F-INJECTIVITY AND FROBENIUS CLOSURE OF IDEALS IN NOETHERIAN RINGS OF CHARACTERISTIC $p > 0$

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Abstract. The main aim of this article is to study the relation between F-injective singularity and the Frobenius closure of parameter ideals in Noetherian rings of positive characteristic. The paper consists of the following themes, including many other topics.

(1) We prove that if every parameter ideal of a Noetherian local ring of prime characteristic $p > 0$ is Frobenius closed, then it is F-injective.
(2) We prove a necessary and sufficient condition for the injectivity of the Frobenius action on $H^i_{\mathfrak{m}}(R)$ for all $i \leq f_{\mathfrak{m}}(R)$, where $f_{\mathfrak{m}}(R)$ is the finiteness dimension of R. As applications, we prove the following results. (a) If the ring is F-injective, then every ideal generated by a filter regular sequence, whose length is equal to the finiteness dimension of the ring, is Frobenius closed. It is a generalization of a recent result of Ma and which is stated for generalized Cohen-Macaulay local rings. (b) Let $(R, \mathfrak{m}, k)$ be a generalized Cohen-Macaulay ring of characteristic $p > 0$. If the Frobenius action is injective on the local cohomology $H^i_{\mathfrak{m}}(R)$ for all $i < \dim R$, then R is Buchsbaum. This gives an unexpected answer to a question of Takagi. We also prove a recent result of Bhatt, Ma and Schwede with an elementary proof.
(3) We consider the problem when the union of two F-injective closed subschemes of a Noetherian $\mathbb{F}_p$-scheme is F-injective. Using this idea, we construct an F-injective local ring R such that R has a parameter ideal that is not Frobenius closed. This result adds a new member to the family of F-singularities.
(4) We give the first ideal-theoretic characterization of F-injectivity in terms the Frobenius closure and the limit closure. We also give an answer to the question about when the Frobenius action on the top local cohomology is injective.

Dedicated to Prof. Shiro Goto on the occasion of his 70th birthday

Contents

Key words and phrases. F-injective ring, F-pure ring, Frobenius closure, filter regular sequence, generalized Cohen-Macaulay ring, local cohomology, limit closure.

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1. Introduction

In this paper, we study the behavior of the Frobenius closure of ideals generated by a system of parameters (which we call a *parameter ideal*) of a given Noetherian ring containing a field of characteristic $p > 0$. Then we investigate how the *F-injectivity* condition is related to the Frobenius closure of parameter ideals. Recall that a local ring $(R, \mathfrak{m})$ of positive characteristic is F-injective if the natural Frobenius action on all local cohomology modules $H^i_{\mathfrak{m}}(R)$ are injective (cf. [12]). F-injective rings together with F-regular, F-rational and F-pure rings are the main objects of the family of singularities defined by the Frobenius map and they are called the *F-singularities*. F-singularities appear in the theory of *tight closure* (cf. [23] for its introduction), which was systematically introduced by Hochster and Huneke around the mid 80's [21] and developed by many researchers, including Hara, Schwede, Smith, Takagi, Watanabe, Yoshida and others. A recent active research of F-singularities is centered around the correspondence with the singularities of the minimal model program. We recommend [41] as an excellent survey for recent developments. It should be noted that the class of F-injective rings is considered to be the largest among other notable classes of F-singularities.

Under mild conditions of rings, F-regularity, F-rationality and F-purity can be checked by computing either the tight closure, or the Frobenius closure of (parameter) ideals. However, there was no known characterization of F-injectivity in terms of a closure operation of (parameter) ideals. If R is Cohen-Macaulay, Fedder proved that R is F-injective if and only if every parameter ideal is Frobenius closed. More than thirty years later after Fedder's work appeared, Ma extended the Fedder's result for the class of generalized Cohen-Macaulay local rings that is class of rings with finite local cohomology $H^i_{\mathfrak{m}}(R)$ for all $i < \dim R$. Therefore, it is quite natural to ask the following question (cf. [29, Remark 3.6]):

Question 1. *Is it true that a local ring is F-injective if and only if every parameter ideal is Frobenius closed?*

Both authors of the present paper are very interested in the works of Ma in [28] and [29]. We started our join work when the first author was able to prove one direction of the above question. Indeed, this is the first main result of this paper and stated as follows.

Main Theorem A (Theorem 3.7). *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. Assume that every parameter ideal is Frobenius closed. Then R is F-injective.*

However, we prove that the converse of this theorem does not hold true by constructing an explicit example. It should be noted that the example comes from our geometrical consideration of patching two F-injective closed subshemes.

Main Theorem B (Example 6.1, Theorem 6.3 and Corollary 6.4). *There exists an F-finite local ring $(R, \mathfrak{m}, k)$ of characteristic $p > 0$ which is non-Cohen-Macaulay, F-injective, but not F-pure. Moreover, R has a parameter ideal that is not Frobenius closed.*

Thus, the above theorems give an answer to Question 1 in a complete form, and it seems reasonable to call $(R, \mathfrak{m}, k)$ *parameter F-closed* if every parameter ideal of R is Frobenius closed (cf. Definition 6.6). It is true that every F-pure local ring is parameter F-closed and every parameter F-closed ring is F-injective as a consequence of our main theorems. However, our example in Main Theorem B may not be optimal, since the ring is not equidimensional. There is some hope that Question 1 has an affirmative answer when the ring is equidimensional with additional mild conditions. We note that a generalized Cohen-Macaulay ring is equidimensional. Inspired by Ma's work on generalized Cohen-Macaulay rings, we study F-injective rings in connection with the finiteness dimension. Recall that the finiteness dimension of R is defined as

$$f_{\mathfrak{m}}(R) := \inf\{i \mid H^i_{\mathfrak{m}}(R) \text{ is not finitely generated}\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

We have the following theorem.

Main Theorem C (Theorem 4.16). *Let $(R, \mathfrak{m})$ be a reduced F-finite local ring of characteristic $p > 0$ with $f_{\mathfrak{m}}(R) = t$ and let $s \leq t$ be a positive integer. Then the following statements are equivalent.*

1. *The Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i \leq s$.*
2. *Every filter regular sequence $x_1, \ldots, x_s$ of both R and $R^{1/p}/R$ generates a Frobenius closed ideal of R and it is a standard sequence on R.*
3. *Every filter regular sequence $x_1, \ldots, x_s$ of both R and $R^{1/p}/R$ generates a Frobenius closed ideal of R.*

We deduce many interesting results from Main Theorem C. First, we generalize Ma's results [29] in terms of finiteness dimension.

Corollary 1 (Theorem 4.19, Corollary 4.20 and Corollary 4.21). *Let $(R, \mathfrak{m}, k)$ be an F-injective local ring with $f_{\mathfrak{m}}(R) = t$. Then every filter regular sequence of length t is a standard sequence, and every ideal generated by a filter regular sequence of length at most t is Frobenius closed. If R is F-injective and generalized Cohen-Macaulay, then every parameter ideal is Frobenius closed and the R is Buchsbaum.*

The last assertion in the above corollary is Ma's affirmative answer to a question of Takagi. Ma's proof follows from his result on the equivalence between the class of F-injective rings and the class of parameter F-closed rings for a generalized Cohen-Macaulay ring R, together with the result of Goto and Ogawa in [15], who showed that if a parameter F-closed local ring (in our terminology) is generalized Cohen-Macaulay, then it is Buchsbaum. Using Main Theorem C and Goto-Ogawa's argument, we prove the following corollary as an unexpected answer to Takagi's question.

Corollary 2 (Corollary 4.24). *Let $(R, \mathfrak{m})$ be a reduced F-finite generalized Cohen-Macaulay local ring with $d = \dim R$. Suppose the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i < d$. Then R is Buchsbaum.*

Finally, we state a result that gives an ideal-theoretic characterization of F-injectivity via the notion of *limit closure*. The notion of limit closure appears naturally when we consider local cohomology as the direct limit of Koszul cohomology for non-Cohen-Macaulay rings. The limit closure of a sequence of elements $x_1, \ldots, x_t$ in a ring R is defined as follows

$$(x_1, \ldots, x_t)^{\lim} = \bigcup_{n>0} \left((x_1^{n+1}, \ldots, x_t^{n+1}) :_R (x_1 \cdots x_t)^n \right)$$

with convention that $(x_1, \ldots, x_t)^{\lim} = 0$ when $t = 0$. Note that this is indeed an ideal of R. We prove the following theorem.

Main Theorem D (Theorem 7.3 and Theorem 7.5)**.** *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and of dimension $d > 0$. Then we have the following statements.*

1. *The Frobenius action on the top local cohomology $H^d_{\mathfrak{m}}(R)$ is injective if and only if $\mathfrak{q}^F \subseteq \mathfrak{q}^{\lim}$ for all parameter ideals $\mathfrak{q}$.*
2. *R is F-injective if and only if for some (and hence any) filter regular sequence $x_1, \ldots, x_d$, we have*
$$(x_1^n, \ldots, x_t^n)^F \subseteq (x_1^n, \ldots, x_t^n)^{\lim}$$
for all $0 \le t \le d$ and for all $n \ge 1$.

Main Theorem D is not only a generalization of Main Theorem A, but it also helps us better understand Main Theorems B and C. As other side topics, we also discuss problems such as non-F-injective locus and the small Cohen-Macaulay module conjecture. Many questions concerning F-injectivity are addressed in the last section.

The main technique of this paper is to analyze the local cohomology modules by filter regular sequence via the Nagel-Schenzel isomorphism (cf. Lemma 3.6). It is worth noting that the notion of filter regular sequence raised from the theory of generalized Cohen-Macaulay ring in [8] has become a powerful tool in many problems of commutative algebra nowadays. The authors hope that the present paper will shed light on the connection between tight closure theory and non-Cohen-Macaulay rings via techniques developed in this article. The structure of this paper is as follows.

In § 2, we introduce notation and give a brief review on Frobenius closure of ideals, Frobenius action, and the local cohomology modules.

In § 3, We recall some standard results on the Frobenius closure and give complete proofs to them for the convenience of readers. An important fact to keep in mind is that the formation of Frobenius closure of ideals and F-injectivity commute with localization. After that, we recall the notion of filter regular sequence. The reader will find that this notion, together with the Nagel-Schenzel isomorphism, will play a prominent role in proving the injectivity-type results on local cohomology modules under the Frobenius action. The main theorem A will be proven in this section (cf. Theorem 3.7).

In § 4, firstly, we review the definition of generalized Cohen-Macaulay rings and Buchsbaum rings. They form a class wider than that of Cohen-Macaulay rings. The notion of standard parameter ideal plays an important role in the theory of generalized Cohen-Macaulay rings. Then we combine these notions with the notion of the finiteness dimension. We prove Main Theorem C in this section (cf. Theorem 4.16). Among many consequences, we recover Ma's result about generalized Cohen-Macaulay F-injective ring (cf. Corollary 4.20), Corollary 1 (cf. Theorem 4.19), Corollary 2 (cf. Corollary 4.24), and a recent result of Bhatt, Ma and Schwede (cf. Theorem 4.26). Another main result is about the existence of a small Cohen-Macaulay module in positive characteristic under a certain condition (cf. Proposition 4.29).

In § 5, we compare F-injective and F-pure rings. In geometrical setting, we consider the problem when the union of two F-injective closed subschemes is again F-injective (cf. Theorem 5.6). Our result is useful in the construction of certain local rings in characteristic $p > 0$.

In § 6, we construct some interesting examples. The main result in this section is that there exists a local ring that is F-injective with a parameter ideal that is not Frobenius closed (cf. Theorem 6.3). Then we define the parameter F-closed rings as a new member of F-singularities.

In § 7, we prove an ideal-theoretic characterization of F-injectivity using the limit closure (cf. Theorem 7.5). Then we consider the injectivity of the Frobenius action on the top local cohomology (cf. Theorem 7.3). The readers are encouraged to ponder on this characterization to shed light on the previous results (some results of [10] may be helpful).

In § 8, we make a list of some open problems for the future research.

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2. Notation and conventions

In this paper, all rings are (Noetherian) commutative with unity. A *local ring* is a commutative Noetherian ring with the unique maximal ideal $\mathfrak{m}$. Denote a local ring by $(R, \mathfrak{m}, k)$. Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$. We say that an ideal $\mathfrak{q} \subseteq R$ is a *parameter ideal* of M, if $\mathfrak{q}$ is generated by a system of parameters of M. We say that a ring R is *equidimensional*, if $\dim R = \dim R/\mathfrak{p}$ for all minimal primes $\mathfrak{p}$ of R. Let $\underline{x} := x_1, \ldots, x_n$ be a sequence of elements in a ring R. For an R-module M, let $H^i(\underline{x}; M)$ denote the *i-th Koszul cohomology module* of M with respect to $\underline{x}$. We employ the convention that $\dim M = -1$ if M is a trivial module.

Let R be a Noetherian ring containing a field of characteristic $p > 0$. Let $F : R \to R$ denote the Frobenius endomorphism (the p-th power map). We say that a Noetherian ring R is *F-finite* if $F : R \to R$ is module-finite. Let us recall the definition of the Frobenius closure with its properties briefly. Let $I = (x_1, \ldots, x_t)$ be an ideal of R. The *Frobenius closure* of I, denoted by I^F, is defined to be the set of all elements of R satisfying the following property: $u \in I^F$ if and only if $u^q \in I^{[q]}$ for $q = p^e \gg 0$, where $I^{[q]} := (x_1^q, \ldots, x_t^q)$. Indeed, I^F is an ideal of R. If I is a parameter ideal of a local ring R, then so is $I^{[q]}$. Let R be a Noetherian ring with I its proper ideal. Then we denote by $H^i_I(R)$ the *i-th local cohomology module* with support at I (cf. [6] and [25] for local cohomology modules). Recall that local cohomology may be computed as the homology of the Čech complex

$$0 \to R \to \bigoplus_{i=1}^{t} R_{x_i} \to \cdots \to R_{x_1 \ldots x_t} \to 0.$$

If R is a Noetherian ring of characteristic $p > 0$ with an ideal $I = (x_1, \ldots, x_t)$, then we have the Frobenius endomorphism $F : R \to R$. It induces a natural Frobenius action $F_* : H^i_I(R) \to H^i_{I^{[p]}}(R) \cong H^i_I(R)$ (cf. [7] for this map). There is a very useful way of describing the top local cohomology. It can be given as the direct limit of Koszul cohomologies

$$H^t_I(R) \cong \varinjlim R/(x_1^n, \ldots, x_t^n).$$

Then for each $\overline{a} \in H^t_I(R)$, which is the canonical image of $a + (x_1^n, \ldots, x_t^n)$, we find that $F_*(\overline{a})$ is the canonical image of $a^p + (x_1^{pn}, \ldots, x_t^{pn})$.

By a *Frobenius action* on local cohomology modules, we always mean the one as defined above. A local ring $(R, \mathfrak{m}, k)$ is *F-injective* if the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i \geq 0$.

Assume that R is a reduced ring of characteristic $p>0$ with minimal prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_r$. Then for all $e\ge 1$ the map $F^e: R\to R; x\mapsto x^{p^e}$ can be identified with the natural inclusion

$$R\hookrightarrow \prod_{i=1}^r R/\mathfrak{p}_i \hookrightarrow \prod_{i=1}^r \overline{Q(R/\mathfrak{p}_i)},$$

where $\overline{Q(R/\mathfrak{p}_i)}$ is the algebraic closure of the quotient field $Q(R/\mathfrak{p}_i)$ of $R/\mathfrak{p}_i$. We define

$$R^{1/p^e} := \big\{x\in \prod_{i=1}^r \overline{Q(R/\mathfrak{p}_i)} \bigm| x^{p^e}\in R\big\}.$$

It is worth noting that R is F-injective if and only if the inclusion $R\hookrightarrow R^{1/p^e}$ induces the injective map of local cohomology $H^i_{\mathfrak{m}}(R)\hookrightarrow H^i_{\mathfrak{m}}(R^{1/p^e})$ for all $i\ge 0$. We will review definitions and results from generalized Cohen-Macaulay rings in § 4.

3. Frobenius action on local cohomology modules

3.1. **Frobenius closure of ideals.** We collect basic known facts on the Frobenius closure of ideals in a Noetherian ring of characteristic $p>0$ for the convenience of readers.

Lemma 3.1. *Let $(R,\mathfrak{m},k)$ be a local ring of characteristic $p>0$. If there is an element $x\in R$ such that (x^n) is Frobenius closed for all $n>0$, then R is reduced.*

Proof. Assume that we have $u^m=0$ for $m>0$ and $u\in R$. Then we have $u^q\in(x^n)^q$ for all $q=p^e\ge m$ and $u\in(x^n)^F=(x^n)$. Hence $u\in\bigcap_{n>0}(x^n)=(0)$ by Krull's intersection theorem. ☐

Lemma 3.2. *Let $x_1,\dots,x_k$ be a sequence of elements in a Noetherian ring R of characteristic $p>0$ such that $x_1,\dots,x_k$ is a regular sequence in any order and $(x_1,\dots,x_k)$ is Frobenius closed. Then $(x_1^{n_1},\dots,x_k^{n_k})$ is Frobenius closed for all integers $n_1,\dots,n_k\ge 1$.*

Proof. It is enough to prove that $(x_1^{n_1},x_2,\dots,x_k)$ is Frobenius closed for all $n_1\ge 1$. We proceed by induction on n_1. The case $n_1=1$ is trivial. For $n_1>1$, let us take $a\in(x_1^{n_1},x_2,\dots,x_k)^F$. Then $a^q\in(x_1^{n_1},x_2,\dots,x_k)^{[q]}\subseteq(x_1,x_2,\dots,x_k)^{[q]}$ for $q=p^e\gg 0$. Therefore $a\in(x_1,x_2,\dots,x_k)^F=(x_1,x_2,\dots,x_k)$. So $a=b_1x_1+\cdots+b_kx_k$. We have

$$a^q=b_1^qx_1^q+\cdots+b_k^qx_k^q\in(x_1^{n_1},x_2,\dots,x_k)^{[q]}$$

and $b_1^qx_1^q\in(x_1^{n_1},x_2,\dots,x_k)^{[q]}$. Hence $b_1^qx_1^q-cx_1^{n_1q}\in(x_2,\dots,x_k)^{[q]}$ for some c. Since $x_1,\dots,x_k$ is a regular sequence in any order, we have $b_1^q-cx_1^{(n_1-1)q}\in(x_2,\dots,x_k)^{[q]}$ and $b_1^q\in(x_1^{n_1-1},x_2,\dots,x_k)^{[q]}$. Therefore, $b_1\in(x_1^{n_1-1},x_2,\dots,x_k)^F=(x_1^{n_1-1},x_2,\dots,x_k)$ by induction hypothesis on n_1. Hence $a=b_1x_1+\cdots+b_kx_k\in(x_1^{n_1},x_2,\dots,x_k)$, as required. ☐

Lemma 3.3. *Frobenius closure commutes with localization. In particular, localization of a Frobenius closed ideal is Frobenius closed.*

Proof. Let $J\subseteq R$ be an ideal. Then $u^q\in J^{[q]}$ for $q=p^e\gg 0$ if and only if $u\in JR^\infty\cap R$, where R^∞ is the perfect closure of R; R^∞ is the direct limit of $\{R\to R\to R\to\cdots\}$, where $R\to R$ is the Frobenius map. Let $\phi: R\to R^\infty$ be the natural ring map and write $JR^\infty\cap R$ for $\phi^{-1}(JR^\infty)\cap R$ for simplicity. Hence $J^F=JR^\infty\cap R$. Let $S\subseteq R$ be a multiplicative set. Since the localization functor is exact, we have

$$S^{-1}R^\infty\cong(S^{-1}R)^\infty.$$

Then we have

$$S^{-1}(J^F)=S^{-1}(JR^\infty\cap R)=J(S^{-1}R^\infty)\cap S^{-1}R=(S^{-1}J)^F,$$

as claimed. ☐

3.2. **Filter regular sequence.** Let us recall the definition of filter regular sequence. We always assume that a module is finitely generated over a ring.

Definition 3.4. Let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m}, k)$ and let $x_1, \ldots, x_t$ be a sequence of elements in R. Then we say that $x_1, \ldots, x_t$ is a *filter regular sequence* on M if the following conditions hold:

(1) We have $(x_1, \ldots, x_t) \subseteq \mathfrak{m}$.
(2) We have $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R\Big(\dfrac{M}{(x_1, \ldots, x_{i-1})M}\Big) \setminus \{\mathfrak{m}\},\ i = 1, \ldots, t$.

For the existence of filter regular sequence, we refer the reader to [33, Remark 4.5]. Note that the filter regular sequence that we just defined is also called an $\mathfrak{m}$-*filter regular sequence* in other literatures.

Lemma 3.5. *Let M be a finitely generated module over $(R, \mathfrak{m}, k)$. Then $x_1, \ldots, x_t \in \mathfrak{m}$ form a filter regular sequence on M if and only if one of the following conditions holds:*

(1) The quotient
$$\frac{\big((x_1, \ldots, x_{i-1})M :_M x_i\big)}{(x_1, \ldots, x_{i-1})}$$
is a finite length R-module for $i = 1, \ldots, t$.
(2) Fix $i \in \mathbb{N}$ with $1 \le i \le t$. Then the sequence
$$\frac{x_1}{1}, \frac{x_2}{1}, \ldots, \frac{x_i}{1}$$
form a regular sequence in $M_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ such that $(x_1, \ldots, x_i) \subseteq \mathfrak{p}$.
(3) The sequence $x_1^{n_1}, \ldots, x_t^{n_t}$ is a filter regular sequence for all $n_1, \ldots, n_t \ge 1$.

Proof. The proof is found in [32, Proposition 2.2]. □

The following result is very useful in this paper [32, Proposition 3.4].

Lemma 3.6 (Nagel-Schenzel isomorphism)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring and let M be a finitely generated R-module. Let $x_1, \ldots, x_t$ be a filter regular sequence on M. Then we have*
$$H^i_\mathfrak{m}(M) \cong \begin{cases} H^i_{(x_1, \ldots, x_t)}(M) & \text{if } i < t \\ H^{i-t}_\mathfrak{m}(H^t_{(x_1, \ldots, x_t)}(M)) & \text{if } i \ge t. \end{cases}$$

3.3. **Frobenius closed parameter ideals I.** Now we prove the following theorem.

Theorem 3.7. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. Set $d = \dim R$. Then we have the following results:*

(1) Assume that $x_1, \ldots, x_t$ is a filter regular sequence on R such that $(x_1^{p^n}, \ldots, x_t^{p^n})$ is Frobenius closed for all $n \ge 0$. Then the Frobenius action on $H^t_\mathfrak{m}(R)$ is injective.
(2) If every parameter ideal of R is Frobenius closed, then R is F-injective.

Proof. (1): Set $I = (x_1, \ldots, x_t)$. Then $I^{[p^n]}$ is Frobenius closed by assumption. Then we have the following commutative diagram:
$$\begin{array}{ccccccc} R/I & \longrightarrow & R/I^{[p]} & \longrightarrow & R/I^{[p^2]} & \longrightarrow & \cdots \\ {\scriptstyle F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & \\ R/I^{[p]} & \longrightarrow & R/I^{[p^2]} & \longrightarrow & R/I^{[p^3]} & \longrightarrow & \cdots \end{array}$$

where each vertical map is the Frobenius and each map in the horizontal direction is multiplication map by $(x_1\cdots x_t)^{p^e-p^{e-1}}$ in the corresponding spot. The direct limits of both lines are $H^t_I(R)$ and the vertical map is exactly the Frobenius action on $H^t_I(R)$. Since $I^{[p^n]}$ is Frobenius closed, each vertical map is injective. Hence the direct limit map is injective. Therefore, the Frobenius acts injectively on $H^t_I(R)$. To show that Frobenius acts injectively on $H^t_{\mathfrak{m}}(R)$, we need Nagel-Schenzel isomorphism:

$$H^t_{\mathfrak{m}}(R)\cong H^0_{\mathfrak{m}}\big(H^t_I(R)\big).$$

Thus, the Frobenius action on $H^t_{\mathfrak{m}}(R)$ is the direct limit of the following direct system

$$\begin{array}{ccccccc} H^0_{\mathfrak{m}}(R/I) & \longrightarrow & H^0_{\mathfrak{m}}(R/I^{[p]}) & \longrightarrow & H^0_{\mathfrak{m}}(R/I^{[p^2]}) & \longrightarrow & \cdots \\ {\scriptstyle F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & \\ H^0_{\mathfrak{m}}(R/I^{[p]}) & \longrightarrow & H^0_{\mathfrak{m}}(R/I^{[p^2]}) & \longrightarrow & H^0_{\mathfrak{m}}(R/I^{[p^3]}) & \longrightarrow & \cdots \end{array}$$

proving that R is F-injective, as claimed.

(2): There exists a filter regular sequence $x_1,\ldots,x_d$ that is a system of parameters of R by prime avoidance lemma. Let $I=(x_1,\ldots,x_t)$. Then by [29, Lemma 3.1], the ideals I and $I^{[p^n]}$ are both Frobenius closed for $0\le t\le d$ and $n\ge 0$. Using this together with the above discussions, we conclude that Frobenius acts injectively on $H^t_{\mathfrak{m}}(R)$. ☐

Remark 3.8. Let $(R,\mathfrak{m},k)$ be a local ring of characteristic $p>0$ with a regular element $x\in\mathfrak{m}$. Assume that every parameter ideal of R/xR is Frobenius closed. Then we claim that the Frobenius action on $H^t_{\mathfrak{m}}(R)$ is injective, where $t=\operatorname{depth}R$. Indeed, R/xR is F-injective by Theorem 3.7 and the claim follows by [22, Lemma A1]. We will slightly generalize this statement as Corollary 3.14.

Corollary 3.9. *Let $(R,\mathfrak{m},k)$ be a Cohen-Macaulay local ring of characteristic $p>0$. Then the following are equivalent:*

(1) Every parameter ideal of R is Frobenius closed.
(2) There is a parameter ideal of R that is Frobenius closed.
(3) R is F-injective.

Proof. (1) ⇒ (2) is trivial, (2) ⇒ (3) by Theorem 3.7 and Lemma 3.2 and (3) ⇒ (1) by [7, Lemma 10.3.20]. ☐

Remark 3.10. Let $(R,\mathfrak{m},k)$ be a reduced F-finite local ring. Then by a theorem of Kunz [27], R is an excellent ring. Hence the $\mathfrak{m}$-adic completion $\widehat{R}$ is also reduced and F-finite. Since R is F-finite, it is known that it is a homomorphic image of a regular local ring by Gabber [13, Remark 13.6]. Hence R admits a dualizing complex.

Lemma 3.11. *Let $(R,\mathfrak{m},k)$ be an F-injective local ring. Then R is a reduced ring. If furthermore R is F-finite, then $R_{\mathfrak{p}}$ is F-injective for all $\mathfrak{p}\in\operatorname{Spec}R$.*

Proof. Using Γ-construction as in [29, Theorem 3.4], we have a faithfully flat extension:

$$R\to\widehat{R}^{\Gamma}\to S:=\widehat{\widehat{R}^{\Gamma}}.$$

and S is shown to be an F-finite F-injective local ring. If we can show that S is reduced, then R is also reduced. So we may henceforth assume that R is F-finite and F-injective. Note that R has a dualizing complex by Remark 3.10. Then the fact that R is reduced is found in [36, Remark 2.6] and the fact that $R_{\mathfrak{p}}$ is F-injective is found in [35, Proposition 4.3]. ☐

The following is of independent interest.

Proposition 3.12. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring and let*

$$U := \{\mathfrak{p} \in \operatorname{Spec} R \mid R_{\mathfrak{p}} \text{ is } F\text{-injective}\}.$$

Then U is a Zariski open subset of $\operatorname{Spec} R$.

Proof. We may present R as a homomorphic image of a regular local ring S of Krull dimension n. Set $d = \dim R$. Then $R_{\mathfrak{p}}$ is F-injective if and only if the natural map

$$H^i_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \to H^i_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{1/p})$$

is injective for all $i \le \dim R_{\mathfrak{p}}$. Let P be the preimage of $\mathfrak{p}$ in S under the surjection $S \twoheadrightarrow R$. Note that $\dim S_P = n - \dim R/\mathfrak{p}$ (S is a catenary domain). By Grothendieck's local duality theorem, the map

$$\operatorname{Ext}^{n-\dim R/\mathfrak{p}-i}_{S_P}(R_{\mathfrak{p}}^{1/p}, S_P) \to \operatorname{Ext}^{n-\dim R/\mathfrak{p}-i}_{S_P}(R_{\mathfrak{p}}, S_P)$$

is surjective for all $i \le \dim R_{\mathfrak{p}}$. For each $i \le d$, we set

$$C_i := \operatorname{Coker}\big(\operatorname{Ext}^{n-i}_S(R^{1/p}, S) \to \operatorname{Ext}^{n-i}_S(R, S)\big).$$

Therefore, $R_{\mathfrak{p}}$ is F-injective if and only if $\mathfrak{p} \notin \bigcup_{i=0}^{d} \operatorname{Supp}(C_i)$, which is a closed subset in $\operatorname{Spec} R$ since C_i is finitely generated for all $i \le d$. □

Proposition 3.13. *Assume that $(R, \mathfrak{m}, k)$ is an F-finite F-injective local ring. Then every ideal generated by a regular sequence is Frobenius closed.*

Proof. By Remark 3.10, R has a dualizing complex. Let $t = \operatorname{depth} R$ and $d = \dim R$. Since the length of every regular sequence of maximal length is equal to t, it is enough to prove that every ideal generated by a regular sequence $x_1, \dots, x_t$ is Frobenius closed by [29, Lemma 3.1]. We proceed by induction on d. The case $d = 1$ follows from Corollary 3.9, since R is Cohen-Macaulay. For $d > 1$, if $t = d$, then we use Corollary 3.9 again. Therefore we can assume henceforth that $t < d$. Set $I = (x_1, \dots, x_t)$. We have

$$H^t_I(R) \cong \varinjlim R/I^{[q]}.$$

Since I is generated by a regular sequence, all the maps in the direct system are injective. Thus, the natural map $R/I \to H^t_I(R)$ is injective. Suppose $a \in R$ satisfies $a^q \in I^{[q]}$ for some $q = p^e$. Let $\overline{a}$ be the image of $a + I \in R/I$ in $H^t_I(R)$. Then $\overline{a}$ is nilpotent under the Frobenius action on $H^t_I(R)$. Let $N \subseteq R/I$ be the cyclic R/I-module generated by $a \in R$. If $\mathfrak{p} \neq \mathfrak{m}$ is a prime ideal such that $I \not\subseteq \mathfrak{p}$, then $N_{\mathfrak{p}} = 0$ quite evidently. Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal such that $I \subseteq \mathfrak{p}$. Then $R_{\mathfrak{p}}$ is F-injective by Lemma 3.11 and $x_1, \dots, x_t$ is also an $R_{\mathfrak{p}}$-regular sequence. By induction hypothesis, $IR_{\mathfrak{p}}$ is Frobenius closed and we have $a \in IR_{\mathfrak{p}}$, which implies that $N_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$ and N is a finite length R/I-module. That is, $a \in I : \mathfrak{m}^k$ for $k \gg 0$. Note that $H^t_{\mathfrak{m}}(R) \cong H^0_{\mathfrak{m}}(H^t_I(R))$. Therefore $\overline{a} \in H^t_{\mathfrak{m}}(R)$. Since R is F-injective, we have $\overline{a} = 0$. Hence the injectivity of $R/I \to H^t_I(R)$ shows that $a \in I$. □

Corollary 3.14. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and let $I = (x, x_2, \dots, x_t)$ be an ideal of R which is generated by a regular sequence. Assume that R/xR is F-injective. Then the Frobenius action on both $H^t_I(R)$ and $H^t_{\mathfrak{m}}(R)$ are injective.*

Proof. By Proposition 3.13, the ideal $(\overline{x_2}, \dots, \overline{x_t}) \subseteq R/xR$ is Frobenius closed. We prove that $I = (x, x_2, \dots, x_d)$ is Frobenius closed. For this, let $u^q \in (x^q, x_2^q, \dots, x_d^q)$ for $u \in R$ and $q = p^e \gg 0$. Then mapping this relation to the quotient ring R/xR and since $\overline{x_2}, \dots, \overline{x_d}$ forms a system of parameters of R/xR, we have $u \in xR + (x_2, \dots, x_d)$ by assumption, showing that I is Frobenius closed. By Lemma 3.2, the Frobenius power $I^{[p^e]}$ is also Frobenius closed.

We consider the commutative diagram

$$\begin{array}{ccccccc} R/I & \longrightarrow & R/I^{[p]} & \longrightarrow & R/I^{[p^2]} & \longrightarrow & \cdots \\ {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow & & \\ R/I^{[p]} & \longrightarrow & R/I^{[p^2]} & \longrightarrow & R/I^{[p^3]} & \longrightarrow & \cdots \end{array}$$

The vertical map is the Frobenius map and the direct limit of the horizontal direction is the local cohomology $H^t_I(R)$ and the Frobenius action on $H^t_I(R)$ is injective. It is clear that a regular sequence is a filter regular sequence and by Nagel-Schenzel isomorphism:

$$H^t_{\mathfrak{m}}(R) \cong H^0_{\mathfrak{m}}(H^t_I(R)),$$

the Frobenius action on $H^t_{\mathfrak{m}}(R)$ is injective. □

Remark 3.15. Assume that R is a weakly normal Noetherian ring of characteristic p. Then we can show that every principal ideal generated by a regular element $x \in R$ is Frobenius closed. To see this, let $y \in (x)^F$. Then $y^q \in (x^q)$ for some $q = p^e$. Hence we have

$$\Big(\big(\frac{y}{x}\big)^{p^{e-1}}\Big)^p = \big(\frac{y}{x}\big)^{p^e} \in R.$$

Considering this relation to belong to the total ring of fractions of R, we must have $\big(\frac{y}{x}\big)^{p^{e-1}} \in R$ by the definition of weak normality. That is, $y^{p^{e-1}} \in (x^{p^{e-1}})$. By induction on $e \geq 0$, it follows that $y \in (x)$, proving that the principal ideal (x) is Frobenius closed. If $(R, \mathfrak{m}, k)$ is an F-finite F-injective local ring, then R is weakly normal by [35, Theorem 4.7].

Recall that an F-finite F-injective local ring $(R, \mathfrak{m}, k)$ is reduced by Lemma 3.11.

Corollary 3.16. *Let $(R, \mathfrak{m}, k)$ be an F-finite F-injective local ring. Let $x_1, \ldots, x_t$ be a regular sequence of R. Then $x_1, \ldots, x_t$ is a regular sequence of $R^{1/q}/R$.*

Proof. For $i = 0, \ldots, t$, let $I_i = (x_1, \ldots, x_i)$. By Proposition 3.13, I_i is Frobenius closed for all $i \leq t$. Thus we have the following short exact sequence

$$0 \to R/I_iR \to R^{1/q}/I_iR^{1/q} \to (R^{1/q}/R)/I_i(R^{1/q}/R) \to 0.$$

For each $i = 1, \ldots, t$ we consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/I_{i-1}R & \longrightarrow & R^{1/q}/I_{i-1}R^{1/q} & \longrightarrow & (R^{1/q}/R)/I_{i-1}(R^{1/q}/R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_i} & & \downarrow{\scriptstyle x_i} & & \downarrow{\scriptstyle x_i} & & \\ 0 & \longrightarrow & R/I_{i-1}R & \longrightarrow & R^{1/q}/I_{i-1}R^{1/q} & \longrightarrow & (R^{1/q}/R)/I_{i-1}(R^{1/q}/R) & \longrightarrow & 0. \end{array}$$

Since $x_1, \ldots, x_t$ is a regular sequence of both R and $R^{1/q}$ we have the following exact sequence

$$0 \to 0 :_{(R^{1/q}/R)/I_{i-1}(R^{1/q}/R)} x_i \to R/I_iR \to R^{1/q}/I_iR^{1/q} \to (R^{1/q}/R)/I_i(R^{1/q}/R) \to 0.$$

Therefore $0 :_{(R^{1/q}/R)/I_{i-1}(R^{1/q}/R)} x_i = 0$ for all $i = 1, \ldots, t$. Thus $x_1, \ldots, x_t$ is a regular sequence of $R^{1/q}/R$. □

We need the following corollary in the sequel.

Corollary 3.17. *Let $(R, \mathfrak{m}, k)$ be an F-finite F-injective local ring. Let $x_1, \ldots, x_t$ be a filter regular sequence of R. Then $x_1, \ldots, x_t$ is a filter regular sequence of $R^{1/q}/R$.*

Proof. $R_{\mathfrak{p}}$ is a reduced F-finite F-injective local ring for $\mathfrak{p} \in \operatorname{Spec} R$ by Lemma 3.11. The corollary follows from Lemma 3.5 together with Corollary 3.16. □

4. Generalized Cohen-Macaulay rings

Let us recall the definition of generalized Cohen-Macaulay modules. Let $\ell_R(M)$ denote the length of an R-module M. Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$ and let $\mathfrak{q}$ be a parameter ideal of M. We denote by $e(\mathfrak{q}, M)$ the multiplicity of M with respect to $\mathfrak{q}$ (cf. [7] for details).

Definition 4.1. Let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m}, k)$ such that $d = \dim M > 0$. Then M is called *generalized Cohen-Macaulay*, if the difference

$$\ell_R(M/\mathfrak{q}M) - e(\mathfrak{q}, M)$$

is bounded above, where $\mathfrak{q}$ ranges over the set of all parameter ideals of M.

The following characterization of generalized Cohen-Macaulay modules play the key role in this section.

Theorem 4.2. *M is generalized Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M)$ is finitely generated for all $i < d$.*

Remark 4.3. Let the notation be as in Definition 4.1.

(1) Under mild conditions of the base ring, M is generalized Cohen-Macaulay if and only if the non-Cohen-Macaulay locus is isolated, and if and only if every system of parameters forms a filter regular sequence (cf. [8]).
(2) Let M be a generalized Cohen-Macaulay R-module over $(R, \mathfrak{m}, k)$ such that $d = \dim M > 0$. Then
$$\ell_R(M/\mathfrak{q}M) - e(\mathfrak{q}, M) \leq \binom{d-1}{i} \ell_R(H^i_{\mathfrak{m}}(M))$$
for every parameter ideal $\mathfrak{q}$ of M.

4.1. Buchsbaum rings and standard sequence.

Definition 4.4 (cf. [42])**.** Let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m}, k)$ such that $d = \dim M > 0$. A parameter ideal $\mathfrak{q}$ of M is called *standard* if

$$\ell_R(M/\mathfrak{q}M) - e(\mathfrak{q}, M) = \binom{d-1}{i} \ell_R(H^i_{\mathfrak{m}}(M)).$$

We say that M is *Buchsbaum*, if every parameter ideal of M is standard.

We will use the following characterization of standard parameter ideal as its definition (cf. Theorem 4.8).

Theorem 4.5. *A parameter ideal $\mathfrak{q} = (x_1, \ldots, x_d)$ of M is standard if and only if for every $i+j < d$, we have the equality:*

$$\mathfrak{q} \cdot H^i_{\mathfrak{m}}\Big(\frac{M}{(x_1, \ldots, x_j)M}\Big) = 0.$$

Proof. See [42, Theorem 2.5]. □

Remark 4.6. Let M be a generalized Cohen-Macaulay module over $(R, \mathfrak{m}, k)$ such that $d = \dim M > 0$ and let $n \in \mathbb{N}$ be a positive integer such that $\mathfrak{m}^n \cdot H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$. Then every parameter element $x \in \mathfrak{m}^{2n}$ of M admits the splitting property, i.e., $H^i_{\mathfrak{m}}(M/xM) \cong H^i_{\mathfrak{m}}(M) \oplus H^{i+1}_{\mathfrak{m}}(M)$ for all $i < d-1$. Furthermore, every parameter ideal contained in $\mathfrak{m}^{2n}$ is standard (cf. [9]).

Proposition 4.7. *Let M be a generalized Cohen-Macaulay module over $(R,\mathfrak{m},k)$ such that $d = \dim M > 0$ and let $\mathfrak{q} = (x_1,\ldots,x_d)$ be a parameter ideal of M. Then*

$$\ell_R(H^i(\mathfrak{q};M)) := \ell_R(H^i(x_1,\ldots,x_d;M)) \leq \sum_{j=0}^{i} \binom{d}{i-j} \ell_R(H^j_{\mathfrak{m}}(M))$$

for all $i < d$.

Proof. See [40, Proposition 1.4]. □

Theorem 4.8. *Let M be a generalized Cohen-Macaulay module over $(R,\mathfrak{m},k)$ such that $d = \dim M > 0$ and let $\mathfrak{q} = (x_1,\ldots,x_d)$ be a parameter ideal of M. Then the following statements are equivalent.*

(1) $\mathfrak{q}$ is a standard parameter ideal of M.
(2) The canonical map $H^i(\mathfrak{q};M) \to H^i_{\mathfrak{m}}(M)$ is surjective for all $i < d$.
(3) $\ell_R(H^i(\mathfrak{q};M)) = \sum_{j=0}^{i} \binom{d}{i-j} \ell_R(H^j_{\mathfrak{m}}(M))$ for all $i < d$.

Proof. (1) ⇔ (2) follows from [42, Theorem 3.4].

(1) ⇒ (3) is well known in Buchsbaum ring theory (see [40, Corollary 1.6]), as every parameter ideal is standard in a Buchsbaum module.

(3) ⇒ (1), by [7, Theorem 4.7.6] we have

$$\begin{aligned}
\ell_R(M/\mathfrak{q}M) - e(\mathfrak{q},M) &= \sum_{i=0}^{d-1} (-1)^{d-1-i} \ell_R(H^i(\mathfrak{q};M)) \\
&= \sum_{i=0}^{d-1} (-1)^{d-1-i} \sum_{j=0}^{i} \binom{d}{i-j} \ell_R(H^j_{\mathfrak{m}}(M)) \\
&= \sum_{j=0}^{d-1} \left(\sum_{i=j}^{d-1} (-1)^{d-1-i} \binom{d}{i-j} \right) \ell_R(H^j_{\mathfrak{m}}(M)) \\
&= \sum_{j=0}^{d-1} \binom{d-1}{j} \ell_R(H^j_{\mathfrak{m}}(M)).
\end{aligned}$$

Therefore, $\mathfrak{q}$ is standard. □

The advantage of using characterizations of generalized Cohen-Macaulay modules and standard parameter ideals via local cohomology as in Theorems 4.2 and 4.5 is that it allows us to consider the problem in a more general context.

Definition 4.9. (cf. [6, Definition 9.1.3]) Let M be a finitely generated module over a local ring $(R,\mathfrak{m},k)$. The *finiteness dimension $f_{\mathfrak{m}}(M)$ of M with respect to* $\mathfrak{m}$ is defined as follows:

$$f_{\mathfrak{m}}(M) := \inf\{i \mid H^i_{\mathfrak{m}}(M) \text{ is not finitely generated}\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

Remark 4.10. Assume that $\dim M = 0$ or $M = 0$ (recall that a trivial module has dimension -1). In this case, $H^i_{\mathfrak{m}}(M)$ is finitely generated for all i and $f_{\mathfrak{m}}(M)$ is equal to ∞. It will be essential to know when the finiteness dimension is a positive integer. We mention the following result. Let $(R,\mathfrak{m},k)$ be a Noetherian local ring and let M be a finitely generated module. If $d = \dim M > 0$, then the local cohomology module $H^d_{\mathfrak{m}}(M)$ is not finitely generated. For the proof of this result, see [6, Corollary 7.3.3].

Definition 4.11. Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$ such that $t = f_{\mathfrak{m}}(M) < \infty$ and let $x_1, \ldots, x_s$, $s \leq t$, be a filter regular sequence on M. Then we say that $x_1, \ldots, x_s$ is a *standard sequence* of M if

$$(x_1, \ldots, x_s) \cdot H^i_{\mathfrak{m}}\Big(\frac{M}{(x_1, \ldots, x_j)M}\Big) = 0$$

for all $i + j < s$.

Remark 4.12. Let the notation be as in Definition 4.9.

(1) Let M be a finitely generated module over $(R, \mathfrak{m}, k)$ such that $d = \dim M > 0$. Then M is generalized Cohen-Macaulay if and only if $f_{\mathfrak{m}}(M) = d$.
(2) Assume that $f_{\mathfrak{m}}(M) < \infty$. By Grothendieck's finiteness theorem, we have
$$f_{\mathfrak{m}}(M) = \min\{\operatorname{depth}(M_{\mathfrak{p}}) + \dim R/\mathfrak{p} \mid \mathfrak{p} \neq \mathfrak{m}\},$$
provided that R is a homomorphic image of a regular local ring (cf. [6, Theorem 9.5.2]).

The following lemma is the reformulation of Proposition 4.7 in our context (cf. [16, Section 3]).

Lemma 4.13. *Let M be a finitely generated module over $(R, \mathfrak{m}, k)$. Let $t = f_{\mathfrak{m}}(M) < \infty$, let $x_1, \ldots, x_s$, $s \leq t$, be a filter regular sequence on M and let $I = (x_1, \ldots, x_s)$. Then*

$$\ell_R(H^i(I; M)) \leq \sum_{j=0}^{i} \binom{s}{i-j} \ell_R(H^j_{\mathfrak{m}}(M))$$

for all $i < s$.

The following theorem is the reformulation of Theorem 4.8 in our context (cf. [16, Proposition 3.15]).

Theorem 4.14. *Let M be a finitely generated module over $(R, \mathfrak{m}, k)$. Let $t = f_{\mathfrak{m}}(M) < \infty$, let $x_1, \ldots, x_s$, $s \leq t$, be a filter regular sequence on M and let $I = (x_1, \ldots, x_s)$. Then the following statements are equivalent.*

(1) $x_1, \ldots, x_s$ is a standard sequence of M.
(2) The canonical map $H^i(I; M) \to H^i_I(M) \cong H^i_{\mathfrak{m}}(M)$ is surjective for all $i < s$, where the isomorphism is due to Nagel and Schenzel.
(3) $\ell_R(H^i(I; M)) = \sum_{j=0}^{i} \binom{s}{i-j} \ell_R(H^j_{\mathfrak{m}}(M))$ for all $i < s$.

4.2. **Frobenius closed parameter ideals II.** Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring. Using Lemma 3.5 (3) and applying the induction for the sequence

$$0 \to R^{1/p}/R \to R^{1/q}/R \to R^{1/q}/R^{1/p} \to 0,$$

we can prove the following lemma.

Lemma 4.15. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring of characteristic $p > 0$ and let $x_1, \ldots, x_s$ be a filter regular sequence of $R^{1/p}/R$. Then it is also a filter regular sequence of $R^{1/q}/R$ for all $q = p^e$.*

The following is the first main result of this section.

Theorem 4.16. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring of characteristic $p > 0$ with $f_{\mathfrak{m}}(R) = t$ and let $0 < s \leq t$ be an integer. Then the following are equivalent.*

(1) The Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i \leq s$.

(2) Every filter regular sequence $x_1,\dots,x_s$ of both R and $R^{1/p}/R$ generates a Frobenius closed ideal of R and it is a standard sequence on R.

(3) Every filter regular sequence $x_1,\dots,x_s$ of both R and $R^{1/p}/R$ generates a Frobenius closed ideal of R.

Proof. (2) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (1) follows from Theorem 3.7.

For (1) $\Rightarrow$ (2), let $x_1,\dots,x_s$, $s\le t$, be a filter regular sequence of both R and $R^{1/p}/R$. We prove that $x_1,\dots,x_s$ is standard and the ideal $I=(x_1,\dots,x_s)$ is Frobenius closed. Let n_0 be a positive integer such that $\mathfrak{m}^{n_0}H^i_{\mathfrak{m}}(R)=0$ for all $i<s$. For each $x\in\mathfrak{m}$ choose $e\ge 1$ such that $q=p^e>n_0$. We have $F^e_*(xH^i_{\mathfrak{m}}(R))=x^qF^e_*(H^i_{\mathfrak{m}}(R))=0$, where F_* is the natural Frobenius action on the local cohomology. By F-injectivity of R, we have $x\cdot H^i_{\mathfrak{m}}(R)=0$ and so $\mathfrak{m}\cdot H^i_{\mathfrak{m}}(R)=0$ for all $i<s$. By [9, Theorem 1.1, Corollary 4.1], it is easy to see that $x_1^q,\dots,x_s^q$ is a standard sequence of R for all $q=p^e\ge 2$. It is equivalent to say that $x_1,\dots,x_s$ is a standard sequence of $R^{1/q}$. Since R is reduced and F-finite, we get a short exact sequence of finitely generated R-modules:

$$0\to R\to R^{1/q}\to R^{1/q}/R\to 0.$$

Because the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i\le s$, each induced homomorphism $H^i_{\mathfrak{m}}(R)\to H^i_{\mathfrak{m}}(R^{1/q})$ is injective for all $i\le s$. Thus we have short exact sequences

$$0\to H^i_{\mathfrak{m}}(R)\to H^i_{\mathfrak{m}}(R^{1/q})\to H^i_{\mathfrak{m}}(R^{1/q}/R)\to 0$$

for all $i<s$. Therefore

$$\ell_R(H^i_{\mathfrak{m}}(R^{1/q}))=\ell_R(H^i_{\mathfrak{m}}(R))+\ell_R(H^i_{\mathfrak{m}}(R^{1/q}/R))$$

for all $i<s$. By Lemma 4.15, $x_1,\dots,x_s$ is a filter regular sequence of $R^{1/q}/R$. Applying Lemma 4.13 and Theorem 4.14, for all $i<s$ we have

$$\begin{aligned}\ell_R(H^i(I;R)) &\le \sum_{j=0}^{i}\binom{s}{i-j}\ell_R(H^j_{\mathfrak{m}}(R))\\ \ell_R(H^i(I;R^{1/q})) &= \sum_{j=0}^{i}\binom{s}{i-j}\ell_R(H^j_{\mathfrak{m}}(R^{1/q}))\\ \ell_R(H^i(I;R^{1/q}/R)) &\le \sum_{j=0}^{i}\binom{s}{i-j}\ell_R(H^j_{\mathfrak{m}}(R^{1/q}/R)),\end{aligned}$$

where the middle equation follows from the fact that $x_1,\dots,x_s$ is a standard sequence of $R^{1/q}$. On the other hand, by applying Koszul cohomology to the sequence

$$0\to R\to R^{1/q}\to R^{1/q}/R\to 0,$$

we have $\ell_R(H^i(I;R^{1/q}))\le \ell_R(H^i(I;R))+\ell_R(H^i(I;R^{1/q}/R))$ for all $i<s$. Therefore,

$$\begin{aligned}\ell_R(H^i(I;R^{1/q})) &= \ell(H^i(I;R))+\ell_R(H^i(I;R^{1/q}/R)) \quad (\star)\\ \ell_R(H^i(I;R)) &= \sum_{j=0}^{i}\binom{s}{i-j}\ell_R(H^j_{\mathfrak{m}}(R)) \quad (\star\star)\end{aligned}$$

for all $i<s$. By ($\star\star$) and Theorem 4.14 we have $x_1,\dots,x_s$ is a standard sequence of R. Applying ($\star$) to the exact sequence of Koszul cohomology:

$$0\to H^0(I;R)\to H^0(I;R^{1/q})\to H^0(I;R^{1/q}/R)\to\cdots$$

$$\to H^{s-1}(I; R^{1/q}/R) \to H^s(I;R) \to H^s(I; R^{1/q}) \to \cdots$$

we have short exact sequences

$$0 \to H^i(I;R) \to H^i(I;R^{1/q}) \to H^i(I;R^{1/q}/R) \to 0$$

for all $i < s$ and the injection

$$0 \to H^s(I;R) \cong R/I \to H^s(I;R^{1/q}) \cong R^{1/q}/IR^{1/q}$$

for $q = p^e$ with $e > 0$. Thus, I is Frobenius closed. □

Remark 4.17. The condition that $x_1, \ldots, x_s$ is a filter regular sequence of $R^{1/p}/R$ is necessary. To see this, take A to be a reduced and F-finite Cohen-Macaulay local ring which is not F-injective. So there exists a parameter ideal $(x_1, \ldots, x_s)$ of A that is not Frobenius closed. Let $R := A[[x]]$. Then R is Cohen-Macaulay and so $f_{\mathfrak{m}}(R) = \dim R$. Let $s = \dim A = f_{\mathfrak{m}}(R) - 1$. The assumption on the injectivity of the Frobenius is clear since $H^i_{\mathfrak{m}}(R) = 0$ for all $i \le s$. However, $x_1, \ldots, x_s$ is a regular sequence of R of length s, which generates a non-Frobenius closed ideal of R.

For each parameter ideal $\mathfrak{q}$, we can find a system of parameters that is a filter regular sequence of both R and $R^{1/p}/R$ by prime avoidance lemma. In the same spirits of the method in the proof of [39, Chapter I, Proposition 1.9], we have the following lemma.

Lemma 4.18. *Let $(R, \mathfrak{m}, k)$ be a local ring and with $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal and $\ell_R(\mathfrak{a}/\mathfrak{m}\mathfrak{a}) = n$. Let $M_1, \ldots, M_k$ be finitely generated R-modules with $\dim M_i = d$ for $i = 1, \ldots, k$. Then we can choose a set of generators $\{x_1, \ldots, x_n\}$ of $\mathfrak{a}$ such that for any subset J of $\{1, \ldots, n\}$ with $\sharp J = d$, the sequence $\{x_j \mid j \in J\}$ forms a filter regular sequence of all M_i, $1 \le i \le k$, in any order.*

The following theorem is the second main result of this section, which can be seen as a generalization of Proposition 3.13.

Theorem 4.19. *Let $(R, \mathfrak{m}, k)$ be an F-injective local ring with $f_{\mathfrak{m}}(R) = t$. Then every filter regular sequence of length at most t is a standard sequence and the ideal generated by it is Frobenius closed.*

Proof. If $\dim R = 0$, then R is a field by assumption and there is nothing to prove. Therefore, we may assume that $\dim R > 0$ and hence $t < \infty$. It is clear that $H^0_{\mathfrak{m}}(R) = 0$, so $\operatorname{depth} R > 0$. Let $x_1, \ldots, x_s$, $s \le t$, be a filter regular sequence of R. Using Γ-construction as in [29, Theorem 3.4], consider a chain of faithfully flat extensions of local rings of the same Krull dimension:

$$R \to \widehat{R}^{\Gamma} \to S := \widehat{\widehat{R}^{\Gamma}}.$$

Then we have

$$\big((x_1, \ldots, x_{i-1}) :_R x_i)/(x_1, \ldots, x_{i-1})\big) \otimes_R S \cong ((x_1, \ldots, x_{i-1}) :_S x_i)/(x_1, \ldots, x_{i-1})$$

and therefore, $x_1, \ldots, x_s$ is a filter regular sequence on R if and only if so is on S. Likewise, since local cohomology commutes with flat base change, $x_1, \ldots, x_s$ is a standard sequence on R if and only if so is on S. Finally, the ideal $(x_1, \ldots, x_s)$ is Frobenius closed if and only if so is $(x_1, \ldots, x_s)S$. Hence, we may assume that R is an F-finite F-injective complete local ring (the fact that R is reduced follows from Lemma 3.11). Moreover by Corollary 3.17, $x_1, \ldots, x_s$ is a filter regular sequence of $R^{1/q}/R$ for all $q = p^e$. The theorem now follows from Theorem 4.16. □

The following is the main result of [29].

Corollary 4.20. *Let $(R, \mathfrak{m}, k)$ be a generalized Cohen-Macaulay ring of positive characteristic $p > 0$. Then R is F-injective if and only if every parameter ideal is Frobenius closed.*

We recover the following corollary which is the main result of [29], giving an answer to a question of Takagi (cf. [26, Open problem A.3]). It should be noted that the proof of [29] relies on [15], but it follows immediately from our proof of Theorems 4.16 and 4.19.

Corollary 4.21. *Let $(R, \mathfrak{m}, k)$ be a generalized Cohen-Macaulay F-injective local ring. Then R is Buchsbaum.*

Proof. Under the stated assumption, every system of parameters of R is a filter regular sequence by Lemma 3.5. The corollary follows from Theorem 4.19. ☐

4.3. **Frobenius closed parameter ideals III.** Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. In order to show that R is F-injective, it is necessary to consider all local cohomology modules, while it suffices to consider all local cohomology modules except for the top one to check the Buchsbaumness on R. It thus seems natural to pose the following question.

Question 2. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite generalized Cohen-Macaulay local ring. Suppose that the Frobenius acts on $H^i_{\mathfrak{m}}(R)$ injectively for all $i < d$. Then is R Buchsbaum?*

We will work with the finiteness dimension, in which the following lemma plays a role. Its proof is the adoptation of that of [42, Theorem 3.4].

Lemma 4.22. *Let $(R, \mathfrak{m}, k)$ be a local ring and let $\mathfrak{a} \subseteq R$ be an $\mathfrak{m}$-primary ideal. Let $s \leq f_{\mathfrak{m}}(R)$ be an integer. Then the following are equivalent.*

1. *The canonical map $H^i(\mathfrak{a}; R) \to H^i_{\mathfrak{m}}(R)$ is surjective for all $i < s$.*
2. *Every filter regular sequence $x_1, \ldots, x_s \in \mathfrak{a}$ is standard on R.*
3. *The ideal $\mathfrak{a}$ has a set of generators $\{a_1, \ldots, a_n\}$ such that for every subset J of $\{1, \ldots, n\}$ with $\sharp J = s$, we have that $\{a_j \,|\, j \in J\}$ forms a filter regular sequence (in any order) and this sequence is standard.*

As observed from the discussions after Theorem 4.16, it is not true that any filter regular sequence of R of length s generates a Frobenius closed ideal when the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i \leq s$. However, we have the following theorem.

Theorem 4.23. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring with $f_{\mathfrak{m}}(R) = t \geq 1$ and let $s \leq t$ be a positive integer. Suppose that the Frobenius action is injective on $H^i_{\mathfrak{m}}(R)$ for all $i < s$. Then every filter regular sequence of length s of R is standard.*

Proof. Put $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$. By Lemma 4.18, we can choose a set of generators $\{a_1, \ldots, a_n\}$ of $\mathfrak{m}$ such that for every subset J of $\{1, \ldots, n\}$ with $\sharp J = s$, we have that $\{a_j \,|\, j \in J\}$ forms a filter regular sequence (in any order) of both R and $R^{1/p}/R$. By Theorem 4.16, we see that $a_1, \ldots, a_{s-1}$ is a standard sequence and the ideal $(a_1, \ldots, a_i)$ is Frobenius closed for all $i < s$. Using the technique of [29, Proposition 3.3], $a_1, \ldots, a_s$ is also a standard sequence. Now the theorem follows from Lemma 4.22. ☐

We obtain the following corollary as an affirmative answer to Question 2.

Corollary 4.24. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite generalized Cohen-Macaulay local ring with $d = \dim R$. Suppose the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i < d$. Then R is Buchsbaum.*

Note that the Buchsbaumness can be characterized in terms of the canonical map $\mathrm{Ext}^i(k, -) \to H^i_{\mathfrak{m}}(-)$ (cf. [39, Corollary 2.16]) or by the truncated dualizing complex (cf. [34, Theorem 2.3]). We need the following generalization for the finiteness dimension. For the details of the truncated dualizing complex $\tau^{>-s}\omega^{\bullet}_R$ and $\tau^{<s}\mathbf{R}\Gamma_{\mathfrak{m}}(R)$, see [34] and [3].

Lemma 4.25. *Let $(R, \mathfrak{m}, k)$ be a local ring and let $s \le f_{\mathfrak{m}}(R)$ be an integer. Then the following are equivalent.*

(1) Every filter regular sequence $x_1, \ldots, x_s$ of R is standard.
(2) The canonical map $H^i(\mathfrak{m}; R) \to H^i_{\mathfrak{m}}(R)$ is surjective for all $i < s$.

Suppose that there is a surjective ring map $(A, \mathfrak{m}, k) \twoheadrightarrow (R, \mathfrak{m}, k)$ with A regular. Then the above conditions are equivalent to each of the following conditions.

(a) The canonical map $\mathrm{Ext}^i_A(k, R) \to H^i_{\mathfrak{m}}(R)$ is surjective for all $i < s$.
(b) The $\tau^{<s}\mathbf{R}\Gamma_{\mathfrak{m}}(R)$ is quasi-isomorphic to a complex of k-vector spaces.
(c) The $\tau^{>-s}\omega^{\bullet}_R$ is quasi-isomorphic to a complex of k-vector spaces.

Proof. The statements (b) and (c) are dual to each other. The lemma follows from Lemma 4.22, [42, Corollary 3.6] and [3, Lemma 3.7]. □

The following theorem is a recent result of Bhatt, Ma and Schwede in [3, Main Theorem A, Theorem 3.4]. While their proof is based on difficult techniques of Bhatt and Scholze (cf. [4]), our proof is elementary.

Theorem 4.26. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring with $f_{\mathfrak{m}}(R) = t \ge 1$ and let $0 < s \le t$ be an integer. Suppose that the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i < s$. Then $\tau^{>-s}\omega^{\bullet}_R$ is quasi-isomorphic to a complex of k-vector spaces. Equivalently, $\tau^{<s}\mathbf{R}\Gamma_{\mathfrak{m}}(R)$ is quasi-isomorphic to a complex of k-vector spaces. In particular, these truncated complexes split into a direct sum of their cohomologies.*

Proof. Just combine Theorem 4.23 and Lemma 4.25. □

4.4. **The small Cohen-Macaulay module conjecture.** We close this section with a link between generalized Cohen-Macaulay F-injective rings and the small Cohen-Macaulay module conjecture of Hochster.

Conjecture 4.27 (The small Cohen-Macaulay module conjecture)**.** *Let $(R, \mathfrak{m}, k)$ be a complete local domain. Then R admits a non-zero finitely generated module M such that* $\dim R = \operatorname{depth} M$.

Let $(R, \mathfrak{m}, k)$ be a complete local domain with $d = \dim R$. Assume that $d = 2$. Then the small Cohen-Macaulay module conjecture is true by letting M equal the integral closure of R in the field of fractions of R. However, Conjecture 4.27 remains open in higher dimensional case. In the case $d = 3$, Hartshorne, Hochster, and Peskine and Szpiro independently proved that if R is an $\mathbb{N}$-graded finitely generated domain over a perfect field $R_0 = K$ of characteristic $p > 0$, then R has a graded maximal Cohen-Macaulay module [25, pages 224, 225]. In fact, we have the following stronger result (cf. [25, Exercise 21.25]).

Theorem 4.28. *Let R be an $\mathbb{N}$-graded domain, finitely generated over a perfect field $R_0 = K$ of characteristic $p > 0$. Suppose that there is a generalized Cohen-Macaulay graded module M such that* $\dim M = \dim R$. *Then R has a maximal Cohen-Macaulay graded R-module.*

The following result is also proved by Bhatt, Ma and Schwede in [3, Lemma 3.9], independently. In fact, the original idea comes from [18, Proposition 6.3.5].

Proposition 4.29. *Let $(R, \mathfrak{m}, k)$ be a complete local domain of characteristic $p > 0$ with perfect residue field. Suppose that R is F-injective and generalized Cohen-Macaulay. Then for any $q = p^e$ with $e > 0$, $R^{1/q}/R$ is a maximal Cohen-Macaulay R-module.*

Proof. As mentioned above, $R^{1/q}/R$ is a finitely generated R-module. From the proof of Theorem 4.19, we have

$$\ell_R(H^i_{\mathfrak{m}}(R^{1/q})) = \ell_R(H^i_{\mathfrak{m}}(R)) + \ell_R(H^i_{\mathfrak{m}}(R^{1/q}/R))$$

for all $i < d$. The R-length of $H^i_{\mathfrak{m}}(R^{1/q})$ is computed as the R^q-length of $H^i_{\mathfrak{m}}(R)$ via an inclusion $R^q \subset R$. Then since $k = R/\mathfrak{m}$ is a perfect field, we find that any simple R-module is also a simple R^q-module which is isomorphic to k. From this, it follows that $\ell_R(H^i_{\mathfrak{m}}(R^{1/q})) = \ell_R(H^i_{\mathfrak{m}}(R))$. So $H^i_{\mathfrak{m}}(R^{1/q}/R) = 0$ for all $i < d$ and we are done. □

5. F-injective, F-pure and stably FH-finite rings

Definition 5.1. Let R be a Noetherian ring of characteristic $p > 0$. Then R is said to be *F-pure* (resp. *F-split*), if the Frobenius endomorphism $R \to R$ is pure (resp. split).

It is easy to see that F-split rings are F-pure. If R is an F-finite ring, then R is F-pure if and only if R is F-split. We will consider F-pure rings which behave better than F-split rings in the non F-finite case. If R is F-pure, then every ideal is Frobenius closed, and the converse holds true under a mild condition (cf. [19]).

Definition 5.2. Let M be an R-module with a Frobenius action F. A submodule N of M is called *F-compatible* if $F(N) \subseteq N$.

In [28], Ma showed that the local cohomology modules of F-pure local rings satisfy certain interesting conditions originally studied in [11].

Definition 5.3. We say that an R-module M with a Frobenius action F is *anti-nilpotent*, if for any F-compatible submodule N, the induced Frobenius action of F on M/N is injective. We say that $(R, \mathfrak{m}, k)$ is *stably FH-finite*, if the local cohomology $H^i_{\mathfrak{m}}(R)$ are anti-nilpotent for all $i \geq 0$.

For later use, we prove the following lemma. We consider M as an $R\{F\}$-module via the Frobenius action F.

Lemma 5.4. *Let R be a Noetherian ring of characteristic $p > 0$. Then*

(1) Let $0 \to W_1 \to W_2 \to W_3 \to 0$ be a short exact sequence of $R\{F\}$-modules. Then W_2 is anti-nilpotent if and only if so are W_1 and W_3.

(2) If $W_1 \to W_2 \xrightarrow{\alpha} W_3$ is an exact sequence of $R\{F\}$-modules such that W_1 is anti-nilpotent and F acts on W_3 injectively, then F acts on W_2 injectively.

Proof. (1) is clear.

(2) We have a short exact sequence:

$$0 \to \mathrm{Ker}(\alpha) \to W_2 \to \mathrm{Im}(\alpha) \to 0.$$

Then F acts on $\mathrm{Ker}(\alpha)$ injectively, since $\mathrm{Ker}(\alpha)$ is an $R\{F\}$-subquotient of W_1 and W_1 is anti-nilpotent. Moreover, F acts on $\mathrm{Im}(\alpha)$ injectively, since $\mathrm{Im}(\alpha)$ is an $R\{F\}$-submodule of W_3 and F acts on W_3 injectively. Now F acts on W_2 injectively by the five lemma. □

It is clear that a stably FH-finite local ring is F-injective. We need the following result in the sequel (cf. [28, Theorems 2.3 and 3.8]).

Theorem 5.5. *Let $(R, \mathfrak{m}, k)$ be an F-pure local ring. Then the local cohomology modules $H^i_{\mathfrak{m}}(R)$ are anti-nilpotent for all $i \geq 0$ i.e. R is stably FH-finite.*

We prove the main result of this section, which is a variation of [35, Proposition 4.8].

Theorem 5.6. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p. Suppose there exist ideals I, J of R such that $R/(I+J)$ is F-pure, R/I and R/J are F-injective. Then $R/(I \cap J)$ is F-injective.*

Proof. There is a short exact sequence

$$0 \to R/(I \cap J) \xrightarrow{p_1} R/I \oplus R/J \xrightarrow{p_2} R/(I+J) \to 0,$$

where $p_1(a) = (a, -a)$ and $p_2(a,b) = a+b$ and this is compatible with the Frobenius. Taking local cohomology, we get

$$\begin{array}{ccccccc} \longrightarrow H^i_{\mathfrak{m}}(R/(I+J)) & \xrightarrow{h} & H^{i+1}_{\mathfrak{m}}(R/(I\cap J)) & \xrightarrow{g} & H^{i+1}_{\mathfrak{m}}(R/I) \oplus H^{i+1}_{\mathfrak{m}}(R/J) & \longrightarrow \\ \downarrow F_3 & & \downarrow F_1 & & \downarrow F_2 & \\ \longrightarrow H^i_{\mathfrak{m}}(R/(I+J)) & \xrightarrow{h} & H^{i+1}_{\mathfrak{m}}(R/(I\cap J)) & \xrightarrow{g} & H^{i+1}_{\mathfrak{m}}(R/I) \oplus H^{i+1}_{\mathfrak{m}}(R/J) & \longrightarrow \end{array}$$

By assumption, $R/(I+J)$ is F-pure and its local cohomology is anti-nilpotent by Theorem 5.5. That is, the Frobenius action on $\mathrm{Im}(h)$ induced by F_3 is injective. Let $\alpha \in \mathrm{Ker}(F_1)$. First assume that $g(\alpha) \neq 0$. Then $F_2(g(\alpha)) \neq 0$, which is a contradiction. Hence we must have $g(\alpha) = 0$. Then $\alpha \in \mathrm{Im}(h)$. Thus by injectivity of the Frobenius, we have $\alpha = 0$. This proves that the map F_1 is injective. Since this is true for all $i \geq 0$, we find that $R/(I \cap J)$ is F-injective. □

With the same method and Lemma 5.4, we have the following theorem.

Theorem 5.7. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p. Suppose that there exist ideals I, J of R such that R/I, R/J and $R/(I+J)$ are stably FH-finite. Then $R/(I \cap J)$ is stably FH-finite.*

Theorems 5.6 and 5.7 are useful for constructing examples of non-Cohen-Macaulay local rings which are F-injective and not F-pure. Examples of such type do not abound in literatures. We explore examples in the next section.

6. Examples

6.1. **Patching F-injective closed subschemes.** The aim of this section is to give an explicit example of F-injective ring with a parameter ideal that is not Frobenius closed. To do that, we need examples of F-injective rings that are neither F-pure nor generalized Cohen-Macaulay (with "patching" from the previous section). The following is a modification of [38, Example 3.2]. For the sake of readers, we produce all necessary computations.

Example 6.1. Let K be a perfect field of characteristic $p > 0$ and let

$$R := K[[U,V,Y,Z,T]]/(T) \cap (UV, UZ, Z(V - Y^2)).$$

This is an F-finite non-equidimensional local ring of dimension 4. Note that

$$(T) \cap (UV, UZ, Z(V - Y^2)) = (TUV, TUZ, TZ(V - Y^2)).$$

and

$$(UV, UZ, Z(V - Y^2)) = (U, V - Y^2) \cap (Z, UV) = (U, V - Y^2) \cap (Z, U) \cap (Z, V).$$

Let u, v, y and z denote the image of U, V, Y and Z in R (or the quotient ring of R), respectively. The only associated prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ such that $R/\mathfrak{p} = 4$ is (t). Let $a := y^2(u^2 - z^4)$. We find that a is a parameter element of R. Since $a \in (u, z)$, we see that $a \in R$ is a zero divisor. Now we prove that the ideal $(a) \subseteq R$ is not Frobenius closed. We have $zvt = zy^2t$ and $y^{2p}u^{2p} = y^{2p}z^{4p}$ in the ring $R/(a)^{[p]}$ and it follows that

$$(y^3z^4t)^p = y^{3p}z^{4p}t^p = y^{3p-2}y^2z^{4p}t^p = y^{3p-2}vz^{4p}t^p = y^{3p-2}u^{2p}vt^p = 0$$

in $R/(a)^{[p]}$. That is, we have $(y^3z^4t)^p \in (a)^{[p]}$. Next consider the equation

$$Y^3Z^4T = A(Y^2(U^2 - Z^4)) + B(TUV) + C(TUZ) + D(TZ(V - Y^2)$$

in $K[[U,V,Y,Z,T]]$. We have $A = TA'$. Then y dividing this equation out by T, we simply get

$$Y^3Z^4 = A'(Y^2(U^2 - Z^4)) + B(UV) + C(UZ) + D(Z(V - Y^2).$$

Taking this equation modulo (U,V), we have $A' = -Y$ and $Y^3U = B(V) + C(Z) + D'(Z(V - Y^2))$, where $D = UD'$. Thus $Y^3U \in (V,Z)$ and this is impossible. Hence $y^3z^4t \notin (a)$ and (a) is not Frobenius closed. We next prove that R is F-injective. We know that y is a regular element of R and

$$R/yR \cong K[[U,V,Z,T]]/(TUV, TUZ, TZV)$$

Apparently, the ideal (TUV, TUZ, TZV) is generated by square-free monomials and R/yR is F-pure (in the case of F-finite rings, F-pure condition is the same as F-split condition) by [20, Proposition 5.38]. Then by [22, Corollary 4.13], R is F-injective. Hence R is a non-equidimensional local ring which is F-injective, but not F-pure. Since a generalized Cohen-Macaulay local ring is necessarily equidimensional, R is not generalized Cohen-Macaulay.

Based on the previous example, we can construct a local ring that is equidimensional, F-injective but not generalized Cohen-Macaulay and not F-pure.

Example 6.2. Let K be a perfect field of characteristic $p > 0$ and let

$$R := K[[U,V,Y,Z,T,S]]/(T,S) \cap (UV, UZ, Z(V - Y^2)).$$

This is an equidimensional local ring of dimension 4. Since R/yR is F-pure, R is F-injective. To show that R is not F-pure, consider

$$Y^3Z^4T = A(Y^2(U^2 - Z^4)) + B(TUV) + C(TUZ) + D(TZ(V - Y^2)$$

$$+E(SUV) + F(SUZ) + G(SZ(V - Y^2)$$

in the ring $K[[U,V,Y,Z,T,S]]$. Taking this equation modulo (S), we get

$$Y^3Z^4T = A(Y^2(U^2 - Z^4)) + B(TUV) + C(TUZ) + D(TZ(V - Y^2)$$

in $K[[U,V,Y,Z,T]]$. As in the previous example, we see that the ideal $(y^2(u^2 - z^4))$ is not Frobenius closed and thus, R is not F-pure. Set $I = (T,S)$ and $J = (UV, UZ, Z(V - Y^2))$. Moreover, both R/I and R/J are Cohen-Macaulay rings and we have $\dim R/I = \dim R/J = 4$ and $\dim R/(I+J) = 2$. Applying the local cohomology for the short exact sequence

$$0 \to R \to R/I \oplus R/J \to R/(I+J) \to 0,$$

we get the following exact sequence

$$0 \to H^2_{\mathfrak{m}}(R/(I+J)) \to H^3_{\mathfrak{m}}(R) \to H^3_{\mathfrak{m}}(R/I) \oplus H^3_{\mathfrak{m}}(R/J) = 0.$$

Thus $H^3_{\mathfrak{m}}(R) \cong H^2_{\mathfrak{m}}(R/(I+J))$ does not have finite length by Grothendieck's non-vanishing theorem (cf. [6, Corollary 7.3.3]). So R is not generalized Cohen-Macaulay.

The above examples prove the following main result of this section.

Theorem 6.3. *Let the ring R and the notation be as in Example 6.1. Then R is F-injective that has a non-Frobenius closed parameter ideal.*

Proof. Let us construct a parameter ideal of R that is not Frobenius closed. Since (t) is the unique minimal associated prime of R such that $\dim R/(t) = 4$, the element $a = y^2(u^2 - z^4)$ is a parameter element of R. Note that $a \in R$ is a zero divisor, because $a \in (u, z)$. Thus, we may extend it to a full system of parameters a, x_2, x_3, x_4 of R. By the Krull intersection theorem, we have

$$(a) = \bigcap_{n \geq 1} (a, x_2^n, x_3^n, x_4^n).$$

By choosing $n \gg 0$, we find that $b = y^3z^4t$ is not contained in the parameter ideal (a, x_2^n, x_3^n, x_4^n). However, we have $b \in (a)^F \subseteq (a, x_2^n, x_3^n, x_4^n)^F$ as demonstrated in Example 6.1. Hence (a, x_2^n, x_3^n, x_4^n) is not Frobenius closed for $n \gg 0$. ☐

We have the following corollary.

Corollary 6.4. *There exists an F-finite local ring $(R, \mathfrak{m}, k)$ of characteristic $p > 0$ which is non-Cohen-Macaulay, F-injective, but not F-pure. Moreover, R has a parameter ideal that is not Frobenius closed.*

Remark 6.5.
(1) We cannot put the ring R of Example 6.2 in Theorem 6.3 to deduce the same conclusion, because $a \in R$ is a not parameter element in this case. In view of Theorem 4.19 and Corollary 4.21, an example of an F-injective local domain with a parameter ideal that is not Frobenius closed looks tricky to construct. We make a useful comment on the construction of Buchsbaum rings. By [14], it is possible to construct a Buchsbaum ring $(R, \mathfrak{m}, k)$ with $d = \dim R$ such that $\ell_R(H^i_{\mathfrak{m}}(R)) = s_i$, where $s_0, \ldots, s_{d-1}$ is any assigned sequence of non-negative integers.
(2) It is worth noting that Theorem 6.3 also claims that the result of Theorem 4.19 is optimal. Indeed, we see that $\dim R = 4$ and $f_{\mathfrak{m}}(R) = \operatorname{depth}(R) = 3$. By the prime avoidance theorem, we can choose a filter regular sequence x_1, x_2, x_3 (so it is a regular sequence) such that x_1, x_2, x_3, a form a system of parameters of R. Note that x_1, x_2, x_3, a form a filter regular sequence. Using the Krull intersection theorem as in the proof of Theorem 6.3 and Lemma 3.5 (3), we can assume that $b \notin (x_1, x_2, x_3, a)$. Therefore, we find that (x_1, x_2, x_3) is Frobenius closed by Theorem 4.19, but (x_1, x_2, x_3, a) is not Frobenius closed.

6.2. **Parameter F-closed rings.** We introduce a new class of F-singularities.

Definition 6.6. Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. We say that R is a *parameter F-closed ring* if every ideal generated by a system of parameters of R is Frobenius closed.

The deformation property fails for parameter F-closed rings. We are grateful to Linquan Ma for bringing this to our attention and the remark below.

Corollary 6.7. *Let R be the ring as in Example 6.1 with a regular element $y \in \mathfrak{m}$. Then R/yR is parameter F-closed, but R is not parameter F-closed.*

Proof. We have proved that R/yR is F-pure. So every parameter ideal of R/yR is Frobenius closed, but the parameter ideal (a, x_2^n, x_3^n, x_4^n) of R is not so for $n \gg 0$ with notation as in Theorem 6.3. ☐

Remark 6.8. We show that the class of parameter F-closed local rings and the class of stably FH-finite local rings, are not related to each other.

(1) Let us consider the local ring as produced in [11, Example 2.16]. This ring is Cohen-Macaulay and F-injective. Hence it is parameter F-closed. However, its local cohomology modules are not anti-nilpotent.

(2) Let the notation be as in Example 6.1. Here we write $R = S/(I \cap J)$, where $I = (U, V - Y^2)$, $J = (TZ, TUV)$, and $S = k[[U, V, Y, Z, T]]$. Notice that S/I, S/J and $S/(I + J)$ are all F-pure, so they are stably FH-finite. By Theorem 5.7, R is stably FH-finite, while R was shown to be not parameter F-closed.

7. A characterization of F-injectivity via limit closure

In this section, we give a sufficient and necessary condition of F-injectivity via the notion of limit closure which is defined for any Noetherian local ring. For some studies of the limit closure, we refer the reader to [10], [24] and [30].

7.1. **Frobenius action on the top local cohomology.**

Definition 7.1. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d.

(1) The *limit closure* of a sequence of elements $\underline{x} = x_1, \ldots, x_t$ in R is defined as follows:

$$(\underline{x})^{\lim} = \bigcup_{n>0} \left((x_1^{n+1}, \ldots, x_t^{n+1}) :_R (x_1 \cdots x_t)^n \right)$$

with convention that $(\underline{x})^{\lim} = 0$ if $t = 0$.

(2) Let $\mathfrak{q}$ be a parameter ideal of R which is generated by a system of parameters $x_1, \ldots, x_d$. In this case, we put $\mathfrak{q}^{\lim} := (\underline{x})^{\lim}$ and this definition makes sense, because the limit closure is independent of the choice of $x_1, \ldots, x_d$.

Remark 7.2. Let the notation be as in Definition 7.1.

(1) The quotient $(\underline{x})^{\lim}/(\underline{x})$ is the kernel of the canonical map $H^t(\underline{x}; R) \cong R/(\underline{x}) \to H^t_{(\underline{x})}(R)$.

(2) We have $(\underline{x}) = (\underline{x})^{\lim}$ if and only if $\underline{x}$ forms a regular sequence.

(3) It is shown that the Hochster monomial conjecture is equivalent to the claim that $\mathfrak{q}^{\lim} \neq R$ for every parameter ideal $\mathfrak{q}$ of R.

For a sequence $\underline{x} = x_1, \ldots, x_t$ in a ring R, we set $\underline{x}^{[n]} := x_1^n, \ldots, x_t^n$ and let $(\underline{x}^{[n]})$ be the ideal generated by the sequence $\underline{x}^{[n]}$. We study the Frobenius action on the top local cohomology.

Theorem 7.3. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ and let $\underline{x} = x_1, \ldots, x_t$ be a sequence of elements of R such that $(\underline{x}) \subseteq \mathfrak{m}$. Then we have the following statements.*

(1) The Frobenius action on the top local cohomology $H^t_{(\underline{x})}(R)$ is injective if and only if $(\underline{x}^{[n]})^F \subseteq (\underline{x}^{[n]})^{\lim}$ for all $n \geq 1$, where $(\underline{x}^{[n]})^F$ is the Frobenius closure of $(\underline{x}^{[n]})$.

(2) The Frobenius action on the top local cohomology $H^d_{\mathfrak{m}}(R)$ is injective if and only if $\mathfrak{q}^F \subseteq \mathfrak{q}^{\lim}$ for all parameter ideals $\mathfrak{q}$.

Proof. As (2) follows immediately from (1) and the fact that every parameter ideal can be generated by a system of parameters which is a filter regular sequence (cf. [33, Remark 4.5]), it is enough to prove (1). As in the proof of Theorem 3.7, we find that the Frobenius action on $H^t_{(\underline{x})}(R)$ is the direct limit of the following commutative diagram:

$$\begin{array}{ccccccc}
R/(\underline{x}) & \longrightarrow & R/(\underline{x}^{[2]}) & \longrightarrow & R/(\underline{x}^{[3]}) & \longrightarrow & \cdots \\
\downarrow F & & \downarrow F & & \downarrow F & & (\star) \\
R/(\underline{x}^{[p]}) & \longrightarrow & R/(\underline{x}^{[2p]}) & \longrightarrow & R/(\underline{x}^{[3p]}) & \longrightarrow & \cdots
\end{array}$$

where each vertical map is the Frobenius and each map in the horizontal direction is multiplication by $(x_1 \cdots x_t)$ or $(x_1 \cdots x_t)^p$ in the corresponding spot. The above diagram induces the following commutative diagram

$$\begin{array}{ccccccc} R/(\underline{x})^{\lim} & \longrightarrow & R/(\underline{x}^{[2]})^{\lim} & \longrightarrow & R/(\underline{x}^{[3]})^{\lim} & \longrightarrow & \cdots \\ \overline{F}\big\downarrow & & \overline{F}\big\downarrow & & \overline{F}\big\downarrow & & (\star\star) \\ R/(\underline{x}^{[p]})^{\lim} & \longrightarrow & R/(\underline{x}^{[2p]})^{\lim} & \longrightarrow & R/(\underline{x}^{[3p]})^{\lim} & \longrightarrow & \cdots \end{array}$$

where each vertical map $\overline{F}$ is induced by F and every horizontal map is injective.

We first prove the "only if" part. Suppose that there is an element $a \in (\underline{x}^{[n]})^F \subseteq (\underline{x})^F$ such that $a \notin (\underline{x}^{[n]})^{\lim}$ for some n. By the commutative diagram $(\star\star)$, we find that the image $\overline{a}$ of $a + (\underline{x}^{[n]})^{\lim}$ in $H^t_{(\underline{x})}(R)$ is non-zero by the injectivity of the horizontal maps. On the other hand, $a \in (\underline{x})^F$ implies that $\overline{a}$ is a nilpotent element under the Frobenius action. Then it contradicts the injectivity of Frobenius action on $H^t_{(\underline{x})}(R)$.

We next prove the "if" part. Suppose that the Frobenius action F_* on $H^t_{(\underline{x})}(R)$ is not injective. Then there is a non-zero element $\overline{a} \in H^t_{(\underline{x})}(R)$ such that $F_*(\overline{a}) = 0$. Applying the exactness of the direct limit for the diagram $(\star)$, there is an element $a \in R$ together with an integer $n > 0$ such that $\overline{a}$ is the canonical image of $a + (\underline{x}^{[n]})$ and $a + (\underline{x}^{[n]})$ is in the kernel of the map $R/(\underline{x}^{[n]}) \xrightarrow{F} R/(\underline{x}^{[np]})$. Therefore, we have $a^p \in (\underline{x}^{[np]})$ and so $a \in (\underline{x}^{[n]})^F$. However, $\overline{a} \neq 0$ implies that $a \notin (\underline{x}^{[n]})^{\lim}$ and this contradicts the assumption $(\underline{x}^{[n]})^F \subseteq (\underline{x}^{[n]})^{\lim}$. $\square$

Corollary 7.4. *Let $(R, \mathfrak{m}, k)$ be an F-injective local ring of characteristic $p > 0$. Then $\mathfrak{q}^F \subseteq \mathfrak{q}^{\lim}$ for all parameter ideals $\mathfrak{q}$.*

7.2. **F-injectivity and Frobenius closure.** We now obtain an ideal-theoretic characterization of F-injectivity which is a generalization of Theorem 3.7.

Theorem 7.5. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ and of dimension $d > 0$. Then R is F-injective if and only if for some (and hence any) filter regular sequence $x_1, \ldots, x_d$, we have*

$$(x_1^n, \ldots, x_t^n)^F \subseteq (x_1^n, \ldots, x_t^n)^{\lim}$$

for all $0 \le t \le d$ and for all $n \ge 1$.

Proof. We first prove the "if" part. The case $t = 0$ implies that $(0)^F \subseteq (0)$, so R is reduced. There is noting to do with $H^0_{\mathfrak{m}}(R)$. Moreover by Theorem 7.3, the Frobenius action on $H^d_{\mathfrak{m}}(R)$ is injective. For $0 < t < d$, the Frobenius action on $H^t_{(x_1,\ldots,x_t)}(R)$ is injective by Theorem 7.3 again. By Nagel-Schenzel's isomorphism, it follows that $H^t_{\mathfrak{m}}(R) \cong H^0_{\mathfrak{m}}\big(H^t_{(x_1,\ldots,x_t)}(R)\big)$ is an F-compatible submodule of $H^t_{(x_1,\ldots,x_t)}(R)$. Hence R is F-injective.

We next prove the "only if" part. As in the proof of Theorem 4.19, using Γ-construction, we may assume that R is an F-finite F-injective local ring. Since R is assumed to be F-injective, it is reduced. Therefore we have the assertion with $t = 0$. We also have $(x_1^n, \ldots, x_d^n)^F \subseteq (x_1^n, \ldots, x_d^n)^{\lim}$ for all $n \ge 1$ by Theorem 7.3. So let us consider the case $0 < t < d$. We prove that the Frobenius action on $H^t_{(x_1,\ldots,x_t)}(R)$ is injective. Then the assertion follows from Theorem 7.3. Suppose that $\overline{a} \in H^t_{(x_1,\ldots,x_t)}(R)$ is nilpotent under the Frobenius action. For all prime ideals $\mathfrak{p} \neq \mathfrak{m}$ and $(x_1, \ldots, x_t) \subseteq \mathfrak{p}$, we see that $R_{\mathfrak{p}}$ is F-injective by Lemma 3.11. On the other hand, $x_1, \ldots, x_t$ is a regular sequence of $R_{\mathfrak{p}}$ by Lemma 3.5 (2). By Proposition 3.13 and Corollary 3.14 together with their proofs, the Frobenius action on $\big(H^t_{(x_1,\ldots,x_t)}(R)\big)_{\mathfrak{p}} \cong H^t_{(x_1,\ldots,x_t)R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is injective. Hence we have $\mathrm{Supp}_R(R \cdot \overline{a}) =$

$\{\mathfrak{m}\}$. Therefore, $\overline{a} \in H^0_{\mathfrak{m}}(H^t_{(x_1,\ldots,x_t)}(R)) \cong H^t_{\mathfrak{m}}(R)$. Now the F-injectivity of R implies that $\overline{a} = 0$. Thus the Frobenius action on $H^t_{(x_1,\ldots,x_t)}(R)$ is injective. The proof is complete. □

We return to the example considered in the previous section.

Remark 7.6. Le $d = \dim R$. Then the *unmixed component* of R, which is denoted by $U_R(0)$, is defined to be the largest submodule of R of dimension less than d. If $(0) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R)} N(\mathfrak{p})$ is a reduced primary decomposition of the zero ideal, then $U_R(0) = \bigcap_{\dim R/\mathfrak{p}=d} N(\mathfrak{p})$. In [10], Cuong and the first author proved the relation: $U_R(0) = \bigcap_{\mathfrak{q}} \mathfrak{q}^{\lim}$, where $\mathfrak{q}$ runs over all parameters ideals. Now let R be the ring of Example 6.1 and Theorem 6.3 and we keep the notation. Then we find that R is not equidimensional, $U_R(0) = (t)$ and the element $b = u^3z^4t$ is contained in $U_R(0)$. At the time of writing this article, the authors do not have an example of R for which $U_R(0) = (0)$.

Remark 7.7. It is known that F-injective singularities in characteristic $p > 0$ have close connections with Du Bois singularities in characteristic 0. This connection was studied intensively by Schwede in [35], where it was proved that in characteristic 0, those singularities of dense F-injective type are Du Bois. It was conjectured that the converse is also true (see [41]). More recently, this conjecture has been found to be equivalent to a certain conjecture in arithmetic geometry. This conjecture is considered to reflect deep arithmetic nature of the Frobenius action on sheaf cohomology. This was first observed in [31] as a weakened version of the *ordinary varieties* due to Bloch and Kato. Then Bhatt, Schwede and Takagi observed its connection with Du Bois and F-injective singularities and proposed the *weak ordinarity conjecture* in [5]. An interesting observation in [5] is that, using Voevodsky's h-topology and the sheafification of the structure sheaf on the site (a category with a Grothendieck topology) associated to h-topology, Gabber has found sheaf theoretic characterizations for both Du Bois and F-injective singularities. It will be interesting to know how our ideal theoretic characterization of F-injectivity is related to the weak ordinarity conjecture.

8. Open problems

We list some open problems in this section.

8.1. Parameter ideals of F-injective rings.

Problem 1. Let I be a Frobenius closed parameter ideal of a local ring $(R, \mathfrak{m}, k)$ of characteristic $p > 0$. Then is $I^{[q]}$ Frobenius closed for all $q = p^e$?

Problem 1 has an affirmative answer when R is Cohen-Macaulay. It seems to be open even when the ring is assumed to be Buchsbaum.

Problem 2. *Suppose that $(R, \mathfrak{m}, k)$ is an F-injective local ring. Then how does one find Frobenius or non-Frobenius closed parameter ideals?*

Problem 3. *Does there exist an equidimensional local ring of characteristic $p > 0$ that is F-injective, but has a parameter ideal that is not Frobenius closed?*

The authors are suspicious of Problem 3. If this is the case, an equidimensional local ring is F-injective if and only if it is parameter F-closed.

Problem 4 (Deformation problem I)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ with a regular element $x \in \mathfrak{m}$. Assume that R/xR is F-injective. Then is R an F-injective local ring?*

For partial answers related to Problem 4, refer to [22].

Problem 5 (Deformation problem II)**.** *Let $(R, \mathfrak{m}, k)$ be an equidimensional local ring of characteristic $p > 0$ with a regular element $x \in \mathfrak{m}$. Assume that R/xR is parameter F-closed. Then is R a parameter F-closed local ring?*

The example in Corollary 6.7 is not equidimensional, while the authors are still suspicious of Problem 5.

Problem 6 (Deformation problem III)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ with a regular element $x \in \mathfrak{m}$. Assume that R/xR is stably FH-finite. Then is R a stably FH-finite?*

Problem 7. *What about the above problems in the graded case?*

Problem 8. *It was shown that the class of parameter F-closed rings is strictly contained in the class of F-injective local rings. This class contains all F-pure local rings. It then becomes a new member in the family of F-singularities. In view of the correspondence between the singularities of the minimal model program and the singularities defined by Frobenius map, what is the class of the singularities in the minimal model program corresponding to parameter F-closed rings?*

8.2. Localization Problem for F-injective rings.

Definition 8.1. Let R be a Noetherian ring of characteristic $p > 0$. Then we say that R are*F-injective* if $R_{\mathfrak{p}}$ are F-injective for all $\mathfrak{p} \in \operatorname{Spec} R$. Let R be a Noetherian K-algebra, where K is a field of characteristic $p > 0$. Then we say that R is *geometrically F-injective over K* if $R \otimes_K L$ are F-injective for all finite field extensions $K \to L$.

One can find comprehensive discussions and many interesting results around the localization problem in [1] and [2]. We state it for F-injective singularities below.

Problem 9 (Localization Problem)**.** *Let $\phi : (R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ be a flat local homomorphism of F-finite local rings of characteristic $p > 0$. Assume that the closed fiber $S/\mathfrak{m}S$ is geometrically F-injective over K, then does all the fibers of ϕ have F-injective singularities?*

Hashimoto gave an affirmative answer to Problem 7 when the closed fiber is assumed to be Cohen-Macaulay [17, Theorem 5.8]. See [37] for a different proof under the stronger assumption that the closed fiber is Gorenstein and F-pure.

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