$, $\gamma_33 + \gamma_44 + \gamma_48 + \gamma_52 = 0$, $\gamma_41 = \gamma_52$. By a direct computation using the above equalities, one gets

$$
\begin{align*}
  p(1;3,4)(S) &\equiv (\gamma_5 + \gamma_25 + \gamma_26 + \gamma_47 + \gamma_48) v_1 + (\gamma_28 + \gamma_47 + \gamma_48 + \gamma_49 + \gamma_50) v_2 \\
  &\quad + (\gamma_7 + \gamma_70 + \gamma_71 + \gamma_49 + \gamma_50) v_3 + \gamma_36 v_4 + \gamma_39 v_5 + \gamma_62 v_7 \equiv 0, \\
  p(2;3,4)(S) &\equiv (\gamma_13 + \gamma_20 + \gamma_25 + \gamma_26 + \gamma_33 + \gamma_36 + \gamma_40 + \gamma_41) v_1 + (\gamma_6 + \gamma_7 \\
  &\quad + \gamma_13 + \gamma_14 + \gamma_28 + \gamma_33 + \gamma_34 + \gamma_36 + \gamma_39) v_2 + (\gamma_14 + \gamma_23 + \gamma_30 + \gamma_31 \\
  &\quad + \gamma_34 + \gamma_39 + \gamma_40 + \gamma_41) v_3 + (\gamma_44 + \gamma_47 + \gamma_48 + \gamma_51 + \gamma_52) v_4 \\
  &\quad + (\gamma_45 + \gamma_49 + \gamma_50 + \gamma_51 + \gamma_52) v_5 + \gamma_65 v_7 \equiv 0.
\end{align*}
$$

So we obtain $\gamma_36 = \gamma_39 = \gamma_62 = \gamma_65 = 0$, $\gamma_5 + \gamma_25 + \gamma_26 + \gamma_47 + \gamma_48 = 0$, $\gamma_28 + \gamma_47 + \gamma_48 + \gamma_49 + \gamma_50 = 0$, $\gamma_13 + \gamma_20 + \gamma_25 + \gamma_26 + \gamma_33 + \gamma_40 + \gamma_41 = 0$. 

Combining (5.4.6.2) and (5.4.6.3) gives

\[
\begin{align*}
\{x, y, D, \theta, \eta, \gamma, b\} = & 0, \\
\{y, \gamma, D, \theta, \eta, \gamma, b\} = & 0, \\
\{\gamma, D, \theta, \eta, \gamma, b\} = & 0.
\end{align*}
\]

Combining the above equalities, one gets \( \gamma_i = 0 \) for \( i \neq 5, 8, 13, 14, 20, 23, 25, 26, 30, 31, 37, 38, 40, 41, 44, 45, 47, 48, 49, 50, 51 \), \( \gamma_i = \gamma_5 \) for \( i = 8, 13, 14, 37, 38 \), \( \gamma_i = \gamma_20 \) for \( i = 23, 44, 45 \), \( \gamma_i = \gamma_25 \) for \( i = 40, 47, 51 \), \( \gamma_i = \gamma_31 \) for \( i = 41, 50, 52 \), \( \gamma_{20} + \gamma_{25} + \gamma_{49} = 0 \), \( \gamma_{30} + \gamma_{26} + \gamma_{31} = 0 \), \( \gamma_{20} + \gamma_{31} + \gamma_{48} = 0 \), \( \gamma_{7} + \gamma_{20} + \gamma_{25} + \gamma_{30} = 0. \)

Substituting the above equalities into the relation (5.4.6.1), we have

\[
\gamma_{25}[\theta_1] + \gamma_{31}[\theta_2] + \gamma_{5}[\theta_3] + \gamma_{20}[\theta_4] = 0, \tag{5.4.6.2}
\]

where

\[
\begin{align*}
\theta_1 = & d_{25} + d_{30} + d_{40} + d_{47} + d_{49} + d_{51}, \\
\theta_2 = & d_{26} + d_{31} + d_{41} + d_{48} + d_{50} + d_{52}, \\
\theta_3 = & d_{5} + d_{13} + d_{14} + d_{26} + d_{30} + d_{37} + d_{38}, \\
\theta_4 = & d_{20} + d_{23} + d_{26} + d_{30} + d_{44} + d_{45} + d_{48} + d_{49}.
\end{align*}
\]

We need to show that \( \gamma_5 = \gamma_{20} = \gamma_{25} = \gamma_{31} = 0. \) The proof is divided into 4 steps.

**Step 1.** The homomorphism \( \varphi_3 \) sends (5.4.6.2) to

\[
\gamma_{25}[\theta_1] + \gamma_{31}[\theta_2] + (\gamma_5 + \gamma_{20})[\theta_3] + \gamma_{20}[\theta_4] = 0. \tag{5.4.6.3}
\]

Combining (5.4.6.2) and (5.4.6.3) gives

\[
\gamma_{20}[\theta_4] = 0.
\]

We prove \( [\theta_3] \neq 0. \) We have \( \varphi_2 \varphi_3([\theta_1]) = [\theta_3]. \) So we need only to prove that \( [\theta_1] \neq 0. \) Suppose \( [\theta_1] = 0. \) Then the polynomial \( \theta_1 \) is hit and we have

\[
\theta_1 = S^q(A) + S^q(B) + S^q(C) + S^q(D),
\]

for some polynomials \( A \in (P^+_{4})_{20}, B \in (P^+_{4})_{19}, C \in (P^+_{4})_{17}, D \in (P^+_{4})_{13}. \)

Let \( (S^q)^3 \) act on the both sides of this equality. Since \( (S^q)^3 S^q = 0 \) and \( (S^q)^3 S^q = 0, \) we get

\[
(S^q)^3(\theta_4) = (S^q)^3 S^q(C) + (S^q)^3 S^q(D).
\]

By a direct computation, we see that the monomial \( x = x_1 x_2^{12} x_3^{2} x_4^{2} x_5^{6} \) is a term of \( (S^q)^3(\theta_4). \) If this monomial is a term of \( (S^q)^3 S^q(y), \) then \( y = x_1 f_1(z) \) with \( z \) a monomial of degree 6 in \( P_3 \) and \( x \) is a term of \( x_1 (S^q)^3 S^q(f_1(z)) = 0. \) So the monomial \( x \) is not a term of \( (S^q)^3 S^q(D) \) for all \( D \in (P^+_{4})_{13}. \)

If this monomial is a term of \( (S^q)^3 S^q(y), \) where the monomial \( y \) is a term of \( C, \) then \( y = x_1 f_1(z) \) with \( z \) a monomial of degree 10 in \( P_3 \) and \( x \) is a term of \( x_1^3(S^q)^3 S^q(f_1(z)) = 0. \) By a direct computation, we see that either \( x_1 x_2^{6} x_3 x_4^{3} \) or \( x_1^3 x_2^{3} x_3^{2} x_4^{3} \) is a term of \( C. \)

If \( x_1 x_2^{6} x_3 x_4^{3} \) is a term of \( C \) then

\[
(S^q)^3(\theta_4) \in (S^q)^3 S^q(C) = (S^q)^3 S^q(D),
\]

where \( C' = x_1^2 x_2^{12} x_3^{2} x_4^{2} x_5^{3} \). The monomial \( x' = x_1^{10} x_2^{6} x_3^{2} x_4^{3} \) is a term of the polynomial \( (S^q)^3(\theta_4) \). If \( x' \) is a term of the polynomial \( (S^q)^3 S^q(y'), \) with \( y' \) a monomial in \( (P^+_{4})_{13}. \) Then \( y' = x_1 a x_2^{b} x_3^{c} x_4^{d} \) with \( a > 7, b > 3, c > 0. \) This contradicts with the fact that \( \deg y' = 13. \) So \( x' \) is not a term of \( (S^q)^3 S^q(D) \) for all \( D \in (P^+_{4})_{13}. \) Hence \( x' \) is a term of \( (S^q)^3 S^q(C') \). By a direct computation,
we see that either \(x_1^7x_2^6x_3x_4^3\) or \(x_1^7x_2^5x_2^2x_4^4\) is a term of \(C\). Since \(x_1^7x_2^6x_3x_4^3\) is not a term of \(C\), the monomial \(x_1^7x_2^6x_3x_4^3\) is a term of \(C\). Then we have
\[
(Sq^2)^3(\theta_1 + Sq^4(x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_2^2x_4^4)) = (Sq^2)^3(Sq^4(C') + Sq^4(D)),
\]
where \(C' = x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_2^2x_4^4\). Now the monomial \(x = x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_2^2x_4^4\) is a term of \((Sq^2)^3(\theta_1 + Sq^4(x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_2^2x_4^4))\).

Hence either \(x_1^7x_2^6x_3x_4^3\) or \(x_1^7x_2^5x_2^2x_4^4\) is a term of \(C\) is a term of \(C''\). On the other hand, the two monomials \(x_1^7x_2^6x_3x_4^3\) and \(x_1^7x_2^5x_2^2x_4^4\) are not the terms of \(C''\). We have a contradiction. Hence one gets \(\gamma_{20} = 0\).

**Step 2.** Since \(\gamma_{20} = 0\), the homomorphism \(\varphi_2\) sends \((5.4.6.3)\) to
\[
\gamma_{25}[\theta_1] + \gamma_{31}[\theta_2] + \gamma_{5}[\theta_3] = 0.
\]
Using \((5.4.6.4)\) and the result in Step 1, we get \(\gamma_5 = 0\).

**Step 3.** The homomorphism \(\varphi_3\) sends \((5.4.6.3)\) to
\[
\gamma_{25}[\theta_4] + \gamma_{31}[\theta_2] = 0.
\]
Using the relation \((5.4.6.5)\) and the result in Step 2, we obtain \(\gamma_{25} = 0\).

**Step 4.** Since \(\varphi_4([\theta_2]) = [\theta_1]\), we have
\[
\gamma_{31}[\theta_1] = 0.
\]
Using this equality and by a same argument as given in Step 3, we get \(\gamma_{31} = 0\).

For \(t > 1\), we have \(|B_4^t(n)| = m(t)|\) with \(m(2) = 95, m(3) = 128\) and \(m(t) = 139\) for \(t \geq 4\). Suppose there is a linear relation
\[
S = \sum_{i=1}^{m(t)} \gamma_i d_i \equiv 0,
\]
with \(\gamma_i \in \mathbb{F}_2\) and \(d_i = d_{n,i}\). A direct computation from the relations \(p_{(j,j)}(S) \equiv 0\), for \((j,j) \in N_4\), we obtain \(\gamma_i = 0\) for all \(i\). The proposition is proved. \(\square\)

### 5.5. The case of degree \(2^{s+t} + 2^s - 2\).

For \(s \geq 1\) and \(t \geq 2\), the space \((QP_4)_n\) was determined in \([32]\). Hence, in this subsection we need only to compute \((QP_4)_n\) for \(n = 2^{s+1} + 2^s - 2\) with \(s > 1\).

Recall that, the homomorphism
\[
\tilde{S_q}^0 : (QP_4)_{2^{s+1} + 2^s - 2} \to (QP_4)_{2^{s+1} + 2^s - 3}
\]
is an epimorphism. Hence we have
\[
(QP_4)_{2m+4} \cong (QP_4)_m \oplus (QP_4)_{2m+4} \oplus (\text{Ker} \tilde{S_q}^0 \cap (QP_4)_{2m+4}),
\]
where \(m = 2^s + 2^{t-1} - 3\). So it suffices to compute \(\text{Ker} \tilde{S_q}^0 \cap (QP_4)_n\) for \(s > 1\).

For \(s > 1\), denote by \(C(s)\) the set of all the following monomials:
\[
\begin{align*}
& x_1^2x_2x_3^2x_4^2 - x_3^2x_4^2x_1^2x_2, \\
& x_1^2x_2x_3^2x_4^2 - x_3^2x_4^2x_1^2x_2, \\
& x_1^2x_2x_3^2x_4^2 - x_3^2x_4^2x_1^2x_2, \\
& x_1^2x_2x_3^2x_4^2 - x_3^2x_4^2x_1^2x_2, \\
& x_1^2x_2x_3^2x_4^2 - x_3^2x_4^2x_1^2x_2,
\end{align*}
\]
For \(s > 2\), denote by \(D(s)\) the set of all the following monomials:
Proposition 5.5.1. For any integer $P$

Proof. \]
\[
\text{The following monomials are strictly inadmissible:}
\]
\[
\text{Set } E(2) = C(2) \cup \{x_i^4 x_j x_3 x_4 \}, \quad E(3) = C(3) \cup D(3) \cup \{x_i^4 x_j^2 x_3 x_4 \} \text{ and } E(s) = C(s) \cup D(s) \cup \{x_i^4 x_j^2 x_3^2 x_4^{s-1} \}, \text{ for } s > 3.
\]

Proposition 5.5.1. For any integer $s > 1$, $E(s) \cup \Phi^0(B_3(n)) \cup \psi(B_3(m))$ is the set of all the admissible monomials for $A$-module $P_4$ in degree $n = 2m + 4$ with $m = 2^s + 2^s - 1 - 3$.

Lemma 5.5.2. Let $x$ be an admissible monomial of degree $n = 2^s t + 2^s - 2$ in $P_4$. If $[x] \in \text{Ker} S_q^0$, then either $\omega(x) = (2^s, 1)$.

Proof. We prove the lemma by induction on $s$. Since $n = 2^s t + 2^s - 2$ is even, we get either $\omega_1(x) = 0$ or $\omega_1(x) = 2$ or $\omega_1(x) = 4$. If $\omega_1(x) = 0$, then $x = S q^2(y)$ for some monomial $y$. If $\omega_1(x) = 4$, then $x = X_{y^2}$ for some monomial $y$. Since $x$ is admissible, $y$ also is admissible. This implies $\text{Ker} S_q^0([x]) = [y] \neq 0$ and we have a contradiction. So $\omega_1(x) = 2$ and $x = x_i x_j y^2$ with $1 \leq i < j \leq 4$, and $y$ a monomial of degree $2^s + 2^s - 2$ in $P_4$. Using Proposition 2.10 we get $\omega_1(x) = 2$ for $1 \leq i \leq s$. Then $x = x^2 y^2$ with $x^2$ monomials in $P_4$ and $\text{deg } z = 2^t - 1$. By a direct computation we see that if $w$ is a monomial such that either $\omega(w) = (2, 1, 3)$ or $\omega(w) = (2, 2, 3)$ or $\omega(w) = (2, 3, 2, 2)$ then $w$ is strictly inadmissible. Now, the lemma follows from this fact, Lemma 5.3.1 and Theorem 2.9. \]

The following is proved by a direct computation.

Lemma 5.5.3. The following monomials are strictly inadmissible:

\[
i) x_i^2 x_j x_m, x_i^3 x_j^2 x_m, x_i^2 x_j x_m^2, 1 \leq i < j < m \leq 4.
\]

\[
ii) x_i x_j x_k x_3 x_4, x_i x_j x_k x_4 x_4, x_i x_j x_k x_3 x_4, x_i x_j x_k x_3 x_4^2, x_i x_j x_k x_3 x_4^3, x_i x_j x_k x_3 x_4^4.
\]

Proof of Proposition 5.5.1. Let $x$ be an admissible monomial of degree $n = 2^s t + 2^s - 2$ in $P_4$ and $[x] \in \text{Ker} S_q^0$. By Lemma 5.5.2, $\omega_1(x) = 2$, for $1 \leq i \leq s$, $\omega_{s+1}(x) = 1$ and $\omega_i(x) = 0$ for $i > s + 1$. By induction on $s$, we see that if $x \notin E(s) \cup \Phi^0(B_3(n))$ then there is a monomial $y$ which is given in one of Lemmas 5.2.3 and 5.5.3 such that $x = wy^2 u$ for some monomial $y$ and positive integer $u$. By Theorem 2.9, $x$ is inadmissible. Hence $\text{Ker} S_q^0$ is spanned by the set $[E(s) \cup \Phi^0(B_3(n))]$ in degree $n = 2^s t + 2^s - 2$. Now, we prove that the set $[E(s) \cup \Phi^0(B_3(n))]$ is linearly independent.

It suffices to prove that the set $[E(s)]$ is linearly independent. For $s = 2$, $|E(2)| = 12$. Suppose there is a linear relation

\[
S = \sum_{i=1}^{12} \gamma_i d_i = 0,
\]

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{10,i}$. A direct computation from the relations $p_{(1j)}(S) = 0$, for $j = 1, 2, 3$, we obtain $\gamma_i = 0$ for all $i$. 

\[
ON THE PETERSON HIT PROBLEM 47
\]
For $s > 2$, $|E(s)| = 26$. Suppose there is a linear relation

$$S = \sum_{i=1}^{26} \gamma_i d_i \equiv 0,$$

(5.2)

with $\gamma_i \in F_2$ and $d_i = d_{n_i}$. A direct computation from the relations $p(r,j)(S) \equiv 0$, for $1 \leq r < j \leq 4$, we obtain $\gamma_i = 0$ for all $i$. The proposition is proved. \(\square\)

5.6. The case of degree $2^{s+t+u} + 2^{s+t} + 2^s - 3$.

First, we determine the $\omega$-vector of an admissible monomial of degree $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$.

Lemma 5.6.1. If $x$ is an admissible monomial of degree $2^{s+t+u} + 2^{s+t} + 2^s - 3$ in $P_4$ then $\omega(x) = (3^s, 2^t, 1^u)$.

Proof. Observe that $z = x_1^{2^{s+t+u}}x_2^{2^{s+t}}x_3^{2^s}$ is the minimal spike of degree $2^{s+t+u} + 2^{s+t} + 2^s - 3$ and $\omega(z) = (3^s, 2^t, 1^u)$. Since $2^{s+t+u} + 2^{s+t} + 2^s - 3$ is odd and $x$ is admissible, using Proposition 2.10 and Theorem 2.12 we get $\omega_1(x) = 3$ for $1 \leq i \leq s$. Set $x' = \prod_{1 \leq i \leq s} X_i^{2^t}$ Then $x = x'y^2$ for some monomial $y$. We have $\omega_j(y) = \omega_{j+s}(x)$ for all $j \geq 1$ and

$$2^{s+t+u} + 2^{s+t} + 2^s - 3 = \text{deg } x = \sum_{i \geq 1} 2^{t-i} \omega_i(x)$$

$$= 3(2^s - 1) + 2^s \sum_{j \geq 1} 2^{j-i} \omega_{j+s}(x)$$

$$= 3.2^s - 3 + 2^s \text{deg } y.$$

This equality implies $\text{deg } y = 2^{s+t+u} + 2^u - 2$. Since $x$ is admissible, using Theorem 2.9 we see that $y$ is also admissible. By a direct computation we see that if $w$ is a monomial such that $\omega(w) = (3^s, 2^t, 1^u)$ then $w$ is strictly inadmissible. Combining this fact, Lemma 5.3.1 Proposition 2.10 and Theorem 2.9 we obtain $\omega(y) = (2^t, 1^u)$. The proposition is proved. \(\square\)

Applying Theorem 1.3 we get the following.

Proposition 5.6.2. Let $s, t, u$ be positive integers. If $s \geq 3$, then $\Phi(B_3(n))$ is a minimal set of generators for $A$-module $P_4$ in degree $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$.

So, we need only to consider the cases $s = 1$ and $s = 2$.

5.6.1. The subcase $s = t = 1$.

For $s = 1, t = 1$, we have $n = 2^{u+2} + 3$. According to Theorem 4.3 we have

$$B_3(n) = \begin{cases} 
\psi(\Phi(B_2(2^{u+1}))), & \text{if } u \neq 2, \\
\psi(\Phi(B_2(8))) \cup \{x_1^2x_2^2x_3^3\}, & \text{if } u = 2.
\end{cases}$$

Proposition 5.6.3.

i) $\Phi(B_3(11)) \cup \{x_1^3x_2^4x_3^3x_4^3, x_1^2x_2^3x_3^2x_4^3\}$ is the set of all the admissible monomials for $A$-module $P_4$ in degree 11.

ii) $\Phi(B_3(19)) \cup \{x_1^7x_2^9x_3^2x_4, x_1^3x_2^3x_3^3, x_1^3x_2^2x_3^4, x_1^3x_2x_3^5, x_1x_2^4x_3^3x_4^2, x_1^3x_2^2x_3^3x_4^4, x_1^2x_2^3x_3^2x_4, x_1^2x_2^2x_3^4, x_1^2x_2x_3^5, x_1x_2^3x_3^2x_4, x_1x_2^2x_3^4, x_1x_2x_3^5\}$ is the set of all the admissible monomials for $A$-module $P_4$ in degree 19.
iii) $\Phi(B_3(n)) \cup \{x_1^3x_2^4x_3^{2u+2-5}, x_1^3x_2^4x_3^{2u+2-5}, x_4, x_1^3x_2^4x_3^{2u+2-7}\}$ is the set of all the admissible monomials for $A$-module $P_4$ in degree $n = 2u+2+3$, with any positive integer $u \geq 3$.

By a direct computation, we can easily obtain the following lemma.

**Lemma 5.6.4.** The following monomials are strictly inadmissible:

i) $x_1^2x_2^3x_3^2x_4^3x_3^j, \ i, j > 1, \ i \neq j, \ x_1^2x_2^3x_3^2x_4^3x_3^j, \ j = 3, 4.$

ii) $x_2^2x_1^3x_2^3x_3^2, \ x_3x_2^3x_4x_2x_4^3, \ x_2x_1^2x_2^3x_3^2x_4^4, \ x_2^2x_1^2x_2x_3^2x_4^2x_3^1, \ x_2^2x_1^2x_2x_3^2, \ x_2x_1^2x_2x_3^2x_4, \ x_2x_1^2x_2x_3^2, \ x_2x_1^2x_2x_3^2, \ x_2x_1^2x_2x_3^2, \ x_2x_1^2x_2x_3^2, \ x_2x_1^2x_2x_3^2, \ i = 1, 2, \ j = 3, 4.$

**Proof of Theorem 5.6.3.** Let $x$ be an admissible monomial of degree $n = 2u+2+3$ in $P_4$. By Lemma 5.6.1, $\omega_1(x) = 3$. So $x = X_iy^2$ with $y$ a monomial of degree $2u+1$.

Since $x$ is admissible, by Theorem 2.9, $y \in B_4(2u+1)$. By a direct computation, we see that if $x = X_iy^2$ with $y \in B_4(2u+1)$ and $x$ not belongs to the set $C_4(n)$ as given in the proposition, then there is a monomial $w$ which is given in one of Lemmas 5.3.3, 5.6.4 such that $x = wy_r^p$ for some monomial $y$ and integer $r > 1$.

By Theorem 2.9, $x$ is inadmissible. Hence $(QP_4)_n$ is spanned by the set $[C_4(n)]$.

Set $|C_4(2u+2+3) \cap P_4^*| = m(u)$, where $m(1) = 32$, $m(2) = 80$, $m(u) = 64$ for all $u > 2$. Suppose that there is a linear relation

$$S = \sum_{i=1}^{m(u)} \gamma_i d_i = 0,$$  \hspace{1cm} (5.6.1)

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{n,i}$. By a direct computation from the relations $p_{(j; i)}(S) = 0$ with $(j; i) \in N_4$, we obtain $\gamma_i = 0$ for all $i$ if $u \neq 2$.

For $u = 2$, $\gamma_j = 0$ for $j = 1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19, 21, 23, 26, 27, 28, 29, 30, 31, 32, 35, 36, 38, 40, 43, 45, 51, 54, 55, 56, 60, 61, 62, 63, 67, 71, 79, 80, and $\gamma_2 = \gamma_i, i = 5, 24, 25, 41, 42, 52, 53, \gamma_{13} = \gamma_i, i = 13, 33, 20, 56, 48, 58, \gamma_{15} = \gamma_i, i = 22, 34, 49, 57, 59, \gamma_{37} = \gamma_i, i = 67, 70, 75, \gamma_{46} = \gamma_i, i = 69, 72, 76, \gamma_{65} = \gamma_i, i = 66, 73, 74, 77, 78, \gamma_6 = \gamma_3, \gamma_2, \gamma_4 = \gamma_3, \gamma_2, \gamma_6 = \gamma_47, \gamma_13, \gamma_65 = \gamma_70, \gamma_{22}, \gamma_63 = \gamma_{37}, \gamma_{13}, \gamma_{64} = \gamma_{46} + \gamma_{22}$.  

Substituting the above equalities into the relation (5.6.1), we have

$$\gamma_{37}[\theta_1] + \gamma_{46}[\theta_2] + \gamma_{13}[\theta_3] + \gamma_{22}[\theta_4] + \gamma_{65}[\theta_5] + \gamma_{2}[\theta_6] = 0,$$  \hspace{1cm} (5.6.2)

where

- $\theta_1 = d_{37} + d_{44} + d_{63} + d_{67} + d_{70} + d_{75}$,
- $\theta_2 = d_{39} + d_{46} + d_{64} + d_{69} + d_{72} + d_{76}$,
- $\theta_3 = d_{13} + d_{20} + d_{33} + d_{47} + d_{48} + d_{56} + d_{58} + d_{63}$,
- $\theta_4 = d_{15} + d_{22} + d_{34} + d_{49} + d_{50} + d_{57} + d_{59} + d_{64}$,
- $\theta_5 = d_{47} + d_{50} + d_{65} + d_{66} + d_{73} + d_{74} + d_{77} + d_{78}$,
- $\theta_6 = d_{2} + d_{5} + d_{24} + d_{25} + d_{39} + d_{41} + d_{42} + d_{43} + d_{52} + d_{53}$.

We need to prove $\gamma_2 = \gamma_{13} = \gamma_{22} = \gamma_{37} = \gamma_{46} = \gamma_{65} = 0$. The proof is divided into 4 steps.

**Step 1.** First we prove $\gamma_{65} = 0$ by showing the polynomial $[\theta] = [\beta_1\theta_1 + \beta_2\theta_2 + \beta_3\theta_3 + \beta_4\theta_4 + \beta_5\theta_5 + \beta_6\theta_6] \neq 0$ for all $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \in \mathbb{F}_2$. Suppose the contrary that this polynomial is hit. Then we have

$$\theta = Sq^1(A) + Sq^2(B) + Sq^4(C) + Sq^8(D),$$
for some polynomials $A, B, C, D$ in $P_1$. Let $(S_q^2)^3$ act on the both sides of this equality. Using the relations $(S_q^2)^3 S_q^1 = 0, (S_q^2)^3 S_q^2 = 0,$ we get

$$(S_q^2)^3 \theta = (S_q^2)^3 S_q^4(C) + (S_q^2)^3 S_q^8(D).$$

The monomial $x_1^{7} x_2^{12} x_3^{3} x_4^{4}$ is a term of $(S_q^2)^3 \theta$. If $x_1^{7} x_2^{12} x_3^{3} x_4^{2}$ is a term of the polynomial $(S_q^2)^3 S_q^1(y)$ with $y$ a monomial of degree 11 in $P_1$, then $y = x_1^7 f_1(z)$ with $z$ a monomial of degree 4 in $P_1$. Then $x_1^{7} x_2^{12} x_3^{3} x_4^{2}$ is a term of $x_1^{7}(S_q^2)^3 S_q^8(f_1(z)) = 0$.

This is a contradiction. So $x_1^{7} x_2^{12} x_3^{3} x_4^{2}$ is not a term of $(S_q^2)^3 S_q^8(D)$ for all $D$. Hence $x_1^{7} x_2^{12} x_3^{3} x_4^{4}$ is a term of $(S_q^2)^3 S_q^4(C)$, then either $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ or $x_1^{7} x_2^{2} x_3^{3} x_4^{4}$ or $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ is a term of $C$.

Suppose $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ is a term of $C$. Then

$$(S_q^2)^3 \theta + S_q^4(x_1^{7} x_2^{3} x_3^{3} x_4^{4}) = (S_q^2)^3 (S_q^4(C') + S_q^8(D)),$$

where $C' = C + x_1^{7} x_2^{3} x_3^{3} x_4^{4}$. We see that the monomial $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ is a term of $(S_q^2)^3 \theta + S_q^4(x_1^{7} x_2^{3} x_3^{3} x_4^{4})$. This monomial is not a term of $(S_q^2)^3 S_q^8(D)$ for all $D$. So it is a term of $(S_q^2)^3 S_q^4(C')$. Then either $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ or $x_1^{7} x_2^{2} x_3^{3} x_4^{4}$ is a term of $C$. Since $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ is a term of $C'$, $x_1^{7} x_2^{2} x_3^{3} x_4^{4}$ is a term of $C'$. Hence we obtain

$$(S_q^2)^3 \theta + S_q^4(x_1^{7} x_2^{3} x_3^{3} x_4^{4} + x_1^{7} x_2^{2} x_3^{3} x_4^{4}) = (S_q^2)^3 (S_q^4(C'') + S_q^8(D)),$$

where $C'' = C + x_1^{7} x_2^{3} x_3^{3} x_4^{4} + x_1^{7} x_2^{2} x_3^{3} x_4^{4}$. Now $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ is a term of

$$(S_q^2)^3 \theta + S_q^4(x_1^{7} x_2^{3} x_3^{3} x_4^{4} + x_1^{7} x_2^{2} x_3^{3} x_4^{4}).$$

So either $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ or $x_1^{7} x_2^{2} x_3^{3} x_4^{4}$ or $x_1^{7} x_2^{2} x_3^{3} x_4^{4}$ is a term of $C''$. Since $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ or $x_1^{7} x_2^{2} x_3^{3} x_4^{4}$ is a summand of $C''$, $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ is a term of $C''$. Hence $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ or $x_1^{7} x_2^{2} x_3^{3} x_4^{4}$ is a term of $C''$ and we have a contradiction.

By a same argument, if either $x_1^{7} x_2^{3} x_3^{3} x_4^{4}$ or $x_1^{7} x_2^{2} x_3^{3} x_4^{4}$ is a term of $C$ then we have also a contradiction. Hence $[\theta] \neq 0$ and $\gamma_{65} = 0$.

**Step 2.** By a direct computation, we see that the homomorphism $\varphi_3$ sends $[5.6.2]$ to

$$\gamma_{37}[\theta_1] + \gamma_{2}[\theta_3] + \gamma_{22}[\theta_4] + \gamma_{46}[\theta_5] + \gamma_{13}[\theta_6] = 0.$$ 

By Step 1, we obtain $\gamma_{46} = 0$.

**Step 3.** The homomorphism $\varphi_2$ sends $[5.6.2]$ to

$$\gamma_{13}[\theta_1] + \gamma_{22}[\theta_2] + \gamma_{37}[\theta_3] + \gamma_{2}[\theta_6] = 0.$$ 

By Step 2, we obtain $\gamma_{22} = 0$.

**Step 4.** Now the homomorphism $\varphi_3$ sends $[5.6.2]$ to $\gamma_{37}[\theta_2] + \gamma_{13}[\theta_4] + \gamma_{2}[\theta_6] = 0$.

Combining Step 2 and Step 3, we obtain $\gamma_{13} = \gamma_{37} = 0$.

Since $\varphi_2([\theta_3]) = [\theta_6]$, we get $\gamma_2 = 0$. So we obtain $\gamma_j = 0$ for all $j$. The proposition follows.

**5.6.2. The subcase $s = 1, t = 2$.**

For $s = 1, t = 2$, we have $n = 2^{u+3} + 7 = 2m + 3$ with $m = 2^{u+2} + 2$. Combining Theorem 1.3 and Theorem 4.3, we have $B_3(n) = \psi(\Phi(B_2(m)))$. where

$$B_2(m) = \begin{cases} 
\{x_1^3 x_2^7, x_1^7 x_2^3\}, & \text{if } u = 1, \\
\{x_1^3 x_2^{2u+2-1}, x_1^7 x_2^{2u+2-5}\}, & \text{if } u > 1.
\end{cases}$$
Proposition 5.6.5.

i) \( \Phi(B_3(23)) \cup F(1) \cup \{ x_1^7x_2^2x_3^4, x_1^2x_2^3x_4^5 \} \) is the set of all the admissible monomials for \( A \)-module \( P_4 \) in degree 23.

ii) \( \Phi(B_3(n)) \cup F(u) \cup \{ x_1^7x_2^2x_3^4x_4^5, x_1^2x_2^3x_3x_4^4 \} \) is the set of all the admissible monomials for \( A \)-module \( P_4 \) in degree \( n = 2u+3+7 \) with any positive integer \( u > 1 \).

By a direct computation, we can easily obtain the following lemma.

Lemma 5.6.6. The following monomials are strictly inadmissible:

i) \( x_2x_1^2, x_2x_1^3x_2^4x_1^2, x_2x_1^2x_2^4x_1^4x_2x_3x_4^3, x_2x_1^3x_2^4x_1^4x_2x_3x_4^3, \quad i = 1, 2, j = 3, 4. \)

ii) \( x_3x_2x_1^2x_2^4x_1^2x_2x_3, x_3x_2x_1^2x_2^4x_1^2x_2x_3, x_3x_2x_1^2x_2^4x_1^2x_2x_3, x_3x_2x_1^2x_2^4x_1^2x_2x_3, i = 1, 2, j = 3, 4. \)

Proof of Proposition 5.6.5. Let \( x \) be an admissible monomial of degree \( n = 2u+3+7 \) in \( P_4 \).

By Lemma 5.6.1 \( \omega_1(x) = 3 \). So \( x = X_1y^2 \) with \( y \) a monomial of degree \( 2u+2 \). Since \( x \) is admissible, by Theorem 2.9 \( y \in B_4(2u+2+2) \).

By a direct computation, we see that if \( x = X_1y^2 \) with \( y \in B_4(2u+2+2) \) and \( x \) not belongs to the set \( C_4(n) \) as given in the proposition, then there is a monomial \( w \) which is given in one of Lemmas 5.6.6 5.3.3 such that \( x = wy^2 \) for some monomial \( y \) and integer \( r > 1 \).

By Theorem 2.9 \( x \) is inadmissible. Hence \( (QP_4)_n \) is spanned by the set \( [C_4(n)] \).

For \( u = 1 \), we have, \( |C_4^+(23) \cap P_4^+| = 99 \). Suppose that there is a linear relation

\[
S = \sum_{i=1}^{99} \gamma_i d_i = 0, \tag{5.6.1}
\]

with \( \gamma_i \in \mathbb{F}_2 \) and \( d_i = d_{23,i} \). By a direct computation from the relations \( p_{(i,j)}(S) \equiv 0 \) with \( (j; J) \in \mathcal{N}_4 \), we obtain \( \gamma_i = 0 \) for all \( i \in E \), with some \( E \subset \mathbb{N}_{99} \) and the relation (5.6.2) becomes

\[
\sum_{i=1}^{15} c_i[\theta_i] = 0, \tag{5.6.2}
\]
where \( c_1 = \gamma_1, c_2 = \gamma_4, c_3 = \gamma_{33}, c_4 = \gamma_{94}, c_5 = \gamma_2, c_6 = \gamma_{22}, c_7 = \gamma_{74}, c_8 = \gamma_{29}, c_9 = \gamma_{81}, c_{10} = \gamma_{68}, c_{11} = \gamma_{10}, c_{12} = \gamma_{43}, c_{13} = \gamma_{54}, c_{14} = \gamma_{79}, c_{15} = \gamma_{11} \) and

\[
\begin{align*}
\theta_1 &= d_1 + d_{17} + d_{37} + d_{49}, \\
\theta_2 &= d_4 + d_{21} + d_{44} + d_{53}, \\
\theta_3 &= d_{33} + d_{36} + d_{72} + d_{73}, \\
\theta_4 &= d_{54} + d_{57} + d_{98} + d_{99}, \\
\theta_5 &= d_2 + d_{19} + d_{40} + d_{51}, \\
\theta_6 &= d_{22} + d_{25} + d_{62} + d_{63}, \\
\theta_7 &= d_{74} + d_{77} + d_{82} + d_{83}, \\
\theta_8 &= d_{12} + d_{14} + d_{26} + d_{29} + d_{66} + d_{67}, \\
\theta_9 &= d_{40} + d_{42} + d_{78} + d_{81} + d_{86} + d_{87}, \\
\theta_{10} &= d_{10} + d_{15} + d_{24} + d_{27} + d_{36} + d_{47} + d_{64} + d_{65}, \\
\theta_{11} &= d_{38} + d_{43} + d_{46} + d_{47} + d_{76} + d_{79} + d_{84} + d_{85}, \\
\theta_{12} &= d_{62} + d_{67} + d_{68} + d_{71} + d_{88} + d_{89} + d_{92} + d_{93}, \\
\theta_{13} &= d_{42} + d_{54} + d_{57} + d_{62} + d_{69} + d_{82} + d_{85} + d_{88} + d_{90}, \\
\theta_{14} &= d_{12} + d_{15} + d_{19} + d_{20} + d_{46} + d_{47} + d_{51} + d_{52} + d_{58} + d_{61} \\
&\quad + d_{64} + d_{66} + d_{67} + d_{70} + d_{84} + d_{87} + d_{89} + d_{91}, \\
\theta_{15} &= d_{11} + d_{12} + d_{18} + d_{20} + d_{24} + d_{25} + d_{26} + d_{27} + d_{38} + d_{40} + d_{45} \\
&\quad + d_{47} + d_{48} + d_{50} + d_{52} + d_{57} + d_{61} + d_{63} + d_{64} + d_{65} + d_{66} \\
&\quad + d_{67} + d_{69} + d_{77} + d_{83} + d_{85} + d_{86} + d_{87} + d_{89} + d_{90}.
\end{align*}
\]

Now, we show that \( c_i = 0 \) for \( i = 1, 2, \ldots, 15 \). The proof is divided into 6 steps.

**Step 1.** Set \( \theta = \theta_1 + \sum_{i=2}^{15} \beta_i \theta_i \) for \( \beta_i \in F_2, i = 2, 3, \ldots, 15 \). We prove that \( [\theta] \neq 0 \). Suppose the contrary that \( \theta \) is hit. Then we have

\[ \theta = Sq^1(A) + Sq^2(B) + Sq^4(C) + Sq^8(D) \]

for some polynomials \( A, B, C, D \in P_4^+ \). Let \((Sq^2)^3\) act to the both sides of the above equality, we obtain

\[ (Sq^2)^3(\theta) = (Sq^2)^3Sq^4(C) + (Sq^2)^3Sq^8(D). \]

By a similar computation as in the proof of Proposition \[5.4.5\], we see that the monomial \( x_1^8 x_2^2 x_3^2 x_4^{15} \) is a term of \((Sq^2)^3(\theta)\). This monomial is not a term of \((Sq^2)^3(Sq^4(C) + Sq^8(D))\) for all polynomials \( C, D \) and we have a contradiction. So \( [\theta] \neq 0 \) and we get \( c_1 = \gamma_1 = 0 \).

By an argument analogous to the previous one, we get \( c_2 = c_3 = c_4 = 0 \). Now, the relation \((5.6.2)\) becomes

\[ \sum_{i=5}^{15} c_i[\theta_i] = 0. \quad (5.6.3) \]

**Step 2.** The homomorphisms

\[ \varphi_1, \varphi_1^2 \varphi_3, \varphi_1 \varphi_3^2 \varphi_4, \varphi_1 \varphi_3 \varphi_2, \varphi_1 \varphi_3^2 \varphi_2 \varphi_3, \varphi_1 \varphi_3^2 \varphi_4 \varphi_2 \varphi_3 \]
send (5.6.3) respectively to

\[ c_{10} \theta_3 = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \]
\[ c_9 \theta_3 = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \]
\[ c_7 \theta_3 = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \]
\[ c_8 \theta_3 = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \]
\[ c_6 \theta_3 = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \]
\[ c_5 \theta_3 = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]). \]

Using the results in Step 1, we get \( c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0 \). So the relation (5.6.3) becomes

\[ c_{11} [\theta_{11}] + c_{12} [\theta_{12}] + c_{13} [\theta_{13}] + c_{14} [\theta_{14}] + c_{15} [\theta_{15}] = 0. \] (5.6.4)

**Step 3.** The homomorphism \( \varphi_1 \) sends (5.6.4) to

\[ c_{13} \theta_6 + (c_{14} + c_{15}) [\theta_7] + (c_{11} + c_{12}) [\theta_{11}] + c_{12} [\theta_{12}] + c_{13} [\theta_{13}] + c_{14} [\theta_{14}] + c_{15} [\theta_{15}] = 0. \]

By Step 2, we get \( c_{13} = 0 \) and \( c_{14} = c_{15} \). So the relation (5.6.4) becomes

\[ c_{11} [\theta_{11}] + c_{12} [\theta_{12}] + c_{14} [\theta_{14}] + c_{14} [\theta_{15}] = 0. \] (5.6.5)

**Step 4.** The homomorphism \( \varphi_3 \) sends (5.6.5) to

\[ c_{11} [\theta_{11}] + c_{13} [\theta_{12}] + (c_{12} + c_{14}) [\theta_{13}] + c_{14} [\theta_{14}] + c_{14} [\theta_{15}] = 0. \]

By Step 3, we get \( c_{12} = c_{14} \). Then the relation (5.6.5) becomes

\[ c_{11} [\theta_{11}] + c_{12} [\theta_{12}] + c_{12} [\theta_{14}] + c_{12} [\theta_{15}] = 0. \] (5.6.6)

**Step 5.** The homomorphism \( \varphi_2 \) sends (5.6.6) to

\[ (c_{11} + c_{12}) [\theta_{12}] + c_{12} [\theta_{14}] + c_{12} [\theta_{15}] = 0. \]

From the result in Step 4, we get \( c_{11} = 0 \). Then the relation (5.6.6) becomes

\[ c_{12} ([\theta_{12}] + [\theta_{14}] + [\theta_{15}]) = 0. \] (5.6.7)

**Step 6.** The homomorphism \( \varphi_1 \) sends (5.6.7) to

\[ c_{12} [\theta_{11}] + c_{12} ([\theta_{12}] + [\theta_{14}] + [\theta_{15}]) = 0. \]

By the result in Step 5, we have \( c_{12} = 0 \). The case \( u = 1 \) of the proposition is completely proved.

For \( u > 1 \), we have \( |C_4(n) \cap P_+^1| = 141 \). Suppose that there is a linear relation

\[ S = \sum_{i=1}^{141} \gamma_i d_i = 0, \] (5.6.8)

with \( \gamma_i \in \mathbb{F}_2 \) and \( d_i = d_{n,i} \in B_+^1(n) \). By a direct computation from the relations \( p_{(j,j)}(S) \equiv 0 \) with \( (j,J) \in \mathcal{N}_4 \), we obtain \( \gamma_i = 0 \) for all \( i \not\in E \), with some \( E \subset \mathbb{N}_{141} \) and the relation (5.6.8) becomes

\[ \sum_{i=1}^{15} c_i [\theta_i] = 0, \] (5.6.9)
where $c_1 = \gamma_1, c_2 = \gamma_6, c_3 = \gamma_{51}, c_4 = \gamma_{136}, c_5 = \gamma_2, c_6 = \gamma_{31}, c_7 = \gamma_{107}, c_8 = \gamma_{40}, c_9 = \gamma_{116}, c_{10} = \gamma_{101}, c_{11} = \gamma_{14}, c_{12} = \gamma_{56}, c_{13} = \gamma_{79}, c_{14} = \gamma_{23}, c_{15} = \gamma_{15}$ and

$$
\begin{align*}
\theta_1 &= d_1 + d_{25} + d_{55} + d_{73}, \\
\theta_2 &= d_6 + d_{30} + d_{66} + d_{78}, \\
\theta_3 &= d_{51} + d_{54} + d_{105} + d_{106}, \\
\theta_4 &= d_7 + d_8 + d_{47} + d_{48}, \\
\theta_5 &= d_2 + d_{27} + d_{58} + d_{75}, \\
\theta_6 &= d_{31} + d_{34} + d_{89} + d_{90}, \\
\theta_7 &= +d_{107} + d_{110} + d_{117} + d_{118}, \\
\theta_8 &= d_{16} + d_{22} + d_{35} + d_{40} + d_{94} + d_{95}, \\
\theta_9 &= d_{58} + d_{64} + d_{111} + d_{116} + d_{122} + d_{123}, \\
\theta_{10} &= d_{59} + d_{95} + d_{101} + d_{104} + d_{124} + d_{127} + d_{129} + d_{130}, \\
\theta_{11} &= d_{14} + d_{19} + d_{33} + d_{36} + d_{68} + d_{69} + d_{91} + d_{92}, \\
\theta_{12} &= d_{56} + d_{61} + d_{68} + d_{69} + d_{109} + d_{112} + d_{119} + d_{120}, \\
\theta_{13} &= d_{67} + d_{69} + d_{79} + d_{82} + d_{89} + d_{90} + d_{117} + d_{118} + d_{124} + d_{125}, \\
\theta_{14} &= d_{16} + d_{23} + d_{27} + d_{29} + d_{70} + d_{71} + d_{72} + d_{75} + d_{77} + d_{83} + d_{88} + d_{94} + d_{95} + d_{122} + d_{123} + d_{126} + d_{127}, \\
\theta_{15} &= d_{15} + d_{19} + d_{26} + d_{27} + d_{33} + d_{34} + d_{35} + d_{36} + d_{58} + d_{61} + d_{68} + d_{69} + d_{70} + d_{74} + d_{75} + d_{82} + d_{83} + d_{91} + d_{92} + d_{109} + d_{110} + d_{111} + d_{112} + d_{119} + d_{120} + d_{125}.
\end{align*}
$$

Now, we prove $c_i = 0$ for $i = 1, 2, \ldots, 15$. The proof is divided into 6 steps.

*Step 1.* First, we prove $c_1 = 0$. Set $\theta = \theta_1 + \sum_{j=2}^{15} c_j \theta_j$. We show that $|\theta| \neq 0$ for all $c_j \in \mathbb{F}_2, j = 2, 3, \ldots, 15$. Suppose the contrary that $\theta$ is hit. Then we have

$$
\theta = \sum_{m=0}^{u+2} S q^{2m} (A_m),
$$

for some polynomials $A_m, m = 0, 1, \ldots, u + 2$. Let $(S q^2)^3$ act on both sides of this equality. Since $(S q^2)^3 S q^1 = 0, (S q^2)^3 S q^2 = 0$, we get

$$
(S q^2)^3 (\theta) = \sum_{m=2}^{u+2} (S q^2)^3 S q^{2m} (A_m).
$$

It is easy to see that the monomial $x = x_1^4 x_2^4 x_3^2 x_4^{2u+3} - 1$ is a term of $(S q^2)^3 (\theta)$, hence it is a term of $(S q^2)^3 S q^{2m} (y)$ for some monomial $y$ of degree $2u+3 - 2^m + 7$ with $m \geq 2$. Then $y = x_2^{2u+3-1} f_2(z)$ with $z$ a monomial of degree $8 - 2^m \leq 4$ in $P_3$ and $x$ is a term of $x_2^{2u+3-1} (S q^2)^3 S q^{2m} (z)$. If $m > 2$ then $S q^{2m} (z) = 0$. If $m = 2$ the $S q^2 (z) = z^2$, hence $(S q^2)^3 S q^{2m} (z) = (S q^2)^3 (z^2) = 0$. So $x$ is not a term of

$$
(S q^2)^3 (\theta) = \sum_{m=2}^{u+2} (S q^2)^3 S q^{2m} (A_m),
$$

for all polynomial $A_m$ with $m > 1$. This is a contradiction. So we get $c_1 = 0$. 


By an argument analogous to the previous one, we get \( c_2 = c_3 = c_4 = 0 \). Then the relation (5.6.9) becomes

\[
\sum_{i=5}^{15} c_i[\theta_i] = 0. \quad (5.6.10)
\]

**Step 2.** The homomorphisms

\( \varphi_1, \varphi_1 \varphi_3, \varphi_1 \varphi_3 \varphi_4, \varphi_1 \varphi_3 \varphi_2, \varphi_1 \varphi_3 \varphi_2 \varphi_4, \varphi_1 \varphi_3 \varphi_4 \varphi_2 \varphi_3 \)

send (5.6.3) respectively to

\[
\begin{align*}
&c_{10}[\theta_3] = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \\
&c_9[\theta_3] = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \\
&c_7[\theta_3] = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \\
&c_8[\theta_3] = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \\
&c_6[\theta_3] = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]), \\
&c_5[\theta_3] = 0 \mod ([\theta_5], [\theta_6], \ldots, [\theta_{15}]).
\end{align*}
\]

By Step 1, we get \( c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0 \). So the relation (5.6.3) becomes

\[
c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0. \quad (5.6.11)
\]

**Step 3.** Applying the homomorphism \( \varphi_1 \) to (5.6.11), we get

\[
c_{13}[\theta_6] + c_{14}[\theta_8] + (c_{11} + c_{12} + c_{15})[\theta_{11}] + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0.
\]

By the results in Step 2, we obtain \( c_{13} = c_{14} = 0 \). Then the relation (5.6.11) becomes

\[
c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{14}[\theta_{15}] = 0. \quad (5.6.12)
\]

**Step 4.** Applying the homomorphism \( \varphi_3 \) to the relation (5.6.12) we obtain

\[
c_{11}[\theta_{11}] + c_{12}[\theta_{13}] + c_{15}[\theta_{15}] = 0.
\]

By the results in Step 3, we get \( c_{12} = 0 \). So the relation (5.6.12) becomes

\[
c_{11}[\theta_{11}] + c_{15}[\theta_{15}] = 0. \quad (5.6.13)
\]

**Step 5.** Applying the homomorphism \( \varphi_2 \) to the relation (5.6.12) one gets

\[
c_{11}[\theta_{13}] + c_{15}[\theta_{15}] = 0.
\]

By Step 4, we get \( c_{10} = \gamma_{41} = 0 \). So the relation (5.6.13) becomes

\[
c_{15}[\theta_{15}] = 0. \quad (5.6.14)
\]

**Step 6.** Applying the homomorphism \( \varphi_1 \) to the relation (5.6.14) we obtain

\[
c_{15}[\theta_{11}] + c_{15}[\theta_{15}] = 0.
\]

By Step 5, we get \( c_{15} \). The proposition is completely proved. \( \square \)

5.6.3. **The subcase** \( s = 1, \ t > 2 \).

For \( s = 1, t > 2 \), we have \( n = 2^{t+u+1} + 2^{t+1} - 1 = 2m + 3 \) with \( m = 2^{t+u} + 2^t - 2 \). From Theorem 4.3 we have \( B_3(n) = \psi(\Phi(B_2(m))) \).
Proposition 5.6.7.
i) \( \Phi(B_3(n)) \cup \{x_1^3x_2^4x_3^{2t+1-5}x_4^3, x_1^2x_2^2x_3^{2t+2-5}x_4^{t+1-3}\} \) is the set of all the admissible monomials for \( A \)-module \( P_4 \) in degree \( n = 2t+2 + 2t+1 - 1 \) with any positive integer \( t > 2 \).

ii) \( \Phi(B_3(n)) \cup A(t,u) \) is the set of all the admissible monomials for \( A \)-module \( P_4 \) in degree \( n = 2t+u+1 + 2t+1 - 1 \) with any positive integers \( t > 2, u > 1 \), where \( A(t,u) \) is the set consisting of 3 monomials:

\[
x_1^3x_2^4x_3^{2t+1-5}x_4^3, x_1^2x_2^2x_3^{2t+u+1-5}x_4^{t+1-3}, x_1^2x_2x_3^{2t+2-5}x_4^{t+u+1-2t+1-3}.
\]

By a direct computation, we can easy obtain the following lemma.

Lemma 5.6.8. The following monomials are strictly inadmissible:

\[
X_3x_1^2x_2^2x_3^5x_4^4, X_3x_1^2x_2^2x_3^4x_4^4, i = 1, 2, X_4x_1^2x_2^2x_3^4x_4^4.
\]

Proof of Proposition 5.6.7. Let \( x \in P_4 \) be an admissible monomial of degree \( n = 2t+u+1 + 2t+1 - 1 \).

By Lemma 5.6.1, \( \omega_1(x) = 3 \). So \( x = X_3y^2 \) with \( y \) a monomial of degree \( 2t+u+1 + 2t - 2 \). Since \( x \) is admissible, by Theorem 2.9 \( y \in B_4(2t+u+1 + 2t - 2) \).

By a direct computation, we see that if \( x = X_3y^2 \) with \( y \in B_4(2t+u+1 + 2t - 2) \) and \( x \) not belongs to the set \( C_4(n) \) as given in the proposition, then there is a monomial \( w \) which is given in one of Lemmas 5.6.8 and 5.3.3 such that \( x = wy^2 \) for some monomial \( y \) and integer \( r > 1 \).

By Theorem 2.9 \( x \) is inadmissible. Hence \( (QP_4)^n \) is spanned by the set \( [C_4(n)] \).

We set \( [C_4(n)] \cap P_4^+ = n(t,u) \) with \( n(t,1) = 84 \) for \( u = 1 \) and \( n(t,u) = 126 \) for \( u > 1 \). Suppose that there is a linear relation

\[
S = \sum_{i=1}^{n(t,u)} \gamma_i d_i = 0,
\]

with \( \gamma_i \in \mathbb{F}_2 \) and \( d_i = d_{n,i} \). By a direct computation from the relations \( p_{(j,1)}(S) \equiv 0 \) with \( (j; J) \in \mathcal{N}_4 \), we obtain \( \gamma_i = 0 \) for all \( i \).

5.6.4. The subcase \( s = 2, t = 1 \).

For \( s = 2, t = 1 \), we have \( n = 2u+3 + 9 \). According to Theorem 4.3 we have

\[
B_3(n) = \begin{cases} \psi^2(\Phi(B_2(2u+1))), & \text{if } u \neq 2, \\ \psi^2(\Phi(B_2(8))) \cup \{x_1^5x_2^9x_3^7\}, & \text{if } u = 2. \end{cases}
\]

Denote by \( G(u) \) the set of 7 monomials:

\[
x_1^3x_2^3x_3^2x_4^7, x_1^3x_2^3x_3^3x_4^5, x_1^3x_2^3x_3^4x_4^4, x_1^3x_2^3x_3^5x_4^3, x_1^3x_2^3x_3^6x_4^2, x_1^3x_2^3x_3^7x_4^1, x_1^3x_2^3x_3^8x_4^0,
\]

Proposition 5.6.9.

i) \( \Phi(B_3(25)) \cup G(1) \cup \{x_1^3x_2^3x_3^3x_4^7\} \) is the set of all the admissible monomials for \( A \)-module \( P_4 \) in degree 25.

ii) \( \Phi(B_3(n)) \cup G(u) \cup H(u) \) is the set of all the admissible monomials for \( A \)-module \( P_4 \) in degree \( n = 2u+3 + 9 \) with any positive integer \( u > 1 \), where \( H(u) \)
is the set consisting of 5 monomials:
\[x_1^3 x_2 x_3^2 x_4^{2u+3} - 12, \ x_1 x_2^2 x_3 x_4^{2u+3} - 12, \ x_2 x_3^2 x_4^{2u+3} - 12,
\[x_1 x_2^2 x_3 x_4^{2u+3} - 13, \ x_1 x_2 x_3^{11} x_4^{2u+3} - 16.\]

The following is proved by a direct computation.

**Lemma 5.6.10.** The following monomials are strictly inadmissible:

i) \(X_3 x_2^2 x_1 x_4^4, X_3 x_2^2 x_1^2 x_4, X_3 x_2 x_4^4, X_3 x_2^4 x_4^4, \), \(i = 1, 2, j = 3, 4.\)

ii) \(X_3 x_1^2 x_2^{12} x_3^4, X_4 x_2^2 x_3^4 x_4^{24}, X_2 x_1^2 x_2^{12} x_3^4 x_4^4, X_4 x_3^2 x_4^{24} x_4^{24}, X_4 x_3 x_2^4 x_3^4 x_4^4, X_3 x_2^4 x_3^4 x_4^4, X_4 x_2^4 x_3^4 x_4^4,\)

\(X_1 x_2^2 x_3^4 x_4^4, X_3 x_2^4 x_3^4 x_4^4, X_4 x_2^4 x_3^4 x_4^4, X_3 x_2^4 x_3^4 x_4^4, X_4 x_2^4 x_3^4 x_4^4,\)

\(X_1 x_2^2 x_3 x_4^4, i = 1, 2, j = 3, 4.\)

**Proof of Proposition 5.6.9.** Let \(x\) be an admissible monomial of degree \(n = 2u+3+9\) in \(P_4\).

By Lemma 5.6.1, \(\omega_1(x) = \omega_2(x) = 3\). So \(x = X_1 x_2^2 y^4\) with \(y\) a monomial of degree \(2u+1\). Since \(x\) is admissible, by Theorem 2.9, \(y \in B_4(2u+u+2 - 2)\).

By a direct computation, we see that if \(x = X_1 x_2^2 y^4\) with \(y \in B_4(2u+u+2 - 2)\) and \(x\) not belongs to the set \(C_4(n)\) given in the proposition, then there is a monomial \(w\) which is given in one of Lemmas 5.6.10, 5.3.3 such that \(x = wy^2\) for some monomial \(y\) and integer \(r > 1\).

By Theorem 2.9, \(x\) is inadmissible. Hence \((QP_4)_n\) is spanned by the set \([C_4(n)]\).

We denote \([C_4(n)] \cap P_4^t = (m(u))\) with \(m(1) = 88, m(2) = 165\) and \(m(u) = 154\) for \(u \geq 3\). Suppose that there is a linear relation

\[S = \sum_{i=1}^{m(u)} \gamma_i d_i = 0,\]

with \(\gamma_i \in \mathbb{F}_2\) and \(d_i = d_{n,i}\). By a direct computation from the relations \(p_{(j;j)}(S) \equiv 0\) with \((j,j) \in N_4\), we obtain \(\gamma_i = 0\) for all \(i\).

\[\Box\]

### 5.6.5. The subcase \(s = 2, t \geq 2.\)

For \(s = 2, t \geq 2\), we have \(n = 2t+u+2 + 2t+2 = 4m + 9\) with \(m = 2t+u+2 + 2t - 2\).

From Theorem 1.3 we have

\[B_3(n) = \psi(2)(\Phi(B_2(m))).\]

Denote by \(B(t,u)\) the set of 8 monomials:

\[x_1^3 x_2 x_3 x_4^{2t+2} - 5, x_1 x_2^2 x_3 x_4^{2t+2} - 5, x_1^2 x_3^2 x_4^{2t+2} - 5, x_1 x_2^2 x_3 x_4^{2t+2} - 5, x_1^2 x_3^2 x_4^{2t+2} - 5, x_1 x_2^2 x_3 x_4^{2t+2} - 5, x_1^2 x_3^2 x_4^{2t+2} - 5, x_1 x_2^2 x_3 x_4^{2t+2} - 5, x_1^2 x_3^2 x_4^{2t+2} - 5,\]

and by \(C(t,u)\) the set of 4 monomials:

\[x_1^3 x_2 x_3 x_4^{2t+3} - 5, x_1 x_2^2 x_3 x_4^{2t+3} - 5, x_1^2 x_3^2 x_4^{2t+3} - 5, x_1 x_2^2 x_3 x_4^{2t+3} - 5, x_1^2 x_3^2 x_4^{2t+3} - 5, x_1 x_2^2 x_3 x_4^{2t+3} - 5, x_1^2 x_3^2 x_4^{2t+3} - 5, x_1 x_2^2 x_3 x_4^{2t+3} - 5, x_1^2 x_3^2 x_4^{2t+3} - 5.\]
Proposition 5.6.11.

i) $\Phi(B_3(n)) \cup B(t, 1)$ is the set of all the admissible monomials for $A$-module $P_4$ in degree $n = 2^{t+3} + 2^{t+2} + 1$.

ii) For any positive integer $t, u > 1$, $\Phi(B_3(n)) \cup B(t, u) \cup C(t, u)$ is the set of all the admissible monomials for $A$-module $P_4$ in degree $n = 2^{t+u+2} + 2^{t+2} + 1$.

By a direct computation, we get the following.

Lemma 5.6.12. The following monomials are strictly inadmissible:

$$X_jX_3^2x_2^{12}x_2^{16}, X_3^3x_1^2x_2^{12}x_2^{16}, X_3^3x_1^2x_2^{12}x_2^{16}, X_3^3x_1^2x_2^{12}x_2^{16}, X_3^3x_1^2x_2^{12}x_2^{16}, X_3^3x_1^2x_2^{12}x_2^{16}, X_3^3x_1^2x_2^{12}x_2^{16}, X_3^3x_1^2x_2^{12}x_2^{16},$$. For any positive integer

Proof of Proposition 5.6.11. Let $x \in P_4$ be an admissible monomial of degree $n = 2^{t+u+2} + 2^{t+2} + 1$. By Lemma 5.6.1 $\omega_1(x) = \omega_2(x) = 3$. So $x = X_iX_jy^4$ with $y$ a monomial of degree $2^{t+u} + 2^t - 2$.

Since $x$ is admissible, by Theorem 2.9, $y \in B_4(2^{t+u} + 2^t - 2)$. By a direct computation, we see that if $x = X_iX_jy^r$ with $y \in B_4(2^{t+u} + 2^t - 2)$ and $x$ not belongs to the set $C_4(n)$ as given in the proposition, then there is a monomial $w$ which is given in one of Lemmas 5.6.12 [5.1.3] such that $x = wy^2$ for some monomial $y$ and integer $r > 1$.

By Theorem 2.9, $x$ is inadmissible. Hence $(QP_4)_n$ is spanned by the set $[C_4(n)]$.

We set $[C_4(n)] \cap P_4^n = m(t, u)$ with $m(t, 1) = 154$ and $m(t, u) = 231$ for $t \geq 2$.

Suppose that there is a linear relation

$$S = \sum_{i=1}^{m(t, u)} \gamma_i d_i = 0,$$

with $\gamma_i \in \mathbb{F}_2$ and $d_i = d_{n, i}$. By a direct computation from the relations $p_{(j; J)}(S) \equiv 0$ with $(j; J) \in \mathcal{N}_4$, we obtain $\gamma_i = 0$ for all $i$. \hfill $\square$

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