Univariate time series

SARIMA processes

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Introduction

If $(X_t)_{t\in\mathbb{Z}}$ is a stationary process and if $(a_i)_{i\in\mathbb{Z}}$ is a sequence such that $\sum_{i\in\mathbb{Z}} |a_i| < +\infty$ then the process $(Y_t)_{t\in\mathbb{Z}}$ defined by:

$$Y_t = \sum_{i\in\mathbb{Z}} a_i X_{t-i}$$

is stationary.
One can rewrite:

$$Y_t = \sum_{i\in\mathbb{Z}} a_i X_{t-i} = \sum_{i\in\mathbb{Z}} a_i B^i X_t = P(B) X_t$$

where $P(B) = \sum_{i\in\mathbb{Z}} a_i B^i$.

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Lag series

P is said to be a lag series if:

$$P(B) = \sum_{i \in \mathbb{Z}} a_i B^i$$

where $\sum_{i \in \mathbb{Z}} |a_i| < +\infty$.

We have:

- The linear combination of two lag series is a lag series.
- The product of two lag series is a lag series.

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Inverse of $I - \lambda B$

We usually need to invert lag poynomials, so we consider the inversion of $I - \lambda B$.
We want to find the stationary process $(Y_t)_{t \in \mathbb{Z}}$ such that:

$$(I - \lambda B)\, Y_t = X_t$$

where $\lambda \in \mathbb{C}$.

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Inverse of $I - \lambda B$: $|\lambda| < 1$ (1/2)

The module root $\frac{1}{\lambda}$ of $1 - \lambda z$ is more than 1.
Consider the sequence $(a_i)_{i \in \mathbb{Z}}$ such that :

$$a_i = \begin{cases} 0 & \text{if } i \in \mathbb{Z} \setminus \mathbb{N} \\ \lambda^i & \text{if } i \in \mathbb{N} \end{cases}$$

The series $\sum_{i=-\infty}^{+\infty} a_i$ is absolutely convergent and:

$$(I - \lambda B) \sum_{i \in \mathbb{N}} \lambda^i B^i = \lambda^0 B^0 = I.$$

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Inverse of $I - \lambda B$: $|\lambda| < 1$ (2/2)

The solution $(Y_t)_{t\in\mathbb{Z}}$ of $(I - \lambda B)\, Y_t = X_t$ is the sum of the general case and of the particular case $(I - \lambda B)\, Y_t = 0$.
We have:

$$\begin{aligned}(I - \lambda B)\, Y_t = 0 &\Leftrightarrow Y_t = \lambda Y_{t-1} \\ &\Leftrightarrow Y_t = c\lambda^t\end{aligned}$$

where $c \in \mathbb{R}$.
Thus the solution is:

$$Y_t = \sum_{i\in\mathbb{N}} \lambda^i X_{t-i} + c\lambda^t.$$

However the only stationary solution is obtained with $c = 0$:

$$Y_t = \sum_{i\in\mathbb{N}} \lambda^i X_{t-i}.$$

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Inverse of $I - \lambda B$: $|\lambda| > 1$

We have:

$$\begin{aligned} Y_t &= (I - \lambda B)^{-1} X_t \\ &= \left[-\lambda B \left(-\frac{1}{\lambda} F + I \right) \right]^{-1} X_t \\ &= \left(-\frac{1}{\lambda} F \right) \sum_{i \in \mathbb{N}} \left(\frac{1}{\lambda} \right)^i F^i X_t \\ &= -\sum_{i=1}^{+\infty} \frac{1}{\lambda^i} X_{t+i}. \end{aligned}$$

Thus:

$$Y_t = -\sum_{i=1}^{+\infty} \frac{1}{\lambda^i} X_{t+i}.$$

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Inverse of $I - \lambda B$: $|\lambda| = 1$

For $|\lambda| = 1$, there is no stationary process $(Y_t)_{t \in \mathbb{Z}}$ such that $(I - \lambda B)\, Y_t = X_t$.

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General case 1/3

Consider the lag series:

$$\Phi(B) = I - \varphi_1 B - \ldots - \varphi_p B^p$$

where $(\varphi_1, \ldots, \varphi_p) \in \mathbb{R}^p$.
We want to find the stationary process $(Y_t)_{t \in \mathbb{Z}}$ such that:

$$\Phi(B) Y_t = X_t.$$

Let $z_j = \frac{1}{\lambda_j}$ with $j \in \{1, \ldots, p\}$ be the roots of $\Phi(z)$.
If one of the roots z_j is on the unit circle then there is no stationary solution.

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General case 2/3

If there is no roots z_j on the unit circle then there is lag series $\Psi(B) = \sum_{i \in \mathbb{Z}} \psi_i B^i$, where $(\psi_i)_{i \in \mathbb{Z}}$ is a real sequence, such that:

- $\Phi(B)\Psi(B) = I$,
- $Y_t = \Psi(B) X_t$ is stationary.

If all the roots z_j are outside the unit circle then we can write:

$$\Psi(B) = \sum_{i \in \mathbb{N}} \psi_i B^i.$$

If all the roots z_j are inside the unit circle then we can write:

$$\Psi(B) = \sum_{i \in \mathbb{N}^*} \psi_i F^i.$$

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General case 3/3

In order to determine Ψ, we can use one of these methods:

- Identification method.
- Partial fraction expansion of $\frac{1}{\Phi(z)}$ and power series expansion.
- Division of 1 by $\Phi(z)$ by increasing power order.

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Definition

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise of variance σ^2.

$(X_t)_{t\in\mathbb{Z}}$ is said to be an autoregressive process or a AR process of order p, written AR(p), if:

- $(X_t)_{t\in\mathbb{Z}}$ is stationary,
- $$\forall t \in \mathbb{Z} : X_t = \sum_{i=1}^{p} \varphi_i X_{t-i} + \varepsilon_t$$

  where $(\varphi_1, \ldots, \varphi_p) \in \mathbb{R}^p$ and $\varphi_p \neq 0$.

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Notation

We generally use the notation $\Phi(B) X_t = \varepsilon_t$ where:

$$\Phi(B) = I - \sum_{i=1}^{p} \varphi_i B^i.$$

Note that:

- Sometimes we find $\Phi(B) = I + \sum_{i=1}^{p} \varphi_i B^i$.
- If $\Phi(B)$ has a root on the unit circle then the process $(X_t)_{t \in \mathbb{Z}}$ isn't stationary, thus it isn't an AR process.

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Canonical representation: introduction 1/2

We suppose that the lag polynomial $\Phi(B)$ is invertible (no root on the unit circle).
Let $z_j = \frac{1}{\lambda_j}$, $j \in \{1, \ldots, p\}$, be the roots of $\Phi(z)$.
If one root z_j is on the unit circle then there is no stationary solution.
Assume that $(z_j)_{j \in \{1,\ldots,r\}}$ are inside the circle unit and that $(z_j)_{j \in \{r+1,\ldots,p\}}$ are outside the unit circle. We can write:

$$\Phi(B) = \prod_{j=1}^{p} (I - \lambda_j B).$$

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Canonical representation: introduction 2/2

This polynomial is invertible but we prefer to have only positive powers of B (the roots must be outside the unit circle).
Thus we prefer to consider the polynomial:

$$\Phi^*(B) = \prod_{j=1}^{r} \left(I - \frac{1}{\lambda_j} B \right) \prod_{j=r+1}^{p} (I - \lambda_j B).$$

Φ^* is obtained from Φ by inverting the roots which are inside the unit circle.
Result: we obtain a new AR model (the canonical version):

$$\Phi^*(B) X_t = \eta_t.$$

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Canonical representation: definition

Let $(X_t)_{t\in\mathbb{Z}}$ an AR(p) process:

$$\Phi(B)X_t = \varepsilon_t.$$

If the roots of Φ are outside the unit circle then we have the canonical representation.
In this case the associated white noise is the innovation.

From now we consider canonical AR processes.

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$MA(\infty)$ representation

Let $(X_t)_{t\in\mathbb{Z}}$ be a canonical AR(p) process:

$$\Phi(B)X_t = \varepsilon_t.$$

It has a $MA(\infty)$ representation:

$$X_t = \Phi^{-1}(B)\varepsilon_t = \varepsilon_t + \sum_{i=1}^{+\infty}\psi_i\varepsilon_{t-i}$$

where $(\psi_i)_{i\in\mathbb{N}}$ is a real sequence.

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Yule-Walker equations

Let $(X_t)_{t\in\mathbb{Z}}$ be a canonical AR(p) process:

$$\forall t \in \mathbb{Z} : X_t = \sum_{i=1}^{p} \varphi_i X_{t-i} + \varepsilon_t.$$

We can obtain the Yule-Walker equations:

$$\begin{cases} \gamma(0) = \dfrac{\sigma^2}{1 - \sum_{i=1}^{p} \varphi_i \rho(i)} \\ \begin{pmatrix} \rho(1) \\ \vdots \\ \rho(p) \end{pmatrix} = R_p \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_p \end{pmatrix} \end{cases}.$$

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Autocorrelations of an AR process

Autocorrelations are solutions of a simple linear recurrence equation of order p.

If the roots of $\Phi(z)$, $z_i = \frac{1}{\lambda_i}$, $i \in \{1, \ldots, p\}$, are real and unique then we have $\rho(h) = \sum_{i=1}^{p} c_i \lambda_i^h$. Autocorrelations exponentially decrease to 0.

In the general case we obtain a damped sine wave.

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Partial autocorrelations of an AR process

If $(X_t)_{t\in\mathbb{Z}}$ is a AR(p) process then its partial autocorrelations are zero after p:

$$\begin{cases} r(p) \neq 0 \\ \forall h \in \mathbb{N}, h \geq p+1 : r(h) = 0 \end{cases} .$$

Conversely it's a necessary and sufficient condition that $(X_t)_{t\in\mathbb{Z}}$ is an AR(p) process.

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Proposition

Let $(X_t)_{t\in\mathbb{Z}}$ be a canonical AR(p) process:

$$\forall t \in \mathbb{Z} : X_t = \sum_{i=1}^{p} \varphi_i X_{t-i} + \varepsilon_t.$$

We have:

$$r(p) = \varphi_p.$$

Note that:

- This proposition applies only to canonical process.
- One can't deduce anything for $r(h)$, $h \in \{2, \ldots, p\}$.

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Definition

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise of variance σ^2.

$(X_t)_{t\in\mathbb{Z}}$ is said to be a MA process of order q, written MA(q), if:

$$\forall t \in \mathbb{Z} : X_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$$

where $(\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$ and $\theta_q \neq 0$.

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Notation

We generally use the notation $X_t = \Theta(B)\varepsilon_t$ where:

$$\Theta(B) = I + \sum_{i=1}^{q} \theta_i B^i.$$

Note that:

- Sometimes we find $\Theta(B) = I - \sum_{i=1}^{q} \theta_i B^i$.
- A MA process is stationnary.

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Canonical representation

Let $(X_t)_{t\in\mathbb{Z}}$ be a MA(q) process:

$$X_t = \Theta(B)\,\varepsilon_t.$$

If the roots of Θ are outside the unit circle then we have the canonical representation.
In this case, the associated white noise is the innovation.

From now we consider canonical MA processes.

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AR (∞) representation

Let $(X_t)_{t\in\mathbb{Z}}$ be a canonical MA(q) process:

$$X_t = \Theta(B)\,\varepsilon_t.$$

It has a AR (∞) representation:

$$\varepsilon_t = \Theta^{-1}(B)\,X_t = X_t + \sum_{i=1}^{+\infty} \pi_i X_{t-i},$$

thus:

$$X_t = -\sum_{i=1}^{+\infty} \pi_i X_{t-i} + \varepsilon_t$$

where $(\pi_i)_{i\in\mathbb{N}}$ is a real sequence.

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Autocorrelations of a MA process

Let $(X_t)_{t\in\mathbb{Z}}$ be a canonical MA(q) process:

$$\forall t \in \mathbb{Z} : X_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

We have:

$$\gamma(0) = \sigma^2 \left(1 + \sum_{i=1}^{q} \theta_i^2\right)$$

and:

$$\forall h \in \mathbb{N}^* : \gamma(h) = \begin{cases} \left(\theta_h + \sum_{i=h+1}^{q} \theta_i \theta_{i-h}\right) \sigma^2 & \text{if } h \in \{1, \ldots, q\} \\ 0 & \text{otherwise} \end{cases}.$$

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Autocorrelations of a MA process

If $(X_t)_{t\in\mathbb{Z}}$ is a process MA(q) process then its autocorrelations are zero after q:

$$\begin{cases} \rho(q) \neq 0 \\ \forall h \in \mathbb{N}, h \geq q+1 : \rho(h) = 0 \end{cases}.$$

Conversely its a necessary and sufficient condition that $(X_t)_{t\in\mathbb{Z}}$ is a MA(q) process.

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Partial autocorrelations of a MA process

Partial autocorrelations are solutions of a simple linear recurrence equation of order q. They decrease to 0.

In the general case, we obtain an exponential decrease or a damped sine wave.

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Definition

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise of variance σ^2.

$(X_t)_{t\in\mathbb{Z}}$ is said to be a ARMA process of order (p, q), written ARMA(p, q), if:

- $(X_t)_{t\in\mathbb{Z}}$ is stationary,
- $\forall t \in \mathbb{Z} : X_t - \sum_{i=1}^{p} \varphi_i X_{t-i} = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$ where $(\varphi_1, \ldots, \varphi_p) \in \mathbb{R}^p$, $\varphi_p \neq 0$, $(\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$ and $\theta_q \neq 0$.

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Notation

We generally use the notation:

$$\Phi(B) X_t = \Theta(B) \varepsilon_t$$

where:

$$\Phi(B) = I - \sum_{i=1}^{p} \varphi_i B^i,$$
$$\Theta(B) = I + \sum_{i=1}^{q} \theta_i B^i.$$

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Remarks

- One can consider non centered ARMA processes $(X_t)_{t\in\mathbb{Z}}$. In this case, resultats are the same withe the process $Y_t = X_t - \mu_X$.
  From now we consider centered ARMA processes.
- An AR(p) process is a ARMA($p, 0$) process.
- A MA(q) process is a ARMA($0, q$) process.

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Representations of an ARMA process

Let $(X_t)_{t\in\mathbb{Z}}$ be an ARMA(p,q) process:

$$\Phi(B)X_t = \Theta(B)\varepsilon_t.$$

The representation is:

- minimal if Φ and Θ have no common root,
- causal if the roots of Φ are outside the unit circle,
- invertible if the roots of Θ are outside the unit circle,
- canonical if the representation is causal and invertible. In this case the associated white noise is the innovation.

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MA (∞) representation of an ARMA process

Let $(X_t)_{t\in\mathbb{Z}}$ be an minimal canonical ARMA(p,q) process:

$$\Phi(B)X_t = \Theta(B)\varepsilon_t.$$

it has a MA (∞) representation:

$$X_t = \Phi^{-1}(B)\Theta(B)\varepsilon_t = \varepsilon_t + \sum_{i=1}^{+\infty}\psi_i\varepsilon_{t-i}$$

where $(\psi_i)_{i\in\mathbb{N}}$ is a real sequence.
With $\psi_i = 0$ for $i < 0$, $\theta_0 = 1$ and $\theta_i = 0$ for $i > q$, we have:

$$\forall i \in \mathbb{N} : \psi_i - \sum_{j=1}^{p}\varphi_j\psi_{i-j} = \theta_i.$$

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AR (∞) representation of an ARMA process

It has an AR (∞) representation:

$$\varepsilon_t = \Theta^{-1}(B)\,\Phi(B)\,X_t = X_t + \sum_{i=1}^{+\infty} \pi_i X_{t-i},$$

thus:

$$X_t = -\sum_{i=1}^{+\infty} \pi_i X_{t-i} + \varepsilon_t$$

where $(\pi_i)_{i\in\mathbb{N}}$ is a real sequence.
With $\pi_i = 0$ for $i < 0$, $\varphi_0 = -1$ and $\varphi_i = 0$ for $i > p$, we have:

$$\forall i \in \mathbb{N} : \pi_i + \sum_{j=1}^{q} \theta_j \pi_{i-j} = -\varphi_i.$$

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Autocorrelations of an ARMA process 1/2

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From the $MA(\infty)$ representation:

$$\gamma(h) = \sigma^2 \sum_{i=0}^{+\infty} \psi_i \psi_{i+h}$$

where $\psi_0 = 1$.

Autocorrelations of an ARMA process 2/2

From the AR equation, for $h \in \mathbb{N}$:

$$\begin{aligned}
&\gamma(h) - \varphi_1 \gamma(h-1) - \ldots - \varphi_p \gamma(h-p) \\
&= \mathrm{Cov}\left(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}, X_{t-h}\right) \\
&= \mathrm{Cov}\left(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}, \sum_{i=0}^{+\infty} \psi_i \varepsilon_{t-h-i}\right) \\
&= \begin{cases} \sigma^2 \sum_{i=0}^{+\infty} \theta_{i+h} \psi_i & \text{if } h \in \{0, \ldots, q\} \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

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Remarks

- Autocorrelations decrease to 0.
  - If $p > q$ then we obtain an exponential decrease or a damped sine wave.
  - If $q \geq p$, the decrease is after the first $q - p$ values.
- There are similar properties for partial autocorrelations.

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Corner method 1/3

There is no simple characterization of ARMA processes based on simple and partial autocorrelations. The corner method comes from autocorrelation matrixes properties. Consider, for $(i,j) \in \mathbb{N}^2$:

$$\Omega_{i,j} = \begin{bmatrix} \rho(i) & \rho(i-1) & \dots & \dots & \rho(i-j+1) \\ \rho(i-1) & \rho(i) & \rho(i-1) & \dots & \rho(i-j) \\ \rho(i-2) & \rho(i-1) & \rho(i) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \rho(i-1) \\ \rho(i-j+1) & \rho(i-j) & \dots & \rho(i-1) & \rho(i) \end{bmatrix}$$

and their determinants:

$$\Delta_{i,j} = \det(\Omega_{i,j}).$$

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Corner method 2/3

For a ARMA(p, q) process, we have:

- $\forall (i, j) \in \mathbb{N}^2, i > q, j > p : \Delta_{i,j} = 0,$
- $\forall (i, j) \in \mathbb{N}^2, i \leq q : \Delta_{i,p} \neq 0,$
- $\forall (i, j) \in \mathbb{N}^2, j \leq p : \Delta_{q,j} \neq 0.$

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Corner method 3/3

For k enough large we represent the matrix $M = (\Delta_{i,j})_{(i,j)\in\{1,\dots,k\}^2}$ and a corner appears:

$$M = \left[\begin{array}{ccc|ccc} \Delta_{1,1} & \cdots & \Delta_{1,p} & \Delta_{1,p+1} & \cdots & \Delta_{1,k} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Delta_{q,1} & \cdots & \Delta_{q,p} & \Delta_{q,p+1} & \cdots & \Delta_{q,k} \\ \hline \Delta_{q+1,1} & \cdots & \Delta_{q+1,p} & & & \\ \vdots & & \vdots & & 0 & \\ \Delta_{k,1} & \cdots & \Delta_{k,p} & & & \end{array}\right].$$

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Spectral density of an ARMA process

Let $(X_t)_{t\in\mathbb{Z}}$ be an ARMA(p,q) process:

$$\Phi(B)X_t = \Theta(B)\varepsilon_t.$$

Its spectral density is:

$$f(\omega) = \frac{\sigma^2}{2\pi}\frac{\left|\Theta\left(e^{-i\omega}\right)\right|^2}{\left|\Phi\left(e^{-i\omega}\right)\right|^2}.$$

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ARMA process estimation: principle

Let $(X_t)_{t\in\mathbb{Z}}$ be a minimal canonical ARMA(p,q) process:

$$\Phi(B)X_t = \Theta(B)\varepsilon_t.$$

The aim is to estimate Φ and Θ, and σ^2.

Estimations from autocorrelations aren't efficient. We use maximum likelihood estimation after a preliminary estimation.

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Preliminary estimation: AR processes 1/2

From Yule-Walker equations:

$$\begin{cases} \begin{pmatrix} \widehat{\varphi}_1 \\ \vdots \\ \widehat{\varphi}_p \end{pmatrix} = \widehat{R}_p^{-1} \begin{pmatrix} \widehat{\rho}(1) \\ \vdots \\ \widehat{\rho}(p) \end{pmatrix} \\ \\ \widehat{\sigma}^2 = \widehat{\gamma}(0) \left(1 - \sum_{i=1}^{p} \widehat{\varphi}_i \widehat{\rho}(i)\right) \end{cases} .$$

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Preliminary estimation: AR processes 2/2

With $\varphi = (\varphi_1, \ldots, \varphi_p)$ and $\widehat{\varphi} = (\widehat{\varphi}_1, \ldots, \widehat{\varphi}_p)$, we have:

$$\sqrt{n}\left(\widehat{\varphi} - \varphi\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2 \Sigma_p^{-1}\right)$$

and:

$$\widehat{\sigma}^2 \xrightarrow{\mathbb{P}} \sigma^2$$

where:

$$\Sigma_p = \begin{bmatrix} \gamma(0) & \gamma(1) & \ldots & \gamma(p-1) \\ \gamma(1) & \gamma(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma(1) \\ \gamma(p-1) & \ldots & \gamma(1) & \gamma(0) \end{bmatrix}.$$

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Preliminary estimation: MA and ARMA processes 1/2

Consider a minimal canonical ARMA(p, q) process.
From the MA (∞) representation:

$$X_t = \varepsilon_t + \sum_{i=1}^{+\infty} \psi_i \varepsilon_{t-i}.$$

We use the innovation algorithm in order to estimate coefficients $(\psi_i)_{i \in \{1,\ldots,n\}}$.

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Preliminary estimation: MA and ARMA processes 2/2

With $\psi_i = 0$ for $i < 0$, $\theta_0 = 1$ and $\theta_i = 0$ for $i > q$, we have:

$$\forall i \in \mathbb{N} : \psi_i - \sum_{j=1}^{p} \varphi_j \psi_{i-j} = \theta_i.$$

We thus obtain a first estimation of $(\varphi_1, \ldots, \varphi_p)$ and $(\theta_1, \ldots, \theta_q)$ from $\left(\widehat{\psi}_1, \ldots, \widehat{\psi}_{p+q}\right)$.

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Maximum likelihood estimation

We assume now that the residuals are a gaussian white noise with variance $,\sigma^2$.
Based on $(X_1, \ldots, X_T)$, the likelihood is:

$$\ell\left(x_1, \ldots, x_T; \varphi_1, \ldots, \varphi_p, \theta_1, \ldots, \theta_q, \sigma^2\right)$$
$$= \frac{1}{(2\pi)^{\frac{T}{2}}} \frac{1}{\sqrt{\det \Sigma_T}} \exp\left(-\frac{1}{2} x^\top \Sigma_T^{-1} x\right)$$

where $x = (x_1, \ldots, x_T)^\top$ and Σ_T is the variance-covariance matrix of $(X_1, \ldots, X_T)$.
It's a nonlinear optimization problem.

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Maximum likelihood estimation

Consider:

$$\forall i \in \{1, \ldots, T\} : \widehat{X}_i = \mathbb{E}\left(X_i \Big/ \mathcal{H}_1^{i-1}(X)\right).$$

We use the innovation algorithm for the one step optimal linear forecast and for forecast error $\varepsilon_i = X_i - \widehat{X}_i$, and their variance $v_{i-1} = \mathbb{E}\left[\left(X_i - \widehat{X}_i\right)^2\right]$.

One don't need to calculate Σ_T^{-1} and $\det \Sigma_T$.

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Maximum likelihood estimation

We have:

$$\begin{pmatrix} X_1 \\ \vdots \\ X_T \end{pmatrix} = C_T \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_T \end{pmatrix}.$$

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Maximum likelihood estimation

$(\varepsilon_i)_{i\in\{1,\dots,T\}} = \left(X_i - \widehat{X}_i\right)_{i\in\{1,\dots,T\}}$ are uncorrelated, the variance-covariance matrix of $(\varepsilon_i)_{i\in\{1,\dots,T\}}$ is:

$$V_T = \begin{bmatrix} v_0 & 0 & \dots & 0 \\ 0 & v_1 & \ddots & \dots \\ \dots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & v_{T-1} \end{bmatrix}.$$

We have $\Sigma_T = C_T V_T (C_T)^\top$ thus:

$$\det \Sigma_T = (\det C_T)^2 \det V_T = v_0 \dots v_{T-1}$$

and:

$$x^\top \Sigma_T^{-1} x = \sum_{i=1}^{T} \frac{(x_i - \widehat{x}_i)^2}{v_{i-1}}.$$

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Maximum likelihood estimation

We have:

$$\ell\left(x_1, \ldots, x_T; \varphi_1, \ldots, \varphi_p, \theta_1, \ldots, \theta_q, \sigma^2\right)$$
$$= \frac{1}{(2\pi)^{\frac{T}{2}}} \frac{1}{\sqrt{v_0 \ldots v_{T-1}}} \exp\left(-\frac{1}{2} \sum_{i=1}^{T} \frac{(x_i - \widehat{x}_i)^2}{v_{i-1}}\right)$$

$\widehat{X}_i$ is recursively obtained with the innovation algorithm.

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Maximum likelihood estimation

Consider:

$$v_{i-1} = \sigma^2 r_{i-1}.$$

We can rewrite:

$$\ell\left(x_1, \ldots, x_T; \varphi_1, \ldots, \varphi_p, \theta_1, \ldots, \theta_q, \sigma^2\right)$$
$$= \frac{1}{(2\pi\sigma^2)^{\frac{T}{2}}} \frac{1}{\sqrt{r_0 \ldots r_{T-1}}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{T} \frac{(x_i - \widehat{x}_i)^2}{r_{i-1}}\right).$$

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Maximum likelihood estimation

Finally we have:

$$\left(\widehat{\varphi}_1, \ldots, \widehat{\varphi}_p, \widehat{\theta}_1, \ldots, \widehat{\theta}_q\right)$$
$$= \underset{(\varphi_1,\ldots,\varphi_p,\theta_1,\ldots,\theta_q)}{\arg\min} \left\{ \ln\left[\frac{1}{T} S\left(\varphi_1, \ldots, \varphi_p, \theta_1, \ldots, \theta_q\right)\right] + \frac{1}{T}\sum_{i=1}^{T} \ln r_i \right\}$$

and:

$$\widehat{\sigma}^2 = \frac{1}{T} S\left(\widehat{\varphi}_1, \ldots, \widehat{\varphi}_p, \widehat{\theta}_1, \ldots, \widehat{\theta}_q\right)$$

where $S\left(\widehat{\varphi}_1, \ldots, \widehat{\varphi}_p, \widehat{\theta}_1, \ldots, \widehat{\theta}_q\right) = \sum_{i=1}^{T} \frac{\left(x_i - \widehat{x}_i\right)^2}{r_{i-1}}$.

Estimators are efficient.

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Model selection: Kullback criterium

Let f_0 be a probability density which is estimated by an element of the family $\mathcal{F}$.
The Kullback criterium measure the difference between the the trues and the estimated probability density:

$$I\left(f_0, \mathcal{F}\right) = \min_{f \in \mathcal{F}} \int \ln\left(\frac{f_0(x)}{f(x)}\right) f_0(x) \mathrm{d}x.$$

This quantity is positive, zero if $f_0 \in \mathcal{F}$.

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Model selection: Kullback criterium

We have:

$$\int \ln\left(\frac{f_0(x)}{f(x)}\right) f_0(x)dx = \mathbb{E}_{f_0}\left[\ln\left(\frac{f_0(X)}{f(X)}\right)\right]$$
$$= \mathbb{E}_{f_0}\left[\ln\left(f_0(X)\right)\right] - \mathbb{E}_{f_0}\left[\ln\left(f(X)\right)\right].$$

We need to minimize $-\mathbb{E}_{f_0}\left[\ln\left(f(X)\right)\right]$.

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Model selection: ARMA case

Based on $(X_1, \ldots, X_T)$, estimators of $-\mathbb{E}_{f_0}\left[\ln\left(f\left(X\right)\right)\right]$ for ARMA(p, q) processes can be written:

$$\widehat{C}\left(f_0, \mathcal{F}\right) = -\frac{1}{T}\ln\left(f\right) + \alpha\left(T\right)\left(p+q\right)$$

where α is a decreasing function.
If f is a gaussian, we obtain:

$$\widehat{C}\left(f_0, \mathcal{F}\right) = \ln\left(\widehat{\sigma}^2\right) + \alpha\left(T\right)\left(p+q\right)$$

where $\widehat{\sigma}^2$ is the estimated variance.

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Model selection: ARMA case

We generally consider:

- Akake criterium
  With $\alpha(T) = \frac{2}{T}$:

$$AIC(p,q) = \ln\left(\widehat{\sigma}^2\right) + 2\frac{p+q}{T}.$$

- Schwarz criterium
  With $\alpha(T) = \frac{\ln(T)}{T}$:

$$BIC(p,q) = \ln\left(\widehat{\sigma}^2\right) + (p+q)\frac{\ln(T)}{T}.$$

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Definition

$(X_t)_{t\in\mathbb{N}}$ is said to be an ARIMA process of order (p, d, q), written ARIMA(p, d, q), if:

$$\Phi(B)\nabla^d X_t = \Theta(B)\varepsilon_t$$

where:

- $\nabla^d = (I - B)^d$,
- $\Phi(B) = I - \varphi_1 B - \ldots - \varphi_p B^p$ where $(\varphi_1, \ldots, \varphi_p) \in \mathbb{R}^p$ and $\varphi_p \neq 0$,
- $\Theta(B) = I + \theta_1 B + \ldots + \theta_q B^q$ where $(\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$ and $\theta_q \neq 0$.

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Remarks

- An ARIMA process isn't defined on $\mathbb{Z}$ but on $\mathbb{N}$ by convention. Initial conditions are fixed with:

$$Z = (X_{-p}, \ldots, X_{-1}, \varepsilon_{-q}, \ldots, \varepsilon_{-1})^\top .$$

- ARIMA models can be applied on times series with a trend.
- $(I - B)^d X_t$ is asympotically equivalent to an ARMA(p, q) process.
- $(X_t)_{t \in \mathbb{N}}$ isn't a stationary process.

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Autoregressive and moving average representations

We can't obtain a AR (∞) nor MA (∞) representation but it's possible to obtain:

- Moving average representation:

$$X_t = \varepsilon_t + \sum_{i=1}^{t} \psi_i \varepsilon_{t-i} + \psi^* (t) Z$$

  where $\psi^* (t)$ is a $p + q$ dimensional vector.

- Autoregressive representation:

$$X_t = -\sum_{i=1}^{t} \pi_i X_{t-i} - \pi^* (t) Z + \varepsilon_t$$

  where $\pi^* (t)$ is a $p + q$ dimensional vector.

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Definition

$(X_t)_{t\in\mathbb{N}}$ is said to be a SARIMA process of order $(p,d,q)(P,D,Q)_s$ if:

$$\Phi(B)\,\Phi'(B^s)\,\nabla^d\nabla_s^D X_t = \Theta(B)\,\Theta'(B^s)\,\varepsilon_t$$

where:

- $\nabla^d = (I-B)^d$,
- $\nabla_s^D = (I-B^s)^D$,
- $\Phi(B) = I - \varphi_1 B - \ldots - \varphi_p B^p$ where $(\varphi_1,\ldots,\varphi_p)\in\mathbb{R}^p$ and $\varphi_p \neq 0$,
- $\Phi'(B) = I - \varphi'_1 B - \ldots - \varphi'_P B^P$ where $(\varphi'_1,\ldots,\varphi'_P)\in\mathbb{R}^P$ and $\varphi'_P \neq 0$,
- $\Theta(B) = I + \theta_1 B + \ldots + \theta_q B^q$ where $(\theta_1,\ldots,\theta_q)\in\mathbb{R}^q$ and $\theta_q \neq 0$,
- $\Theta'(B) = I + \theta'_1 B + \ldots + \theta'_Q B^Q$ where $(\theta'_1,\ldots,\theta'_Q)\in\mathbb{R}^Q$ and $\theta'_Q \neq 0$.

We write: SARIMA$(p,d,q)(P,D,Q)_s$.

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Remarks

- SARIMA models can be applied on times series with a trend and a seasonality.
- Estimation of a SARIMA model: estimation of an ARMA model on the differenced time series.

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Remark

The method is similar for SARIMA models than for ARIMA models.

To forecast X_{T+h} (with $h \in \mathbb{N}^*$) based on $(X_1, \ldots, X_T)$, we can use the autoregressive or the moving average representation.

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Forecasting with the autoregressive representation

$$
\begin{aligned}
&\widehat{X}_T(h) \\
&= \mathbb{E}\left(X_{T+h} \Big/ \mathcal{H}_1^{T*}(X)\right) \\
&= \mathbb{E}\left(-\sum_{i=1}^{T+h} \pi_i X_{T+h-i} - \pi^*(T+h) Z + \varepsilon_{T+h} \Big/ \mathcal{H}_1^{T*}(X)\right) \\
&= \mathbb{E}\left(-\sum_{i=1}^{h-1} \pi_i X_{T+h-i} - \sum_{i=h}^{T+h} \pi_i X_{T+h-i} - \pi^*(T+h) Z + \varepsilon_{T+h} \Big/ \mathcal{H}_1^{T*}(X)\right) \\
&= -\sum_{i=1}^{h-1} \pi_i \widehat{X}_T(h-i) - \sum_{i=1}^{h-1} \pi_i X_{T+h-i} - \pi^*(T+h) Z \\
&\simeq -\sum_{i=1}^{h-1} \pi_i \widehat{X}_T(h-i) - \sum_{i=1}^{h-1} \pi_i X_{T+h-i} \text{ car } \pi^*(T) \overset{T \to +\infty}{\longrightarrow} 0.
\end{aligned}
$$

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Forecasting with the moving average representation

$$
\begin{aligned}
\widehat{X}_T(h) &= \mathbb{E}\left(X_{T+h} \Big/ \mathcal{H}_1^{T*}(X)\right) \\
&= \mathbb{E}\left(\varepsilon_{T+h} + \sum_{i=1}^{T+h} \psi_i \varepsilon_{T+h-i} + \psi^*(T+h) Z \Big/ \mathcal{H}_1^{T*}(X)\right) \\
&= \sum_{i=h-1}^{T+h} \psi_i \varepsilon_{T+h-i} + \psi^*(T+h) Z \\
&\simeq \sum_{i=h-1}^{T+h} \psi_i \varepsilon_{T+h-i} \text{ car } \psi^*(T) \overset{T \to +\infty}{\longrightarrow} 0.
\end{aligned}
$$

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