SEMIGROUPS, INTERPOLATIONS AND STABILITY OF SOLUTIONS TO FLUID FLOW PROBLEMS

THIEU HUY NGUYEN

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Let me start with the intertwinement between boundedness and periodicity for solutions to differential equations. Such a relation was discovered for the first time by Massera [20]. Since then the method posed by Massera has became a folklore methodology for periodic solutions to Ordinary Differential Equations (which roughly said that if an ODE has a bounded solution then it has a periodic one). We would like to note that such a methodology has been extended to various types of equations in Banach spaces, and we refer the reader to Zubelevich [33] for a survey on the state of the art and a nice generalization of such Massera-type Theorems.

When invoking the Massera's methodology in researching for the periodic solutions to fluid flow problems in unbounded domains, one of the main difficulties, as announced by Maremonti [17], is lying in the following important theorem related to bounded solutions of Navier-Stokes equations in unbounded (in all directions) domains saying that

"**Theorem A**. *Denote by $f(t,x)$ the body force and $u(t,x)$ a solution to the Navier-Stokes equations $u_t - \Delta u + (u\cdot\nabla)u + \nabla p = f$; X and Y two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. If $f(t,\cdot)\in X$ with $\|f(t,\cdot)\|_X$ uniformly bounded in time, then $u(t,\cdot)\in Y$ with $\|u(t,\cdot)\|_Y$ uniformly bounded in the time.*"

If the domain Ω is bounded (in some direction), then using the Poincaré inequality and some compact embeddings it is convenient to prove the validity of Theorem A. The situation becomes more complicated when one considers the unbounded domain Ω in all directions since the Poincaré inequality is no longer true and compact embeddings are not valid. Therefore, some new approaches have been introduced to overcome this difficulty.

Maremonti [17, 18] and Maremonti-Padula [19] used some geometric properties of the domains such as the symmetry of Ω and/or the smallness of the complement $\mathbb{R}^d\setminus\Omega$ to show the validity of Theorem A. Galdi and Sohr [5] discovered the fact that the specific structures of the phase-spaces X and Y played important roles when looking for bounded solutions (and also periodic ones) to Navier-Stokes equations in exterior domains. Consequently, they introduced in [5] some relevant function spaces featuring the decay of the solutions at spatial infinity to prove Theorem A on an exterior domain without restricted conditions on the domain. The last approach that we would like to mention was given by Yamazaki [31], and exploited the interpolation features of the weak-L^d spaces to prove the existence of bounded (in time) weak mild solutions of Navier-Stokes equations on exterior domains for each bounded external force. This approach has then been extended in [22] to obtain bounded strong mild solutions in weak-L^3 spaces of Navier-Stokes equations around rotating obstacles.

In this short lecture, I will take a survey of our recent results published in [8, 22, 23] on boundedness, periodicity and stability of solutions to fluid flow problems in unbounded (in all directions) domains. We start from a general framework to study the Theorem A on an unbounded domain Ω, namely, we consider the general semi-linear equations on Ω of the form

$$\begin{cases} u_t + Au = \mathbb{P}\mathrm{div}(G(u)+F(t)) \\ \quad u(0) = u_0, \end{cases} \tag{1}$$

where $-A$ generates a C_0-semigroup $(e^{-tA})_{t\geq 0}$ on $L^d_{\sigma,w}(\Omega)$, $\mathbb{P}$ is Helmholtz projection; G is a nonlinear and local Lipschitz operator acting from $L^d_{\sigma,w}(\Omega)$ into $L^{d/2}_{\sigma,w}(\Omega)^{d^2}$, and $F(\cdot)$ is a time-dependent second-order tensor in $L^{d/2}_{\sigma,w}(\Omega)^{d^2}$. Under assumptions on $L^p - L^q$ smoothing properties of $(e^{-tA})_{t\geq 0}$ and local Lipschitz properties of G, and using the interpolation techniques combined with differential inequalities and fixed point arguments we are able to prove the existence of bounded (in time t) solutions to (1) for each bounded tensor $F(\cdot)$. Then, we can use either ergodic approach (see [22]) or topology arguments (see [8]) to prove the existence of periodic solutions to fluid flow problems. Moreover, our methods can be extended to obtain the stability of bounded as well as periodic solutions to such problem.

In our strategy we consider the *mild solution* to (1) i.e. a function u satisfying the following integral equation

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-\tau)A}\mathbb{P}\mathrm{div}(G(u) + F(\tau))d\tau. \tag{2}$$

We would like to note that in case $-A$ generates a bounded analytic semigroup $(e^{-tA})_{t\geq 0}$, by the standard method one can see that the mild solution to (1) is also the classical solution in the sense of [1, Prop. 3.1.16]. The converse is clearly true for all (not necessarily analytic) C_0-semigroup $(e^{-tA})_{t\geq 0}$.

To show the existence and uniqueness of the bounded and periodic mild solution to (1) we need the following space of bounded continuous functions with values in a Banach space X (with norm $\|\cdot\|_X$) defined as

$$C_b(\mathbb{R}_+, X) := \{v : \mathbb{R}_+ \to X \mid v \text{ is continuous and } \sup_{t\in\mathbb{R}_+} \|v(t)\|_X < \infty\} \tag{3}$$

endowed with the norm

$$\|v\|_{\infty,X} := \sup_{t\in\mathbb{R}_+} \|v(t)\|_X.$$

Assumption 1. We suppose that the operator $-A$ and its dual $-A'$ generate bounded C_0-semigroups $(e^{-tA})_{t\geq 0}$ and $(e^{-tA'})_{t\geq 0}$ (respectively) satisfying the following $L^p - L^q$ smoothing estimates.

(1) For some $r > d$:

$$\|e^{-tA}x\|_{r,w} \leqslant Mt^{-\frac{d}{2}(\frac{1}{d}-\frac{1}{r})}\|x\|_{d,w}. \tag{4}$$

(2) For all $1 < p < \frac{d}{d-2}$:

$$\|\nabla e^{-tA'}x\|_{\frac{d}{d-2},1} \leqslant Mt^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{p}-\frac{d-2}{d})}\|x\|_{p,\infty}. \tag{5}$$

(3) For the number $r > d$ appearing in Item (1):

$$\|\nabla e^{-tA'}x\|_{\frac{d}{d-2},1} \leqslant Mt^{-\frac{3}{2}+\frac{d}{2r}}\|x\|_{\frac{r}{r-1},1}. \tag{6}$$

The following lemma is one of the keys in our strategy. It asserts the boundedness of the mild solutions to the linearized problem of Equation (1). Its proof is relying on the interpolation functor and duality estimates.

Lemma 2 ([8]). *For $F \in C_b(\mathbb{R}_+, L^{d/2}_{\sigma,w}(\Omega))^{d^2}$ and $u_0 \in L^d_{\sigma,w}(\Omega)$ we have that the function u defined by*

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}\mathbb{P}\mathrm{div}F(\tau)d\tau \tag{7}$$

belongs to $C_b(\mathbb{R}_+, L^d_{\sigma,w}(\Omega))$ and satisfies

$$\|u\|_{\infty,d,w} \leqslant M\|u_0\|_{d,w} + \widetilde{M}\|F\|_{\infty,\frac{d}{2},w} \tag{8}$$

for positive constants M and $\widetilde{M}$ independent of u_0 and F.

Using topological arguments we can invoke Massera's methodology to obtain the existence and uniqueness of periodic solutions of the linearized equation of (1) in the following theorem.

Theorem 3 ([8]). *Let Assumption 1 hold. Then, for a T-periodic function $F \in C_b(\mathbb{R}_+, L^{d/2}_{\sigma,w}(\Omega))^{d^2}$, there exists a unique $u_0 \in L^d_{\sigma,w}(\Omega)$ such that the function u defined by (7) is a T-periodic, and moreover*

$$\|u\|_{L^\infty(\mathbb{R}_+;Y)} \leqslant \tilde{M}(M+1)\|F\|_{L^\infty(\mathbb{R}_+;X)}. \tag{9}$$

Remark 4. i) In [8] we obtain a more general result of the existence and uniqueness of periodic solutions to linear equations in general interpolation spaces that can be applied to various types of linearized equations appearing fluid dynamics as well as to diffusion equations in rough domains with rough coefficient and Ornstein-Uhlenbeck equations. For simplicity of presentation we present here the version on the weak-L^d spaces over exterior domains as in the above theorem.

ii) In [22] we used an ergodic approach to prove the existence and uniqueness of periodic solutions to Stokes equations around a rotating obstacle.

Theorem 5 ([8, 23])**.** *Let $F \in C_b(\mathbb{R}_+, L_w^{d/2}(\Omega)^{d^2})$. Suppose that $G : L^d_{\sigma,w}(\Omega) \to L_w^{d/2}(\Omega)^{d^2}$ satisfies*

$$\begin{aligned} &(1)\ G(0) = 0, \text{ and} \\ &(2)\ \|G(v_1) - G(v_2)\|_{d/2,w} \leqslant (\kappa + \|v_1\|_{d,w} + \|v_2\|_{d,w})\|v_1 - v_2\|_{d,w} \text{ for all } v_1, v_2 \in L^d_{\sigma,w}(\Omega), \end{aligned} \tag{10}$$

where $\kappa \geq 0$ is a constant; $-A$ satisfies Assumption 1, and $u_0 \in L^d_{\sigma,w}(\Omega)$.

Then, the following assertions hold.

i) If $\kappa, \|u_0\|_{d,\omega}, \|F\|_{\infty,\frac{d}{2},w}$ and ρ are small enough, the problem (1) has a unique mild solution $\hat{u}$ in the ball $B_\rho := \{v \in C_b(\mathbb{R}_+, L^d_{\sigma,w}(\Omega)) : \|v\|_{\infty,d,w} \leqslant \rho\}$.

ii) If in addition, F is T-periodic, then there exists a unique T-periodic mild solution to (1).

Similarly as mentioned in Remark 4, in [8] we have proved a more general version of the above theorem for general semi-linear equations in general interpolation spaces that can be applied to show the existence and uniqueness of periodic solutions to Navier-Stokes-Oseen flow, the Navier-Stokes flow past rotating obstacles, and, in the geophysical setting, for Ornstein-Uhlenbeck and various diffusion equations with rough coefficients.

Lastly, using interpolation functors together with differential inequalities and rescaling techniques we obtained in [23] the following theorem on the stability of mild solutions to (1).

Theorem 6 ([23])**.** *Under the conditions of Theorem 5 we consider Equation (1) on an exterior domain $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) with a C^3-boundary. For the number $r > d$ appearing in Assumption 1 we suppose that*

(a) *for all $1 < p < \frac{dr}{dr-r-d}$:*

$$\|\nabla e^{-tA'}x\|_{\frac{dr}{dr-r-d},1} \leqslant Mt^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{p}-\frac{dr-r-d}{dr})}\|x\|_{p,\infty} \text{ for all } x \in L^d_{\sigma,w}(\Omega), \tag{11}$$

(b) *G satisfies*

$$\|G(v_1) - G(v_2)\|_{\frac{dr}{d+r},w} \leqslant (\kappa + \|v_1\|_{d,w} + \|v_2\|_{d,w})\|v_1 - v_2\|_{r,w} \text{ for } v_1, v_2 \in L^d_{\sigma,w}(\Omega) \cap L^r_{\sigma,w}(\Omega) \tag{12}$$

with a small constant $\kappa \geq 0$.

Then, the small mild solution $\hat{u}$ of (1) is stable in the sense that for any other mild solution $u \in C_b(\mathbb{R}_+, L^d_{\sigma,w}(\Omega))$ of (1) such that $\|u(0) - \hat{u}(0)\|_{d,w}$ is small enough we have

$$\|u(t) - \hat{u}(t)\|_{r,w} \leqslant \frac{C}{t^{\frac{1}{2}-\frac{d}{2r}}} \text{ for all } t > 0, \tag{13}$$

with the number $r > d$ as in (11).

This abstract result can be applied to establish the stability results on Navier-Stokes flows on exterior domains (Yamazaki [31]) and/or around rotating obstacles ([22]), and to obtain the existence and polynomial stability of bounded solutions to Navier-Stokes-Oseen equations on exterior domains in our recent work [23].

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Thieu Huy Nguyen, School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Vien Toan ung dung va Tin hoc, Dai hoc Bach khoa Hanoi, 1 Dai Co Viet, Hanoi, Vietnam
E-mail address: `huy.nguyenthieu@hust.edu.vn`