THE LOCALLY $\mathcal{F}$-APPROXIMATION PROPERTY OF BOUNDED HYPERCONVEX DOMAINS

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Abstract. In this paper, we study the local property of bounded hyperconvex domains $\Omega$ which we can approximative each plurisubharmonic function $u \in \mathcal{F}(\Omega)$ by an increasing sequence of plurisubharmonic functions defined on strictly larger domains.

1. INTRODUCTION

Hed [10] give in 2012 the following definition of the $\mathcal{F}$-approximation property of bounded hyperconvex domains.

Definition 1.1. A bounded hyperconvex domain $\Omega$ in $\mathbb{C}^n$ has the $\mathcal{F}$-approximation property if there exists a sequence of hyperconvex domains $\{\Omega_j\}$ such that $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ and we can approximate each function $u \in \mathcal{F}(\Omega)$ by an increasing sequence of functions $u_j \in \mathcal{F}(\Omega_j)$ quasi everywhere on $\Omega$.

The first result in this direction is the theorem of Benelkourchi [2] in 2006 about the approximation of plurisubharmonic functions. Cegrell and Hed [6] proved in 2008 that a sufficient condition for $\Omega$ to have the $\mathcal{F}$-approximation property is that one single function in the class $\mathcal{N}(\Omega)$ can be approximated with functions in $\mathcal{N}(\Omega_j)$. Hed [9] proved in 2010 that if $\Omega$ has the $\mathcal{F}$-approximation property then we can approximate each function with given boundary values $u \in \mathcal{F}(\Omega, f|_{\partial \Omega})$ by an increasing sequence of functions $u_j \in \mathcal{F}(\Omega_j, f|_{\partial \Omega_j})$ a.e. on $\Omega$. Later, Benelkourchi [3] studied in 2011 the approximation of plurisubharmonic functions in the weighted energy class. Amal [1] studied in 2014 the approximation of plurisubharmonic functions in the weighted energy class with given boundary values. Recently, Hong [11] proved in 2015 a generalization of Cegrell and Hed’s theorem.

The purpose of this paper is to study the local property of the $\mathcal{F}$-approximation property. Namely, we prove the following theorem.

Theorem 1.2. Let $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ be bounded hyperconvex domains in $\mathbb{C}^n$ such that $\Omega = \bigcap_{j=1}^{\infty} \Omega_j$. Then $\Omega$ has the $\mathcal{F}$-approximation property if only if $\Omega$ has the locally $\mathcal{F}$-approximation property, i.e., for every $z \in \partial \Omega$ there exists a neighborhood $U_z$ of $z$ such that $\Omega \cap U_z$ has the $\mathcal{F}$-approximation property.

This result is proved using the $\mathcal{F}$-plurisubharmonic functions and the technique of Coltoiu and Mihalache [7].

The organization of the paper is as follows. In Section 2 we recall some notions of pluripotential theory which is necessary for the next results of the paper. In Section 3 we prove the main result of the paper.

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2. Preliminaries

Some elements of pluripotential theory that will be used throughout the paper can be found in [1]-[15]. Let $\Omega$ be a domain in $\mathbb{C}^n$. We denote by $PSH(\Omega)$ ($PSH^{-}(\Omega)$) the family of plurisubharmonic (negative plurisubharmonic) functions.

2.1. Cegrell’s classes

We recall some Cegrell’s classes of plurisubharmonic functions. Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$, i.e. a connected, bounded open subset of $\mathbb{C}^n$ such that there exists a negative plurisubharmonic function $\rho$ such that $\{z \in \Omega : \rho(z) < -c\} \subset \Omega$, $\forall c > 0$. Put

$$E_0(\Omega) = \{\varphi \in PSH^{-}(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial \Omega} \varphi(z) = 0, \int_\Omega (dd^c \varphi)^n < \infty\},$$

$$\mathcal{F}(\Omega) = \{\varphi \in PSH^{-}(\Omega) : \exists \varphi_j \downarrow \varphi, \sup_j \int_\Omega (dd^c \varphi_j)^n < \infty\}$$

and

$$E(\Omega) = \{\varphi \in PSH^{-}(\Omega) : \forall G \subset \Omega, \exists u_G \in \mathcal{F}(\Omega), u = u_G \text{ on } G\}.$$

Let $\varphi \in E(\Omega)$ and let $\{\Omega_j\}$ a fundamental sequence of $\Omega$, i.e, $\Omega_j$ be strictly pseudoconvex domains such that $\Omega_j \subset \Omega_{j+1} \subset \Omega$ and $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Put

$$\varphi^j = \sup\{u \in PSH(\Omega) : u \leq \varphi \text{ on } \Omega \setminus \Omega_j\}$$

and

$$N(\Omega) = \{\varphi \in E(\Omega) : \varphi^j \not\rightarrow 0 \text{ a. e. in } \Omega\}.$$

2.2. The plurifine topology

The plurifine topology $\mathcal{F}$ on open subsets of $\mathbb{C}^n$ is the weakest topology in which all plurisubharmonic functions are continuous. Notions pertaining to the plurifine topology are indicated with the prefix $\mathcal{F}$ and notions pertaining to the fine topology are indicated with $C^n$. For a set $A \subset \mathbb{C}^n$ we write $\overline{A}$ for the closure of $A$ in the one point compactification of $\mathbb{C}^n$, $\overline{A}^\mathcal{F}$ for the $\mathcal{F}$-closure of $A$ and $\partial_F A$ for the $\mathcal{F}$-boundary of $A$. We denote by $\mathcal{F}-PSH(\Omega)$ the set of $\mathcal{F}$-plurisubharmonic functions on an $\mathcal{F}$-open set $\Omega$.

Note that if $\Omega$ be an open subsets of $\mathbb{C}^n$ then $\mathcal{F}-PSH(\Omega) = PSH(\Omega)$.

3. Proof of Theorem 1.2

First, we need the following auxiliary result. The idea of the proof is to use the $\mathcal{F}$-plurisubharmonic functions.

Lemma 3.1. Let $\Omega \subset \mathbb{C}^n$ be bounded hyperconvex domains. Assume that there exists a sequence of bounded hyperconvex domains $\{\Omega_j\}$ such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ and $\overline{\Omega} = \bigcap_{j=1}^\infty \Omega_j$. Then the following statements are equivalent.

(a) if $u \in E_0(\Omega)$ and define $u_j := \sup\{\varphi \in PSH^{-}(\Omega_j) : \varphi \leq u \text{ in } \Omega\}$ then $1_{\Omega_j} u_j$ converges uniformly to $1_{\Omega} u$ in $\mathbb{C}^n$.

(b) there exists $u_j \in PSH^{-}(\Omega_j)$ such that $(\sup_j u_j)^* \in N(\Omega)$.

(c) there exists $u \in N(\Omega)$, $u_j \in PSH^{-}(\Omega_j)$ such that $u_j \rightarrow u$ a. e. in $\Omega$.

(d) $\Omega$ has the $\mathcal{F}$-approximation property.
Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is obvious. (c) $\Rightarrow$ (d): see [6]. We prove (d) $\Rightarrow$ (a). Let $u \in \mathcal{E}_0(\Omega)$. Since $\Omega$ has the $\mathcal{F}$-approximation property so there exists a sequence of hyperconvex domains $\{U_j\}$ and sequence of functions $\psi_j \in \mathcal{F}(U_j)$ such that $\Omega \Subset U_{j+1} \Subset U_j$ and $\psi_j \not> u$ a.e. in $\Omega$. Without loss of generality we can assume that $\Omega_j \subset U_j$. Put

$$ u_j := \sup\{\varphi \in PSH^-(\Omega_j) : \varphi \leq u \text{ in } \Omega\}. $$

It is clear that $u_j \in \mathcal{E}_0(\Omega_j)$ and $u_j \leq u_{j+1}$ in $\Omega_{j+1}$. We claim that $u_j$ is maximal plurisubharmonic function in a open neighborhood of $\Omega_j \setminus \Omega$. Indeed, put $\delta = \sup\Omega_{j+1} u_j$. Since $\Omega_{j+1} \Subset \Omega_j$ and $u_j \in \mathcal{E}_0(\Omega_j)$ so $\delta < 0$. Put

$$ G_j := \Omega_j \setminus \{\Omega \cap \{u < \delta/2\}\}. $$

Since $\{u < \delta/2\} \Subset \Omega$ so $G_j$ be a open neighborhood of $\Omega_j \setminus \Omega$. Since $\{u > \delta/2\} \cap \Omega \subset \{u_j < u\} \cap \Omega$ so from Theorem 1.1 in [11] we have $(dd^c u_j)^n = 0$ in $G_j$. Hence, $u_j$ is maximal plurisubharmonic function in $G_j$. This proves the claim.

Since $\psi_j \leq u_j \leq u$ in $\Omega$ so $u_j \not> u$ a.e. in $\Omega$. Choose $\psi \in \mathcal{F}(\Omega)$ such that $u_j \not> u$ in $\Omega \setminus \{\psi = -\infty\}$. Put $\Omega' := \Omega \setminus \{\psi = -\infty\}$. Let $k \in \mathbb{N}^*$. Since $\{u \leq -\frac{1}{k}\} \Subset \Omega$ and

$$ \{u_j \leq -\frac{1}{k}\} \cap \Omega' \searrow \{u \leq -\frac{1}{k}\} \cap \Omega' $$

as $j \to +\infty$ so there exists an increasing sequence $\{j_k\}$ such that $\{u_{j_k} \leq -\frac{1}{k}\} \cap \Omega' \Subset \Omega$ for all $k$. By replacing $\{u_j\}$ with its subsequence if necessary, we can assume that

$$ \{u_j \leq -\frac{1}{j}\} \cap \Omega' \Subset \Omega $$

for every $j \geq 1$. Put

$$ v_j = \begin{cases} u_j & \text{in } \{u_j \geq -\frac{1}{j}\} \cap \Omega' \\ \max(u_j, u - \frac{1}{j}) & \text{in } \{u_j < -\frac{1}{j}\} \cap \Omega'. \end{cases} $$

Since $u - \frac{1}{j} < -\frac{1}{j} = u_j$ in $\{u_j = -\frac{1}{j}\}$ so by Proposition 2.3 in [13] we have $v_j$ is $\mathcal{F}$-plurisubharmonic function in $\Omega'$. Since $\{\psi = -\infty\}$ is pluripolar and $\mathcal{F}$-closed in $\Omega$ so by Theorem 3.7 in [12] the function

$$ v_j^*(z) := \mathcal{F} \text{-lim sup}_{\Omega' \ni \zeta \to z} v_j(\zeta), \ z \in \Omega $$

is $\mathcal{F}$-plurisubharmonic function in $\Omega$. Since $\Omega$ be open subset of $\mathbb{C}^n$ so from Proposition 2.14 in [12] we have $v_j^* \in \mathcal{P}SH^-(\Omega)$.

We claim that $u_j = v_j^* \in \Omega$. Indeed, since $\{\psi = -\infty\}$ is a pluripolar subset of $\Omega$ and $u_j = v_j$ in $\Omega \setminus \{u_j < -\frac{1}{j}\} \cap \Omega'$ so $u_j = v_j^* \in \Omega \setminus \{u_j < -\frac{1}{j}\} \cap \Omega'$. Put

$$ \varphi = \begin{cases} v_j^* & \text{in } \Omega \\ u_j & \text{in } \Omega \setminus \Omega. \end{cases} $$

Then, $\varphi \in \mathcal{P}SH^-(\Omega_j)$ and $\varphi \leq u$ in $\Omega$. Hence, $\varphi \leq u_j$ in $\Omega_j$. Moreover, since $\varphi = v_j^* \geq u_j$ in $\Omega$ so $u_j = v_j^* \in \Omega$. This proves the claim. Since $u - \frac{1}{j} \leq v_j \leq u$ in $\Omega'$ so $u - \frac{1}{j} \leq u_j \leq u$ in $\Omega$. Moreover, since $u_j$ is maximal plurisubharmonic function in a open neighborhood of $\Omega_j \setminus \Omega$ and $u_j \geq -\frac{1}{j}$ in $\partial(\Omega_j \setminus \Omega)$ so $u_j \geq -\frac{1}{j}$ in $\Omega_j \setminus \Omega$. Therefore,

$$ l_{\Omega_j} u - \frac{1}{j} \leq l_{\Omega_j} u_j \leq l_{\Omega_j} u $$

in $\mathbb{C}^n$. Hence, $l_{\Omega_j} u_j$ converges uniformly to $l_{\Omega_j} u$ in $\mathbb{C}^n$. The proof is complete. \(\square\)
Remark 3.2. Let $\Omega \subset \Omega_{j+1} \subset \Omega_j$ be bounded open subsets of $\mathbb{C}^n$ such that $\Omega$ has the $\mathcal{F}$-approximation property and $\bigcap_{j=1}^{\infty} \Omega_j \subset \Omega$. If $u \in \mathcal{E}_0(\Omega)$ and

$$u_j := \sup\{\varphi \in PSH^- (\Omega_j) : \varphi \leq u \text{ in } \Omega\}$$

then $1_{\Omega_j}u_j$ converges uniformly to $1_{\Omega}u$ in $\mathbb{C}^n$. Indeed, since $\Omega$ has the $\mathcal{F}$-approximation property so there exists a sequence of hyperconvex domains $\{U_j\}$ such that $\Omega \Subset U_{j+1} \Subset U_j$ and $\bigcap_{j=1}^{\infty} U_j = \Omega$. Without loss of generality we can assume that $\Omega_j \subset U_j$. Put $v_j := \sup\{\varphi \in PSH^-(U_j) : \varphi \leq u \text{ in } \Omega\}$. Since $v_j \leq u_j$ in $\Omega_j$ so $1_{U_j}v_j \leq 1_{\Omega_j}u_j \leq 1_{\Omega}u$ in $\mathbb{C}^n$. By Lemma 3.1 we have $1_{U_j}v_j$ converges uniformly to $1_{\Omega}u$ in $\mathbb{C}^n$. Hence, $1_{\Omega_j}u_j$ converges uniformly to $1_{\Omega}u$ in $\mathbb{C}^n$.

We now give the proof of theorem 1.2. The idea of the proof is taken from [7] (also see [8], [15]).

**Proof of theorem 1.2.** The necessity is obvious. We prove the sufficiency. Let $U''_j \Subset U_j$, $j = 1, \ldots, m$ be open subsets such that $U_j \cap \Omega$ has the $\mathcal{F}$-approximation property and $\partial \Omega \Subset \bigcup_{j=1}^{m} U''_j$. Without loss of generality we can assume that $\Omega_1 \cap \Omega \Subset \bigcup_{j=1}^{m} U''_j$. Let $w^j \in \mathcal{E}_0(\Omega \cap U_j)$ and define

$$w_j^k := \sup\{\varphi \in PSH^-(\Omega_k \cap U_j) : \varphi \leq w^j \text{ in } \Omega \cap U_j\}.$$ 

Without loss of generality we can assume that $-1 \leq w_j^k \leq 0$ for all $j = 1, \ldots, m$ and for any $k \in \mathbb{N}^*$. From the proof of Theorem 1 in [7] (also see the proof of Proposition 3.2 in [8]) there exists a convex continuous increasing function $\tau : (-\infty, 0) \to (0, +\infty)$ and a positive number $\varepsilon_0 \in (0, 1)$ such that $\lim_{x \to 0} \tau(x) = +\infty$ and

$$|\tau(u^j - \varepsilon) - \tau(u^k - \varepsilon)| \leq 1 \text{ in } U_j \cap U_k \cap \Omega$$

for all $k, j = 1, \ldots, m$ and for any $\varepsilon \in (0, \varepsilon_0)$. Let $\{\varepsilon_j\} \subset (0, \varepsilon_0)$ such that $\varepsilon_j \searrow 0$. Since $\tau$ is continuous function so there exists a decreasing sequence of positive real numbers $\{\delta_j\}$ such that $\delta_j \searrow 0$ and

$$\tau(x - \varepsilon_j) - \tau(x - \varepsilon_j - \delta) \leq \min \left( \frac{\tau(-\varepsilon_j - \delta - 1)}{\varepsilon_j}, 1 \right)$$

for any $x \in [-1, 0]$, for any $\delta \in (0, \delta_j)$. By Remark 3.2 we have $1_{\Omega \cap U_j}w_j^k$ converges uniformly to $1_{\Omega \cap U_j}w^j$ in $\mathbb{C}^n$. Hence, by replacing $\{w_j^k\}$ with a subsequence if necessary, we can assume that

$$1_{\Omega \cap U_j}w^j - \delta_k \leq 1_{\Omega \cap U_j}w_j^k \leq 1_{\Omega \cap U_j}w^j$$

in $\mathbb{C}^n$. Therefore,

$$|\tau(w^j_{\delta} - \varepsilon_{\delta}) - \tau(u^k_{\delta} - \varepsilon_{\delta})| \leq 3$$

in $U_j \cap U_k \cap \Omega_k$ for any $k, j = 1, \ldots, m$. Choose $\chi_j \in C^0_0(\mathbb{C}^n)$ satisfying $0 \leq \chi_j \leq 1$, supp $\chi_j \Subset U'_j$ and $\chi_j = 1$ on a neighborhood of $U''_j$. Let $A > 0$ so large that $|z|^2 + A < 0$ on $\Omega_1$ and that $\chi_j(z) + A|z|^2$ is plurisubharmonic in $\mathbb{C}^n$ for every $j = 1, \ldots, m$. Put

$$v^j_{\delta}(z) = \tau(w^j_{\delta}(z) - \varepsilon_{\delta}) + 3(\chi_j(z) + A|z|^2 - A^2 - 1), \quad z \in \Omega_k \cap U_j$$

and

$$v_{\delta}(z) = \max \left\{ \frac{v^j_{\delta}(z)}{\tau(\varepsilon_{\delta})} - 1 : z \in U'_j \right\}.$$
Since \( v^j_h \in PSH(\Omega_h \cap U_j) \) and \( v^j_h \leq v^j_k \) in \( \partial U'_j \cap U''_k \cap \Omega_h \), so \( v_h \) is a negative plurisubharmonic function in \( \Omega_h \cap \bigcup_{j=1}^m U''_j \). Put \( \Omega' = \Omega \cap \bigcup_{j=1}^m U''_j \) and define

\[
v = \left( \sup_{h \geq 1} v_h \right)^*
\]

in \( \Omega' \). Then \( v \in PSH(\Omega') \). We claim that \( v < 0 \) in \( \Omega' \). Indeed, let \( G \subset \Omega' \) be an open set. Choose \( \delta > 0 \) such that \( U'_j \cap G \subset \{ w^j < -\delta \} \cap U'_j \) for any \( j = 1, \ldots, m \). Since \( w^j \leq w_j \) in \( \Omega \cap U_j \) so

\[
v_h(z) \leq \max \left\{ \frac{\tau(u^j_{\varepsilon h})}{\tau(-\varepsilon h)} - 1 : z \in \mathbb{H}'_j \right\}
\]

\[
\leq \frac{\tau(-\delta - \varepsilon h)}{\tau(-\varepsilon h)} - 1
\]

for all \( z \in G \). Hence, \( v < 0 \) in \( G \). This proves the claim. Let \( K \subset \Omega \) be an open subset of \( \Omega \) such that \( \partial K \subset \Omega' \) and \( \Omega \setminus K \subset \Omega' \). Put \( B = \sup_{\partial K} v < 0 \) and define

\[
w = \begin{cases} B & \text{in } K \\ \max(v, B) & \text{in } \Omega \setminus K. \end{cases}
\]

Then \( w \in PSH^-(\Omega) \). We claim that \( w \in \mathcal{N}(\Omega) \). Indeed, let \( \varepsilon > 0 \). Choose \( h \in \mathbb{N} \) such that \( \frac{\lambda(\varepsilon h)}{\frac{3(\varepsilon h)}{2} + 1} < \frac{\delta}{2} \) and \( (1 + \frac{\varepsilon}{h})(1 - \delta) < 1 \). Choose \( \varepsilon' > \varepsilon_h \) such that \( (1 + \frac{\varepsilon}{h})(1 - \delta) < \varepsilon' ) < \varepsilon_h \). Then, we have

\[
\{ w < -\varepsilon \} \cap \Omega \subset \{ \{ v < -\varepsilon \} \cap \Omega' \} \cup K
\]

\[
\subset \bigcup_{j=1}^m \left( \left\{ \frac{v^j_h}{\varepsilon_h} - 1 < -\varepsilon \right\} \cap \Omega \cap U_j \right) \cup K
\]

\[
\subset \bigcup_{j=1}^m \left( \left\{ \frac{\tau(u^j_{\varepsilon h}) - 3(A^2 + 1)}{\tau(-\varepsilon h)} < 1 - \varepsilon \right\} \cap \Omega \cap U_j \right) \cup K
\]

\[
\subset \bigcup_{j=1}^m \left( \left\{ \frac{\tau(u^j_{\varepsilon h})}{\tau(-\varepsilon h)} < 1 - \varepsilon \right\} \cap \Omega \cap U_j \right) \cup K.
\]

Since

\[
\tau(x - \varepsilon_h) \leq \tau(x - \varepsilon - \delta) + \frac{\tau(-\varepsilon_h - \delta - 1)}{h} \leq \left( 1 + \frac{1}{h} \right) \tau(x - \varepsilon - \delta)
\]

for all \( x \in [-1, 0] \), for any \( \delta \in (0, \delta_h] \) so

\[
\left( 1 + \frac{1}{h} \right) \tau(w^j_{\varepsilon h}) \geq \tau(u^j - \varepsilon_h)
\]

in \( \Omega \cap U_j \). Hence,

\[
\{ w < -\varepsilon \} \cap \Omega \subset \bigcup_{j=1}^m \left( \left\{ \frac{\tau(u^j - \varepsilon_h)}{\tau(-\varepsilon h)} < \left( 1 + \frac{1}{h} \right) \left( 1 - \frac{\varepsilon}{2} \right) \right\} \cap \Omega \cap U_j \right) \cup K
\]

\[
\subset \bigcup_{j=1}^m \left( \left\{ \tau(w^j - \varepsilon_h) < \tau(-\varepsilon'_{h_j}) \right\} \cap \Omega \cap U_j \right) \cup K.
\]
\[ \subset \bigcup_{j=1}^{m} \left( \{ w_j < \varepsilon_h - \varepsilon_j' \} \cap \Omega \cap U_j \right) \cup K. \]

Since \( \{ w_j < \varepsilon_h - \varepsilon_j' \} \cap \Omega \cap U_j \subseteq \Omega \) for all \( j = 1, \ldots, m \) so \( \{ w < -\varepsilon \} \cap \Omega \subseteq \Omega \). It follows that \( w \in \mathcal{N}(\Omega) \). This proves the claim. Now put

\[ w_j = \begin{cases} B & \text{in } K \\ \max(v_j, B) & \text{in } \Omega_j \setminus K. \end{cases} \]

Then, \( w_j \in PSH^-(\Omega_j) \) and \( \left( \sup_j w_j \right)^* = w \in \mathcal{N}(\Omega) \). Hence, by Lemma 3.1 we get \( \Omega \) has the \( \mathcal{F} \)-approximation property. The proof is complete. \( \square \)

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