

DEFORMATION QUANTIZATION AND VECTOR FIELDS

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Abstract

In this short note I discuss the problem, if it is possible to extend the action of a Lie algebra \mathfrak{g} of Poisson vector fields on a Poisson manifold M, π (here π is the Poisson bivector) to an action of the same Lie algebra on the deformation quantization of the algebra $C^\infty(M)$ by differentiations. In the paper I give a list of cohomological obstructions, (taking values in Hochschild and Lichnerowicz-Poisson cohomology of M), that vanish iff such deformation of \mathfrak{g} exists.

Let A be a Poisson algebra, i.e. a commutative algebra A over a characteristic 0 field (usually, over \mathbb{C} or \mathbb{R} , equipped with an antisymmetric bilinear bracket $\{f, g\}$, $f, g \in A$ (called *Poisson bracket*), which verifies the Jacobi identity and the Leibniz rule with respect to the multiplication. An important particular case of such algebras is given by the algebra of smooth functions on a Poisson manifold M, π , where π is a bivector field on M , such that its Schouten bracket with itself vanishes $[\pi, \pi] = 0$. In this case the Poisson bracket is given by the formula

$$\{f, g\} = \pi(df, dg).$$

The notion of *deformation quantization of a Poisson algebra* (in case, when $A = C^\infty(M)$), one speaks about the *quantization of Poisson Manifold M* has been the subject of an extensive study for the last 30 years, culminating with Kontsevich's theorem in 1997, see [1, ?]. The main problem, solved for the manifolds by Kontsevich, is the existence (and uniqueness up to an equivalence relation) of the deformed associative product $*$ on the algebra of formal power series in \hbar with coefficients in A , such that

$$f * g = fg + \frac{\hbar}{2}\{f, g\} + o(\hbar),$$

where $o(\hbar)$ denotes the terms of quadratic and higher order in \hbar .

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On the other hand, there are many important geometric and algebraic structures, associated with a Poisson manifold and it is natural to ask, if it is possible (and how) to transfer them from the classical algebra $C^\infty(M)$ to the quantized one. In this paper I discuss some important particular cases of this general situation. Namely, we begin with a simple question *whether it is possible to transfer to $\mathcal{A} = (C^\infty(M)[[\hbar]], *)$ a Poisson vector field X* , i.e. a vector field on M , which commutes with the Poisson brackets:

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\},$$

which is equivalent to the equality $L_X\pi = 0$. The transferred object is naturally assumed to be a derivative \mathcal{X} of the latter (noncommutative) algebra, such that

$$\mathcal{X}(f) = X(f) + \sum_{n=1}^{\infty} \hbar^n \mathcal{X}_n(f),$$

for some differential operators \mathcal{X}_n .

As a natural extension of this question, we consider the following problem: let \mathfrak{g} be a Lie algebra, acting on M by Poisson vector fields. We ask, *if it is possible to extend this action to an action of \mathfrak{g} on the noncommutative algebra \mathcal{A} by derivatives*. This question is a bit more complicated than the previous one since now we are obliged to take care not only of the derivative properties of the operator \mathcal{X} , but also of the commutators of the derivatives $[\mathcal{X}, \mathcal{Y}]$ for any two elements $X, Y \in \mathfrak{g}$. In both cases we give an inductive construction, giving an exhaustive list of cohomological obstructions, which vanish iff the problem has a solution.

1 Preliminaries: Hochschild cohomology and deformations

Let M, π be a Poisson manifold. Then we shall always assume that the deformation quantization of its functions algebra is given by the following formal series in \hbar (c.f. [3]):

$$f * g = fg + \frac{\hbar}{2}\{f, g\} + \sum_{k=2}^{\infty} \hbar^k B_k(f, g), \quad (1)$$

for all $f, g \in C^\infty(M)$. Here $B_k(f, g)$ are some linear (over numbers) differential operators of f and g . The associativity condition

$$(f * g) * h = f * (g * h), \quad (2)$$

for all $f, g, h \in C^\infty(M)$ can be extended as a series of partial differential equations on operators B_k . These equations are rather complicated. In order to have a more convenient view on this problem it is better to consider the Hochschild complex version of this equation.

Recall (the book [4] is the main reference for this subject), that for an algebra A , its Hochschild cohomology is defined as the cohomology of the complex

$$C^*(A) = \bigoplus_{n \geq 0} \text{Hom}(A^{\otimes n}, A),$$

with differential

$$\begin{aligned} \delta\varphi(f_1, \dots, f_{p+1}) &= f_1\varphi(f_2, \dots, f_{p+1}) + \sum_{i=1}^p (-1)^i \varphi(f_1, \dots, f_i f_{i+1}, \dots, f_{p+1}) \\ &\quad + (-1)^{p+1} \varphi(f_1, \dots, f_p) f_{p+1}. \end{aligned}$$

Here $\varphi \in C^p(A) = \text{Hom}(A^{\otimes p}, A)$ and $f_1, \dots, f_{p+1} \in A$ are arbitrary elements. In the important particular case, when $A = C^\infty(M)$ for a smooth manifold M , one often reduces this complex to the so-called *local Hochschild cohomology complex* $C_{loc}(C^\infty(M))$, in which the spaces of linear maps $\text{Hom}(A^{\otimes n}, A)$ are replaced with the spaces of *local cochains*, $\text{Hom}_{loc}(A^{\otimes n}, A)$, given by the polydifferential operators on functions; recall, that a map $\varphi : A^{\otimes p} \rightarrow A$ is called polydifferential operator, if for any $k = 1, \dots, p$ and any $f_i \in C^\infty(M)$, $i = 1, \dots, \widehat{k}, \dots, p$ (here and elsewhere the hat $\widehat{}$ over an element means that this element is missing), the map

$$\varphi_k(f) = \varphi(f_1, \dots, f_{k-1}, f, f_{k+1}, \dots, f_p) : C^\infty(M) \rightarrow C^\infty(M)$$

is a linear (over the field) differential operator.

Unlike the usual Hochschild cohomology of $C^\infty(M)$, its local cohomology can be easily calculated, see for instance [5]: the resulting theorem, usually called (a cohomological version of) *Hochschild-Kostant-Rosenberg* theorem, says that the following map induces an isomorphism in cohomology

$$\begin{aligned} \chi : \Gamma(\Lambda^* TM) &\rightarrow C_{loc}^*(C^\infty(M)) \\ \chi(\Phi)(f_1, \dots, f_p) &= \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma \Phi(df_{\sigma(1)}, df_{\sigma(2)}, \dots, df_{\sigma(p)}), \end{aligned}$$

for any polyvector field $\Phi \in \Gamma(\lambda^p TM)$ and any functions $f_1, \dots, f_p \in C^\infty(M)$; here on the left hand side we have zero differential, and the Hochschild differential δ on the right. In particular local Hochschild cohomology of the algebra $C^\infty(M)$ is equal to the space of polyvector fields: $H_{loc}^*(C^\infty(M)) = \Gamma(\Lambda^* TM)$. Below we shall usually omit the adjective local, when speaking about the Hochschild cohomology of the smooth functions of a manifold.

Hochschild complex of an algebra A bears many additional algebraic structures. Two most important of them are the cup-product and Gerstenhaber bracket. The cup-product $C^p(A) \otimes C^q(A) \rightarrow C^{p+q}(A)$ is determined by the formula:

$$(\varphi \cup \psi)(f_1, \dots, f_{p+q}) = \varphi(f_1, \dots, f_p)\psi(f_{p+1}, \dots, f_{p+q}),$$

where $\varphi \in C^p(A)$, $\psi \in C^q(A)$. This is an associative product; differential δ verifies the graded Leibniz rule with respect to this product:

$$\delta(\varphi \cup \psi) = \delta(\varphi) \cup \psi + (-1)^p \varphi \cup \delta(\psi).$$

The Gerstenhaber bracket (c.f. [3]) is a map $[\cdot, \cdot] : C^p(A) \otimes C^q(A) \rightarrow C^{p+q-1}(A)$, determined by the formula

$$[\varphi, \psi] = \sum_{k=1}^p (-1)^{k(q-1)} \varphi \circ_k \psi - (-1)^{(p-1)(q-1)} \sum_{l=1}^q (-1)^{l(p-1)} \psi \circ_l \varphi,$$

where the composition maps \circ_k are defined by the formulas:

$$\varphi \circ_k \psi(f_1, \dots, f_{p+q-1}) = \varphi(f_1, \dots, f_{k-1}, \psi(f_k, \dots, f_{k+q-1}), f_{k+q}, \dots, f_{p+q-1}),$$

i.e. the values of ψ is substituted as an argument into φ . The map $[\cdot, \cdot]$ is skew-symmetric with respect to shifted dimension:

$$[\varphi, \psi] = -(-1)^{(p-1)(q-1)}[\psi, \varphi]$$

and direct computations show that it verifies the graded Jacobi identity

$$[\varphi, [\psi, \omega]] = [[\varphi, \psi], \omega] + (-1)^{(p-1)(q-1)}[\psi, [\varphi, \omega]],$$

or, in more symmetric form:

$$(-1)^{(p-1)(r-1)}[\varphi, [\psi, \omega]] + (-1)^{(q-1)(p-1)}[\psi, [\omega, \varphi]] + (-1)^{(r-1)(q-1)}[\omega, [\varphi, \psi]] = 0$$

for any $\omega \in C^r(A)$. Observe, that if $\mu : A \otimes A \rightarrow A$ is the product map, i.e. if we regard the product in A as an element in $C^2(A)$, then one can define the differential δ by the formula

$$\delta(\varphi) = -[\mu, \varphi],$$

thus it follows from the Jacobi identity that a skew-symmetric version of Leibniz rule holds for δ with respect to the bracket $[\cdot, \cdot]$:

$$\delta[\varphi, \psi] = [\delta\varphi, \psi] + (-1)^{p-1}[\varphi, \delta\psi].$$

It is clear, that these two operations preserve the space of local cochains. One can now improve the statement of the Hochschild-Kostant-Rosenberg theorem as follows: *the product and the bracket in (local) Hochschild cohomology of the algebra $C^\infty(M)$, induced from \cup -product and the bracket $[\cdot, \cdot]$ on the local complex coincide with the wedge-product and the Schouten bracket on polyvector fields respectively.* Recall, that the Schouten bracket is the unique bracket on vector field, that verifies the Leibniz rule with respect to wedge product and is given by the commutator on usual vector fields.

It is now easy to write down the conditions, that guarantee the associativity of the $*$ -product in terms of the operations in Hochschild complex: first of all we interpret the bidifferential operators B_k as elements in $C^2(C^\infty(M))$. To make our notation shorter we shall also put $B_1(f, g) = \frac{1}{2}\{f, g\} = \chi(\pi)$ (here χ denotes the Hochschild-Kostant-Rosenberg antisymmetrization map, see above). One now obtains the equations by comparing the coefficients with the same power in \hbar on both sides of (2) using the definitions of Gerstenhaber bracket and its properties, listed above. The first few equations are

$$\begin{aligned} \delta B_1 &= 0; \\ \delta B_2 &= -\frac{1}{2}([B_1, B_1]) \\ \delta B_3 &= -[B_1, B_2] = -\frac{1}{2}([B_1, B_2] + [B_2, B_1]), \end{aligned} \tag{3}$$

and so on. If we consider the formal power series $B = \sum_{k=1}^{\infty} \hbar^k B_k$ as an element in $C^*(C^\infty(M))[[\hbar]]$ and extend all the operations in Hochschild complex to this module in an evident way, then we can write all these equalities in a rather concise way:

$$\delta B + \frac{1}{2}[B, B] = 0. \tag{4}$$

This equation is usually called *the Maurer-Cartan equation*.

We conclude this section, by recalling, the the existence (and uniqueness up to an equivalence) of the solution of the Maurer-Cartan equation (4) with any first term B_1 given by a Poisson bivector (i.e. $B_1(f, g) = \frac{1}{2}\{f, g\}$), is guaranteed by the well-known Kontsevich's formality theorem, see [1, ?]. In what follows we shall assume, that such a solution B is fixed and denote by B_k its coefficients.

2 From vector field to derivation

We begin with the problem of extending a Poisson vector field to a derivative of the deformed algebra. So let $X = X_0$ be a Poisson vector field on M, π ; this is equivalent to the condition that X verifies Leibniz rule with respect to the Poisson bracket $\{, \}$, i.e.

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}. \quad (5)$$

We are looking for a formal power series operator

$$\mathcal{X} = \sum_{k=0}^{\infty} \hbar^k X_k,$$

where $X_k : C^\infty(M) \rightarrow C^\infty(M)$, $k \geq 1$ are some differential operators (and $X_0 = X$); it is our purpose to find the series \mathcal{J} such, that

$$\mathcal{X}(f * g) = \mathcal{X}(f) * g + f * \mathcal{X}(g). \quad (6)$$

Using the decomposition (1) we can rearrange this graded relation in the form of a series of equations, beginning with:

$$X_0(fg) - fX_0(g) - X_0(f)g = 0,$$

which holds, since X_0 is a vector field;

$$X_1(fg) - fX_1(g) - X_1(f)g = \frac{1}{2}(\{X_0(f), g\} + \{f, X_0(g)\} - X_0(\{f, g\})),$$

which can be easily fulfilled: recall, that X is a differentiation of Poisson bracket (see (5)) so the right hand side vanishes; now it is enough to take an arbitrary vector field as X_1 . Next:

$$\begin{aligned} X_2(fg) - fX_2(g) - X_2(f)g &= \frac{1}{2}(\{X_1(f), g\} + \{f, X_1(g)\} - X_1(\{f, g\})) \\ &\quad + B_2(X_0(f), g) + B_2(f, X_0(g)) - X_0(B_2(f, g)) \end{aligned} \tag{7}$$

The left hand side of this equality is equal to the opposite of Hochschild differential of X_2 . On the other hand, the expression on the right of this formula can be interpreted as the sum of two Gerstenhaber brackets:

$$\begin{aligned} \frac{1}{2}(\{X_1(f), g\} + \{f, X_1(g)\} - X_1(\{f, g\})) &= [B_1, X_1](f, g), \\ B_2(X_0(f), g) + B_2(f, X_0(g)) - X_0(B_2(f, g)) &= [B_2, X_0](f, g). \end{aligned}$$

So, if we apply Hochschild differential to the right hand side of this formula, we shall obtain:

$$\begin{aligned} \delta([B_1, X_1] + [B_2, X_0]) &= [\delta(B_1), X_1] - [B_1, \delta(X_1)] \\ &\quad + [\delta(B_2), X_0] - [B_2, \delta(X_0)] \\ &= [\delta(B_2), X_0], \end{aligned}$$

since X_0 , X_1 and B_1 are Hochschild cocycles (the latter follows from the Leibniz rule for a Poisson bracket; c.f. also the first equality in (3)). On the other hand, since $*$ is an associative product, we have from (3)

$$\delta(B_2) = -\frac{1}{2}[B_1, B_1].$$

Since X_0 is a symmetry of B_1 (the latter being given by Poisson bivector), it follows from Jacobi identity that $[[B_1, B_1], X_0] = 0$; so the right hand side of the last equation vanishes, and we conclude, that the right hand side is a Hochschild cocycle.

Recall, that for a Poisson manifold M , π its *Lichnerowicz-Poisson* cohomology is defined as the cohomology of the complex $\Gamma(\Lambda^*TM)$ with differential d_π . Using the identifications of previous section, we can say, that d_π is equal to the map in Hochschild cohomology, induced by the Gerstenhaber bracket

with $\chi(\pi)$. If we apply this map to the Hochschild cocycle $[B_2, X_0]$, we obtain from Jacobi identity, invariance of π with respect to $X = X_0$ and the Maurer-Cartan equation:

$$[B_1, [B_2, X_0]] = [[B_1, B_2], X_0] = [\delta B_3, X_0] = \delta([B_3, X_0]).$$

The last equality follows from the fact, that X_0 is closed 1-cochain. Thus the following is true:

Proposition 2.1. *One can find the derivative \mathcal{X} up to the second degree in \hbar iff the class of $[B_2, X_0] \in H^2(C^\infty(M))$ belongs to the image of*

$$d_\pi : H^1(C^\infty(M)) \rightarrow H^2(C^\infty(M)), \text{ where } d_\pi(Y) = [\pi, Y],$$

for a vector field $X \in H^1(C^\infty(M))$. Here π is the bivector, which defines the Poisson structure and the brackets on the right denote the Schouten brackets on polyvector fields. Or, more accurately, it vanishes, iff the class of $[B_2, X_0]$ in the Lichnerowicz-Poisson cohomology of M vanishes.

Now we are going to proceed by induction in the powers of \hbar . To make the pattern clear we begin with the next degree: we suppose that X_0 , X_1 and X_2 have been chosen so that equation (6) holds up to the second degree in \hbar . Thus, the first non-zero term, that we should consider is:

$$X_3(fg) - fX_3(g) - X_3(f)g = [B_1, X_2](f, g) + [B_2, X_1](f, g) + [B_3, X_0](f, g). \quad (8)$$

First, we show, that the right hand side of this equation is a Hochschild cocycle. Recall, that

$$\begin{aligned} \delta X_0 &= 0, & \delta X_1 &= 0, & \delta X_2 &= -[B_1, X_1] - [B_2, X_0], \\ \delta B_1 &= 0, & \delta B_2 &= -\frac{1}{2}[B_1, B_1], & \delta B_3 &= -[B_2, B_1]. \end{aligned}$$

So we have, using this and the graded skew symmetry of Gerstenhaber bracket

$$\begin{aligned} \delta([B_1, X_2] + [B_2, X_1] + [B_3, X_0]) &= [B_1, [B_1, X_1]] + [B_1, [B_2, X_0]] \\ &\quad + \frac{1}{2}[X_1, [B_1, B_1]] + [X_0, [B_2, B_1]] \end{aligned}$$

Using the Jacobi identity for Gerstenhaber bracket and its skew symmetry we have:

$$\begin{aligned}\frac{1}{2}[X_1, [B_1, B_1]] + [B_1[B_1, X_1]] &= \frac{1}{2}([X_1, [B_1, B_1]] + 2[B_1, [B_1, X_1]]) \\ &= \frac{1}{2}([B_1, [B_1, X_1]] - [B_1, [X_1, B_1]] + [X_1, [B_1, B_1]]) = 0\end{aligned}$$

Similarly, with the help of Jacobi identity and the fact, that X_0 is a symmetry of the Poisson bracket, we have

$$[B_1, [B_2, X_0]] + [X_0, [B_2, B_1]] = [B_1, [B_2, X_0]] - [B_2, [X_0, B_1]] + [X_0, [B_2, B_1]] = 0.$$

So the claim is true. Further, one can show, that Gerstenhaber bracket of this element with the Poisson bivector is exact with respect to the Hochschild boundary:

$$\begin{aligned}[B_1, [B_1, X_2]] + [B_2, X_1] + [B_3, X_0] &= [B_1, [B_1, X_2]] + [[B_1, B_2], X_1] - [B_2, [B_1, X_1]] + [[B_1, B_3], X_0] \\ &= \frac{1}{2}[[B_1, B_1], X_2] - \frac{1}{2}[\delta B_3, X_1] + [B_2, \delta X_2] \\ &\quad + [B_2, [B_2, X_0]] - [\delta B_4, X_0] + \frac{1}{2}[[B_2, B_2], X_0] \\ &= -([\delta B_2, X_2] - [B_2, \delta X_2]) + \frac{1}{2}([\delta B_3, X_1] + [\delta B_4, X_0]) \\ &\quad + \frac{1}{2}[[B_2, B_2], X_0] + [B_2, [B_2, X_0]] \\ &= \delta([B_2, X_2] + [B_3, X_1] + [B_4, X_0]).\end{aligned}$$

Observe, that we can perturb the last chosen element X_2 by any vector field X' without spoiling its cohomological properties: this will not change its Hochschild coboundary, so the previous equation (7) will not be violated. On the other hand, the element on the right hand side of the equation (8) will be perturbed by a Poisson-exact element $d_\pi X$. Thus, we conclude, that the statement of the theorem remains intact: *the existence of X_3 depends on the triviality of the class of $[B_1, X_2] + [B_2, X_1] + [B_3, X_0]$ in Poisson cohomology.*

Now the general construction is clear: we begin by supposing that the terms X_0, X_1, \dots, X_n have been chosen so, that the equality (6) holds up to degree n in \hbar . Then the following stage is given by an operator X_{n+1} ,

verifying the equality:

$$\delta X_{n+1} = - \sum_{k=1}^{n+1} [B_k, X_{n+1-k}]. \quad (9)$$

Then by inductive hypothesis we have the following differentiation properties:

$$\delta X_k = - \sum_{j=1}^k [B_j, X_{k-j}],$$

and, since the multiplication is associative

$$\delta B_k = - \frac{1}{2} \sum_{i=1}^{k-1} [B_i, B_{k-i}].$$

Using these two equations, we see that the right hand side of equation (9) is a cocycle:

$$\begin{aligned} \delta \left(\sum_{k=1}^{n+1} [B_k, X_{n+1-k}] \right) &= \sum_{k=1}^{n+1} ([\delta B_k, X_{n+1-k}] - [B_k, \delta X_{n+1-k}]) \\ &= - \sum_{k=1}^{n+1} \left(\frac{1}{2} \left[\sum_{i=1}^{k-1} [B_i, B_{k-i}], X_{n+1-k} \right] - [B_k, \sum_{j=1}^{n+1-k} [B_j, X_{n+1-k-j}]] \right) \\ &= \frac{1}{2} \sum_{p+q+r=n+1} ([X_r, [B_p, B_q]] + 2[B_p, [B_q, X_r]]) = 0, \end{aligned}$$

where the last equality follows from Jacobi identity. Further, just like in the case of X_2 we can reduce the question of finding the extensions X_{n+1} , $n \geq 2$ to the same form as for X_2 and X_3 . Namely, observe, that adding a vector field X' to X_n does not change the relation, which determines it (since $\delta X' = 0$). On the other hand, this perturbation turns the right hand side of equation (9) into

$$[B_1, X'] + \sum_{k=1}^{n+1} [B_k, X_{n+1-k}].$$

Both terms, as we know, are closed Hochschild cochains, and the first one (after passing to cohomology) has the form $d_\pi(X')$, where d_π is Lichnerowicz's Poisson cohomology differential. Thus, we conclude:

Proposition 2.2. *One can find a continuation X_{n+1} of the deformed symmetry, if and only if the right hand side of equality (9) gives a trivial element in Lichnerowicz's Poisson cohomology.*

To prove this, we need just to show, that the element on the right is closed with respect to d_π , when we pass to cohomology. But this follows easily from the relations (modulo exact Hochschild cochains):

$$[B_1, X_n] = \sum_{k=2}^{n+1} [B_k, X_{n+1-k}]$$

and

$$[B_1, B_n] = \frac{1}{2} \sum_{i=2}^{k-2} [B_i, B_{k-i}].$$

These are just the relations we gave earlier, where we omit the Hochschild differential (since it in any case shall vanish on the level of cohomology).

2.1 Deformation of a Lie algebra action

Let \mathfrak{g} be a Lie algebra, acting on a Poisson manifold M , i.e. represented in the Lie algebra $D_\pi^1(C^\infty(M))$ of Poisson vector fields on M that is vector fields, commuting with the Poisson bivector π . The question is: is it possible to extend this representation to a representation of \mathfrak{g} by derivations of the quantized algebra? In this section we assume, that the bidifferential operators B_{2k} , $k \geq 1$ are symmetric (in particular, this is the case of the operators, constructed by Kontsevich's formula, see [1]).

In order to answer this question, we consider this map in a generic form

$$\Phi = \varphi_0 + \hbar\varphi_1 + \hbar^2\varphi_2 + \dots : \mathfrak{g} \rightarrow C_{loc}^1(C^\infty(M))[[\hbar]].$$

We need to find Φ such that the conditions above would hold, i.e. that it is a representation of \mathfrak{g} in derivations of $(C^\infty(M)[[\hbar]], *)$. Just like in the previous section, one can start reasoning inductively: we assume, that the 0-degree part of Φ is given by a representation $\varphi_0 : \mathfrak{g} \rightarrow Vect_\pi(M)$ of \mathfrak{g} in the Lie algebra of Poisson vector fields on M , π . It is clear, that this map verifies both conditions (i.e. that its image consist of derivations of the deformed algebra and that it is a representation of \mathfrak{g}) up to degree 1 in parameter \hbar . Then we look for a "correction term" $\varphi_1 : \mathfrak{g} \rightarrow C_{loc}^1(C^\infty(M))$; the map φ_1

should be such, that the sum $\varphi_0 + \hbar\varphi_1$ verifies the above mentioned conditions up to degree 2 in \hbar . So when we restrict our attention to the degrees less, or equal than 2 in \hbar , we obtain the following two equalities:

$$\delta\varphi_1(\xi) = 0, \quad [\varphi_1(\xi), \varphi_0(\eta)] + [\varphi_0(\xi), \varphi_1(\eta)] - \varphi_1([\xi, \eta]) = 0$$

for all elements $\xi, \eta \in \mathfrak{g}$. Here, as before, the $[\cdot, \cdot]$ denote the Gerstenhaber brackets. It follows from the first equality, that φ_1 should take value in Hochschild cocycles. Similarly, the second equation here is equal to the Chevalley differential $\partial_{\mathfrak{g}}(\varphi_1)(\xi, \eta)$ of the map φ_1 viewed as an element of Chevalley complex of \mathfrak{g} with values in in the complex of Hochschild cochains on which \mathfrak{g} acts via the representation φ_0 . Thus, the first stage of deformation can be achieved by choosing an arbitrary 1-cocycle in the complex $C^*(\mathfrak{g}, C^*(C^\infty(M)))$ (zero cocycle can also be a choice).

Now, the next stage gives the following equations on the element φ_2 , the next term in the series $\varphi_0 + \hbar\varphi_1 + \hbar^2\varphi_2 + \dots$:

$$\begin{aligned} \delta(\varphi_2(\xi)) &= [B_1, \varphi_1(\xi)] + [B_2, \varphi_0(\xi)], \\ \partial_{\mathfrak{g}}(\varphi_2)(\xi, \eta) &= [\varphi_1(\xi), \varphi_1(\eta)]. \end{aligned}$$

Once again, this equalities should hold for any $\xi, \eta \in \mathfrak{g}$. The right hand side of the first equation is closed with respect to the Hochschild differential δ (this can be proved by the same calculation as above). It is also closed with respect to the Chevalley differential $\partial_{\mathfrak{g}}$: we put

$$\omega_2^1 = [B_1, \varphi_1] + [B_2, \varphi_0] : \mathfrak{g} \rightarrow C^2(C^\infty(M)),$$

then we compute

$$\begin{aligned} \partial_{\mathfrak{g}}(\omega_2^1)(\xi, \eta) &= \varphi_0(\xi)([B_1, \varphi_1(\eta)] + [B_2, \varphi_0(\eta)]) - \varphi_0(\eta)([B_1, \varphi_1(\xi)] \\ &\quad + [B_2, \varphi_0(\xi)]) - [B_1, \varphi_1([\xi, \eta])] - [B_2, \varphi_0([\xi, \eta])] \\ &= [\varphi_0(\xi), [B_1, \varphi_1(\eta)] + [B_2, \varphi_0(\eta)]] - [\varphi_0(\eta), [B_1, \varphi_1(\xi)]] \\ &\quad - [\varphi_0(\eta), [B_2, \varphi_0(\xi)]] - [B_1, [\varphi_1(\xi), \varphi_0(\eta)]] \\ &\quad - [B_1, [\varphi_0(\xi), \varphi_1(\eta)]] - [B_2, [\varphi_0(\xi), \varphi_0(\eta)]] = 0. \end{aligned}$$

The last equality here follows from the skew-antisymmetry and Jacobi identity. The right hand side of the second equality (which we denote as ω_2^2) is

clearly closed with respect to the Hochschild differential; Chevalley differential $\partial_{\mathfrak{g}}$, applied to it gives:

$$\begin{aligned}\partial_{\mathfrak{g}}(\omega_2^1)(\xi, \eta, \zeta) &= [\varphi_0(\xi), [\varphi_1(\eta), \varphi_1(\zeta)]] - [\varphi_0(\eta), [\varphi_1(\xi), \varphi_1(\zeta)]] \\ &\quad + [\varphi_0, (\zeta)[\varphi_1(\xi), \varphi_1(\eta)]] + [\varphi_1([\xi, \eta]), \varphi_1(\zeta)] \\ &\quad - [\varphi_1([\xi, \zeta]), \varphi_1(\eta)] + [\varphi_1([\eta, \zeta]), \varphi_1(\xi)],\end{aligned}$$

which is equal to 0, because of the Jacobi identity and the assumption, that φ_1 is a Chevalley cocycle. Thus, the sum $\omega_2^1 + \omega_2^2$ is a closed element in the bicomplex $C^*(\mathfrak{g}, C^*(C^\infty(M)))$, the Chevalley complex of \mathfrak{g} with coefficients in the Hochschild complex of $C^\infty(M)$. We need to choose φ_1 so, that the cohomology class of this element is equal to 0. This should be done so, that φ_1 would remain closed with respect to both Hochschild and Chevalley differentials. The first condition means, that we can only add a Chevalley 1-cochain on \mathfrak{g} with values in vector fields on M , while the second condition says, that this correction term should be closed with respect to ∂_g . In other words, we can add to φ_1 an arbitrary Chevalley 1-cocycle $\psi : \mathfrak{g} \rightarrow Vect(M)$.

This correction term changes the first equation for φ_2 by adding a new term of the form $[B_1, \psi(\xi)]$. When we pass to Hochschild homology, this term will turn into the Lichnerowicz's Poisson cohomology differential. Thus, we can interpret the first equation as follows: consider the double complex $C^*(\mathfrak{g}, CP^*(M))$, i.e. the Chevalley complex of \mathfrak{g} with coefficients in the Lichnerowicz's complex of M . Then the element $[B_2, \varphi_0]$ in $C^1(\mathfrak{g}, C^2(C^\infty(M)))$ is closed with respect to both differentials (to see this, just observe, that the terms in its differentials above kill each other, and do not interfere with the differentials of $[B_1, \varphi_1]$), in particular, with respect to the Hochschild differential. Thus, it induces an element in $C^*(\mathfrak{g}, CP^*(M))$, closed with respect to the Chevalley differential. An easy calculation, similar to the computations from the previous sections, shows that d_π vanishes on it too. Thus, it gives an element $\tilde{\omega}_2$ in the bicomplex cohomology, i.e. in $H^3(\mathfrak{g}, CP^*(M))$. Then $\tilde{\omega}_2$ is equal to zero, iff one can find an element $c = c^0 + c^1 + c^2$ in $C^0(\mathfrak{g}, CP^2(M)) \oplus C^1(\mathfrak{g}, CP^1(M)) \oplus C^2(\mathfrak{g}, CP^0(M))$, such that $\partial_{\mathfrak{g}}c + d_\pi c = [B_2, \varphi_0]$. Comparing the bidegrees on both sides, we see, that

$$d_\pi c^0 = 0, \quad \partial_{\mathfrak{g}}c^0 + d_\pi c^1 = [B_2, \varphi_0], \quad \partial_{\mathfrak{g}}c^1 + d_\pi c^2 = 0 \quad \text{and} \quad \partial_{\mathfrak{g}}c^2 = 0.$$

This is a bit less than what one should look for: in fact, we need $c^0 = c^2 = 0$.

In this case we would have

$$d_\pi c^1 = [B_2, \varphi_0], \quad \partial_{\mathfrak{g}} c^1 = 0,$$

where c^1 is a Chevalley 1-cochain on \mathfrak{g} with values in vector fields on M . Taking $\varphi_1 = -c^1$, we conclude, that the Hochschild class of $[B_1, \varphi_1] + [B_2, \varphi_0]$ is equal to zero in this case, hence we can find φ_2 , verifying the equation

$$\delta(\varphi_2(\xi)) = [B_1, \varphi_1(\xi)] + [B_2, \varphi_0(\xi)]. \quad (10)$$

Thus, we should consider the sub-bicomplex

$$\tilde{C}^*(\mathfrak{g}, CP^*(M)) = \bigoplus_{p,q>1} C^p(\mathfrak{g}, CP^q(M)) \subseteq C^*(\mathfrak{g}, CP^*(M)).$$

We conclude, that *there exists an extension φ_2 , verifying the equality (10), if the class of $[B_2, \varphi_0]$ in the cohomology of $\tilde{C}^*(\mathfrak{g}, CP^*(M))$ is equal to 0.*

Let us now suppose, that the equation (10) holds and consider the second equality on φ_2 i.e.

$$\partial_{\mathfrak{g}}(\varphi_2)(\xi, \eta) = [\varphi_1(\xi), \varphi_1(\eta)]. \quad (11)$$

It is easy to see, that the expression on the right hand side is closed with respect to the Hochschild differential. On the other hand, if we apply Hochschild differential to the left hand side, we shall get 0, because

$$\delta(\partial_{\mathfrak{g}}(\varphi_2)) = -\partial_{\mathfrak{g}}(\delta\varphi_2) = -\partial_{\mathfrak{g}}([B_1, \varphi_1(\xi)] + [B_2, \varphi_0(\xi)]) = 0.$$

Thus, we can pass to the Hochschild cohomology on both sides. Consider the corresponding element in $C^*(\mathfrak{g}, CP^*(M))$ (i.e. the difference between the cohomology classes from the left and the right side of equation (11)). Arguing just like in the previous section, one can show that it is closed with respect to both differentials of this complex. On the other hand, we cannot change φ_1 otherwise, but by adding a d_π -closed 1-cocycle on \mathfrak{g} , if we don't want to spoil the equality (10). Thus, the only choice for such correction cocycle is given by the formula $d_\pi(f(\xi))$, where $f : \mathfrak{g} \rightarrow C^\infty(M)$ is a $C^\infty(M)$ -valued 1-dimensional \mathfrak{g} -cocycle (i.e. the value of this map will be in the space of Hamiltonian vector fields on M). This operation will not change the cohomology class of the element in $C^*(\mathfrak{g}, CP^*(M))$ since modulo closed (with respect to Hochschild differential) elements we have

$$[d_\pi f(\xi), \varphi_1(\eta)] = [[B_1, f(\xi)], \varphi_1(\eta)] = -[B_1, \varphi_1(\eta)(f(\xi))] + [f(\xi), [B_2, \varphi_0(\eta)]],$$

where the last term is equal to 0, since B_2 , and hence $[B_2, \varphi_0(\eta)]$ is symmetric bidifferential operator; so

$$[B_2, \varphi_0(\eta)](f(\xi), g) - [B_2, \varphi_0(\eta)](g, f(\xi)) = 0$$

for all g . Similarly, we can change φ_2 only by a 1-cochain $c : \mathfrak{g} \rightarrow CP^1(M)$, i.e. by a cochain with values in vector fields (so that the Hochschild differential of φ_1 remains unchanged).

Thus, we conclude, that the question, whether it is possible to choose φ_2 verifying (10) so that the condition (11) holds, can be reduced to the following: choose arbitrary φ_2 , verifying (10), then consider the difference $\partial_{\mathfrak{g}}\varphi_2 - [\varphi_1, \varphi_1]$ as an element in $H^2(\mathfrak{g}, CP^1(M))$. If this element is trivial, then we can change φ_2 as needed.

Now, we can pass in a similar way to the case $n = 3$: then we have the following two equations

$$\delta\varphi_3(\xi) = [B_1, \varphi_2(\xi)] + [B_2, \varphi_1(\xi)] + [B_3, \varphi_0(\xi)] \quad (12)$$

$$\partial_{\mathfrak{g}}\varphi_3(\xi, \eta) = [\varphi_2(\xi), \varphi_1(\eta)] + [\varphi_2(\eta), \varphi_1(\xi)]. \quad (13)$$

Now we want to make (12) hold without disrupting (10) and (11). This means, that we can change φ_2 only by adding to it a closed 1 \mathfrak{g} -cochain with values in $CP^1(M)$. On the other hand, reasoning as above, we see, that the right hand side of equation (12) is closed with respect to the Hochschild differential δ and (when we pass to the cohomology) with respect to the Poisson differential d_π and Chevalley differential $\partial_{\mathfrak{g}}$. Thus, as before we conclude: *one can choose φ_3 , so that the equality (12) would hold, if the class of the right hand side of this equation in the cohomology of bicomplex $\hat{C}^*(\mathfrak{g}, CP^*(M))$ vanishes.*

Further, as before, changing φ_3 by a $CP^1(M)$ -valued 1 \mathfrak{g} -cochain, we see, that *one can choose φ_3 so, that (13) would hold, if the class of the difference $\partial_{\mathfrak{g}}\varphi_3(\xi, \eta) - [\varphi_2(\xi), \varphi_1(\eta)] + [\varphi_2(\eta), \varphi_1(\xi)]$ in Chevalley cohomology $H^2(\mathfrak{g}, CP^1(M))$ is trivial.*

Finally, reasoning by induction we obtain the following general statement:

Proposition 2.3. *Suppose, that we have found the maps $\varphi_1, \varphi_2, \dots, \varphi_n$ so that the conditions on $\Phi_n = \varphi_0 + \hbar\varphi_1 + \dots + \hbar^n\varphi_n$ hold up to \hbar^n . The one can choose φ_{n+1} , so that for the map $\Phi_n + \hbar^{n+1}\varphi_{n+1}$ the first condition (i.e. that this map is derivation) would hold up to degree $n+1$ in \hbar , if the class of*

$$\omega'_n(\xi) = [B_1, \varphi_n(\xi)] + [B_2, \varphi_{n-1}(\xi)] + \dots + [B_{n+1}, \varphi_0(\xi)]$$

in the cohomology of bicomplex $\tilde{C}^*(\mathfrak{g}, CP^*(M))$ vanishes. Further, one can choose this same φ_{n+1} so, that the second condition (i.e. that this map is a representation of \mathfrak{g}) would also hold up to \hbar^{n+1} , if the class of the element

$$\begin{aligned} \omega_n'' = & \partial_{\mathfrak{g}}\varphi_{n+1}(\xi, \eta) - [\varphi_n(\xi), \varphi_1(\eta)] - [\varphi_{n-1}(\xi), \varphi_2(\eta)] - \cdots - [\varphi(\xi), \varphi_1(\eta)] \\ & + [\varphi_n(\eta), \varphi_1(\xi)] - [\varphi_{n-1}(\eta), \varphi_2(\xi)] - \cdots - [\varphi(\eta), \varphi_1(\xi)] \end{aligned}$$

in Chevalley cohomology $H^2(\mathfrak{g}, CP^1(M))$ is trivial.

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