Univariate time series

Discrete univariate stochastic processes

V. Lefieux

Table of contents

Table of contents

Time series

Let $(x_t)_{t \in \mathcal{T}}$ be a sequence of observations (for example in the fields of economics, life sciences, physics...).

Each observation x_t, in $\mathbb{R}^d$, is recorded at a specific time $t \in \mathcal{T}$.

$(x_t)_{t \in \mathcal{T}}$ is called time series.

Discrete and continuous time series

Consider the set of times $\mathcal{T}$.

- $\mathcal{T}$ is a countable set (in general $\mathcal{T} \subset \mathbb{Z}$) : discrete time series. For example : when observations are made at fixed time intervals.
- $\mathcal{T}$ isn't a countable set (in general an interval of $\mathbb{R}$) : continous time series.

Univariate and multivariate time series

Consider $x_t \in \mathbb{R}^d$.

- $d = 1$: univariate time series.
- $d > 1$: multivariate time series.

Remark

In this course "time series" refers to univariate discrete time series.

Probabilistic model

- Observation x_t is considered as the realization of a random variable X_t.
- Time series $(x_t)_{t \in \mathcal{T}}$ is considered as the realization of a stochastic process $(X_t)_{t \in \mathcal{T}}$.

Random variable

X is said to be a random variable if the X function :

$$\begin{aligned} X : (\Omega, \mathcal{A}, \mathbb{P}) &\to (\Omega', \mathcal{A}') \\ \omega &\mapsto X(\omega) \end{aligned}$$

is measurable, that means :

$$\forall A' \in \mathcal{A}' : X^{-1}(A') \in \mathcal{A}.$$

Stochastic process

X is a stochastic process if the function :

$$\begin{aligned} X : (\Omega, \mathcal{A}, \mathbb{P}) \times \mathcal{T} &\to (\Omega', \mathcal{A}') \\ (\omega, t) &\mapsto X(\omega, t) = X_t(\omega) \end{aligned}$$

is such that for all $t \in \mathcal{T}$, X_t is a random variable on the probabilistic space $(\Omega, \mathcal{A}, \mathbb{P})$.

$(\Omega', \mathcal{A}')$, a measurable space, is called state space.

Remark

We consider stochastic processes with :

- $(\Omega', \mathcal{A}') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$,
- $\mathcal{T} = \mathbb{N}$ or $\mathcal{T} = \mathbb{Z}$.

Stochastic process distribution

A stochastic process can be considered as a random variable X taking values in the product probabilistic space $(\Omega', \mathcal{A}')^{\otimes \mathcal{T}}$.

The distribution of the process X is the distribution of the random variable on $(\Omega', \mathcal{A}')^{\otimes \mathcal{T}} = \left((\Omega')^t, (\mathcal{A}')^{\otimes \mathcal{T}} \right)$.

Distributions of $(X_{t_1}, \ldots, X_{t_k})$ with $(t_1, \ldots, t_k) \in \mathcal{T}^k$ ($k \in \mathbb{N}^*$) : finite-dimensional distributions of X.

White noises

- $(\varepsilon_t)_{t\in\mathbb{Z}}$ is said to be an i.i.d noise (strongly white noise) if $(\varepsilon_t)_{t\in\mathbb{Z}}$ are i.i.d and :

$$\begin{aligned} \forall t \in \mathbb{Z} : \mathbb{E}\left(\varepsilon_t\right) &= 0 \ , \\ \mathbb{E}\left(\varepsilon_t^2\right) &= \sigma^2 \ . \end{aligned}$$

  (in this course : $\sigma^2 > 0$).
- A gaussian white noise is an i.i.d noise with distribution $\mathcal{N}\left(0, \sigma^2\right)$.
- $(\varepsilon_t)_{t\in\mathbb{Z}}$ is said to be a white noise (weakly white noise) if $(\varepsilon_t)_{t\in\mathbb{Z}}$ are square–integrable random variables (in $\mathcal{L}^2$) and :

$$\begin{aligned} \forall t \in \mathbb{Z} : \mathbb{E}\left(\varepsilon_t\right) &= 0 \ , \\ \mathbb{E}\left(\varepsilon_t^2\right) &= \sigma^2 \ , \\ \forall \left(t, t'\right) \in \mathbb{Z}^2 / t \neq t' \ : \mathrm{Cov}\left(\varepsilon_t, \varepsilon_{t'}\right) &= 0 \ . \end{aligned}$$

Random walk

Let $(\varepsilon_t)_{t\in\mathbb{N}}$ a white noise.

$(S_t)_{t\in\mathbb{N}}$ is said to be a random walk if :

- $S_0 = 0$,
- $\forall t \in \mathbb{N}^* : S_t = \sum_{i=1}^{t} \varepsilon_i$.

Markov chain

$(x_t)_{t\in\mathbb{Z}}$ is said to be a Markov chain (order p) if :

$$\forall t \in \mathbb{Z} : \mathcal{L}\left(X_t/\left(X_i\right)_{i<t}\right) = \mathcal{L}\left(X_t/X_{t-1}\dots X_{t-p}\right).$$

Gaussian process

$(X_t)_{t\in\mathbb{Z}}$ is said to be a gaussian process if all the margin distributions are gaussian :

$$\forall k \in \mathbb{N}^*, \forall (t_1, \ldots, t_k) \in \mathbb{Z}^k : (X_{t_1}, \ldots, X_{t_k})^\top \text{ is gaussian.}$$

Table of contents

$\mathcal{L}^2$ space

$\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ is the set of all square integrable random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$, that means $\int X^2 \mathrm{d}\mathbb{P} < +\infty$.

$\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ is an Hilbert space (complete inner-product space) with the (almost) inner product :

$$\langle X, Y \rangle = \mathbb{E}(XY)$$

and the induced norm :

$$\|X\|_{\mathcal{L}^2} = \left[\mathbb{E}\left(X^2\right)\right]^{\frac{1}{2}}.$$

X and Y are said to be orthogonal random variables if :

$$\langle X, Y \rangle = 0.$$

Cauchy-Schwarz inequality

Let X et Y two random variables in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
We have :

$$\|XY\|_{\mathcal{L}^1} \leq \|X\|_{\mathcal{L}^2} . \|Y\|_{\mathcal{L}^2}$$

so :

$$\mathbb{E}(|XY|) \leq \left[\mathbb{E}\left(X^2\right)\right]^{\frac{1}{2}} \left[\mathbb{E}\left(Y^2\right)\right]^{\frac{1}{2}} .$$

Projection theorem

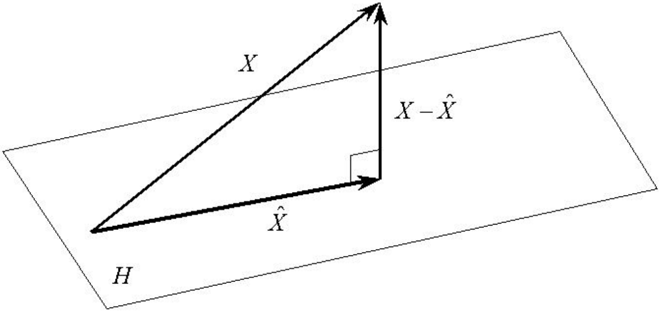
Let $\mathcal{H}$ be a subspace of $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
Consider $X \in \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
There is an unique random variable $\widehat{X} \in \mathcal{H}$ such that :

$$\left\|X - \widehat{X}\right\|_{\mathcal{L}^2} = \min_{Y \in \mathcal{H}} \|X - Y\|_{\mathcal{L}^2}.$$

$\widehat{X}$ is the projection of X on $\mathcal{H}$, also written $\Pi_{\mathcal{H}}(X)$.

We have $\widehat{X} \in \mathcal{H}$ and $X - \widehat{X} \in \mathcal{H}^{\perp}$.

Projection theorem



Mean square convergence

Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
Let X be a random variable in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
$(X_n)_{n \in \mathbb{N}}$ is said to converge in mean square (converge in $\mathcal{L}^2$) towards X, and we note $X_n \overset{\mathcal{L}^2}{\to} X$, if :

$$\|X_n - X\|_{\mathcal{L}^2} \xrightarrow{n \to +\infty} 0$$

Proposition

Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of random variables in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
If $\sum_{i=-m}^{n} X_i$ converges in mean square (towards $\sum_{i=-\infty}^{+\infty} X_i$) then :

$$\mathbb{E}\left(\sum_{i=-\infty}^{+\infty} X_i\right) = \sum_{i=-\infty}^{+\infty} \mathbb{E}(X_i).$$

Proposition

Let $(X_n)_{n\in\mathbb{Z}}$ et $(Y_n)_{n\in\mathbb{Z}}$ be two sequences of random variables in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.
If $\sum_{i=-m}^{n} X_i$ et $\sum_{j=-m'}^{n'} Y_j$ converge in mean square then :

$$\mathbb{E}\left(\sum_{i=-\infty}^{+\infty} X_i \sum_{j=-\infty}^{+\infty} Y_j\right) = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \mathbb{E}\left(X_i Y_j\right).$$

and :

$$\mathrm{Cov}\left(\sum_{i=-\infty}^{+\infty} X_i, \sum_{j=-\infty}^{+\infty} Y_j\right) = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \mathrm{Cov}\left(X_i, Y_j\right).$$

Second order process

$(X_t)_{t\in\mathbb{Z}}$ is said to be a second-order process if :

$$\forall t \in \mathbb{Z} : \mathbb{E}\left(X_t^2\right) < +\infty.$$

Table of contents

Strictly stationary processes

$(X_t)_{t\in\mathbb{Z}}$ is said to be strictly stationary if the joint distribution of $(X_{t_1},\dots,X_{t_k})$ is equal to the distribution of $(X_{t_1+h},\dots,X_{t_k+h})$, for $k\in\mathbb{N}^\star$, $(t_1,\dots,t_k)\in\mathbb{Z}^k$ and $h\in\mathbb{Z}$.

(Weakly) stationary processes

A second-order process $(x_t)_{t\in\mathbb{Z}}$ is weakly stationary, if the expectation $\mathbb{E}\left(x_t\right)$ and the (auto)covariances $\mathrm{Cov}(X_s, X_t)$ are time-shifted invariant :

- $\forall t \in \mathbb{Z} : \mathbb{E}\left(x_t\right) = \mu$
- $\forall \left(s, t\right) \in \mathbb{Z}^2, \forall h \in \mathbb{Z} :$

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+h}, X_{t+h}).$$

In this case we have :

$$\mathrm{Cov}\left(X_s, X_t\right) = \gamma\left(t - s\right).$$

Remarks

- It's easier to consider weak than strict stationary.
- Stationarity isn't a so simple concept...
- The sum of two stationary processes isn't necessarily stationary.
- In this course "stationary process" refers to weakly stationary process.

Proposition

A second-order strictly stationary process is also weakly stationary.

Weak stationarity doesn't imply strict stationarity (except for gaussian processes).

Examples

- An i.i.d white noise is strictly stationary.
- A (weakly) white noise is (weakly) stationary.
- A random walk isn't stationary.
- Time series with a trend and/or a seasonality can't be represented by stationary processes.

Linear filter

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Let $(a_i)_{t\in\mathbb{Z}}$ be a sequence such that $\sum_{i\in\mathbb{Z}} |a_i| < +\infty$.
The process $(Y_t)_{t\in\mathbb{Z}}$ defined by $Y_t = \sum_{i\in\mathbb{Z}} a_i X_{t-i}$, is stationary and :

- $\mu_Y = \mu_X \sum_{i\in\mathbb{Z}} a_i$.
- Considering $\gamma_X(h) = \mathrm{Cov}(X_t, X_{t-h})$:

$$\gamma_Y(h) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} a_i a_j \gamma_X(h+i-j).$$

The transformation
$X_t \mapsto Y_t = \left(\sum_{i\in\mathbb{Z}} a_i B^i\right) X_t = \sum_{i\in\mathbb{Z}} a_i X_{t-i}$,
with $\sum_{i\in\mathbb{Z}} |a_i| < +\infty$, is called linear filter.

Some non-stationarities

- $(X_t)_{t\in\mathbb{Z}}$ is a non-stationary TS (Trend Stationary) process if we can write :

$$X_t = f(t) + Y_t$$

  where f is a deterministic function and $(Y_t)_{t\in\mathbb{Z}}$ is a stationary process.

- $(X_t)_{t\in\mathbb{Z}}$ is a non-stationary DS (Difference Stationary) process if the process becomes stationary after being differenced d times : $\nabla^d X_t = (I - B)^d X_t$ (where $BX_t = X_{t-1}$) is a stationary process.

Table of contents

Definition

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
The autocovariance function of X is the following γ function :

$$\forall h \in \mathbb{Z} : \gamma(h) = \mathrm{Cov}(X_t, X_{t-h}).$$

Properties

- $\gamma(0) \geq 0$
- $\forall h \in \mathbb{Z} : |\gamma(h)| \leq \gamma(0)$.
- γ is even :

$$\forall h \in \mathbb{Z} : \gamma(-h) = \gamma(h).$$

- γ is a nonnegative definite function :

$$\forall n \in \mathbb{N}^*, \forall (a_i)_{i \in \{1,\ldots,n\}} \in \mathbb{R}^n : \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \gamma(i-j) \geq 0.$$

Property

If a function γ checks :

- $\gamma(-h) = \gamma(h)$,
- $\forall n \in \mathbb{N}^*, \forall (a_i)_{i \in \{1,\ldots,n\}} \in \mathbb{R}^n$: $\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \gamma(i-j) \geq 0$,

then it is an autocovariance function.

Table of contents

Definition

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
We call (simple) autocorrelation function of X the following function ρ :

$$\forall h \in \mathbb{Z} : \rho(h) = \text{Corr}(X_t, X_{t-h}) = \frac{\gamma(h)}{\gamma(0)}.$$

We have $\rho(0) = 1$.

Autocorrelation matrix

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
The autocorrelation matrix of $(X_t, \ldots, X_{t-h+1})$ is :

$$R_h = \begin{bmatrix} 1 & \rho(1) & \ldots & \rho(h-1) \\ \rho(1) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho(1) \\ \rho(h-1) & \ldots & \rho(1) & 1 \end{bmatrix}.$$

We have :

$$R_h = \left[\begin{array}{ccc|c} & & & \rho(h-1) \\ & R_{h-1} & & \vdots \\ & & & \rho(1) \\ \hline \rho(h-1) & \ldots & \rho(1) & 1 \end{array}\right]$$

Property

The two following assertions are equivalent :

1. ρ is a nonnegative definite function.
2. $\forall h \in \mathbb{N}^* : \det R_h \geq 0$

The second condition implies for example :

- $\det R_2 \geq 0 \Leftrightarrow \rho^2(1) \leq 1.$
- $\det R_3 \geq 0 \Leftrightarrow [1 - \rho(2)] \left[1 + \rho(2) - 2\rho^2(1)\right] \geq 0.$

Partial autocorrelation function

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Let $\mathcal{H}_{t-h+1}^{t-1}(X)$ be the space spanned by linear combinations of $(X_i)_{i\in\{t-h+1,\dots,t-1\}}$ and 1, $h\in\mathbb{N}\setminus\{0,1\}$.
Let $\mathbb{E}\left(X_t\,/\mathcal{H}_{t-h+1}^{t-1}(X)\right)$ and $\mathbb{E}\left(X_{t-h}\,/\mathcal{H}_{t-h+1}^{t-1}(X)\right)$ be respectively the linear regressions of X_t and X_{t-h} on $X_{t-1},\dots,X_{t-h+1}$.
We call partial autocorrelation function of X the function r such that $r(0)=1$, $r(1)=\rho(1)$ and :

$$
\begin{aligned}
\forall h\in\mathbb{N}\setminus\{0,1\} : r(h) &= \mathrm{Corr}\left(X_t, X_{t-h}\,/X_{t-1},\dots,X_{t-h+1}\right)\\
&= \frac{\mathrm{Cov}\left(\varepsilon_t,\varepsilon_{t-h}\right)}{\mathrm{Var}\left(\varepsilon_t\right)}
\end{aligned}
$$

where :

$$
\begin{aligned}
\varepsilon_t &= X_t - \mathbb{E}\left(X_t\,/\mathcal{H}_{t-h+1}^{t-1}(X)\right),\\
\varepsilon_{t-h} &= X_{t-h} - \mathbb{E}\left(X_{t-h}\,/\mathcal{H}_{t-h+1}^{t-1}(X)\right).
\end{aligned}
$$

Theorem

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Consider the linear regression of X_t on $X_{t-1},\ldots,X_{t-h}$, $h\in\mathbb{N}^*$:

$$\begin{aligned} X_t &= \mathbb{E}\left(X_t \,/\mathcal{H}_{t-h}^{t-1}(X)\right) + \varepsilon_t \\ &= \sum_{i=1}^{h} a_i(h)\, X_{t-i} + \varepsilon_t \end{aligned}$$

where :

- $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise with variance σ^2,
- $\forall i \in \{1,\ldots,h\} : \mathbb{E}\left(\varepsilon_t X_{t-i}\right) = 0.$

The last coefficient is such that $a_h(h) = r(h)$.

Property

Consider the same regression :

$$X_t = \sum_{i=1}^{h} a_i\,(h)\,X_{t-i} + \varepsilon_t$$

We have :

$$\begin{pmatrix} \rho\,(1) \\ \vdots \\ \rho\,(h) \end{pmatrix} = R_h \begin{pmatrix} a_1\,(h) \\ \vdots \\ a_h\,(h) \end{pmatrix}.$$

We can estimate $(a_1\,(h)\,, \ldots, a_h\,(h))$ based on an estimation of $(\rho\,(1)\,, \ldots, \rho\,(h))$, and so have an estimation of de $r\,(h)$.

Durbin-Levinson algorithm

With the Durbin-Levinson algorithm, partial autocorrelations can be recursively computed from the following equations :

- $a_1(1) = \rho(1)$,
- $\forall h \in \mathbb{N} \setminus \{0,1\}, \forall i \in \{1,\ldots,h-1\}$:

$$a_i(h) = a_i(h-1) - a_h(h)\, a_{h-i}(h-1),$$

- $\forall h \in \mathbb{N} \setminus \{0,1\}$:

$$a_h(h) = \frac{\rho(h) - \sum_{i=1}^{h-1} \rho(h-i)\, a_i(h-1)}{1 - \sum_{i=1}^{h-1} \rho(i)\, a_i(h-1)}.$$

Table of contents

Estimation of the mean

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Based on $(X_1,\ldots,X_T)$, $\overline{X}_T$ is a consistent and unbiased estimator of $\mathbb{E}(X)=\mu$:

$$\overline{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t.$$

We have :

$$\begin{aligned}\mathbb{E}\left(\overline{X}_T\right) &= \mu,\\ \mathbb{E}\left[\left(\overline{X}_T-\mu\right)^2\right] &= \mathrm{Var}\left(\overline{X}_T\right)\\ &= \frac{1}{T}\sum_{|h|<T}\left(1-\frac{|h|}{T}\right)\gamma(h).\end{aligned}$$

Property

- If $\gamma(h) \overset{h\to+\infty}{\longrightarrow} 0$ then :

$$\mathrm{Var}\left(\overline{X}_T\right) \overset{T\to+\infty}{\longrightarrow} 0.$$

- If $\sum_{h=-\infty}^{+\infty} |\gamma(h)| < +\infty$ then :

$$T\,\mathrm{Var}\left(\overline{X}_T\right) \overset{T\to+\infty}{\longrightarrow} \sum_{h=-\infty}^{+\infty} \gamma(h).$$

Estimation of the autocorrelation functions

$\forall h \in \{1, \ldots, T-1\}$:

$$\widehat{\rho}(h) = \frac{\sum_{t=h+1}^{T} \left(X_t - \overline{X}_T\right)\left(X_{t-h} - \overline{X}_T\right)}{\sum_{t=1}^{T} \left(X_t - \overline{X}_T\right)^2}.$$

Remarks

- $\widehat{\gamma}(h)$ and $\widehat{\rho}(h)$ are consistent but biased estimators.
- In general we consider that we can estimate the $\frac{T}{4}$ first autocorrelations.
- Partial autocorrelation function estimation is obtained with the Durbin-Levinson algorithm.
- Warning : calculations are done by softwares even if the process isn't stationary.

Table of contents

Portmanteau test

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Consider the test :

$$\begin{cases} H_0 : (X_t)_{t\in\mathbb{Z}} \text{ is a white noise} \\ H_1 : (X_t)_{t\in\mathbb{Z}} \text{ isn't a white noise} \end{cases}.$$

Based on $(X_1, \ldots, X_T)$, the Portmanteau statistic is :

$$Q_k = T \sum_{h=1}^{k} \widehat{\rho}^2 (h)$$

Q_k converges to the χ_k^2 distribution.
So we reject the null hypothesis at the α level if $Q_k > \chi_k^2 (1-\alpha)$.
One can find other statistics, such the Ljung–Box one :

$$Q_k^* = T (T+2) \sum_{h=1}^{k} \frac{\widehat{\rho}^2 (h)}{T-h}.$$

Shapiro-Wilk test

Consider the test :

$$\begin{cases} H_0 : (X_1, \ldots, X_n) \text{ is gaussian} \\ H_1 : (X_1, \ldots, X_n) \text{ isn't gaussian} \end{cases}.$$

The Shapiro-Wilk statistic is :

$$W = \frac{\left(\sum_{i=1}^{\left[\frac{n}{2}\right]} a_i \left(X_{(n-i+1)} - X_{(i)}\right)\right)^2}{\sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2}$$

where $X_{(i)}$ is the i-th order statistic and $[x]$ the integer part of x. Coefficients $(a_i)_{i\in\{1,\ldots,n\}}$ are computed in softwares . We reject the null hypothesis at the α level if :

$$W < W_{n,\alpha}^{threshold}.$$

$W_{n,\alpha}^{threshold}$ can be found in statistics tables or softwares (with the p-value).

Table of contents

Stochastic processes

Second-order processes

Stationary processes

Autocovariance function

Autocorrelation functions

Estimation of the mean and autocorrelation functions

Tests for randomness of the residuals

Spectral density

Definition

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process with γ as autocovariance function.
If $\sum_{h=-\infty}^{+\infty} |\gamma(h)| < +\infty$, we define the spectral density of $(X_t)_{t\in\mathbb{Z}}$ as the function f :

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{+\infty} \gamma(h) \exp(-ih\omega).$$

f is a continous, nonnegative, even and 2π-periodic function.

Property

If f is the spectral density of $(X_t)_{t\in\mathbb{Z}}$ then :

$$\gamma(h) = \int_{-\pi}^{\pi} f(\omega) \exp(ih\omega)\, \mathrm{d}\omega.$$

Example

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise with variance σ^2.
We have :

$$\gamma_\varepsilon(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases}.$$

So :

$$f_\varepsilon(\omega) = \frac{\sigma^2}{2\pi}.$$

Moreover, if the spectral density doesn't depend on the frequency then the associated process is a white noise.

Linear filter spectral density

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process with a spectral density. Consider the linear filter process $(Y_t)_{t\in\mathbb{Z}}$ such that :

$$Y_t = \sum_{i=-\infty}^{+\infty} a_i X_{t-i}$$

where :

$$\sum_{i=-\infty}^{+\infty} |a_i| < +\infty.$$

Then :

$$f_Y(\omega) = f_X(\omega) \left| \sum_{j=-\infty}^{+\infty} a_j e^{-i\omega j} \right|^2 .$$

Periodogram

Based on $(X_1, \ldots, X_T)$, we call periodogram the following function :

$$I_T(\omega) = \frac{1}{T}\left|\sum_{t=1}^{T} X_t e^{-it\omega}\right|^2 .$$

If the spectral density of $(X_t)_{t\in\mathbb{Z}}$ exists, then $\frac{1}{2\pi} I_T(\omega)$ is an unbiased but non consistent estimation of this spectral density.

Discrete spectral average estimator

We consider the discrete spectral average estimator :

$$\widehat{f}(\omega) = \frac{1}{2\pi} \sum_{|j| \leq m_T} W_T(j)\, I_T\left(g(T,\omega) + \frac{2\pi j}{T}\right)$$

where :

- $g(T,\omega)$ is the multiple of $\dfrac{2\pi}{T}$ closest to ω,
- $m_T \xrightarrow{T\to+\infty} +\infty$, $\dfrac{m_T}{T} \xrightarrow{T\to+\infty} 0$,
- $\forall j \in \mathbb{Z}$: $W_T(j) \geq 0$ et $W_T(-j) = W_T(j)$,
- $\sum_{|j|\leq m_T} W_T(j) = 1$; $\sum_{|j|\leq m_T} W_T^2(j) \xrightarrow{T\to+\infty} 0$.

Inverse autocovariance function

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process with γ as autocovariance function.
We call inverse autocovariance function of $(X_t)_{t\in\mathbb{Z}}$ the autocovariance function associated to the inverse spectral density $1/f$:

$$\frac{1}{f(\omega)} = \frac{1}{2\pi}\sum_{h=-\infty}^{+\infty}\gamma^i(h)\exp(-ih\omega)$$

where :

$$\forall h\in\mathbb{Z} : \gamma^i(h) = \int_{-\pi}^{\pi}\frac{1}{f(\omega)}\exp(ih\omega)\,\mathrm{d}\omega.$$

Inverse autocorrelation function

We call Inverse autocorrelation function the following function :

$$\forall h \in \mathbb{Z} : \rho^i(h) = \frac{\gamma^i(h)}{\gamma^i(0)}.$$