Reliability
Part 2: System Reliability

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Plan

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   - Path and Cut
   - Pivotal decomposition

2. Systems with independents components
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   - Static Reliability
   - Nonrepairable systems

Reliability Block Diagram

Aim. Representing the logical structure of a system composed with n components.

We say that we have a **Series System** if the failure of one component implies the failure of the whole system. In other words, the system works if and only if all the components work.

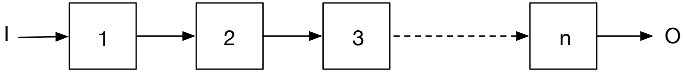

Figure: Block Diagram of a Series System

We say that we have a **Parallel System** if the failure of all the components is necessary to imply the failure of the system. So the system works if at least one component works.

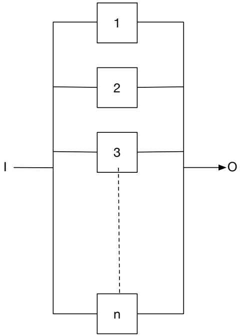

Figure: Block Diagram of a Parallel System

In some cases the system is subdivided into modules. A **module** is a set of component where only one line enters and one line goes out. It can happen that in all the modules, the components are in series or in parallel.

Thus, the system can have series in parallel.

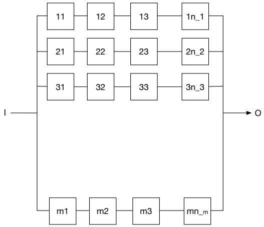

Figure: Block Diagram of a Series-Parallel System

Or parallels in series.

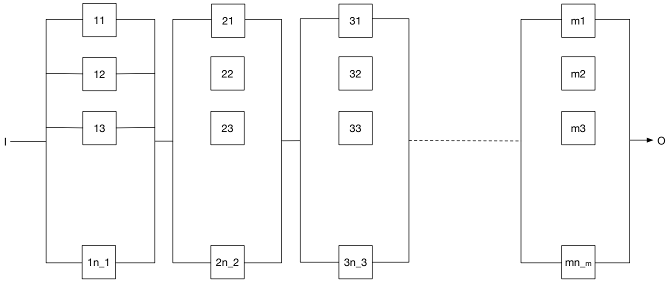

Figure: Block Diagram of a Parallel-Series System

An other important case is the **system** k-**out-of**-n where the system works if at least k components among n work.

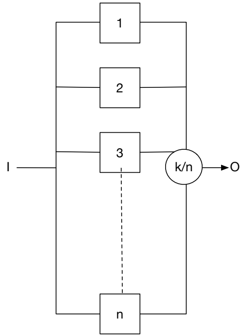

Figure: System k-out-of-n

All the Reliability block diagrams are not of the above kinds. Here are some examples which are not series-parallel, parallel-series or k-out-of-n.

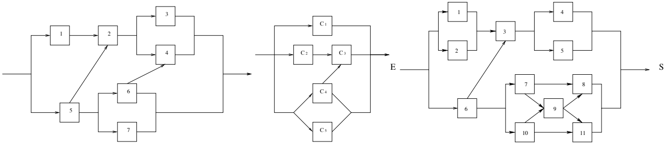

Figure: Other examples of Block Diagram

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Structure function

We consider a system of n components, each component having only **two states: in functioning or failed**.
Let $x_1, \ldots, x_n$ be the states of the n components with the notation

$$x_i = \begin{cases} 1 & \text{if component } i \quad \text{is functionning} \\ 0 & \text{if component } i \quad \text{is in a failed state} \end{cases}, \text{ for } i = 1 \ldots, n.$$

The vector $\underline{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ is called the **state vector**.

We assume that the system has also only two states (in functioning or failed) described by the **structure function**:

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if the system} \quad \text{is functionning} \\ 0 & \text{if the system} \quad \text{is in a failed state} \end{cases}.$$

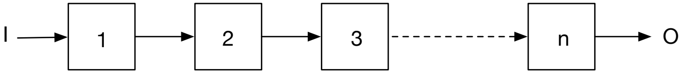

Figure: Block Diagram of a Series System

The structure function of a series system is:

$$\phi(\underline{x}) = \prod_{i=1}^{n} x_i = \min_{i=1,\ldots,n} x_i.$$

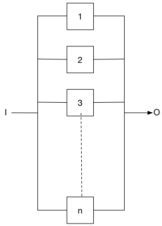

Figure: Block Diagram of a Parallel System

The structure function of a parallel system is:

$$\phi(\underline{x}) = 1 - \prod_{i=1}^{n}(1 - x_i) = \max_{i=1,\ldots,n} x_i.$$

This structure function is sometimes written:

$$\coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n}(1 - x_i).$$

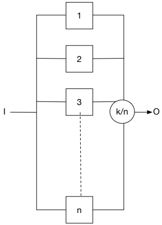

Figure: System k-out-of-n

The structure function of a k-out-of-n system is:

$$\phi(\underline{x}) = \mathbb{1}_{\sum_{i=1}^{n} x_i \geq k}.$$

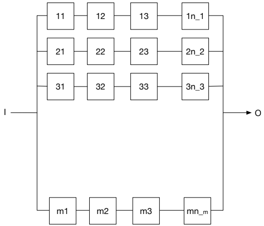

Figure: Block Diagram of a Series-Parallel System

The structure function of a Series-Parallel System is:

$$\phi(\underline{x}) = \coprod_{i=1}^{m} \prod_{j=1}^{n_m} x_{ij} = 1 - \prod_{i=1}^{m}(1 - \prod_{j=1}^{n_m} x_{ij}),$$

where x_{ij} is the state of the component j in the series i.

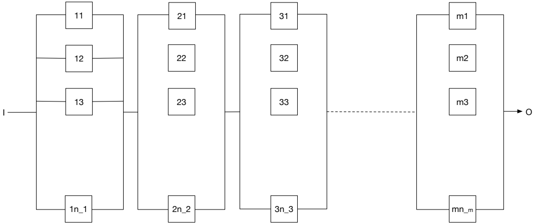

Figure: Block Diagram of a Parallel-Series System

The structure function of a Parallel-Series System is:

$$\phi(\underline{x}) = \prod_{i=1}^{m} \coprod_{j=1}^{n_m} x_{ij} = \prod_{i=1}^{m} (1 - \prod_{j=1}^{n_m} (1 - x_{ij})),$$

where x_{ij} is the state of the component j in the parallel i.

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Coherent structures

Definition

A component i is said **irrelevant** *if we have:*

$$\phi(1_i, \underline{x_{-i}}) = \phi(0_i, \underline{x_{-i}}), \text{ for all } x_{-i},$$

where x_{-i} is the state vector without component i and 1_i (resp. 0_i) means that the component i is functioning (resp. in failure).

In the other case, the component is said **relevant**.

We can leave out the irrelevant components of a system in establishing the structure function of the system.

Note that the property to be relevant or irrelevant depends on the system function considered. A component can be relevant for a specific function and not for the others...

Definition

A system of n components is said coherent if:

- *all the components are relevant,*
- *the structure function is nondecreasing, i.e.*

$$(\forall i = 1, \ldots, n, x_i \leq y_i) \Rightarrow \phi(\underline{x}) \leq \phi(\underline{y}).$$

Theorem

The structure function of a coherent system is such that:

$$\phi(\underline{0}) = 0 \text{ and } \phi(\underline{1}) = 1.$$

Theorem

The structure function of a coherent system is such that:

$$\min_{i=1,\ldots,n} x_i = \prod_{i=1}^{n} x_i \leq \phi(\underline{x}) \leq \coprod_{i=1}^{n} x_i = \max_{i=1,\ldots,n} x_i$$

A coherent system is functioning at least as well as a series system and as most as well as a parallel system.

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Path and Cut

Definition

A subset $\mathcal{P} \subset \{1, \ldots, n\}$ of components is said a **path** *if the system works when all the components of $\mathcal{P}$ are functioning. The path $\mathcal{P}$ is said* **minimal** *if it losses its path property as soon as one of its components is taken off.*

Definition

A subset $\mathcal{P} \subset \{1, \ldots, n\}$ of components is said a **cut** *if the system fails when all the components of $\mathcal{P}$ are failed. The cut $\mathcal{P}$ is said* **minimal** *if it losses its cut property as soon as one of its components is taken off.*

Theorem

Let $(\mathcal{P}_1, \ldots, \mathcal{P}_p)$ be the set of all the minimum paths of a system. We have

$$\phi(\underline{x}) = \coprod_{i=1}^{p} \prod_{j \in \mathcal{P}_i} x_j = \max_{i=1,\ldots,p} \min_{j \in \mathcal{P}_i} x_j$$

If $(\mathcal{C}_1, \ldots, \mathcal{C}_q)$ is the set of all the minimum cuts of a system. We have

$$\phi(\underline{x}) = \prod_{i=1}^{q} \coprod_{j \in \mathcal{C}_i} x_j = \min_{i=1,\ldots,q} \max_{j \in \mathcal{C}_i} x_j$$

Thus, all coherent system can be seen as a series-parallel system and also as a parallel-series system, using the set of minimal paths or minimal cuts respectively.

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Pivotal decomposition

It is sometimes useful to decompose the structure function of a system with respect to a given component.

Definition

A pivotal decomposition of the structure function of a system is obtained thanks to the formula

$$\phi(\underline{x}) = x_i \phi(1_i, \underline{x_{-i}}) + (1 - x_i)\phi(0_i, \underline{x_{-i}}).$$

Example.

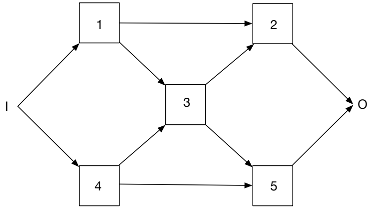

Figure: Block Diagram of the bridge structure

In this case, the structure function of the system can be written

$$\begin{aligned}\phi(\underline{x}) &= x_3\,(1-(1-x_1)(1-x_4))\,(1-(1-x_2)(1-x_5)) \\ &+ (1-x_3)\,(1-(1-x_1x_2)(1-x_4x_5))\end{aligned}$$

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Random and time dependent states

Consider a system with n components where failures occur randomly. Write $X_i(t)$, for $i = 1, \ldots, n$, the Bernoulli r.v. modeling the state of the component i at time t:

$$X_i(t) = \begin{cases} 1 & \text{if the component } i \text{ is functioning at time } t \\ 0 & \text{if the component } i \text{ is in failure at time } t \end{cases}.$$

So the state vector $\underline{X}$ and the structure function $\phi(\underline{X})$ are also random variables.

Definition

The components of a system are said to be **independent** *if the coordinates of the random vector $\underline{X}$ are stochastically independent.*

Write, for all $t > 0$ and all $i = 1, \ldots, n$,

$$p_i(t) = P(X_i(t) = 1) \text{ and } p_S(t) = P(\phi(\underline{X}(t)) = 1).$$

Since the above r.v. are Bernoulli, we have:

$$p_i(t) = \mathbb{E}X_i(t) \text{ and } p_S(t) = \mathbb{E}(\phi(\underline{X}(t))).$$

Definition

- *In case of* **repairable components** *the probabilities $p_i(t)$ and $p_S(t)$ are the* **availability** *functions of the components and of the system respectively.*
- *In case of* **nonrepairable components**, *these probabilities are the* **Reliability** *functions of the components and the system respectively.*

In case of nonrepairable components, we write:

$$R_i(t) = p_i(t) \text{ and } R_S(t) = p_S(t) = P(\phi(\underline{X}(t)) = 1).$$

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Static Reliability

Here, we suppose the **time t fixed** and the system repairaible or not. The aim is to find the link between the probabilities $p_i(t)$ and $p_S(t)$.

Theorem

If the components are independent, there exists a function ψ from $[0,1]^n$ to $[0,1]$ such that:

$$p_S(t) = \psi(p_1(t), \ldots, p_n(t)),$$

i.e.

$$\mathbb{E}(\phi(\underline{X}(t))) = \psi(\mathbb{E}X_1(t), \ldots, \mathbb{E}X_n(t)).$$

Even if their **expressions are sometimes equal**, the function ψ and the structure function ϕ are **not the same**, first because of their domains of definition and sets of values which differ.

Examples.

- Series system

$$\psi(p_1(t), \ldots, p_n(t)) = \prod_{i=1}^{n} p_i(t).$$

  One can note that $\prod_{i=1}^{n} p_i(t) \leq \min_{i=1,\ldots,n} p_i(t)$ and thus the system is at most reliable as the least reliable component.

- Parallel system

$$\psi(p_1(t), \ldots, p_n(t)) = \coprod_{i=1}^{n} p_i(t) = 1 - \prod_{i=1}^{n}(1 - p_i(t)).$$

- Series-Parallel System

$$\psi(p_1(t),\ldots,p_n(t)) = \coprod_{i=1}^{m}\prod_{j=1}^{n_m} p_{ij}(t) = 1-\prod_{i=1}^{m}(1-\prod_{j=1}^{n_m} p_{ij}(t)),$$

- Parallel-Series System

$$\psi(p_1(t),\ldots,p_n(t)) = \prod_{i=1}^{m}\coprod_{j=1}^{n_m} p_{ij}(t) = \prod_{i=1}^{m}(1-\prod_{j=1}^{n_m}(1-p_{ij}(t))),$$

- k-out-of-n System with identically distributed components

$$\begin{aligned}\psi(p(t),\ldots,p(t)) &= P\left(\sum_{i=1}^{n} X_i(t) \geq k\right)\\ &= \sum_{j=k}^{n} C_n^k p^j(t)(1-p(t))^{n-j}.\end{aligned}$$

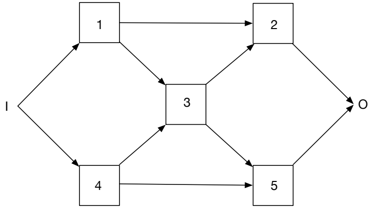

Figure: Block Diagram of the bridge structure

Static Reliability of the bridge structure.

$$\begin{aligned}\psi(p_1(t),\ldots,p_n(t)) = &\ p_3(t)\,(1-(1-p_1(t))(1-p_4(t)))\\ &\times(1-(1-p_2(t))(1-p_5(t)))\\ &+(1-p_3(t))\,(1-(1-p_1(t)p_2(t))(1-p_4(t)p_5(t)))\end{aligned}$$

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Nonrepairable systems

Now let us consider a nonrepairable system and let us study its reliability as a function of t.
Let T_i, for $i = 1, \ldots, n$, be the lifetime of component i. We have, for all t

$$X_i(t) = 1 \iff T_i > t.$$

If T_S is the system lifetime, we have:

$$T_S = \inf\{t \geq 0; \phi(\underline{X}(t)) = 0\}$$

and

$$R_S(t) = P(T_S > t) = P(\phi(\underline{X}(t)) = 1).$$

Theorem

With ψ the function introduced in Static Reliability, we have

$$R_S(t) = \psi(R_1(t), \ldots, R_n(t)).$$

Case of a Series System.

- We have

$$R_S(t) = \prod_{i=1}^{n} R_i(t).$$

  In a Series System the reliabilities are multiplied.

- This result can be refund using the relation:

$$T_S = \min_{i=1,\ldots,n} T_i.$$

- If $\lambda_i(\cdot)$ is the hazard function of component i, for $i = 1, \ldots, n$, we have:

$$R_S(t) = \exp\left(- \int_0^t \sum_{i=1}^{n} \lambda_i(s) ds \right)$$

and thus the system hazard rate function is

$$\lambda_S(t) = \sum_{i=1}^{n} \lambda_i(t).$$

- If the component lifetimes are exponentially distributed with parameters $\lambda_1, \ldots, \lambda_n$, the System lifetime is also exponentially distributed with parameter

$$\lambda_S = \sum_{i=1}^{n} \lambda_i.$$

This is a well-known property of minimum of exponential random variables.

- The MTBF of a Series System with constant rate $\lambda_1, \ldots, \lambda_n$ is

$$MTBF = \frac{1}{\sum_{i=1}^{n} \lambda_i}.$$

Case of a Parallel System.

- We have

$$\begin{aligned} R_S(t) &= \prod_{i=1}^{n} R_i(t) \\ \Longleftrightarrow \quad 1 - R_S(t) &= \prod_{i=1}^{n} (1 - R_i(t)). \end{aligned}$$

In parallel, the c.d.f. are multiplied.