Multivariate time series

VAR processes

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Strict stationarity

Let consider the process $(X_t)_{t\in\mathbb{Z}}$ with $X_t = (X_{1,t}, \ldots, X_{d,t})^\top \in \mathbb{R}^d$, $d \in \mathbb{N}^*$.
$(X_t)_{t\in\mathbb{Z}}$ is said to be strictly stationary if $\forall k \in \mathbb{N}^*, \forall (t_1, \ldots, t_k) \in \mathbb{Z}^k, \forall h \in \mathbb{Z}$:

$$d\left(X_{t_1}, \ldots, X_{t_k}\right) = d\left(X_{t_1+h}, \ldots, X_{t_k+h}\right)$$

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Stationarity

A second order process $(X_t)_{t\in\mathbb{Z}}$ is said to be (weakly) stationary if its expectation $\mathbb{E}(X_t)$ and its covariances $\mathrm{Cov}(X_s, X_t)$ are time shifted invariant.
In this case we write:

- $\forall t \in \mathbb{Z} : \mathbb{E}(X_t) = \mu \ (\in \mathbb{R}^d)$.
- $\forall h \in \mathbb{Z}$:

$$\begin{aligned}\Gamma(h) &= \mathrm{Cov}(X_t, X_{t-h}) \\ &= \mathbb{E}\left[(X_t - \mu)(X_{t-h} - \mu)^\top\right] \\ &= \mathbb{E}\left(X_t X_{t-h}^\top\right) - \mu\mu^\top.\end{aligned}$$

$\gamma_{ij}(h)$ refers to the (i,j)-th element of the matrix $\Gamma(h)$.

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Autocorrelations

Autocorrelations of a stationary process $(X_t)_{t\in\mathbb{Z}}$ are defined by:

$$\forall h \in \mathbb{Z} : \rho(h) = \mathrm{Corr}\left(X_t, X_{t-h}\right) = D^{-1}\Gamma(h)D^{-1}$$

where $D = \mathrm{diag}\left(\sqrt{\gamma_{11}(0)}, \ldots, \sqrt{\gamma_{dd}(0)}\right)$ (vector of the standard deviations of the components of X_t).
$\rho_{ij}(h)$ refers to the (i,j)-th element of the matrix $\rho(h)$.
Each component $(X_{i,t})_{t\in\mathbb{Z}}$ has $\gamma_{ii}(h)$ as autocovariance function and $\rho_{ii}(h)$ as autocorrelation function:

$$\forall h \in \mathbb{Z} : \gamma_{ii}\left(-h\right) = \gamma_{ii}\left(h\right),$$
$$\rho_{ii}\left(-h\right) = \rho_{ii}(h).$$

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Crossed covariances and crossed correlations 1/2

The crossed covariances and crossed correlations of $(X_{i,t})_{t\in\mathbb{Z}}$ and $(X_{j,t})_{t\in\mathbb{Z}}$ are:

$$\gamma_{ij}(h) = \mathrm{Cov}\left(X_{i,t}, X_{j,t-h}\right),$$
$$\rho_{ij}(h) = \mathrm{Corr}\left(X_{i,t}, X_{j,t-h}\right).$$

These quantities aren't symmetric in general.

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Crossed covariances and crossed correlations 2/2

If there exists a value $h \in \mathbb{N}^*$ such that $\gamma_{ij}(h) \neq 0$ then $(X_{j,t})_{t\in\mathbb{Z}}$ causes $(X_{i,t})_{t\in\mathbb{Z}}$.

If there exists a value $h \in \mathbb{N}^*$ such that $\gamma_{ij}(-h) \neq 0$ then we say $(X_{i,t})_{t\in\mathbb{Z}}$ causes $(X_{j,t})_{t\in\mathbb{Z}}$.

The two situations can coexist.

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Proposition

If $(X_t)_{t\in\mathbb{Z}}$ is stationary then the univariate processes $(X_{1t})_{t\in\mathbb{Z}}, \ldots, (X_{dt})_{t\in\mathbb{Z}}$ are also stationary.

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Propertie

We have:

$$
\begin{aligned}
\Gamma_X(-h) &= \mathbb{E}\left[(X_t-\mu)(X_{t+h}-\mu)^\top\right] \\
&= \left(\mathbb{E}\left[(X_{t+h}-\mu)^\top(X_t-\mu)\right]\right)^\top \\
&= \left(\mathbb{E}\left[(X_{t+h-h}-\mu)^\top(X_{t-h}-\mu)\right]\right)^\top \\
&= \left(\mathbb{E}\left[(X_t-\mu)^\top(X_{t-h}-\mu)\right]\right)^\top \\
&= \Gamma_X(h)^\top.
\end{aligned}
$$

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Weakly white noise

$(\varepsilon_t)_{t\in\mathbb{Z}}$ is said to be a (weakly) white noise with variance-covariance matrix Ω and we write $\varepsilon_t \sim WN(0,\Omega)$ if:

- $\forall t \in \mathbb{Z} : \mathbb{E}(\varepsilon_t) = 0$,
- $\Gamma_\varepsilon(0) = \Omega$,
- $\forall h \in \mathbb{Z}^* : \Gamma_\varepsilon(h) = 0$.

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Moments estimation

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Based on $(X_1, \ldots, X_T)$, we define the estimators:

$$\widehat{\mu} = \overline{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$$

and

$$\forall h \in \mathbb{Z} : \widehat{\Gamma}(h) = \frac{1}{T-h}\sum_{t=h+1}^{T} \left(X_t - \overline{X}_T\right)\left(X_{t-h} - \overline{X}_T\right)^\top,$$

$$\widehat{\rho}(h) = \widehat{D}^{-1}\widehat{\Gamma}(h)\widehat{D}^{-1}.$$

These estimators are consistent (ergodic theory).

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Wold decomposition

If $(X_t)_{t\in\mathbb{Z}}$ is a regular stationary process then there exists a unique decomposition (named Wold decomposition):

$$\forall t \in \mathbb{Z} : X_t = \varepsilon_t + \sum_{i=1}^{+\infty} \|\Psi_i\| \, \varepsilon_{t-i} + V_t$$

where:

- $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise,
- $(V_t)_{t\in\mathbb{Z}}$ is a singular process uncorrelated to $(X_t)_{t\in\mathbb{Z}}$,
- $\sum_{i=1}^{+\infty} \|\Psi_i\| < +\infty$, with for example the Frobenius norm: $\|\Psi_i\|_F^2 = \mathrm{Tr}\left(\Psi_i \Psi_i^\top\right)$.

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Definition

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise with Ω as variance-covariance matrix.
$(X_t)_{t\in\mathbb{Z}}$ is said to be a VAR process of order p, written VAR(p), if:

$$\forall t \in \mathbb{Z} : X_t = \nu + \sum_{i=1}^{p} \Phi_i X_{t-i} + \varepsilon_t$$

where $(\Phi_1, \ldots, \Phi_p)$ are p matrixes of dimensions $d \times d$ and $\nu \in \mathbb{R}^d$.

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Notation

One can write:

$$\left(I - \sum_{i=1}^{p} \Phi_i B^i\right) X_t = \nu + \varepsilon_t$$

that is:

$$\mathbf{\Phi}(B) X_t = \nu + \varepsilon_t$$

where:

$$\mathbf{\Phi}(B) = I - \sum_{i=1}^{p} \Phi_i B^i.$$

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Stability condition

If:

$$\forall z \,/|z| \leq 1 : \det\left(\mathrm{I}_d - \sum_{i=1}^{p} \Phi_i z^i\right) \neq 0,$$

then the process $(X_t)_{t\in\mathbb{Z}}$ is stationary, canonical and ergodic. Then $(\varepsilon_t)_{t\in\mathbb{Z}}$ is the innovation process:

$$\forall h \in \mathbb{N}^* : \mathrm{Cov}(\varepsilon_t, X_{t-h}) = 0.$$

We consider here that the stability condition is checked.

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VAR(1) case 1/2

Consider a centered VAR(1) process:

$$\forall t \in \mathbb{Z} : X_t = \Phi X_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \Omega)$.
We have for $(s, t) \in \mathbb{Z}^2$ such that $s < t$:

$$\begin{aligned} X_t &= \Phi X_{t-1} + \varepsilon_t \\ &= \Phi^2 X_{t-2} + \Phi \varepsilon_{t-1} + \varepsilon_t \\ &\vdots \\ &= \Phi^s X_{t-s} + \sum_{i=0}^{s-1} \Phi^i \varepsilon_{t-i}. \end{aligned}$$

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VAR(1) case 2/2

In order to have independency between X_t and X_{t-s} when $s \to +\infty$, one must have:

$$\Phi^s \xrightarrow{s \to +\infty} 0.$$

The eigenvalues λ of Φ must be inside the unit circle. Then:

$$\forall \lambda \,/ |\lambda| \geq 1 : \det\left(\lambda \, \mathrm{I}_d - \Phi\right) \neq 0$$

that is:

$$\forall z \,/ |z| \leq 1 : z^{-d} \det\left(\mathrm{I}_d - z\Phi\right) \neq 0$$

with $z = \frac{1}{\lambda}$.

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Mean

We have:

$$\mu = \nu + \sum_{i=1}^{p} \Phi_i \mu + 0$$

that is:

$$\mu = \left(\mathsf{I}_d - \sum_{i=1}^{p} \Phi_i \right)^{-1} \nu.$$

One can write:

$$X_t - \mu = \sum_{i=1}^{p} \Phi_i \left(X_{t-i} - \mu \right) + \varepsilon_t.$$

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Autocovariances 1/2

Assume here $\mu = 0$ (centered process).
We have:

$$X_t X_t^\top = \sum_{i=1}^{p} \Phi_i X_{t-i} X_t^\top + \varepsilon_t X_t^\top$$

$$\Rightarrow \mathbb{E}\left(X_t X_t^\top\right) = \sum_{i=1}^{p} \Phi_i \mathbb{E}\left(X_{t-i} X_t^\top\right) + \mathbb{E}\left(\varepsilon_t X_t^\top\right)$$

$$\Rightarrow \mathbb{E}\left(\varepsilon_t X_t^\top\right) = \sum_{i=1}^{p} \Phi_i \mathbb{E}\left(\varepsilon_t X_{t-i}^\top\right) + \mathbb{E}\left(\varepsilon_t \varepsilon_t^\top\right) = \Omega$$

$$\Rightarrow \Gamma_X(0) = \sum_{i=1}^{p} \Phi_i \Gamma_X(-i) + \Omega.$$

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Autocovariances 2/2

For $h \in \mathbb{N}^*$:

$$X_t X_{t-h}^\top = \sum_{i=1}^{p} \Phi_i X_{t-i} X_{t-h}^\top + \varepsilon_t X_{t-h}^\top$$

$$\Rightarrow \mathbb{E}\left(X_t X_{t-h}^\top\right) = \sum_{i=1}^{p} \Phi_i \mathbb{E}\left(X_{t-i} X_{t-h}^\top\right) + \mathbb{E}\left(\varepsilon_t X_{t-h}^\top\right)$$

$$\Rightarrow \Gamma_X(h) = \sum_{i=1}^{p} \Phi_i \Gamma_X(h-i).$$

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VAR(1) autocovariances

For a centered VAR(1) process:

$$\forall t \in \mathbb{Z} : X_t = \Phi X_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN\left(0, \Omega\right)$.
We have:

$$\Gamma_X(0) = \Phi \Gamma_X\left(-1\right) + \Omega$$

and:

$$\forall h \in \mathbb{N}^* : \Gamma_X(h) = \Phi \Gamma_X\left(h-1\right).$$

So:

$$\begin{aligned}
\Gamma_X(0) &= \Phi \Gamma_X\left(-1\right) + \Omega \\
&= \Phi \Gamma_X(1)^\top + \Omega \\
&= \Phi \Gamma_X(0)^\top \Phi^\top + \Omega \\
&= \Phi \Gamma_X(0) \Phi^\top + \Omega.
\end{aligned}$$

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Vectorization and Kronecker product

The Kronecker product of 2 matrixes A ($n \times p$ dimensional matrix) and B is:

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1p}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{np}B \end{bmatrix}$$

Vectorization (stacking operation) of a matrix A with dimensions $n \times p$ is:

$$\text{vec}\,(A) = (a_{11}, \dots, a_{n1}, \dots, a_{1p}, \dots, a_{np})^\top$$

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Proposition

We have:

- $A \otimes B \neq B \otimes A$.
- $(A \otimes B)^\top = A^\top \otimes B^\top$.
- $A \otimes (B + C) = A \otimes B + A \otimes C$.
- If A and B are invertible matrixes: $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
- With dimensions conditions: $(A \otimes B)(C \otimes D) = AC \otimes BD$.
- Eigenvalues of $A \otimes B$ are the products of A and B eigenvalues.
- $\mathrm{vec}(A + B) = \mathrm{vec}(A) + \mathrm{vec}(B)$.
- $\mathrm{vec}(ABC) = \left(C^\top \otimes A\right) \mathrm{vec}(B)$.

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Consequence

We have:

$$\begin{aligned}\operatorname{vec}\left(\Gamma_X(0)\right) &= \operatorname{vec}\left(\Phi\Gamma_X(0)\Phi^\top\right) + \operatorname{vec}\left(\Omega\right) \\ &= \left(\Phi \otimes \Phi\right)\operatorname{vec}\left(\Gamma_X(0)\right) + \operatorname{vec}\left(\Omega\right).\end{aligned}$$

So:

$$\operatorname{vec}\left(\Gamma_X(0)\right) = \left(\mathrm{I}_{d^2} - \Phi \otimes \Phi\right)^{-1}\operatorname{vec}\left(\Omega\right).$$

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VAR(1) representation of a VAR(p) process 1/2

For a centered VAR(p) process, we can write:

$$X_t^* = \Phi^* X_{t-1}^* + \varepsilon_t^*$$

where:

$$X_t^* = \begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-p+1} \end{pmatrix}, \Phi^* = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \cdots & \Phi_{p-1} & \Phi_p \\ I_d & 0 & 0 & \cdots & \cdots & 0 \\ 0 & I_d & 0 & \ddots & & 0 \\ \vdots & 0 & I_d & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_d & 0 \end{bmatrix}, \varepsilon_t^* = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

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VAR(1) representation of a VAR(p) process 2/2

The variance-covariance matrix of ε_t^* is:

$$\Gamma_{X^*}(0) = \begin{bmatrix} \Gamma_X(0) & \Gamma_X(1) & \dots & \Gamma_X(p-1) \\ \Gamma_X(1)^\top & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Gamma_X(1) \\ \Gamma_X(p-1)^\top & \dots & \Gamma_X(1)^\top & \Gamma_X(0) \end{bmatrix}.$$

We have:

$$\mathrm{vec}\left(\Gamma_{X^*}(0)\right) = \left(\mathrm{I}_{(pd)^2} - \Phi^* \otimes \Phi^*\right)^{-1} \mathrm{vec}\left(\Omega^*\right)$$

where $\Omega^* = \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix}$ is a $pd \times pd$ dimensional matrix.

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VMA (∞) representation of a VAR(p) process

With the stability condition:

$$\begin{aligned} X_t &= \mathbf{\Phi}^{-1}(B)\,\varepsilon_t \\ &= \mu + \varepsilon_t + \sum_{i=1}^{+\infty} \Psi_i \varepsilon_{t-i} \end{aligned}$$

where $\sum_{i=1}^{+\infty} \|\Psi_i\| < +\infty$ ($(\Psi_i)_{i\in\mathbb{N}^*}$ are $d \times d$ dimensional matrixes).

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VAR processes forecasting 1/3

We want to forecast $X_{T+\ell}$ (with $\ell \in \mathbb{N}^*$) from $(X_i)_{i \leq T}$.
Let $\mathcal{F}_{-\infty}^{T}(X)$ be the set spanned by $(X_i)_{i \leq T}$:

$$\begin{aligned}
\widehat{X}_T(\ell) &= \mathbb{E}\left(X_{T+\ell} \Big/ \mathcal{F}_{-\infty}^{T}(X)\right) \\
&= \mathbb{E}\left(\nu + \sum_{i=1}^{p} \Phi_i X_{T+\ell-i} + \varepsilon_{T+\ell} \Big/ \mathcal{F}_{-\infty}^{T}(X)\right) \\
&= \nu + \sum_{i=1}^{\ell-1} \Phi_i \widehat{X}_T(\ell - i) + \sum_{i=\ell}^{p} \Phi_i X_{T+\ell-i} + 0.
\end{aligned}$$

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VAR processes forecasting 2/3

From VMA (∞) representation:

$$\begin{aligned}\widehat{X}_T(\ell) &= \mathbb{E}\left(X_{T+\ell} \Big/ \mathcal{F}_{-\infty}^{T}(X)\right) \\ &= \mathbb{E}\left(\mu + \varepsilon_{T+\ell} + \sum_{i=1}^{+\infty} \Psi_i \varepsilon_{T+\ell-i} \Big/ \mathcal{F}_{-\infty}^{T}(X)\right) \\ &= \mu + \sum_{i=\ell}^{+\infty} \Psi_i \varepsilon_{T+\ell-i}.\end{aligned}$$

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VAR processes forecasting 3/3

MSE (Mean Square Error) is:

$$\begin{aligned}\mathrm{MSE}(\ell) &= \mathrm{Var}\left(X_{T+l} - \widehat{X}_T(\ell)\right) \\ &= \mathrm{Var}\left(\varepsilon_{T+l} + \sum_{i=1}^{\ell-1} \Psi_i \varepsilon_{T+l-i}\right) \\ &= \Omega + \sum_{i=1}^{\ell-1} \Psi_i \Omega \Psi_i^\top .\end{aligned}$$

We obtain the linear recurrence equation:

$$\mathrm{MSE}(\ell) = \mathrm{MSE}(\ell - 1) + \Psi_{\ell-1} \Omega \Psi_{\ell-1}^\top$$

with $\mathrm{MSE}(0) = 0$.
One can obtain forecast intervals under normality distribution.

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Least square estimation 1/3

Consider a centered VAR(p) process:

$$\forall t \in \mathbb{Z} : X_t = \sum_{i=1}^{p} \Phi_i X_{t-i} + \varepsilon_t.$$

Based on $(X_1, \ldots, X_T)$, with initial values $(X_{1-p}, \ldots, X_0)$, we have:

$$\mathbf{Y} = \Phi \mathbb{X} + \boldsymbol{\varepsilon}$$

where:

$$\mathbf{Y} = \begin{pmatrix} X_1 & \ldots & X_T \end{pmatrix},$$

$$\Phi = \begin{pmatrix} \Phi_1 & \ldots & \Phi_p \end{pmatrix},$$

$$\mathbb{X} = \begin{bmatrix} X_0 & \ldots & X_{T-1} \\ \vdots & & \vdots \\ X_{1-p} & \ldots & X_{T-p} \end{bmatrix},$$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & \ldots & \varepsilon_T \end{pmatrix}.$$

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Least square estimation 2/3

So:

$$\mathrm{vec}\left(\mathbf{Y}\right) = \left(\mathbb{X}^\top \otimes \mathsf{I}_d\right) \mathrm{vec}\left(\Phi\right) + \mathrm{vec}\left(\varepsilon\right).$$

Least square estimator minimizes:

$$\left[\mathrm{vec}\left(\varepsilon\right)\right]^\top \mathrm{vec}\left(\varepsilon\right).$$

Least square estimator is:

$$\begin{aligned}
\mathrm{vec}\left(\widehat{\Phi}\right) &= \left[\left(\mathbb{X}^\top \otimes \mathsf{I}_d\right)^\top \left(\mathbb{X}^\top \otimes \mathsf{I}_d\right)\right]^{-1} \left(\mathbb{X}^\top \otimes \mathsf{I}_d\right)^\top \mathrm{vec}\left(\mathbf{Y}\right) \\
&= \left[\left(\mathbb{X}\mathbb{X}^\top\right)^{-1} \otimes \mathsf{I}_d\right] \mathrm{vec}\left(\mathbf{Y}\mathbb{X}^\top\right) \\
&= \mathrm{vec}\left(\mathbf{Y}\mathbb{X}^\top \left(\mathbb{X}\mathbb{X}^\top\right)^{-1}\right).
\end{aligned}$$

Thus:

$$\widehat{\Phi} = \mathbf{Y}\mathbb{X}^\top \left(\mathbb{X}\mathbb{X}^\top\right)^{-1}.$$

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Least square estimation 3/3

Under stationarity and ergodicity assumptions, the least square estimator is asymptotically normal (that allows significance tests):

$$\sqrt{T}\left(\mathrm{vec}\left(\widehat{\Phi}\right) - \mathrm{vec}\left(\Phi\right)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \lim_{T\to+\infty}\left(\Gamma_{\mathbf{Y}}(0)^{-1} \otimes \Omega\right)\right).$$

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One use information criterium:

$$\mathrm{AIC} = \ln\left(\det\left(\widehat{\Omega}(p)\right)\right) + \frac{2}{T}pd^2,$$

$$\mathrm{BIC} = \ln\left(\det\left(\widehat{\Omega}(p)\right)\right) + \frac{\ln(T)}{T}pd^2.$$

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Notation

We use the notation:

$$\mathbb{E}_L\left(X/A\right)$$

for the expectation of X given the set spanned by all the linear combinations of the components of A (and 1).

One can also find the notations $EL\left(X\,/A\right)$, $\widehat{\mathbb{E}}\left(X\,/A\right)$ or $\widehat{P}\left(X\,/A\right)$.

We also use the notation:

$$\underline{X_t} = \left\{X_i, i \in \mathbb{Z}, i \leq t\right\}.$$

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Causality

Let $X = (X_t)_{t\in\mathbb{Z}}$ and $Y = (Y_t)_{t\in\mathbb{Z}}$ be two multivariate processes (dimensions can be different).
Y doesn't cause X at time t if:

$$\forall \ell \in \mathbb{N}^* : \mathbb{E}_L\left(X_{t+\ell}/\underline{X_t}, \underline{Y_t}\right) = \mathbb{E}_L\left(X_{t+\ell}/\underline{X_t}\right).$$

We use the notation:

$$C_{Y \not\to X}.$$

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Instantaneous causality

Y doesn't instantaneously cause X at time t if:

$$\mathbb{E}_L\left(X_{t+1}/\underline{X_t}, \underline{Y_{t+1}}\right) = \mathbb{E}_L\left(X_{t+1}/\underline{X_t}, \underline{Y_t}\right).$$

We use the notation:

$$C_{Y \neq X}.$$

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Remark

If the processes are stationary then there is equivalence between causality at a time t and causality for all times.

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Definition

One can also use the following characterisation.
Let consider the error of the linear forecast of X given A:

$$\varepsilon_L\left(X_t/A\right) = X_t - \mathbb{E}_L\left(X/A\right).$$

We define:

- Y doesn't cause X at time t if:

$$\forall \ell \in \mathbb{N}^* : \mathrm{Var}\left(\varepsilon_L\left(X_{t+\ell}/\underline{X_t}, \underline{Y_t}\right)\right) = \mathrm{Var}\left(\varepsilon_L\left(X_{t+\ell}/\underline{X_t}\right)\right).$$

- Y doesn't cause X instantaneously at time t if:

$$\mathrm{Var}\left(\varepsilon_L\left(X_{t+1}/\underline{X_t}, \underline{Y_{t+1}}\right)\right) = \mathrm{Var}\left(\varepsilon_L\left(X_{t+1}/\underline{X_t}, \underline{Y_t}\right)\right).$$

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Property

- Y doesn't instantaneously cause X at time t iff:

$$\mathrm{Cov}\left(X_{t+1} - \mathbb{E}_L\left(X_{t+1}/\underline{X_t}, \underline{Y_t}\right), Y_{t+1} - \mathbb{E}_L\left(Y_{t+1}/\underline{X_t}, \underline{Y_t}\right)\right) = 0.$$

- We have:

$$C_{Y-X} \Leftrightarrow C_{X-Y}.$$

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Example

Let $(u_t)_{t\in\mathbb{Z}}$ and $(v_t)_{t\in\mathbb{Z}}$ be two independent white noises.
Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be two processes such that:

$$\begin{aligned} X_t &= u_t + av_t + bv_{t-1}, \\ Y_t &= v_t \end{aligned}$$

where $(a, b) \in \mathbb{R}^2$.
We have:

$$\begin{aligned} \mathbb{E}_L\left(X_{t+1}/\underline{X_t}, \underline{Y_t}\right) &= bY_t, \\ \mathbb{E}_L\left(X_{t+1}/\underline{X_t}, \underline{Y_{t+1}}\right) &= aY_{t+1} + bY_t. \end{aligned}$$

Thus:

- Y doesn't cause X at time t iff $b = 0$,
- Y doesn't instantaneously cause X at time t iff $a = 0$.

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Let $X = (X_t)_{t\in\mathbb{Z}}$ and $Y = (Y_t)_{t\in\mathbb{Z}}$ be two multivariate (with different or same dimensions) processes such that $(X_t, Y_t)^\top$ is a VAR(p) process:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \sum_{i=1}^{p} \begin{bmatrix} \Phi_i^{XX} & \Phi_i^{XY} \\ \Phi_i^{YX} & \Phi_i^{YY} \end{bmatrix} \begin{pmatrix} X_{t-i} \\ Y_{t-i} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix}.$$

We have:

$$\begin{aligned} C_{Y\not\to X} &\Leftrightarrow \forall i \in \{1,\dots,p\} : \Phi_i^{XY} = 0, \\ C_{X\not\to Y} &\Leftrightarrow \forall i \in \{1,\dots,p\} : \Phi_i^{YX} = 0, \\ C_{Y\not\neq X} &\Leftrightarrow \mathrm{Cov}\left(\varepsilon_t, \eta_t\right) = 0. \end{aligned}$$

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Example 1/2

Let $(X_t)_{t\in\mathbb{Z}}$, $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ be three univariate processes (in $\mathbb{R}$) such that $(X_t, Y_t, Z_t)^\top$ is a VAR (1) process:

$$\begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} X_{t-1} \\ Y_{t-1} \\ Z_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \eta_t \\ \xi_t \end{pmatrix}$$

with:

$$\mathrm{Var}\begin{pmatrix} \varepsilon_t \\ \eta_t \\ \xi_t \end{pmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Example 2/2

We have:

$$C_{(Y,Z)\not\to X},$$
$$C_{X\to(Y,Z)},$$
$$C_{(X,Z)\to Y},$$
$$C_{Y\to(X,Z)},$$
$$C_{Z\not\to(X,Y)},$$
$$C_{(X,Y)\to Z}$$

and:

$$C_{X-(Y,Z)},$$
$$C_{Y-(X,Z)},$$
$$C_{Z\not-(X,Y)}.$$

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Causality test 1/2

A test of causality between two multivariate processes $X = (X_t)_{t\in\mathbb{Z}}$ and $Y = (Y_t)_{t\in\mathbb{Z}}$ such that (X_t, Y_t) is a VAR(p) process, is equivalent to a significance test for some coefficients of the matrix $(\Phi_1, \ldots, \Phi_p)$:

$$H_0 : R\,\mathrm{vec}\,(\Phi) = 0$$
$$H_1 : R\,\mathrm{vec}\,(\Phi) \neq 0$$

where R is a r ranked matrix.
We know that:

$$\sqrt{T}\left(\mathrm{vec}\left(\widehat{\Phi}\right) - \mathrm{vec}\,(\Phi)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Gamma^{-1} \otimes \Omega\right).$$

Thus:

$$\sqrt{T}\left(R\,\mathrm{vec}\left(\widehat{\Phi}\right) - R\,\mathrm{vec}\,(\Phi)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, R\left(\Gamma^{-1} \otimes \Omega\right) R^{\top}\right).$$

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Causality test 2/2

Under H_0:

$$T\left(R\,\mathrm{vec}\left(\widehat{\Phi}\right)\right)^{\top}\left[R\left(\Gamma^{-1}\otimes\Omega\right)R^{\top}\right]^{-1}\left(R\,\mathrm{vec}\left(\widehat{\Phi}\right)\right)\xrightarrow{\mathcal{L}}\chi^2\left(r\right)$$

One use this Wald test with estimators:

$$\widehat{\Gamma}=\frac{1}{T}\mathbb{X}\mathbb{X}^{\top},$$

$$\widehat{\Omega}=\frac{1}{T-dp-1}\sum_{t=1}^{T}\widehat{\varepsilon}_t\widehat{\varepsilon}_t^{\top}.$$

Asymptotic distribution is still $\chi^2\left(r\right)$.

One can also test instantaneously causality.

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