Univariate time series

Discrete univariate stochastic processes

V. Lefieux

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Second-order processes
Stationary processes
Autocovariance function
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Time series

Let $(x_t)_{t \in \mathcal{T}}$ be a sequence of observations (for example in the fields of economics, life sciences, physics...).

Each observation x_t, in $\mathbb{R}^d$, is recorded at a specific time $t \in \mathcal{T}$.

$(x_t)_{t \in \mathcal{T}}$ is called time series.

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Discrete and continuous time series

Consider the set of times $\mathcal{T}$.

- $\mathcal{T}$ is a countable set (in general $\mathcal{T} \subset \mathbb{Z}$): discrete time series (for example when observations are made at fixed time intervals).
- $\mathcal{T}$ isn't a countable set (in general an interval of $\mathbb{R}$): continous time series.

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Univariate and multivariate time series

Consider $x_t \in \mathbb{R}^d$.

- $d = 1$: univariate time series.
- $d > 1$: multivariate time series.

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Remark

In this course "time series" refers to univariate discrete time series.

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Probabilistic model

- Observation x_t is considered as the realization of a random variable X_t.
- Time series $(x_t)_{t \in \mathcal{T}}$ is considered as the realization of a stochastic process $(X_t)_{t \in \mathcal{T}}$.

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Random variable

X is said to be a random variable if the X function:

$$\begin{aligned} X : (\Omega, \mathcal{A}, \mathbb{P}) &\to (\Omega', \mathcal{A}') \\ \omega &\mapsto X(\omega) \end{aligned}$$

is measurable, that means:

$$\forall A' \in \mathcal{A}' : X^{-1}(A') \in \mathcal{A}.$$

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Stochastic process

X is a stochastic process if the function:

$$X : (\Omega, \mathcal{A}, \mathbb{P}) \times \mathcal{T} \to (\Omega', \mathcal{A}')$$
$$(\omega, t) \mapsto X(\omega, t) = X_t(\omega)$$

is such that for all $t \in \mathcal{T}$, X_t is a random variable on the probabilistic space $(\Omega, \mathcal{A}, \mathbb{P})$.

$(\Omega', \mathcal{A}')$, a measurable space, is called state space.

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Remark

We consider stochastic processes with:

- $(\Omega', \mathcal{A}') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$,
- $\mathcal{T} = \mathbb{N}$ or $\mathcal{T} = \mathbb{Z}$.

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Stochastic process distribution

A stochastic process can be considered as a random variable X taking values in the product probabilistic space $(\Omega', \mathcal{A}')^{\otimes \mathcal{T}}$.

The distribution of the process X is the distribution of the random variable on $(\Omega', \mathcal{A}')^{\otimes \mathcal{T}} = \left((\Omega')^t, (\mathcal{A}')^{\otimes \mathcal{T}}\right)$.

Distributions of $(X_{t_1}, \ldots, X_{t_k})$ with $(t_1, \ldots, t_k) \in \mathcal{T}^k$ $(k \in \mathbb{N}^*)$: finite-dimensional distributions of X.

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White noises

- $(\varepsilon_t)_{t\in\mathbb{Z}}$ is said to be an i.i.d noise (strongly white noise) if $(\varepsilon_t)_{t\in\mathbb{Z}}$ are i.i.d and:

$$\forall t \in \mathbb{Z} : \mathbb{E}(\varepsilon_t) = 0,$$
$$\mathbb{E}\left(\varepsilon_t^2\right) = \sigma^2.$$

  (in this course: $\sigma^2 > 0$).
- A gaussian white noise is an i.i.d noise with distribution $\mathcal{N}\left(0, \sigma^2\right)$.
- $(\varepsilon_t)_{t\in\mathbb{Z}}$ is said to be a white noise (weakly white noise) if $(\varepsilon_t)_{t\in\mathbb{Z}}$ are squareintegrable random variables (in $\mathcal{L}^2$) and:

$$\forall t \in \mathbb{Z} : \mathbb{E}(\varepsilon_t) = 0 \ ,$$
$$\mathbb{E}\left(\varepsilon_t^2\right) = \sigma^2 \ ,$$
$$\forall \left(t, t'\right) \in \mathbb{Z}^2 / t \neq t' : \mathrm{Cov}\left(\varepsilon_t, \varepsilon_{t'}\right) = 0.$$

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Random walk

Let $(\varepsilon_t)_{t\in\mathbb{N}}$ a white noise.

$(S_t)_{t\in\mathbb{N}}$ is said to be a random walk if:

- $S_0 = 0$,
- $\forall t \in \mathbb{N}^* : S_t = \sum_{i=1}^{t} \varepsilon_i$.

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Markov chain

$(X_t)_{t\in\mathbb{Z}}$ is said to be a Markov chain of order p if:

$$\forall t \in \mathbb{Z} : d\left(X_t / (X_i)_{i<t}\right) = d\left(X_t / X_{t-1} \dots X_{t-p}\right).$$

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Gaussian process

$(X_t)_{t\in\mathbb{Z}}$ is said to be a gaussian process if all the margin distributions are gaussian:

$$\forall k \in \mathbb{N}^*, \forall (t_1, \ldots, t_k) \in \mathbb{Z}^k : (X_{t_1}, \ldots, X_{t_k})^\top \text{ is gaussian.}$$

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$\mathcal{L}^2$ space

$\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ is the set of all square integrable random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$, that means $\int X^2 \mathrm{d}\mathbb{P} < +\infty$.

$\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ is an Hilbert space (complete inner-product space) with the (almost) inner product :

$$\langle X, Y \rangle = \mathbb{E}(XY)$$

and the induced norm:

$$\|X\|_{\mathcal{L}^2} = \left[\mathbb{E}\left(X^2\right)\right]^{\frac{1}{2}}.$$

X and Y are said to be orthogonal random variables if:

$$\langle X, Y \rangle = 0.$$

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Cauchy-Schwarz inequality

Let X et Y two random variables in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
We have:

$$\|XY\|_{\mathcal{L}^1} \leq \|X\|_{\mathcal{L}^2} \cdot \|Y\|_{\mathcal{L}^2}$$

that is:

$$\mathbb{E}(|XY|) \leq \left[\mathbb{E}\left(X^2\right)\right]^{\frac{1}{2}} \left[\mathbb{E}\left(Y^2\right)\right]^{\frac{1}{2}}.$$

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Projection theorem

Let $\mathcal{H}$ be a subspace of $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
Consider $X \in \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
There is an unique random variable $\widehat{X} \in \mathcal{H}$ such that:

$$\left\| X - \widehat{X} \right\|_{\mathcal{L}^2} = \min_{Y \in \mathcal{H}} \|X - Y\|_{\mathcal{L}^2}.$$

$\widehat{X}$ is the projection of X on $\mathcal{H}$, also written $\Pi_{\mathcal{H}}(X)$.

We have $\widehat{X} \in \mathcal{H}$ and $X - \widehat{X} \in \mathcal{H}^{\perp}$.

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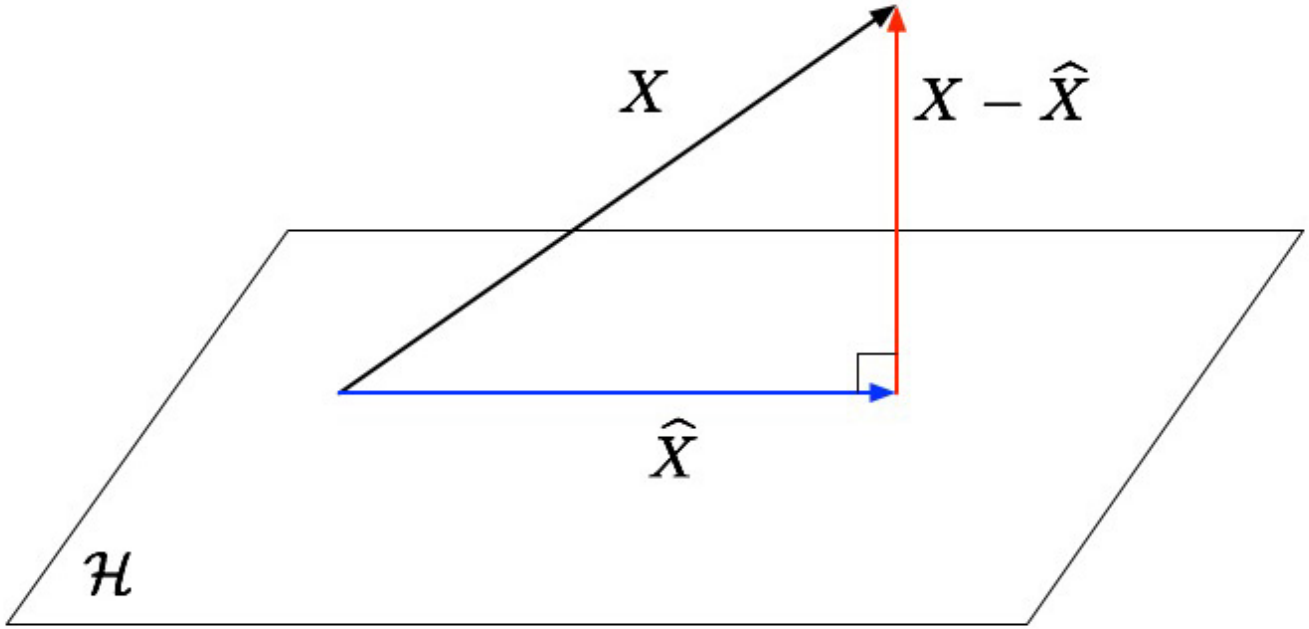
Projection theorem



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Mean square convergence

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
Let X be a random variable in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
$(X_n)_{n\in\mathbb{N}}$ is said to converge in mean square (converge in $\mathcal{L}^2$) to X, and we note $X_n \overset{\mathcal{L}^2}{\to} X$, if:

$$\|X_n - X\|_{\mathcal{L}^2} \xrightarrow{n\to+\infty} 0$$

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Proposition

Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of random variables in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
If $\sum_{i=-m}^{n} X_i$ converges in mean square (to $\sum_{i=-\infty}^{+\infty} X_i$) then:

$$\mathbb{E}\left(\sum_{i=-\infty}^{+\infty} X_i\right) = \sum_{i=-\infty}^{+\infty} \mathbb{E}(X_i).$$

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Proposition

Let $(X_n)_{n\in\mathbb{Z}}$ et $(Y_n)_{n\in\mathbb{Z}}$ be two sequences of random variables in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
If $\sum_{i=-m}^{n} X_i$ et $\sum_{j=-m'}^{n'} Y_j$ converge in mean square then:

$$\mathbb{E}\left(\sum_{i=-\infty}^{+\infty} X_i \sum_{j=-\infty}^{+\infty} Y_j\right) = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \mathbb{E}\left(X_i Y_j\right).$$

and:

$$\mathrm{Cov}\left(\sum_{i=-\infty}^{+\infty} X_i, \sum_{j=-\infty}^{+\infty} Y_j\right) = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \mathrm{Cov}\left(X_i, Y_j\right).$$

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Second order process

$(X_t)_{t\in\mathbb{Z}}$ is said to be a second-order process if:

$$\forall t \in \mathbb{Z} : \mathbb{E}\left(X_t^2\right) < +\infty.$$

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Strictly stationary processes

$(X_t)_{t\in\mathbb{Z}}$ is said to be strictly stationary if the joint distribution of $(X_{t_1}, \ldots, X_{t_k})$ is equal to the distribution of $(X_{t_1+h}, \ldots, X_{t_k+h})$, for $k \in \mathbb{N}^\star$, $(t_1, \ldots, t_k) \in \mathbb{Z}^k$ and $h \in \mathbb{Z}$.

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(Weakly) stationary processes

A second-order process $(X_t)_{t\in\mathbb{Z}}$ is weakly stationary, if the expectation $\mathbb{E}(x_t)$ and the (auto)covariances $\mathrm{Cov}(X_s, X_t)$ are time-shifted invariant:

- $\forall t \in \mathbb{Z} : \mathbb{E}(x_t) = \mu$
- $\forall (s,t) \in \mathbb{Z}^2, \forall h \in \mathbb{Z} :$

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+h}, X_{t+h}).$$

In this case we have:

$$\mathrm{Cov}(X_s, X_t) = \gamma(t-s).$$

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Remarks

- It's easier to consider weak than strict stationary.
- Stationarity isn't thus a simple concept. . .
- The sum of two stationary processes isn't necessarily stationary.
- In this course "stationary process" refers to weakly stationary process.

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Proposition

A second-order strictly stationary process is also weakly stationary.

Weak stationarity doesn't imply strict stationarity (except for gaussian processes).

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Examples

- An i.i.d white noise is strictly stationary.
- A (weakly) white noise is (weakly) stationary.
- A random walk isn't stationary.
- Time series with a trend and/or a seasonality can't be represented by stationary processes.

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Linear filter

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Let $(a_i)_{t\in\mathbb{Z}}$ be a sequence such that $\sum_{i\in\mathbb{Z}} |a_i| < +\infty$.
The process $(Y_t)_{t\in\mathbb{Z}}$ defined by $Y_t = \sum_{i\in\mathbb{Z}} a_i X_{t-i}$, is stationary and:

- $\mu_Y = \mu_X \sum_{i\in\mathbb{Z}} a_i$.
- Considering $\gamma_X(h) = \mathrm{Cov}(X_t, X_{t-h})$:

$$\gamma_Y(h) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} a_i a_j \gamma_X(h+i-j).$$

The transformation
$X_t \mapsto Y_t = \left(\sum_{i\in\mathbb{Z}} a_i B^i\right) X_t = \sum_{i\in\mathbb{Z}} a_i X_{t-i}$,
with $\sum_{i\in\mathbb{Z}} |a_i| < +\infty$, is called linear filter.

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Some non stationarities

- $(X_t)_{t\in\mathbb{Z}}$ is a non-stationary TS (Trend Stationary) process if we can write:

$$X_t = f(t) + Y_t$$

  where f is a deterministic function and $(Y_t)_{t\in\mathbb{Z}}$ is a stationary process.
- $(X_t)_{t\in\mathbb{Z}}$ is a non-stationary DS (Difference Stationary) process if the process becomes stationary after being differenced d times: $\nabla^d X_t = (I - B)^d X_t$ (where $BX_t = X_{t-1}$) is a stationary process.

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Definition

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
The autocovariance function of X is:

$$\forall h \in \mathbb{Z} : \gamma(h) = \mathrm{Cov}(X_t, X_{t-h}).$$

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Properties

- $\gamma(0) \geq 0$
- $\forall h \in \mathbb{Z} : |\gamma(h)| \leq \gamma(0)$.
- γ is even:
$$\forall h \in \mathbb{Z} : \gamma(-h) = \gamma(h).$$
- γ is a nonnegative definite function:
$$\forall n \in \mathbb{N}^*, \forall (a_i)_{i \in \{1,\ldots,n\}} \in \mathbb{R}^n : \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \gamma(i-j) \geq 0.$$

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Property

If a function γ checks:

- $\gamma(-h) = \gamma(h)$,
- $\forall n \in \mathbb{N}^*, \forall (a_i)_{i \in \{1,\ldots,n\}} \in \mathbb{R}^n$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \gamma(i-j) \geq 0$$

then it is an autocovariance function.

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Definition

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
We call (simple) autocorrelation function of X the following function ρ:

$$\forall h \in \mathbb{Z} : \rho(h) = \mathrm{Corr}(X_t, X_{t-h}) = \frac{\gamma(h)}{\gamma(0)}.$$

We have $\rho(0) = 1$.

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Autocorrelation matrix

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
The autocorrelation matrix of $(X_t, \ldots, X_{t-h+1})$ is:

$$R_h = \begin{bmatrix} 1 & \rho(1) & \ldots & \rho(h-1) \\ \rho(1) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho(1) \\ \rho(h-1) & \ldots & \rho(1) & 1 \end{bmatrix}.$$

Notice that:

$$R_h = \left[\begin{array}{ccc|c} & & & \rho(h-1) \\ & R_{h-1} & & \vdots \\ & & & \rho(1) \\ \hline \rho(h-1) & \ldots & \rho(1) & 1 \end{array}\right]$$

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Property

The two following assertions are equivalent:

1. ρ is a nonnegative definite function.
2. $\forall h \in \mathbb{N}^* : \det R_h \geq 0$

The second condition implies for example:

- $\det R_2 \geq 0 \Leftrightarrow \rho^2(1) \leq 1$.
- $\det R_3 \geq 0 \Leftrightarrow [1 - \rho(2)] \left[1 + \rho(2) - 2\rho^2(1)\right] \geq 0$.

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Partial autocorrelation function

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Let $\mathcal{H}_{t-h+1}^{t-1}(X)$ be the space spanned by linear combinations of $(X_i)_{i\in\{t-h+1,\dots,t-1\}}$ and 1, $h \in \mathbb{N}\setminus\{0,1\}$.
Let $\mathbb{E}\left(X_t \,/\mathcal{H}_{t-h+1}^{t-1}(X)\right)$ and $\mathbb{E}\left(X_{t-h} \,/\mathcal{H}_{t-h+1}^{t-1}(X)\right)$ be respectively the linear regressions of X_t and X_{t-h} on $X_{t-1},\dots,X_{t-h+1}$.
We call partial autocorrelation function of X the function r such that $r(0)=1$, $r(1)=\rho(1)$ and $\forall h \in \mathbb{N}\setminus\{0,1\}$:

$$
\begin{aligned}
&r(h)\\
&= \mathrm{Corr}\left(X_t, X_{t-h} \,/ X_{t-1},\dots,X_{t-h+1}\right)\\
&= \frac{\mathrm{Cov}\left(X_t - \mathbb{E}\left(X_t \,/\mathcal{H}_{t-h+1}^{t-1}(X)\right), X_{t-h} - \mathbb{E}\left(X_{t-h} \,/\mathcal{H}_{t-h+1}^{t-1}(X)\right)\right)}{\sqrt{\mathrm{Var}\left(X_t - \mathbb{E}\left(X_t \,/\mathcal{H}_{t-h+1}^{t-1}(X)\right)\right)\mathrm{Var}\left(X_{t-h} - \mathbb{E}\left(X_{t-h} \,/\mathcal{H}_{t-h+1}^{t-1}(X)\right)\right)}}.
\end{aligned}
$$

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Theorem

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Consider the linear regression of X_t on $X_{t-1}, \ldots, X_{t-h}$ for $h \in \mathbb{N}^*$:

$$\begin{aligned} X_t &= \mathbb{E}\left(X_t \,/ \mathcal{H}_{t-h}^{t-1}(X)\right) + \varepsilon_t \\ &= \sum_{i=1}^{h} a_i(h) X_{t-i} + \varepsilon_t \end{aligned}$$

where:

- $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise with variance σ^2,
- $\forall i \in \{1, \ldots, h\} : \mathbb{E}(\varepsilon_t X_{t-i}) = 0$.

The last coefficient is such that $a_h(h) = r(h)$.

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Property

Considering the same regression:

$$X_t = \sum_{i=1}^{h} a_i(h) X_{t-i} + \varepsilon_t,$$

we have:

$$\begin{pmatrix} \rho(1) \\ \vdots \\ \rho(h) \end{pmatrix} = R_h \begin{pmatrix} a_1(h) \\ \vdots \\ a_h(h) \end{pmatrix}.$$

We can estimate $(a_1(h), \ldots, a_h(h))$ based on an estimation of $(\rho(1), \ldots, \rho(h))$, and thus have an estimation of de $r(h)$.

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Durbin-Levinson algorithm

With the Durbin-Levinson algorithm, partial autocorrelations can be recursively computed from the following equations:

- $a_1(1) = \rho(1)$,
- $\forall h \in \mathbb{N} \setminus \{0, 1\}, \forall i \in \{1, \ldots, h-1\}$:

$$a_i(h) = a_i(h-1) - a_h(h)\, a_{h-i}(h-1),$$

- $\forall h \in \mathbb{N} \setminus \{0, 1\}$:

$$a_h(h) = \frac{\rho(h) - \sum_{i=1}^{h-1} \rho(h-i)\, a_i(h-1)}{1 - \sum_{i=1}^{h-1} \rho(i)\, a_i(h-1)}.$$

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Estimation of the mean

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Based on $(X_1, \ldots, X_T)$, $\overline{X}_T$ is a consistent and unbiased estimator of $\mathbb{E}(X) = \mu$:

$$\overline{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t.$$

We have:

$$\begin{aligned}
\mathbb{E}\left(\overline{X}_T\right) &= \mu, \\
\mathbb{E}\left[\left(\overline{X}_T - \mu\right)^2\right] &= \mathrm{Var}\left(\overline{X}_T\right) \\
&= \frac{1}{T}\sum_{|h|<T}\left(1 - \frac{|h|}{T}\right)\gamma(h).
\end{aligned}$$

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Property

- If $\gamma(h) \xrightarrow{h \to +\infty} 0$ then:

$$\mathrm{Var}\left(\overline{X}_T\right) \xrightarrow{T \to +\infty} 0.$$

- If $\sum_{h=-\infty}^{+\infty} |\gamma(h)| < +\infty$ then:

$$T\,\mathrm{Var}\left(\overline{X}_T\right) \xrightarrow{T \to +\infty} \sum_{h=-\infty}^{+\infty} \gamma(h).$$

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Estimation of the autocorrelation functions

$\forall h \in \{1, \ldots, T-1\}$:

$$\widehat{\rho}(h) = \frac{\sum_{t=h+1}^{T} \left(X_t - \overline{X}_T\right)\left(X_{t-h} - \overline{X}_T\right)}{\sum_{t=1}^{T} \left(X_t - \overline{X}_T\right)^2}.$$

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Remarks

- $\widehat{\gamma}(h)$ and $\widehat{\rho}(h)$ are consistent but biased estimators.
- In general we consider that we can estimate the $\frac{T}{4}$ first autocorrelations.
- Partial autocorrelation function estimation is obtained with the Durbin-Levinson algorithm.
- Warning: calculations are done by softwares even if the process isn't stationary.

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Portmanteau test

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Consider the test:

$$\begin{cases} H_0 : (X_t)_{t\in\mathbb{Z}} \text{ is a white noise} \\ H_1 : (X_t)_{t\in\mathbb{Z}} \text{ isn't a white noise} \end{cases}.$$

Based on $(X_1, \ldots, X_T)$, the Portmanteau statistic is:

$$Q_k = T \sum_{h=1}^{k} \widehat{\rho}^2(h)$$

Under H_0: $Q_k \xrightarrow{d} \chi^2_k$.
So we reject the null hypothesis at the α level if $Q_k > \chi^2_k(1-\alpha)$.
One can find other statistics, such the LjungBox one:

$$Q_k^* = T(T+2) \sum_{h=1}^{k} \frac{\widehat{\rho}^2(h)}{T-h}.$$

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Shapiro-Wilk test

Consider the test:

$$\begin{cases} H_0 : (X_1, \ldots, X_n) \text{ comes from a normal distribution} \\ H_1 : (X_1, \ldots, X_n) \text{ doesn't come from a normal distribution} \end{cases}$$

The Shapiro-Wilk statistic is:

$$W = \frac{\left(\sum_{i=1}^{n} a_i X_{(i)}\right)^2}{\sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2}$$

where $X_{(i)}$ is the i-th order statistic. Coefficients $(a_i)_{i \in \{1,\ldots,n\}}$ are constants generated from the means, variances and covariances of the order statistics of a sample of size n from a normal distribution.
We reject the null hypothesis at the α level if:

$$W < W_{n,\alpha}^{threshold}.$$

$W_{n,\alpha}^{threshold}$ is obtained with Monte Carlo simulations.

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Definition

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process with γ as autocovariance function.
If $\sum_{h=-\infty}^{+\infty} |\gamma(h)| < +\infty$, we define the spectral density of $(X_t)_{t\in\mathbb{Z}}$ as the function f:

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{+\infty} \gamma(h) \exp(-ih\omega).$$

f is a continous, nonnegative, even and 2π-periodic function.

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Property

If f is the spectral density of $(X_t)_{t\in\mathbb{Z}}$ then:

$$\gamma(h) = \int_{-\pi}^{\pi} f(\omega) \exp(ih\omega) \, \mathrm{d}\omega.$$

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Example

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise with variance σ^2.
We have:

$$\gamma_\varepsilon(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases}.$$

So:

$$f_\varepsilon(\omega) = \frac{\sigma^2}{2\pi}.$$

Moreover, if the spectral density doesn't depend on the frequency then the associated process is a white noise.

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Linear filter spectral density

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process with a spectral density. Consider the process $(Y_t)_{t\in\mathbb{Z}}$ such that:

$$Y_t = \sum_{i=-\infty}^{+\infty} a_i X_{t-i}$$

where:

$$\sum_{i=-\infty}^{+\infty} |a_i| < +\infty.$$

Then:

$$f_Y(\omega) = f_X(\omega) \left| \sum_{j=-\infty}^{+\infty} a_j e^{-i\omega j} \right|^2 .$$

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Periodogram

Based on $(X_1, \ldots, X_T)$, we call periodogram the following function:

$$I_T(\omega) = \frac{1}{T} \left| \sum_{t=1}^{T} X_t e^{-it\omega} \right|^2 .$$

If the spectral density of $(X_t)_{t \in \mathbb{Z}}$ exists, then $\frac{1}{2\pi} I_T(\omega)$ is an unbiased but non consistent estimation of its spectral density.

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Discrete spectral average estimator

We consider the discrete spectral average estimator:

$$\widehat{f}(\omega) = \frac{1}{2\pi} \sum_{|j| \leq m_T} W_T(j) \, I_T\left(g(T,\omega) + \frac{2\pi j}{T}\right)$$

where:

- $g(T,\omega)$ is the multiple of $\frac{2\pi}{T}$ closest to ω,
- $m_T \xrightarrow{T \to +\infty} +\infty$, $\frac{m_T}{T} \xrightarrow{T \to +\infty} 0$,
- $\forall j \in \mathbb{Z} : W_T(j) \geq 0$ and $W_T(-j) = W_T(j)$,
- $\sum_{|j| \leq m_T} W_T(j) = 1$, $\sum_{|j| \leq m_T} W_T^2(j) \xrightarrow{T \to +\infty} 0$.

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Inverse autocovariance function

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process with γ as autocovariance function.
We call inverse autocovariance function of $(X_t)_{t\in\mathbb{Z}}$ the autocovariance function associated to the inverse spectral density $1/f$:

$$\frac{1}{f(\omega)} = \frac{1}{2\pi} \sum_{h=-\infty}^{+\infty} \gamma^i(h) \exp(-ih\omega)$$

where:

$$\forall h \in \mathbb{Z} : \gamma^i(h) = \int_{-\pi}^{\pi} \frac{1}{f(\omega)} \exp(ih\omega)\, \mathrm{d}\omega.$$

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Inverse autocorrelation function

We call inverse autocorrelation function the following function:

$$\forall h \in \mathbb{Z} : \rho^i(h) = \frac{\gamma^i(h)}{\gamma^i(0)}.$$

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