# Reliability
# System Reliability (cont'd)

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# Plan

# Plan

# Repairable systems

We consider a system subject to recurrent failures. We suppose that when it fails, it is repaired to a functioning state. We assume that the repair time is negligible.

Let us denote by $(T_n)_{n\geq 0}$ :

$$0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_n \leq \cdots$$

the successive times of failures.

This series of increasing times is called a **Point Process**.

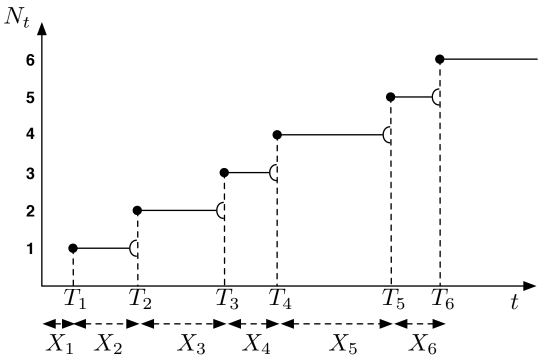


Figure: Different representations of the failure times of a repairable system

# Inter-arrival times

The times between failures

$$X_1 = T_1,\ X_2 = T_2 - T_1, \ldots, X_n = T_n - T_{n-1}, \ldots$$

are called the **inter-arrival times**.

In general, the **Mean Time Between Failures** (MTBF) depends on the index $i$, i.e. $MTBF_i = \mathbb{E}X_i$.

Of course, it is equivalent to know (or modelize) the Point process $(T_n)_{n\geq 0}$ or the series of inter-arrival times $(X_n)_{n\geq 1}$ since we have:

$$T_n = X_1 + X_2 + \ldots + X_n, \forall n \geq 1.$$

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# Counting Process

One can also consider the process $N = (N_t)_{t \geq 0}$ defined on $\mathbb{R}^+$ by

$$N_t = \sum_{n=1}^{+\infty} \mathbb{1}_{T_n \leq t}$$

and which, at every time $t$, gives the number of failures observed on $[0, t]$.

$N = (N_t)_{t \geq 0}$ is called the **Counting Process** associated to the point process $(T_n)_{n \geq 0}$.

Again, it's equivalent to know the distribution of the point process $(T_n)_{n\geq 0}$ or the distribution of the counting process $N=(N_t)_{t\geq 0}$ since we have:

$$P(T_1 \leq t_1, T_2 \leq t_2, \ldots, T_n \leq t_n) = P(N_{t_1} \geq 1, N_{t_2} \geq 2, \cdots, N_{t_n} \geq n).$$

## Definition

*A* **Counting Process** *is a stochastic process which satisfies:*

1. $N_t \geq 0$, *for all* $t \geq 0$,
2. $N_t$ *is integer valued,*
3. *If* $s < t$, *then* $N_s \leq N_t$,
4. *for* $s < t$, *the integer* $N_t - N_s$ *represents the number of events (here failures) observed on the interval* $]s, t]$.

The trajectories of a counting process are increasing step functions, right continuous with left limits.

We have of course: $N_0 = 0$.

# Plan

# Main Properties of a Counting Process

- A counting process is said with **independent increments** if

$$\forall k \in \mathbb{N}^*, \forall 0 < t_1 < \cdots < t_k, \text{the r.v.}$$
$$N_{t_1}, N_{t_2} - N_{t_1} \ldots, N_{t_k} - N_{t_{k-1}}$$

  are **mutually independent**.

  This implies that the number of failures on distinct intervals are independent. It might be a strong assumption...

- A counting process is said with **stationary increments** if the distribution of the number of events (failures) on an interval $]s,t]$ depends only on the length $t-s$ of this interval.
  In other words, the distribution of the r.v. $N_t - N_s$ and $N_{t+h} - N_{s+h}$ are the same, for all $h$.

- A counting process is said **regular** if, for all $t \geq 0$ and $\Delta t > 0$:
$$P(N_{t+\Delta t} - N_t \geq 2) = o(\Delta t).$$
That is to say:
$$\lim_{\Delta t \to 0} \frac{P(N_{t+\Delta t} - N_t \geq 2)}{\Delta t} = 0.$$
In this case, no more than one failure can be observed at time $t$.

# Plan

# Rate of Occurence of Failure

## Definition

*The* **mean function** *of the counting process* $N = (N_t)_{t \geq 0}$ *is defined by:*

$$m(t) = \mathbb{E}(N_t), \forall t \geq 0.$$

Remark. The mean function is not necessarily discontinuous.

## Definition

*The* **rate** *of the counting process* $N = (N_t)_{t \geq 0}$ *is defined by:*

$$w(t) = m'(t) = \lim_{\Delta t \to 0} \frac{m(t + \Delta t) - m(t)}{\Delta t} = 0.$$

*In Reliability, this process is called the* **RoCoF** *(Rate of Occurence of Failure). It represents the mean number of failure per unit at time* $t$.

From the definition of the RoCoF one can write:

$$m(t) = \int_0^t w(s)ds.$$

**In case of a regular process** we have, for "small" $t$:

$$w(t) \approx \frac{P(N_{t+\Delta t} - N_t = 1)}{\Delta t}.$$

This is why in some books the RoCoF is defined as

$$w(t) = \lim_{\Delta t \to 0} \frac{P(N_{t+\Delta t} - N_t = 1)}{\Delta t}.$$

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# Failure intensity

Let us denote $\mathcal{F}_t$ the history of the counting process up to time $t$.

- Roughly speaking, this history is given by the trajectory $\{s \mapsto N_s, \text{ for all } s \leq t\}$ of the process, i.e. the record of all the failure times before time $t$.
- Mathematically, this history is defined by the $\sigma$-field $\mathcal{F}_t = \sigma(N_s, \forall s \leq t)$. Since the trajectory of a counting process is a step function, we also have: $\mathcal{F}_t = \sigma(N_t, T_1, \ldots, T_{N_t})$.
- The family $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ is called a **Filtration** with main property the inclusion $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

## Definition

*The* **conditional rate** *of the counting process $N = (N_t)_{t\geq 0}$ is defined by:*

$$\lambda_t = \lim_{\Delta t \to 0} \frac{P(N_{t+\Delta t} - N_t = 1 | \mathcal{F}_{t^-})}{\Delta t}.$$

*In Reliability, this process is called the* **Failure intensity**.

One can write:

$$P(N_{t+\Delta t} - N_t = 1 | \mathcal{F}_{t^-}) = \lambda_t \Delta t + o(\Delta t).$$

Remind that, in case of a regular process,

$$P(N_{t+\Delta t} - N_t = 1) = w(t)\Delta t + o(\Delta t),$$

**Remark**. Note that the rate function $t \mapsto w(t)$ is deterministic, whereas the conditional rate $t \mapsto \lambda_t$ is generally stochastic.

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# Maintenance efficiency

Remind that we observe a system subject to recurrent failures. After each failure, a repair of the system is operated. The question is to know in which state is the system after repair.

Mathematically, one wonders to know

- the distribution of the inter-arrival time $X_i$, after the failures and repairs at times $T_1, \ldots, T_{t_{i-1}}$,
- or equivalently the failure intensity function $\lambda_t$ for $t \geq T_{t_{i-1}}$.

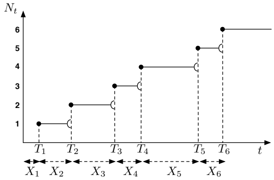


Figure: Occurence of the failure times of a repairable system

Thus the question under consideration is to judge the **maintenance efficiency**. Different models of repair efficiency are considered by engineers, including:

- As Good As New (AGAN): after a repair, the system is as new. This is **Perfect Repair** $\Rightarrow$ Homogeneous Poisson Processes, Renewal processes.
- As Bad As Old (ABAO): after repair, the system is in the same state as it was before failure. This is **Minimal Repair** $\Rightarrow$ Non Homogeneous Poisson Processes
- Some times, the system is between these extreme cases. This is **Imperfect Repair**
  - Brown et Proschan (BP)
  - Virtual Ages
  - ...

We will consider successively these different models.

# Plan

# Plan

# Definitions of an Homogeneous Poisson Processes

We give three equivalent definitions of an Homogeneous Poisson Processes.

## Definition (1)

*A Point Process $(T_n)_{n\geq 0}$, or equivalently its Counting Process $N = (N_t)_{t\geq 0}$, is called an* **Homogenous Poisson Process (HPP)** *with intensity $\lambda$ if the inter-arrival times $(X_n)_{n\geq 1}$ are i.i.d. with exponential distribution $\mathcal{E}(\lambda)$.*

The link with the Poisson distribution is given by the following Proposition.

## Proposition

*Let $N = (N_t)_{t\geq 0}$ be an HPP with intensity $\lambda$. For all $t > 0$, the r.v. $N_t$ has a Poisson distribution with parameter $\lambda t$, i.e.*

$$P(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Let us recall a result about the distribution of the order statistics

$$X_{(1)} < \ldots < X_{(n)}$$

of a sample $(X_1, \ldots, X_n)$.

## Lemma

*Let $X_1, \ldots, X_n$ be $n$ i.i.d. continuous r.v. with probability density function (p.d.f.) $f$. Let $X_{(1)} < \cdots < X_{(n)}$ be the $n$ associated order statistics. The multivariate p.d.f. of the random vector $(X_{(1)}, \ldots, X_{(n)})$ is*

$$f_{(X_{(1)},\ldots,X_{(n)})}(u_1, \ldots, u_n) = n! \prod_{i=1}^{n} f(u_i) \mathbb{1}_{u_1 < u_2 < \cdots < u_n}.$$

## Proposition

*Let $(T_n)_{n\geq 0}$ be an HPP with Counting Process $(N_t)_{t\geq 0}$. We have the following results.*

i) *For all $n \geq 1$, the p.d.f. of the random vector $(T_1, \ldots, T_n)$ is*

$$\lambda^n e^{-\lambda t_n} \mathbb{1}_{t_1 < t_2 < \cdots < t_n}.$$

ii) *For all $n \geq 1$, the conditional distribution of $(T_1, \ldots, T_n)$ given $\{N_t = n\}$ is the same as the distribution of the order statistics of $n$ i.i.d. r.v. with uniform distribution on $[0, t]$, i.e.*

$$f_{(T_1,\ldots,T_n)}^{N_t=n}(t_1, \ldots, t_n) = \frac{n!}{t^n} \mathbb{1}_{t_1 < t_2 < \cdots < t_n \leq t}.$$

## Definition (2)

*Let $N = (N_t)_{t \geq 0}$ be a Counting Process. It is an HPP with intensity $\lambda$ if*

i) $N_0 = 0$;

ii) *$N$ has independent increments;*

iii) *for all $0 \leq s < t$, the r.v. $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t - s)$.*

## Definition (3)

*Let* $N = (N_t)_{t \geq 0}$ *be a Counting Process. It is an HPP with intensity* $\lambda$ *if*

i) $N_0 = 0$,

ii) $N$ *has stationary and independent increments,*

iii) $N$ *is regular,*

iv) $P(N_{t+\Delta t} - N_t = 1) = \lambda \Delta t + o(\Delta t)$.

# Plan

# Properties

## Proposition

*Let $N = (N_t)_{t \geq 0}$ be an HPP with intensity $\lambda$.*

- *The RoCoF is $w(t) = \lambda$, thus is constant.*
- *The failure intensity is equal to the RoCoF, i.e. $\lambda_t = \lambda$. Thus it is deterministic and constant.*
- *The mean number of failures in time interval $]s, t]$ is $w(t) - w(s) = \lambda(t - s)$.*
- *The MTBF doesn't depend on the index $i$ of the inter-arrival times and is equal to $1/\lambda = \mathbb{E}X_i$.*
- *The distribution of the nth failure $T_n$ is Gamma$(n, \lambda)$, i.e. with p.d.f.:*

$$f_{T_n}(t) = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}, \text{ for all } t \geq 0.$$

- *$N$ is a Markov Process.*

# Sum and Decomposition of HPP

## Proposition

- *Suppose that $N^1$ and $N^2$ are two independent HPP with parameter $\lambda_1$ and $\lambda_2$. Then $N^1 + N^2$ is an HPP with parameter $\lambda_1 + \lambda_2$.*
- *Reversely, if $N$ is an HPP where the occurence are of type 1 with constant (over time) probability $p$ and of type 2 with probability $1 - p$, then the counting process $N_1$ (resp. $N_2$) of the number of events of type1 (resp. type 2) is an HPP with parameter $p\lambda$ (resp. $(1-p)\lambda$). The two HPP $N_1$ and $N_2$ are furthermore independent.*

The first point could correspond to the observation of two types of failures on a system, where a failure of type 1 occurs at a rate $\lambda_1$, independently to the other type which appears at a rate $\lambda_2$.

The second point could modelize the situation where, in the failure occurrences of a system, we want to distinguish different causes or different severities, with constant over time probabilities of being of each type.

Note that the previous results can be generalized to a number $n$ of sum or decomposition of HPP.

# Compound HPP

## Proposition

*Suppose that $N$ is an HPP with parameter $\lambda$ and that at each occurence time $T_i$ of the counting process, we observe a random variable $C_i$, for $i \in \mathbb{N}$. If $(C_i)$ is a series of i.i.d. random variables with mean $c$ and variance $\sigma^2$, then, for all $t \geq 0$, the r.v.:*

$$S_t = \sum_{i=1}^{N_t} C_i$$

*has* **mean** $\mu\lambda t$ *and* **variance** $= \lambda(\mu^2 + \sigma^2)t$.

The random variables $(C_i)$ could represent the repair costs after failure.

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# Observations

There are two different types of observation for a Counting process.

- Either we observe the process on a fixed interval $[0, t]$.

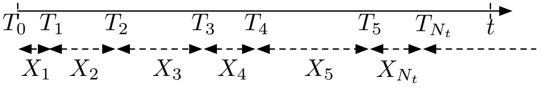


Figure: Observation of a counting process on a fixed interval $[0, t]$

- Or observe it up to the occurence of the $n$th failure.

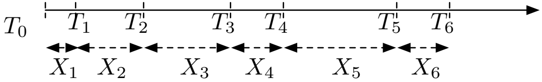


Figure: Observation of the $n = 6$ first occurrences of a counting process

# Maximum Likelihood Estimation

## Proposition

*Depending on the type of observation we have, the likelihood of a trajectory of a HPP is*

- *in case of Type 1 censoring, i.e. when the process is observed on an interval* $[0, t]$*:*

$$\lambda^n e^{-\lambda t} \mathbb{1}_{t_1 < t_2 < \cdots < t_n < t},$$

- *in case of Type 2 censoring, i.e. when we observe the $n$ first failures on the system:*

$$\lambda^n e^{-\lambda t_n} \mathbb{1}_{t_1 < t_2 < \cdots < t_n}.$$

Let us denote by **TTT**, the **Total Time on Test**:

$$TTT = \begin{cases} t & \text{with Type I obsservations} \\ T_n & \text{with Type II observations.} \end{cases}$$

In both cases, the **Maximum Likelihood Estimator** of $\lambda$ can be written as

$$\hat{\lambda}_{MLE} = \frac{N_{TTT}}{TTT}.$$

These two estimators are nothing but what we have obtained through the maximum likelihood estimation of lifetime data with exponential distribution with censored (resp. complete) data.

# Asymptotic Properties of the MLE

Let us consider separately the two cases given by the type of observation we have.

**Under Type I observation**, the time $t$ is fixed and $N_t$ random. We can prove the following result.

## Theorem

*Let $(N_t)_{t\geq 0}$ be an HPP with parameter $\lambda$. We have:*

$$i) \qquad p.s. \lim_{t\to+\infty} \frac{N_t}{t} = \lambda$$

$$ii) \qquad \sqrt{t}\left(\frac{N_t}{t} - \lambda\right) \xrightarrow{\mathcal{L}} N(0,\lambda), \;\; when\ t \to +\infty.$$

Thus the MLE $\hat{\lambda}_{MLE}$ under Type I is **asymptotically unbiased and normal**.

We can deduce asymptotic confidence intervals for $\lambda$ from this convergence in distribution.

## Proposition

*An asymptotic confidence interval for $\lambda$ is under Type II observation of an HPP is*

$$\left[\frac{N_t}{t} - \frac{\sqrt{N_t}}{t} z_{1-\frac{\alpha}{2}}, \frac{N_t}{t} + \frac{\sqrt{N_t}}{t} z_{1-\frac{\alpha}{2}}\right],$$

*where $z_\alpha$ denotes the $\alpha$-quantiles of the $N(0,1)$ distribution.*

**Under Type II observation**, the number of observation $n$ is fixed but the $T_n$ is random. We can prove the following result.

## Theorem

*Let $N$ be an HPP with parameter $\lambda$. We have:*

$$i) \qquad p.s. \lim_{n \to +\infty} \frac{n}{T_n} = \lambda$$

$$ii) \qquad \sqrt{n}\left(\frac{n}{T_n} - \lambda\right) \xrightarrow{\mathcal{L}} N(0, \lambda^2), \; when \; n \to +\infty.$$

Thus the MLE $\hat{\lambda}_{MLE}$ under Type I is also **asymptotically unbiased and normal**.

## Proposition

*The following results holds under Type II observation of an HPP.*

- *An asymptotic confidence interval for $\lambda$ is*

$$\left[\frac{\hat{\lambda}_{MLE}}{1+\frac{1}{\sqrt{n}}z_{1-\frac{\alpha}{2}}}, \frac{\hat{\lambda}_{MLE}}{1-\frac{1}{\sqrt{n}}z_{1-\frac{\alpha}{2}}}\right].$$

- *Fix $\lambda_0$ a limit value under which the system is assumed to be reliable. A test of level $\alpha$ of $H_0 : \lambda \geq \lambda_0$ against $H_1 : \lambda < \lambda_0$ is given by the rule*

$$\text{Reject } H_0 \text{ if } \hat{\lambda}_{MLE} < \lambda_0 + \frac{\lambda_0}{\sqrt{n}} z_\alpha.$$

# Non Asymptotic Properties of the MLE

Here we study the properties of

$$\hat{\lambda}_{MLE} = \frac{N_t}{t}$$

for fixed $n$.

We will see that, again, the results differ according to the type of observation we have: Type I or Type II censoring.

# MLE Properties under Type 1 observations

Remind that, under Type 1 observation, we observe the Counting Process on a fixed interval $[0, t]$. The number of failures $N_t$ in this interval is random.

We have seen that in case of an HPP, the counting process $N_t$ is such that:

$$N_t \sim \mathcal{P}(\lambda t), \text{ for all } t > 0.$$

We deduce from this that:

- $\hat{\lambda}_{MLE}$ is **unbiased**,
- its variance is equal to $\lambda/t$,
- it is an **efficient** estimator of $\lambda$,
- it is the **best (minimal variance) unbiased estimator** of $\lambda$.

# MLE Properties under Type 2 observations

Under Type 2 observation, we observe the Counting Process up to a number of occurence (failures). This number is fixed, equal to $n$, but the time of observation $T_n$ is random.

Remind that the expression of the MLE of $\lambda$ in this case is

$$\hat{\lambda}_{MLE} = \frac{n}{T_n}$$

We can prove that

$$\mathbb{E}(\hat{\lambda}_{MLE}) = \frac{n-1}{n}\lambda,$$

which proves that the estimator is **biased** but asymptotically unbiased.

One can prove that

$$\tilde{\lambda} = \frac{n-1}{T_n}$$

is:

- **unbiased**,
- **not efficient**, but asymptotically efficient.
- it is the **best (minimal variance) unbiased estimator** of $\lambda$

# Exact Confidence interval for $\lambda$

Here also, we have to consider separately the two types of observation.

In **Type I** case, it is hard to find **exact** confidence interval or exact statistical test for the parameter $\lambda$. This is due to the fact that the distribution of $N_t$ is discrete.

However, there exists rather simple approximative confidence intervals. See references.

Things are easier in case of **Type II** observations, thanks to the distribution of $T_n$ which is Gamma$(n, \lambda)$.

## Proposition

- *An exact confidence interval for $\lambda$ in case of Type I observation is:*

$$\left[\frac{\chi^2_{\frac{\alpha}{2}}(2n)}{nT_n}, \frac{\chi^2_{1-\frac{\alpha}{2}}(2n)}{nT_n}\right].$$

- *Fix $\lambda_0$ a limit value under which the system is assumed to be reliable. A test of level exactly equal to $\alpha$ of $H_0 : \lambda \geq \lambda_0$ against $H_1 : \lambda < \lambda_0$ is given by the rule*

$$\textit{Reject } H_0 \textit{ if } \; 2\lambda_0 T_n > \chi^2_{1-\alpha}(2n).$$

# Plan

# Maintenance Planification

Remind that we have supposed that

- after each failure a **corrective maintenance** is performed,
- in case of an HPP, after each maintenance, the system is considered as new.

This is what we call AGAN corrective maintenance.

Let us consider the question of the interest to perform a preventive maintenance on the system before that the next failure occurs and to replace the system by a new one.

We know that the exponential distribution satisfies the **memoryless property**, that is:

$$\begin{aligned} P(X \leqslant x + x_0 \;/\; X > x_0) &= P(X \leqslant x),\ \forall (x, x_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ \Leftrightarrow \mathcal{L}(\tau_{x_0}) &= \mathcal{L}(X),\ \forall x_0 \in \mathbb{R}^+. \end{aligned}$$

This induces that there is **theoretically no reason** to perform preventive maintenances between corrective maintenance in case of an HPP model.