Orbits of rotor-router operation and stationary distribution of random walks on directed graphs*

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July 23, 2014

Abstract

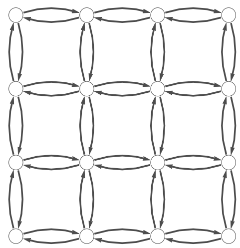
The rotor-router model is a popular deterministic analogue of random walk. In this paper we prove that all orbits of the rotor-router operation have the same size on a strongly connected directed graph (digraph) and give a formula for the size. By using this formula we address the following open question about orbits of the rotor-router operation: Is there an infinite family of non-Eulerian strongly connected digraphs such that the rotor-router operation on each digraph has a single orbit?

It turns out that on a strongly connected digraph the stationary distribution of the random walk coincides with the frequency of vertices in a rotor walk. In this sense a rotor walk can simulate a random walk. This gives a first similarity between two models on (finite) digraphs. We also study the random walk on the set of single-chip-and-rotor states which is induced by the random walk on a strongly connected digraph. We show that its stationary distribution is unique and uniform on the set of recurrent states. This means that recurrent states occur at the same almost sure frequency when the chip performs a random walk.

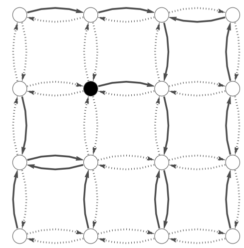
1 Introduction

The rotor-router model is a popular deterministic analogue of random walk that was discovered firstly by Priezzhev, D. Dhar et al. as a model of self organized criticality under the name "Eulerian walkers" [9]. The model has become popular recently because it shows many surprising properties which are similar to those of random walk [1, 2, 3, 5]. The model was studied mostly on $\mathbb{Z}^d$ with the problems similar to those of the random walk. Although the model was defined firstly on (finite) graphs, there are not many known results on this class of graphs, in particular a similarity between the two models on digraphs is still unknown.

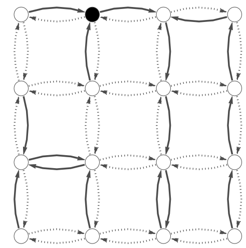
*This paper was partially sponsored by Vietnam Institute for Advanced Study in Mathematics (VIASM) and the Vietnamese National Foundation for Science and Technology Development (NAFOSTED)



(a) A grid graph



(b) A single-chip-and-rotor state (the plane edges for rotor configuration, and the black vertex indicates the location of the chip)



(c) Resulting single-chip-and-rotor state

Fig. 1

Let $G = (V, E)$ be a connected digraph. For each vertex v the set of the edges emanating from v is equipped with a cyclic ordering. We denote by e^+ the next edge of edge e in this order. A vertex s of G is called *sink* if its outdegree is 0. A *rotor configuration* ρ is a map from the set of non-sink vertices of G to E such that for each non-sink vertex v of G $\rho(v)$ is an edge emanating from v. We start with a rotor configuration and a chip placed on some vertex of G. When a chip is at a non-sink vertex v, routing chip at v with respect to a rotor configuration ρ means the process of updating $\rho(v)$ to $\rho(v)^+$, and then the chip moves along the updated edge $\rho(v)$ to the head. The chip is now at the head of the edge $\rho(v)$. We define a *single-chip-and-rotor state* (often briefly *state*) to be a pair (v, ρ) of a vertex and a rotor configuration ρ of G. The vertex v in (v, ρ) indicates the location of the chip in G. When v is not a sink, by routing the chip at v we obtain a new state (v', ρ'). This procedure is called *rotor-router operation*. Look at Figure 1 for an illustration of the rotor-router operation. In this example the acyclic ordering at each vertex is adapted to the counter-clockwise rotation. When the chip is at a sink, it stays at the sink forever, and therefore the rotor-router operation fixes such states. A sequence of vertices of G indicating the consecutive locations of the chip is called a *rotor walk*.

If G has no sink, a state (v, ρ) is *recurrent* if starting from (v, ρ) and after some steps (positive number of steps) of iterating the rotor-router operation we obtain (v, ρ) again. The *orbit* of a recurrent state is the set of all states which are reachable from the recurrent state by iterating the rotor-router operation. Holroyd et al. gave a characterization for recurrent states [4]. By investigating orbits of recurrent states on an Eulerian digraph the authors observed that sizes of orbits are extremely short while number of recurrent states is typically exponential in number of vertices. They asked whether there is an infinite family of non-Eulerian strongly connected digraphs such that all recurrent states of each digraph in the family are in a single orbit. An immediate fact from the results in [4, 9] is that all orbits have the same size on an Eulerian digraph, namely $|E|$. So it is natural and important to ask whether this fact also holds for general digraphs. For this problem we have the following main result.

Theorem 1. *Let $G = (V, E)$ be a strongly connected digraph, and c be a recurrent state of G. Then the size of the orbit of c is $\frac{1}{M}\sum_{v\in V} deg^+_G(v)\mathcal{T}_G(v)$, where $\mathcal{T}_G(v)$ denotes the number of oriented spanning trees of G rooted at v and M denotes the greatest common divisor of the numbers in $\{\mathcal{T}_G(v) : v \in V\}$. As a corollary, the number of orbits is M.*

Note that the value $\mathcal{T}_G(v)$ can be computed efficiently by using the *matrix-tree theorem* [10]. Thus one can compute the size of an orbit efficiently without listing all states in an orbit. Although the orbits depend on the choice of cyclic orderings, it is interesting that the size of orbits is independent of the choice of cyclic orderings. All recurrent states are in a single orbit if and only if $M = 1$. By doing computer simulations on random digraph $G(n, p)$ with $p \in (0, 1)$ fixed, we observe that $M_{n,p} = 1$ occurs with a high frequency when n is sufficiently large. This observation contrasts with the observation on Eulerian digraphs when one sees the orbits are extremely short [4, 9].

Question. *Let $p \in (0, 1)$ be fixed. Is $\boldsymbol{Pr}\{M_{n,p} = 1\} \to 1$ as $n \to \infty$?*

By using Theorem 1 we give a positive answer for the open question of Holroyd et al. in [4].

Theorem 2. *There is an infinite family of non-Eulerian strongly connected digraphs G_n such that for each n all recurrent states of G_n are in a single orbit.*

For G being a connected digraph such that $deg^+_G(v) \geq 1$ for any $v \in V$ *the random walk* on G is a process of moving the chip on V for which the chip at a vertex v chooses an edge e emanating from v at random, and then moves to the head of e. This process is a Markov chain on V. A random sequence of vertices of G indicating the consecutive locations of the chip in this process is called a *random walk*. The *stationary distribution* π on V is an important characteristic which can be thought of as almost sure frequency of vertices in a random walk. If G is strongly connected, the stationary distribution π of G is given by $\pi(v) = \frac{\mathcal{T}_G(v)deg^+_G(v)}{\sum_{w\in V}\mathcal{T}_G(w)deg^+_G(w)}$ for any $v \in V$ [7]. Let $(X_0, X_1, X_2, \dots)$ be a random walk. It follows from the ergodic theorem that $\mathbf{Pr}\left\{\lim_{t\to\infty}\frac{\sum_{0\leq i\leq t-1}\mathbf{1}_{\{X_i=v\}}}{t} = \pi(v)\right\} = 1$ for any $v \in V$, where $\mathbf{1}_A$ denotes the indicator function, for which $\mathbf{1}_A(x) = 1$ if $x \in A$, and $\mathbf{1}_A(x) = 0$ otherwise [6].

For G being strongly connected let $(v_i)_{i=0}^{\infty}$ be a rotor walk. As we will show in the proof of Theorem 1 the number of occurences of the chip at a vertex v in an orbit is $\frac{1}{M}\mathcal{T}_G(v)deg^+_G(v)$. This implies that in a rotor walk the chip visits a vertex v with the frequency $\lim_{t\to\infty} \frac{\sum_{0\le i\le t-1} \mathbf{1}_{\{v_i=v\}}}{t} = \frac{\mathcal{T}_G(v)deg^+_G(v)}{\sum_{w\in V}\mathcal{T}_G(w)deg^+_G(w)}$. This frequency concides with $\pi(v)$. Therefore a rotor walk can be used to simulate a random walk in this sense. It would be interesting to explore properties of random walks by investigating properties of rotor walks.

We also consider a natural non-deterministic variant of the rotor-router model on a strongly connected digraph G in which the cyclic orderings are relaxed. This variant can be considered as an intermediate model between the random walk and the rotor-router model. In the variant the chip chooses a neighbor at random and move to this neighbor. Thus there are many possible next states for each state. In other words we have a random walk on the digraph $\mathcal{S}$ of states which is defined by: The set of vertices of $\mathcal{S}$ is the set of states of G, and a pair $((v,\rho),(v',\rho'))$ of states is an edge of $\mathcal{S}$ if $\rho(w)=\rho'(w)$ for any $w\neq v$, and $\rho'(v)=(v,v')$. Typically, the digraph $\mathcal{S}$ has very large numbers of vertices and edges. Studying the stationary distribution of $\mathcal{S}$ could be extremely complicated. Nevertheless, we will show that the stationary distribution of $\mathcal{S}$ is unique and uniform on the set of recurrent states of G. More precisely, we will prove the following theorem.

Theorem 3. *The digraph $\mathcal{S}$ has a unique stationary distribution $\bar{\pi}$ which is given by*

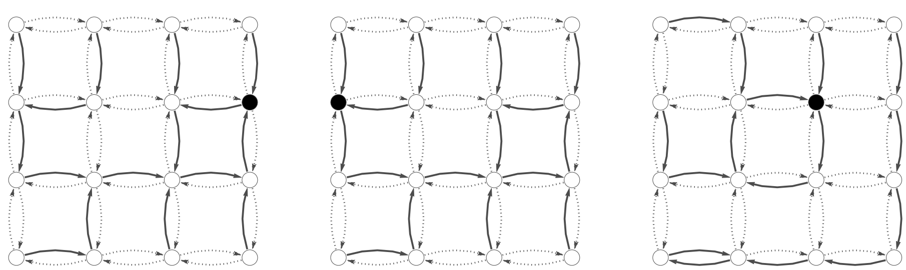
$$\bar{\pi}(v,\rho) = \begin{cases} \frac{1}{\sum_{v\in V}\mathcal{T}_G(v)deg^+_G(v)} & \text{if } (v,\rho) \text{ is a recurrent state of } G \\ 0 & \text{otherwise} \end{cases}$$

The chip alsmost surely visits all vertices of G after a finite number of steps of moving. After this point one only gets recurrent states when the chip continue the walk. The above theorem implies an interesting fact that the recurrent states of G occur at the same frequency when the chip performs a random walk on G.

The structure of this paper is as follows. In Section 2 we will give some background on the rotor-router model and the random walk. The definitions and the results on the rotor-router model we present in this section are mainly from [4]. We also give an equivalent condition for the uniqueness of the stationary distribution on digraphs, which is more intuitive than the one presented in [6]. In Section 3 we will give a proof for Theorem 1 and use this result to give a proof for Theorem 2. In the last section we study the stationary distribution of the random walk on the digraph of states of a strongly connected digraph. This section is devoted to a proof for Theorem 3.

2 Background on rotor-router model and random walk

In this paper all digraphs are assumed to be loopless, and the multi-edges are allowed. For a digraph G we denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. In this section we work with a digraph $G = (V, E)$. The outdegree (resp. indegree) of a vertex v is denoted by $deg_G^+(v)$ (resp. $deg_G^-(v)$). For two distinct vertices v and v' we denote by $a_G(v, v')$ the number of edges connecting v to v'. A *walk* in G is an alternating sequence of vertices and edges $v_0, e_0, v_1, e_1, \ldots, v_{k-1}, e_{k-1}, v_k$ such that for each $i \leq k-1$ we have v_i and v_{i+1} are the tail and the head of e_i, respectively. A *path* is a walk in which all vertices are distinct. For simplicity we often represent a walk (or path) by $e_0, e_1, \ldots, e_{k-1}$, or $v_0, v_1, v_2, \ldots, v_k$ if there is no danger of confusion. A subgraph T of G is called *oriented spanning tree* of G rooted at a vertex s of G if s has outdegree 0 in T for every vertex v of G there is unique path from v to s in T. If G has no sink, a single-chip-and-rotor state (w, ρ) is called a *unicycle* if the subgraph of G induced by the edges in $\{\rho(v) : v \in V\}$ contains a unicycle and w lies on this cycle. Observe that the rotor-router operation takes unicycles to unicycles. Look at Figure 2 for examples of unicycles and non-unicycles. For



(a) A unicycle (b) A non-unicycle (c) A non-unicycle

Fig. 2

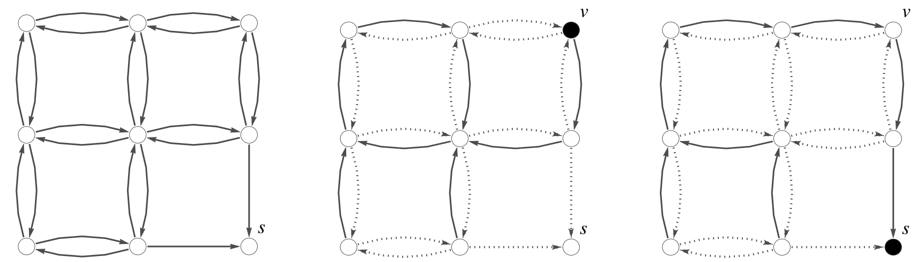
a characterization of recurrent states we have the following lemma.

Lemma 1. *[4] Let $G = (V, E)$ be a strongly connected digraph. A state (w, ρ) is recurrent if and only if (w, ρ) is a unicycle.*

Fix a linear order $v_1 < v_2 < \cdots < v_n$ on V, where $n = |V|$. The $n \times n$ matrix given by

$$\Delta_{i,j} = \begin{cases} -a_G(v_i, v_j) & \text{if } i \neq j \\ deg_G^+(v_i) & \text{if } i = j, \end{cases}$$

is called the *Laplacian* matrix of G. Let $j \in \{1, 2, \ldots, n\}$ be an arbitrary and Δ' be the matrix which is obtained from Δ by deleting the j^{th} row and the j^{th} column. We define the equivalence relation $\sim$ on $\mathbb{Z}^{n-1}$ by $c_1 \sim c_2$ iff there is $z \in \mathbb{Z}^{n-1}$ such that $c_1 - c_2 = z\Delta'$. We recall the matrix-tree theorem.



(a) A digraph with a global sink s

(b) A rotor configuration ρ with a chip at vertex v

(c) When the chip arrives at the sink: $E_v\rho$ (plane edges)

Fig. 3

Theorem 4. *[10] The number of oriented spanning trees of G rooted at v_j is equal to the number of equivalence classes of $\sim$, and therefore equal to $Det(\Delta')$.*

It follows from the theorem that the value $\mathcal{T}_G(v)$ can be computed efficiently by using the Laplacian matrix.

A vertex s of G is called a *global sink* of G if s has outdegree 0 and for every vertex v of G there is a path from v to s. If G has a global sink s, a rotor configuration ρ on G is called *acyclic* if the subgraph of G induced by the edges in $\{\rho(v) : v \neq s\}$ is acyclic. Observe that if ρ is acyclic then $\{\rho(v) : v \neq s\}$ is an oriented spanning tree of G rooted at s. The *chip-addition operator* E_v is the procedure of adding one chip to a vertex v of G and routing this chip until it arrives at the sink. This procedure results the rotor configuration ρ', and we write $E_v\rho = \rho'$. Look at Figure 3 for an illustration of the chip-addition operator.

Lemma 2. *[4] Let $G = (V, E)$ be a digraph with a global sink s. Then the chip-addition operator is commutative. Moreover, for each $v \in V$ the operator E_v is a permutation on the set of acyclic rotor configurations of G.*

If G has a global sink s, a chip configuration on G is a map from $V\setminus\{s\}$ to $\mathbb{N}$. The commutative property of the chip-addition operator allows us to define the action of the set of chip configurations c on the set of rotor configurations of G by $c(\rho) := \prod_{v\in V\setminus\{s\}} E_v^{c(v)}\rho$. The following implies a bijective proof for the matrix-tree theorem.

Lemma 3. *[4] Let G be a digraph with a global sink s, ρ be an acyclic rotor configuration on G, and σ_1, σ_2 be two chip configurations of G. Then $\sigma_1(\rho) = \sigma_2(\rho)$ if and only if σ_1 and σ_2 are in the same equivalence class.*

If $deg^+_G(v) \geq 1$ for any $v \in V$, the $n \times n$ matrix P given by

$$P_{i,j} = \begin{cases} \frac{a_G(v_i,v_j)}{deg^+_G(v_i)} & i \neq j \\ 0 & \text{otherwise} \end{cases}$$

is called *transition matrix* of G. A probability distribution π on V is called *stationary distribution* if $\pi P = P$, where π is considered as a row vector whose entries are adapted to the linear order. The condition for the uniqueness of stationary distribution is given in [6]. We present a more intuitive equivalent condition for the uniqueness of the stationary distribution.

Lemma 4. *Let $G = (V, E)$ be a digraph such that $deg^+_G(v) \geq 1$ for any $v \in V$. The stationary distribution of G is unique if and only if there exists a vertex v such that for any vertex w there is a path in G from w to v, or, equivalently $\mathcal{T}_G(v) \geq 1$.*

Proof. An *essential communicating class* of G is a strongly connected component C of G such that for any edge e of G if the tail of e is in C then its head is also in C. It follows from [6] that the stationary distribution is unique if and only if G has a unique essential communicating class.

Let $\mathcal{H}$ be the digraph defined as follows. The vertices of $\mathcal{H}$ is the set of strongly connected components of G. Two distinct strongly connected components C_1, C_2 are connected by an edge in $\mathcal{H}$ if there is an edge in G connecting a vertex in C_1 to a vertex in C_2.

We have the graph $\mathcal{H}$ is acyclic, and every strongly connected component of G whose outdegree 0 in $\mathcal{H}$ is an essential communicating class. This implies that G has a unique stationary distribution if and only if the graph $\mathcal{H}$ has a unique vertex of outdegree 0.

If $\mathcal{H}$ has a unique vertex of outdegree 0, let C denote this vertex. Then for any vertex D of $\mathcal{H}$ there is a path from D to C in $\mathcal{H}$. Let v be a vertex of G in C. It follows that for any vertex w of G there is a path in G from w to v.

If $\mathcal{H}$ has two vertices of outdegree 0, say C_1, C_2. Let v be an arbitrary vertex of G. Then there exists $C_i, i \in \{1, 2\}$ such that $v \notin C_i$. Let $w \in C_i$. There is no path from w to v in G since there is no edge in G from C_i to the outside of C_i. This concludes the proof. □

3 Orbits of rotor-router operation

In this section we work with a connected digraph $G = (V, E)$. For simplicity we use the notations $deg^+(v), deg^-(v)$ and $a(v, v')$ to stand for $deg^+_G(v), deg^-_G(v)$ and $a_G(v, v')$, respectively. Fix a linear order $v_1 < v_2 < \cdots < v_n$ on V, where $n = |V|$, and let Δ denote the Laplacian matrix of G with respect to this order. For each vertex v let $\mathcal{T}(v)$ denote the number of oriented spanning trees of G rooted at v. Let M denote the greatest common divisor of the numbers in $\{\mathcal{T}(v) : v \in V\}$. The following will be important in the proof Theorem 1. [1]

Lemma 5. $(\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_n))\Delta = \mathbf{0}$*, where* $\mathbf{0}$ *denotes the row vector in* $\mathbb{Z}^n$ *whose entries are* 0.

[1] This result was mentioned in [8] with a reference to a work which was in progress. However we could not find the result in that work. So we decide to give a proof for this fact.

Proof. Let $D_{i,j}$ denote the matrix that is obtained from Δ by deleting the i^{th} row and j^{th} column. We claim that $det(D_{i,j}) = (-1)^{i+j}\mathcal{T}(v_i)$. Clearly, by the matrix-tree theorem the claim holds for $i = j$. So we assume that $i \neq j$. If suffices to show that $det(D_{2,1}) = -\mathcal{T}(v_2)$ since otherwise we can repeatedly switch between rows and between columns so that we obtain a new Laplacian matrix with respect to an linear order on V in which v_j and v_i are the first and second elements in this order, respectively. Then we continue the proof with this matrix. Let Δ' denote the matrix obtained from Δ by deleting the second row and the second column. Since the sum of all columns of $\Delta^{'}$ is equal to minus the first column of $D_{2,1}$, and the other columns of $D_{2,1}$ are the same as those of $\Delta^{'}$, we have $det(\Delta^{'}) = -det(D_{2,1})$. By the matrix-tree theorem we have $det(\Delta^{'}) = \mathcal{T}(v_2)$, therefore $det(D_{2,1}) = -\mathcal{T}(v_2)$.

Since $det(\Delta) = 0$, for any $j \in \{1, 2, \ldots, n\}$ we have

$$\begin{aligned} 0 = \det(\Delta) &= \sum_{1\leq i\leq n} (-1)^{i+j}\Delta_{i,j} det(D_{i,j}) \\ &= \sum_{1\leq i\leq n} \mathcal{T}(v_i)\Delta_{i,j} = (\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_n))(\Delta_{1,j}, \Delta_{2,j}, \ldots, \Delta_{n,j})^\top \end{aligned}$$

This implies that $(\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_n))\Delta = \mathbf{0}$. □

From now until the end of this section we assume G to be strongly connected. This assumption implies that $\mathcal{T}(v) \geq 1$ for any $v \in V$.

Corollary 1. *The vector* $\frac{1}{M}(\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_n))$ *is a generator of the kernel of the operator* $z \mapsto z\Delta$ *in* $(\mathbb{Z}^n, +)$.

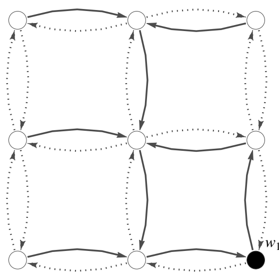
Proof. We consider the operator $z \mapsto z\Delta$ in the vector space $\mathbb{Q}^n$ over the field $\mathbb{Q}$. Since Δ has rank $n-1$, the kernel has dimension 1 in $\mathbb{Q}^n$. By Lemma 5 the vector $(\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_n))$ is in the kernel. Thus for any vector $z \in \mathbb{Z}^n$ such that $z\Delta = 0$ there exists $q \in \mathbb{Q}$ such that $z = q(\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_n))$. Since M is the greatest common divisor of the numbers $\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_n)$, we have $qM \in \mathbb{Z}$. This implies that $\frac{1}{M}(\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_n))$ is a generator of the kernel of $z \mapsto z\Delta$ in $(\mathbb{Z}^n, +)$. □

Lemma 6. *For* $i \in \{1, 2, \ldots, n\}$ *let* $\Delta^{'}$ *denote the matrix obtained from* Δ *by deleting the* i^{th} *column. Then the order of* $\Delta^{'}_i$ *in the quotient group* $(\mathbb{Z}^{n-1}, +)/ < \{\Delta^{'}_j : j \neq i\} >$ *is* $\frac{\mathcal{T}(v_i)}{M}$.

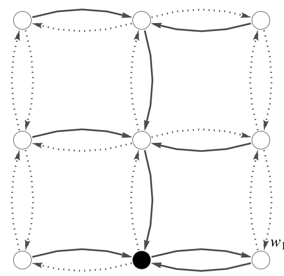
Proof. Clearly, the order of $\Delta^{'}_i$ in $(\mathbb{Z}^{n-1}, +)/ < \{\Delta^{'}_j : j \neq i\} >$ is the smallest positive integer p_i such that there exist integers $p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$ such that $p_i\Delta^{'}_i = \sum_{j\neq i} p_j\Delta^{'}_j$, equivalently

$$(-p_1, -p_2, \ldots, -p_{i-1}, p_i, -p_{i+1}, \ldots, -p_n)\Delta = \mathbf{0}$$

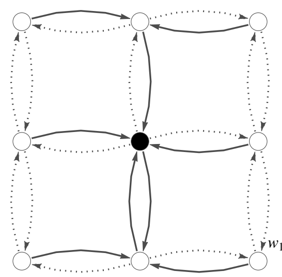
It follows from Corollary 1 that $p_i = \frac{\mathcal{T}(v_i)}{M}$ □



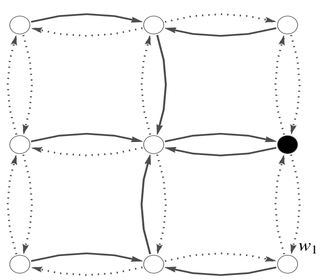
(a) $(w_1, \rho_{i_j}) = (w_{i_j}, \rho_{i_j})$



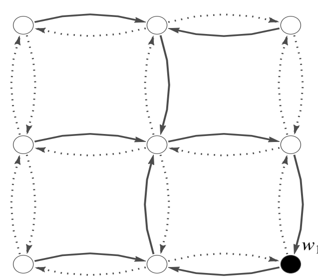
(b) $(w_{i_j+1}, \rho_{i_j+1})$



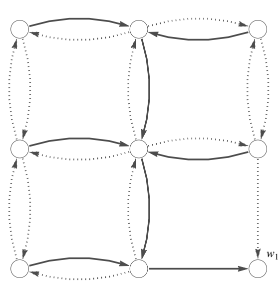
(c) $(w_{i_j+2}, \rho_{i_j+2})$



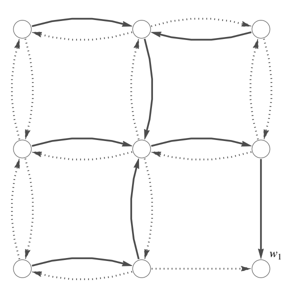
(d) $(w_{i_j+3}, \rho_{i_j+3})$



(e) $(w_{i_j+4}, \rho_{i_j+4}) = (w_1, \rho_{i_{j+1}})$



(f) $\overline{\rho_{i_j}}$



(g) $\overline{\rho_{i_{j+1}}}$

Fig. 4

Proof of Theorem 1. Let (w_1, ρ_1) be an arbitrary unicycle of G. Let (w_1, ρ_1), $(w_2, \rho_2), (w_3, \rho_3), \ldots$ be the infinite sequence of states such that for any $i \geq 1$ the state (w_{i+1}, ρ_{i+1}) is obtained from the state (w_i, ρ_i) by applying the rotor-router operation. By collecting all states (w_i, ρ_i) with $w_i = w_1$ we obtain the subsequence $(w_1, \rho_{i_1}), (w_1, \rho_{i_2}), (w_1, \rho_{i_3}), \ldots$. Note that $1 = i_1$. For each ρ_{i_j} let u_j denote the head of $\rho_{i_j}(w_1)$. Let $e_1, e_2, \ldots, e_k$, where $k = deg^+(w_1)$, be an enumeration of the edges emanating from w_1 such that $e_1 = \rho_1(w_1)$ and $e_{i+1} = e_i^+$ for any $i < k$, and $e_1 = e_k^+$.

Let $\overline{G}$ denote the graph obtained from G by deleting all edges emanating from w_1, and for each ρ_{i_j} let $\overline{\rho_{i_j}}$ denote the restriction of ρ_{i_j} on $\overline{G}$. Note that $\overline{\rho_{i_j}}$ is an acyclic rotor configuration of $\overline{G}$ (See Figure 4). It follows from the definition of the chip addition operator that $\overline{\rho_{i_{j+1}}} = E_{u_{j+1}}\overline{\rho_{i_j}}$. For each $q > 1$ we define the chip configuration $c_q : V\backslash\{w_1\} \to \mathbb{N}$ by for any $v \in V\backslash\{w_1\}$ $c_q(v)$ is the number of occurrences of v in the sequence $u_2, u_3, \ldots, u_q$. The above identity implies that $\overline{\rho_{i_q}} = c_q(\overline{\rho_{i_1}})$. Let $\Delta^{'}$ be the matrix that is obtained from Δ by deleting the column corresponding to w_1. We have $\rho_{i_q} = \rho_{i_1}$ if and only if the following conditions hold

- the configuration c_q is in the same equivalence class as $\mathbf{0}$ in $\overline{G}$. This fact follows from Lemma 3.
- $c_q = -p\Delta'_{w_1}$ for some p, where Δ'_{w_1} denotes the row of Δ' corresponding to the vertex w_1. This follows the fact that the sequence $\rho_{i_1}(w_1), \rho_{i_2}(w_1)$, $, \rho_{i_3}(w_1) \ldots$ is exactly the periodic sequence $e_1, e_2, \ldots, e_k, e_1, e_2, \ldots, e_k, \ldots$ Note that $\rho_{i_2}(w_1), \rho_{i_3}(w_1), \ldots, \rho_{i_q}(w_1)$ is a periodic sequence of length pk, namely $\underbrace{e_2, e_3, \ldots, e_k, e_1, \ldots, e_2, e_3 \ldots, e_k, e_1}_{\text{length } pk}$.

Thus $1 + pk$ is the smallest q satisfying $\rho_{i_1} = \rho_{i_q}$, where p is the order of Δ'_{w_1} in $\mathbb{Z}^{n-1}/ < \{\Delta'_v : v \in V\backslash\{w_1\}\} >$. By Lemma 6 we have $p = \frac{1}{M}\mathcal{T}(w_1)$. It follows that in the orbit $\{(w_i, \rho_i) : 1 \leq i \leq i_{1+pk} - 1\}$ the number of times the chip passes through w_1 is $\frac{1}{M}deg^+(w_1)\mathcal{T}(w_1)$. Since this fact also holds for other vertices, the size of orbit is $\frac{1}{M}\sum\limits_{v\in V} deg^+(v)\mathcal{T}(v)$.

Since the number of unicycles is $\sum\limits_{v\in V} deg^+(v)\mathcal{T}(v)$, it follows that the number of orbits of the rotor-router operation is M. □

If G is an Eulerian digraph then the numbers of oriented spanning trees $\mathcal{T}(v), v \in V$ are the same since $\mathcal{T}(v)$ is equal to the order of the sandpile group of G with sink v and the sandpile group is independent of the choice of sink [4]. Thus $M = \mathcal{T}(v_1) = \mathcal{T}(v_2) = \cdots = \mathcal{T}(v_n)$. By Theorem 1 each orbit of the rotor-router operation has size $\sum\limits_{v\in V} deg^+(v) = |E|$. We recover the result in [4, 9].

Proposition 1. *[4, 9] Let G be an Eulerian digraph with m edges. Starting from a unicycle (w, ρ) the chip traverses each edge exactly once before returning to (w, ρ) for the first time.*

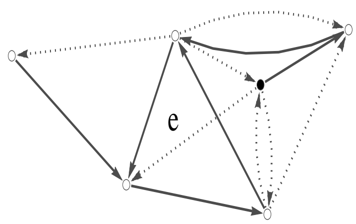
Proof of Theorem 2. For each $n \geq 3$ let G_n be the strongly connected digraph given by $V(G_n) := \{1, 2, \ldots, n\}$ and $E(G_n) := \{(i, i{+}1) : 1 \leq i \leq n{-}1\} \cup \{(i, 1) : 2 \leq i \leq n\}$. Since $deg^+_{G_n}(1) = 1$ and $deg^-_{G_n}(1) = n-1$, G_n is not Eulerian. Since G_n has exactly one oriented spanning tree rooted at n, namely the subgraph induced by the edges in $\{(i, i{+}1) : 1 \leq i \leq n{-}1\}$, we have $\mathcal{T}_{G_n}(n) = 1$, therefore $M_{G_n} = 1$. By Theorem 1 all unicycles are in a single orbit. □

The formula in Theorem 1 is very useful because one can use it to compute size of an orbit effeciently without listing all unicycles in an orbit. As we saw above, size of orbits on a strongly connected digraph is often large while it is extremely short on an Eulerian digraph. If orbit size is too large (resp. too small) then number of orbits is too small (resp. too large). Thus one would expect to see an infinite family of strongly connected digraphs G_n on which the rotor-router operation behaves moderately, i.e. both the orbit size and the number of orbits grow exponentially in the number of vertices and in the number of edges. By using Theorem 1 we construct easily such a family of digraphs as follows. For $n \geq 1$ the graph G_n has the vertex set $\{1, 2, \ldots, n + 1\}$, and for each $i \in \{1, 2, \ldots, n\}$ there are two edges connecting i to $i + 1$ and four edges connecting $i + 1$ to i in G_n. It is easy to see that $\mathcal{T}_{G_n}(i) = 4^{n+1-i} \times 2^{i-1} = 2^{2n+1-i}$ for any $i \in \{1, 2, \ldots, n + 1\}$. Therefore we have $M_{G_n} = 2^n$. It follows from Theorem 1 that the number of orbits is 2^n and the size of orbits is greater than $\frac{\mathcal{T}_{G_n}(1)}{2^n} = 2^n$. Thus the family of digraphs G_n has the desired property.

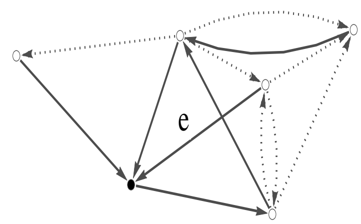
4 Random walks on set of single-chip-and-rotor states

In this section we work with a strongly connected digraph $G = (V, E)$. We consider a natural non-deterministic variant of the rotor-router model in which the cyclic orderings are relaxed. When the chip is at the state (v, ρ), it chooses an edge e emanating from v at random and moves along this edge to the head. We arrive at the new state (v', ρ'), where v' is the head of e, $\rho'(v) = e$ and $\rho'(w) = \rho(w)$ for any vertex $w \neq v$ (See Figure 5). This means that when the chip performs a random walk on G with an initial state, it induces a random walk on the digraph $\mathcal{S}$ of states of G, and vice-versa. The graph $\mathcal{S}$ is defined as follows. The vertex set of $V(\mathcal{S})$ is the set of all states of G and the edge set $E(\mathcal{S})$ is the set of all pairs $((v, \rho), (v', \rho'))$ of states of G such that $\rho(w) = \rho'(w)$ for any $w \neq v$, and $\rho'(v)$ is an edge of G whose head v'. We observe that if (v, ρ) is a unicycle then (v', ρ') is also a unicycle. The following means that when the chip visits all vertices of G, we arrive at unicycles.

Proposition 2. *Let $((v_i, \rho_i))_{i=1}^k$ be a walk in $\mathcal{S}$ for some k. If $V \subseteq \{v_1, v_2, \ldots, v_k\}$ then (v_k, ρ_k) is a unicycle.*



(a) A state of G (plane edges and the chip in black)



(b) The new state if the chip chooses the edge e to move

Fig. 5

Proof. We say that two functions $f, f' : X \to Y$ agree on a subset X' of X if $f(x) = f'(x)$ for any $x \in X'$. Let $((v_i, \rho_i))_{i=1}^{k}$ be an arbitrary walk in $\mathcal{S}$ and $j \in \{1, 2, \ldots, k\}$ be arbitrary. We claim that there is a path with edges in $\{\rho(v_i) : 1 \le i \le k\}$ from v_j to v_k.

We prove the claim by induction on k. Clearly, the claim holds if $k = 1$. We consider the case $k \ge 2$. The claim is trivial if $j = k$. So we assume otherwise $j \le k - 1$. By the inductive assumption there is a path P in $\{\rho_{k-1}(v_i) : 1 \le i \le k-1\}$ from v_j to v_{k-1}. Note that this path may pass through v_k if $v_k \in \{v_1, v_2, \ldots, v_{k-2}\}$. Since ρ_{k-1} and ρ_k agree on $V \setminus \{v_{k-1}\}$, P is also a path in $\{\rho_k(v_i) : 1 \le i \le k-1\}$. Clearly, $(P, \rho_k(v_{k-1}))$ is a walk in $\{\rho_k(v_i) : 1 \le i \le k\}$ from v_j to v_k. Therefore there is a path in $\{\rho_k(v_i) : 1 \le i \le k\}$ from v_j to v_k.

If $V \subseteq \{v_1, v_2, \ldots, v_k\}$, the claim implies that for any vertex $w \in V$ there is a path in $\{\rho_k(v_i) : 1 \le i \le k\}$ from w to v_k. Therefore (v_k, ρ_k) is a unicycle. □

Typically, the digraph $\mathcal{S}$ has very large numbers of vertices and edges. However we will show that the stationary distribution is unique and uniform on the unicycles of G. The following lemma implies that the set of unicycles is a unique essential communicating class of $\mathcal{S}$.

Lemma 7. *Let (w, ρ) be a unicycle and (u, σ) be a state. Then there is a path in $\mathcal{S}$ from (u, σ) to (w, ρ).*

Proof. We prove the lemma by induction on $|V|$. The assertion is trivial if $|V| = 1$. We consider the case $|V| \ge 2$. Since G is strongly connected digraph, the chip can move to any vertex by a path in G. So we can assume that $u = w$. Let $\mathcal{C}$ denote the cycle in $(V, \{\rho(v) : v \in V\})$. Starting from (u, σ) and letting the chip traverse the cycle $\mathcal{C}$ in one round, we arrive at the state (u, σ') such that σ' and ρ agree on the vertices in the cycle. So we can assume that ρ and σ agree on the vertices of the cycle. Let C denote the set of vertices of $\mathcal{C}$. For two vertices $v, v' \in C$ let $P(v, v')$ denote the sequence $(e_1, e_2, \ldots, e_k)$ of edges in the cycle $\mathcal{C}$ such that the chip traverses the edges $e_1, e_2, \ldots, e_k$ in this order to move from v to v' (See Figure 6).

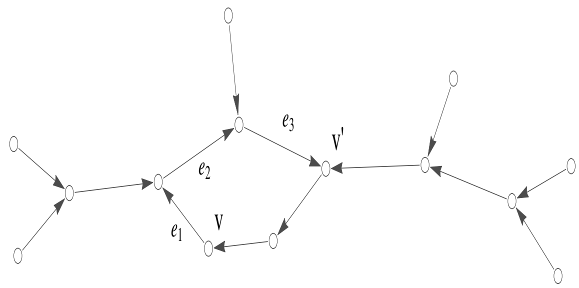


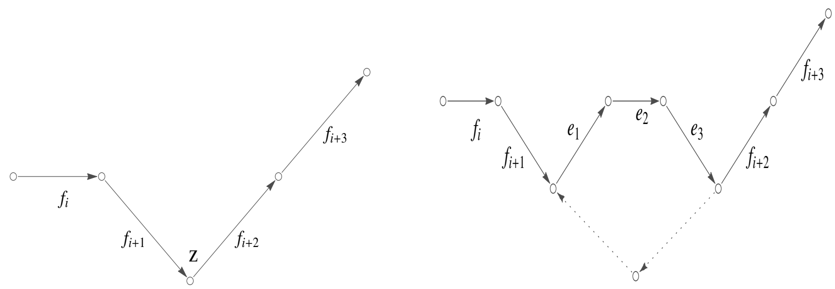
Fig. 6 - $P(v, v') = (e_1, e_2, e_3)$

Let H be the digraph which is obtained from G by removing all edges of G connecting between the vertices in C and then gluing all vertices in C. Let z be the vertex resulting from the gluing. An edge e of G not connecting between vertices in C is an edge of H. Note that for an edge e of H if head (resp. tail) of e is in C in G then head (resp. tail) of e is z in H. Gluing vertices does not make a graph lose the strong connectivity. The digraph G is strongly connected, so is H. If H has exactly one vertex then we are done since ρ and σ agree on C. We assume otherwise that H has at least two vertices.

Since H is strongly connected, there is an edge e' in H whose tail z. The states (w, ρ) and (u, σ) have two corresponding states (z, ρ') and (z, σ') in H which are defined as follows. $\rho'(z) = \sigma'(z) = e'$, $\rho'(v) = \rho(v)$ and $\sigma'(v) = \sigma(v)$ for any $v \in V\backslash C$. Note that ρ' is a unicycle of H. By the inductive assumption there is a walk $(f_1, f_2, \ldots, f_k)$ in H, where $f_i \in E(H)$, such that starting from (z, σ') we arrive at the state (z, ρ') when the chip traverses this walk. By using this walk we will construct a walk in G so that starting from (u, σ) we arrive at (w, ρ) when the chip traverses this walk.

We observe that if the head of f_i is different from the tail of f_{i+1} in G then both the head of f_i and the tail of f_{i+1} are in C. Based on this observation we construct a walk of chip in G as follows. It is possible that u is not the tail of e' in G. If this case happens, we firstly let the chip move from u to the tail of e' by a path in $\mathcal{C}$. The chip then traverses the edges $f_1, f_2, \ldots, f_k$ in this order in G normally. However it is not always possible for the chip to move in this way since the head of f_i may not be equal to the tail of f_{i+1}. Whenever this case happens, we let the chip traverse the path $P(v, v')$ from the head of f_i to the tail of f_{i+1}, and then continue the process normally, where v and v' are the head of f_i and the tail of f_{i+1}, respectively, (See Figure 7).

Let $(f'_1, f'_2, \ldots, f'_p)$ denote the walk in the above construction, where f'_i is an edge of G, and let (u'', σ'') denote the resulting state. Clearly, we have $\sigma''(v) = \rho'(v)$ for any $v \in V\backslash C$, therefore $\sigma''(v) = \rho(v)$ for any $v \in V\backslash C$. Since starting from (z, σ') the chip arrives at the vertex z when it traverses the edges $f_1, f_2, \ldots, f_k$ in this order in H, it follows that the chip arrives at a vertex in C if the chip traverses the edges $f'_1, f'_2, \ldots, f'_p$ in this order in G. This implies



(a) a walk in H (b) the corresponding walk in G

Fig. 7

that u'' is in C. By letting the chip traverse the cycle $\mathcal{C}$ in at least one round we arrive at the state (u''', σ''') such that $u''' = w$, σ''' and ρ agree on C. Thus $(u''', \sigma''') = (w, \rho)$. □

Proof of Theorem 3. It follows from Lemma 7 that the set of unicycles of G is a unique essential communicating class of $\mathcal{S}$. By Lemma 4 the digraph $\mathcal{S}$ has a unique stationary distribution.

Let (w, ρ) be an arbitrary unicycle of G. Let $\overleftrightarrow{w}$ be the vertex of G such that $\rho(\overleftrightarrow{w}) = w$ and both w and $\overleftrightarrow{w}$ lie on the cycle in the digraph $(V, \{\rho(v) : v \in V\})$ (See Figure 8). Let (w', ρ') be a state such that $((w', \rho'), (w, \rho)) \in E(\mathcal{S})$. We claim that if (w', ρ') is a unicycle of G then $w' = \overleftrightarrow{w}$. We assume otherwise that $w' \neq \overleftrightarrow{w}$. Let $\mathcal{C}$ denote the cycle in $(V, \{\rho(v) : v \in V\})$. We distinguish the following cases.

- w' lies on the cycle $\mathcal{C}$.
  Since (w, ρ) is obtained from (w', ρ') by one step of the chip moving, we have $w = \rho(w')$. Since both w' and $\overleftrightarrow{w}$ lie on the cycle $\mathcal{C}$, it follows that $w' = \overleftrightarrow{w}$. This contradicts the assumption.

- w' does not lie on the cycle $\mathcal{C}$.
  This implies that ρ' and ρ agree on the vertices in the cycle $\mathcal{C}$, therefore $\mathcal{C}$ is a cycle in $(V, \{\rho'(v) : v \in V\})$. Thus (w', ρ') is not a unicycle, a contradiction.

Let $(\overleftrightarrow{w}, \rho')$ be a state of G. Clearly, $(\overleftrightarrow{w}, \rho')$ is connected to (w, ρ) by an edge in $\mathcal{S}$ if and only if ρ and ρ' agree on every vertex of G distinct from $\overleftrightarrow{w}$. Moreover if ρ and ρ' agree on every vertex distinct from $\overleftrightarrow{w}$ then $(\overleftrightarrow{w}, \rho')$ is a unicycle since $(\overleftrightarrow{w}, \rho)$ is a unicycle. It follows from the claim that the set of all unicycles of G connecting to (w, ρ) in $\mathcal{S}$ is the $\{(\overleftrightarrow{w}, \rho') : \rho'$ and ρ agree on $V \backslash \{\overleftrightarrow{w}\}\}$. Note that this set has $deg^+_G(\overleftrightarrow{w})$ elements.

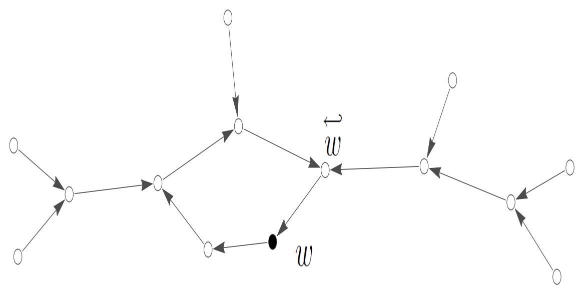


Fig. 8 - A unicycle (w, ρ)

We observe that the outdegree of each state (w, ρ) in $\mathcal{S}$ is $deg_G^+(w)$. Let U denote the set of unicycles of G and let $\mathcal{H}$ denote the subgraph of $\mathcal{S}$ induced by U. It suffices to show that $\bar{\pi}_{|U}$ is the stationary distribution of $\mathcal{H}$. The digraph $\mathcal{H}$ has the following property. If $(X,Y) \in E(\mathcal{S})$ then $deg_{\mathcal{H}}^+(X) = deg_{\mathcal{H}}^-(Y)$. Let Y be an arbitrary vertex of $\mathcal{H}$. We have $\sum\limits_{X \in U, (X,Y) \in E(\mathcal{H})} \frac{1}{deg_{\mathcal{H}}^+(X)} \bar{\pi}(X) = \frac{1}{deg_{\mathcal{H}}^-(Y)} \sum\limits_{X \in U, (X,Y) \in E(\mathcal{H})} \bar{\pi}(X) = \frac{1}{deg_{\mathcal{H}}^-(Y)} deg_{\mathcal{H}}^-(Y) \bar{\pi}(Y) = \bar{\pi}(Y)$. This implies that $\bar{\pi}_{|U}$ is the stationary distribution of $\mathcal{H}$, therefore $\bar{\pi}$ is the stationary distribution of $\mathcal{S}$. □

Acknowledgements. We would like to thank L. Levine and M. Farrell for their useful comments on the paper.

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