Necessary Optimality Conditions for $\varepsilon$e−Pareto Solutions in Vector Optimization with Empty Interior Ordering Cones

Truong Q. Bao · Suvendu R. Pattanaik

Abstract We present new necessary optimality conditions for $\varepsilon$e–Pareto optimal solutions of constrained vector optimization problems with empty interior ordering cones. We use the dual-space approach based on advanced tools of variational analysis and generalized differentiation. It allows us not implement any scalarization technique while be able to establish necessary results for nonconvex and nonsolid ordering cones.

Keywords vector optimization · coderivatives of set-valued mappings · subdifferentials of functionals · necessary optimality conditions · necessary suboptimality conditions · Pareto efficiency

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1 Introduction

Vector optimization is an important area of optimization theory. It has a wide range of applications in economics, business and numerous engineering disciplines.

Let $Z$ be a linear space partially ordered by a proper, closed, convex and pointed cone $\Theta$ which is called by an ordering cone. Denote this order by “$\leq_{\Theta}$”. Its ordering relation is described by

$$z_1 \leq_{\Theta} z_2 \quad \text{if and only if} \quad z_2 - z_1 \in \Theta \quad \text{for all} \quad z_1, z_2 \in Z.$$ (1.1)

Let $\Xi$ be a subset of $Z$ and $\bar{z} \in \Xi$. We say that $\bar{z}$ is a Pareto minimal point to $\Xi$ with respect to $\Theta$ if there is no element $z \in \Xi \setminus \{\bar{z}\}$ such that $z \in \Xi$ and $z \leq_{\Theta} \bar{z}$, i.e.,

$$\Xi \cap (\bar{z} - \Theta) = \{\bar{z}\}. \quad (1.2)$$

It is undoubt that vector optimization has its root in economics and equilibrium theory where the standard optimality concept in multiobjective optimization was initially introduced by F.Y. Edgeworth (1881) and V. Pareto (1906) via the usage of utility functions. Then it has been extended to the classical notion of

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Northern Michigan University
Department of Mathematics & Computer Science
Tel.: +01 (906) 227-1610
E-mail: btruong@nmu.edu

University of Coimbra, Portugal
Department of Mathematics
Tel.: +35 (123) 979-3061
E-mail: suvendu.pattanaik@gmail.com
Pareto efficiency/optimality defined via an ordering cone. However, the majority of publications on vector optimization concern a weak version of the ordering relation “≤θ” denoted by <θ and defined by replacing the cone Θ in (1.1) by its interior counterpart, and a weak Pareto minimal point to Ξ with respect to Θ is defined by replacing the optimality condition (1.2) by its weak counterpart Ξ ∩ (z − int Θ) = ∅ provided that the ordering cone Θ enjoys the nonempty interiority condition,

\[ \text{int } Θ \neq \emptyset. \]  

(1.3)

The most important advantage of the weak optimality notion is the possibility to study its optimality conditions by using the separation theorem and its generalized forms under the nonempty interiority assumption of Θ. Proceeding in this way allows us to convert the vector optimization problem under consideration to a scalar one which has been extensively studied. Employing to the later problem the known optimality results in scalar optimization we obtain optimality conditions for the former vector-valued problem. This approach is known as the scalarization approach.

However, the nonempty interiority condition (1.3) is such a serious restriction in infinite dimensional spaces since the class of topological vector spaces (in particular, Banach spaces) whose natural ordering cones have nonempty interior is not very broad, see, e.g., [14,15]. In particular, a Lebesgue space, either ℓ_\\text{p} or L_p for some 1 ≤ p < ∞, has an empty interior for the natural ordering cone. Fix p ∈ [1,∞) and consider the sequence space

\[ ℓ_\\text{p} := \{ \{x_i\}_{i\in N} | x_i \in \mathbb{R} \text{ for all } i \in N \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty \}. \]

It is a separable Asplund space equipped with the norm

\[ \| \{x_i\} \| := \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}. \]

The so-called natural ordering cone of ℓ_\\text{p} is given by

\[ Θ := \{ \{x\} \in ℓ_\\text{p} | x_i \geq 0 \text{ for all } i \in N \}. \]

It has an empty interior; see [14, Example 1.48]. Motivated by this fact, it has been utmost important for us to investigate issues in vector optimization when the ordering cone of an ordered space has an empty interior.

Working with nonsolid ordering cones we can use a number kinds of proper efficiency; the original concept was introduced by Kuhn and Tucker and modified by Geoffrion and later it was formulated in many general frameworks by Benson, Borwein, Borwein and Zhuang, Hartley, Henig, Hurwicz, etc.; see [3,8,11]. The idea lies on the requirement of the existence of a nonempty interior cone containing the given ordering cone such that the point in question is a weak Pareto point with respect to the new solid cone. Therefore, all the scalarization techniques successfully applied to weak optimality can be employed to proper efficiency as well. Recently, Bao and Mordukhovich have challenged Pareto optimality by using the so-called (dual-space) variational approach. They studied optimality issues of vector/set-valued optimization problems based on variational principles and relied on none scalarization schemes. The reader is referred to [2,3] and the references therein for the development of the dual-space variational approach. In another direction, Durea, Dutta and Tammer have investigated the ε-e-Pareto minimality concept. Given Ξ and Θ as above, a number ε > 0 and a direction e ∈ Θ. We say that \( \bar{z} \in Ξ \) is an ε-e-minimal point to Ξ if

\[ Ξ \cap (\bar{z} - \Theta - \varepsilon e) = \emptyset \iff (Ξ + \varepsilon e) \cap (\bar{z} - \Theta) = \emptyset. \]  

(1.4)

It is not difficult to check that an ε-e-minimal point to a set might not be a Pareto minimal point to that set in general. An ε-e-minimal point of Ξ can be viewed as an ε-optimal point of Ξ along the direction e in the sense that the set obtained by pushing the set Ξ along the direction e a distance ε completely leave the set of all the less-than-\( \bar{z} \) points. In [10] the authors presented an important relationship between ε-e-Pareto
minimal points and Pareto minimal points. This kind of \( \varepsilon \)-minimal points seems to be more suitable to study optimization problems from the practical viewpoint.

Note that when modeling various optimization problems related to traffic flow, equilibrium flows, economics, etc. nonconvex ordering cones and general references being not partial orders have been involved; see [4,5,7,9,13,17,19] and the preferences therein. In [9,13], the authors investigated a special class of nonconvex cones described as a union of a family of convex cones. Let us provide such simple cones. Given a Banach space \( Z \) and two linear independent vectors \( x \) and \( y \) in \( Z \). The cone
\[
\Theta := \text{cone}(x) \cup \text{cone}(y)
\]
is a nonconvex cone and has an empty interior. In \( \mathbb{R}^2 \) the cone
\[
\Theta := \{ (x,y) \in \mathbb{R}^2_+ \mid x y = 0 \}
\]
is nonsolid and nonconvex. In [5,17] the authors needed an ordering set containing the origin which was require neither convexity nor nonempty interiority assumptions in general. In [5,7,17] the authors allowed general preferences which had no transitivity property. In [12], the authors studied concepts of nondominatedness and/or optimality with respect to variable ordering structures with applications in image registration, portfolio optimization and location theory.

The main goal of this paper is to establish efficient necessary conditions for \( \varepsilon \)-Pareto minimizers to constrained vector optimization problems by using advanced tools and techniques in Variational Analysis and Generalized Differentiation systematically developed in [16,17]. The most significance of our necessary results related to ordering sets (not necessarily ordering cones) is that the ordering sets might not be either nonconvex or nonsolid. Our results recaptures the corresponding necessary results for convex ordering cones; in particular, those obtained in [10] by Durea et al. for nonsolid ordering cones.

The rest of the paper is organized as follow. In Section 2, we recall the notions of normal cones to sets, coderivatives of set-valued mappings and subdifferentials of functions and preliminary results in Variational Analysis and Generalized Differentiation. In Section 3 we establish new necessary conditions for \( \varepsilon \)-Pareto optimal solutions of multiobjective optimization problems with and without geometric constraints, where the image space is ordered by a generalized order determined via an ordering set \( \Theta \) containing the origin. We do not require that the ordering set is convex or it has a nonempty interior. Section 4 provides refined results formulated in terms of Fréchet differential objects under the conventional ordering cones in vector optimization.

2 Preliminaries

Throughout this paper, we use standard notation of Variational Analysis in [16] and Vector Optimization in [15,14,20].

For a Banach space \( X \), we denote its norm by \( \| \cdot \| \), its dual space equipped with the weak* topology \( w^* \) by \( X^* \), and the canonical pairing between \( X \) and \( X^* \) by \( \langle \cdot, \cdot \rangle \). The notations \( B \) and \( B^* \) stand for the closed unit ball of \( X \) and \( X^* \), respectively.

Let \( \varphi : X \to \mathbb{R} \cup \{ +\infty \} \) be an extended-real-valued function. The expressions \( \text{dom} \varphi := \{ x \in X \mid \varphi(x) < \infty \} \), \( \text{gph} \varphi := \{ (x,\varphi(x)) \mid x \in \text{dom} \varphi \} \) and \( \text{epi} \varphi := \{ (x,\alpha) \mid \alpha \geq \varphi(x) \} \) denote the domain, the graph and the epigraph of \( \varphi \), respectively.

Given a vector space \( Z \) and \( \Theta \) is a subset in \( Z \) containing the origin, i.e., \( 0 \in \Theta \). The set \( \Theta \) is called an ordering set of \( Z \). Consider an extended Pareto relation on \( Z \) induced by \( \Theta \) via the relation in (1.1). We extend the terminologies in vector optimization from ordering cones to non-conical ordering sets.

Given a set \( \Xi \in Z \). A point \( \bar{z} \in \Xi \) is said to be an (extended) minimal point of \( \Xi \) with respect to \( \Theta \) if \( (\Xi - \bar{z}) \cap -\Theta = \{0\} \). A point \( \bar{z} \) is said to be an \( \varepsilon \)-minimal point of \( \Xi \) with respect to \( \Theta \) if \( (\Xi + \varepsilon e - \bar{z}) \cap -\Theta = \emptyset \).
Given a vector-valued function \( f : X \to Z \) and a nonempty set \( \Omega \subset X \). A feasible solution \( \bar{x} \in \Omega \) is said to be a Pareto optimal solution (respectively, an \( \varepsilon \)-optimal solution) of \( f \) over \( \Omega \) with respect to \( \Theta \) if the image of \( f \) at \( \bar{x} \) denoted by \( \varepsilon := f(\bar{x}) \) is a minimal point (respectively, an \( \varepsilon \)-minimal point) to the image set of \( f \) over \( \Omega \) denoted by \( f(\Omega) := \{ f(x) \mid x \in \Omega \} \) with respect to \( \Theta \). We do not mention the set \( \Omega \) if \( \Omega = X \).

In other words, we simply say that \( \bar{x} \) is an optimal or \( \varepsilon \)-optimal solution of \( f \) with respect to \( \Theta \).

Since the main results of this paper require the Asplund property of the spaces in question, all the primal spaces under consideration are assumed to be Asplund unless otherwise stated. Recall that a Banach space is Asplund if every convex continuous function \( \varphi : U \to \mathbb{R} \) defined on an open convex subset \( U \) of \( X \) is Fréchet differentiable on a dense subset of \( U \). The class of Asplund spaces is quite broad including all reflexive Banach spaces and all Banach spaces with a separable dual; in particular, \( c_0 \) and \( \ell^p \), \( \ell^p[0,1] \) for \( 1 < p < \infty \) are Asplund spaces, but \( \ell_1 \) and \( \ell_\infty \) are not Asplund spaces; see, e.g., [1,16].

We formulate basic generalized differential constructions in Asplund spaces enjoying a full calculus. For their useful modifications in general Banach spaces, see [16, Chapter 1].

**Definition 1 (Fréchet normal cones).** Let \( \Omega \) be a nonempty subset of a Banach space \( X \). Given \( x, \bar{x} \in \Omega \), the set

\[
\hat{N}_x(x; \Omega) := \left\{ x^* \in X^* \mid \limsup_{(u,x) \to (\bar{x},x), u \in \Omega} \frac{\langle x^*, u-x \rangle}{\| u-x \|} \leq \varepsilon \right\}
\]

is called as the \( \varepsilon \)-Fréchet normal cone to \( \Omega \) at \( x \). When \( \varepsilon = 0 \), it reduces to the Fréchet or regular normal cone to \( \Omega \) at \( x \) and we write \( \hat{N}(x; \Omega) \) instead of \( \hat{N}_0(x; \Omega) \). If \( x \notin \text{cl} \Omega \), then \( \hat{N}(x; \Omega) = \emptyset \).

**Definition 2 (basic normals).** Let \( \Omega \) be a closed set of an Asplund space \( X \) and \( \bar{x} \in \Omega \). A dual element \( x^* \) is said to be a basic normal to a set \( \hat{\Omega} \subset X \) at \( \bar{x} \) if there are sequences \( x_k \to \bar{x} \) with \( x_k \in \Omega \), and \( x_k^* \rightharpoonup x^* \) with \( x_k^* \in \hat{N}(x_k, \Omega) \) for all \( k \in \mathbb{N} \). The collection of all the basic normals denoted by

\[
N(x, \Omega) = \limsup_{x \to \bar{x}} \hat{N}(x; \Omega)
\]

is known as the basic/limiting/Mordukhovich normal cone to \( \hat{\Omega} \) at \( \bar{x} \).

Considering now a set-valued mapping \( F : X \rightrightarrows Z \) between Asplund spaces with the domain \( \text{dom} F := \{ x \in X \mid F(x) \neq \emptyset \} \) and the graph \( \text{gph} F := \{ (x,z) \in X \times Z \mid z \in F(x) \} \), we recall the following coderivatives of \( F \) from the book [16].

**Definition 3 (coderivatives of set-valued mappings).** Given \( F : X \rightrightarrows Z \) and \( (\bar{x}, \bar{z}) \in \text{dom} F \).

- The Fréchet coderivative of \( F \) at \( (\bar{x}, \bar{z}) \) is a multifunction \( \hat{D}^*F(\bar{x}, \bar{z}) : Z^* \rightrightarrows X^* \) with the values

\[
\hat{D}^*F(\bar{x}, \bar{z})(z^*) := \left\{ x^* \in X^* \mid (x^*, -z^*) \in \hat{N}(\bar{x}, \bar{z}, \text{gph} F) \right\}
\]

- The normal coderivative of \( F \) at \( (\bar{x}, \bar{z}) \in \text{gph} F \) is a multifunction \( \hat{D}^N_F(\bar{x}, \bar{z}) : Z^* \rightrightarrows X^* \) with the values

\[
\hat{D}^N_F(\bar{x}, \bar{z})(z^*) := \left\{ x^* \in X^* \mid (x^*, -z^*) \in N((\bar{x}, \bar{z}), \text{gph} F) \right\}
\]

- The mixed coderivative of \( F \) at \( (\bar{x}, \bar{z}) \in \text{gph} F \) is a multifunction \( \hat{D}^M_F(\bar{x}, \bar{z}) : Z^* \rightrightarrows X^* \) with the values

\[
\hat{D}^M_F(\bar{x}, \bar{z})(z^*) := \left\{ x^* \in X^* \mid (x^*, -z^*) \in \hat{N}(\bar{x}, \bar{z}, \text{gph} F), \forall k \in \mathbb{N} \right\}
\]
It is obvious from the definitions of coderivatives the validity of the inclusions
\[ \widehat{D}^*F(\bar{x}, \bar{z})(\bar{z}^*) \subset D^*_MF(\bar{x}, \bar{z})(\bar{z}^*) \subset D^*_NF(\bar{x}, \bar{z})(\bar{z}^*) \]
for all $\bar{z}^* \in Z^*$. They hold as equalities and they are equal to $\{\nabla f(\bar{x})^* \bar{z}^*\}$ provided that $f$ is strictly differentiable at $\bar{x}$.

Considering now a function $f : X \to Z$ between Asplund spaces, where the image space is ordered by an ordering set $\Theta$. Associated with $f$ a set-valued mapping $E_f : X \to Z$, namely the epigraphical multifunction of $F$ with respect to $\Theta$, defined by
\[ E_f(x) := f(x) + \Theta. \] (2.5)
The epigraph of $f$ with respect to $\Theta$ is defined by
\[ \text{epi } f := \text{gph } E_f = \{(x, z) \in X \times Z | z \in f(x) + \Theta\}; \]
we omit $\Theta$ in the epigraphical notations $E_{f,\Theta}$ and $\text{epi } f$ for simplicity.

The subdifferentials of vector-valued functions (indeed, set-valued mappings) were introduced by Bao and Mordukhovich by mimicking the Mordukhovich nonconvex subdifferentials of extended-real-valued functionals. In the view of coderivatives of set-valued mappings, the following subdifferential constructions of $f$ are nothing but the corresponding coderivatives of its epigraphical multifunction $E_f$. See [2,3,5,16] for the development of subdifferentials for set-valued mappings.

**Definition 4 (subdifferentials of vector-valued functions).** Given $f : X \to Z$, $0 \in \Theta \subset Z$, $E_f : X \to Z$ and $\bar{x} \in \text{dom } f$

- The Fréchet subdifferential of $f$ at $\bar{x}$ in the direction $\bar{z}^* \in Z^*$ (with $\|\bar{z}^*\| = 1$) is defined by
  \[ \widehat{D}f(\bar{x})(\bar{z}^*) := \widehat{D}^*E_f(\bar{x}, f(\bar{x}))(\bar{z}^*). \]
- The (basic) subdifferential of $f$ at $\bar{x}$ in the direction $\bar{z}^* \in Z^*$ (with $\|\bar{z}^*\| = 1$) is defined by
  \[ \partial f(\bar{x})(\bar{z}^*) := D^*_NE_f(\bar{x}, f(\bar{x}))(\bar{z}^*). \]
- The singular subdifferential of $f$ at $\bar{x}$ is defined by
  \[ \partial^\infty f(\bar{x}) := D^*_NE_f(\bar{x}, f(\bar{x}))(0). \]

Obviously, the subdifferentials $\widehat{D}\varphi(\bar{x})(1), \partial \varphi(\bar{x})(1)$ and $\partial^\infty \varphi(\bar{x})$ of an extended-real-valued function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ reduce to the Fréchet subdifferential and the Mordukhovich nonconvex subdifferentials.

In order to establish the relation between the coderivative of a vector-valued function $f : X \to Z$ and the subdifferential of the scalarization of $f$ in the direction $\bar{z}^* \in Z^*$ (or a weight) $(f, \bar{z}^*) : X \to \mathbb{R}$ given by $(f, \bar{z}^*)(x) := (f(x), \bar{z}^*)$ the function $f$ needs to have Lipschitzian behaviors.

**Definition 5 (strictly Lipschitzian).** Let $f : X \to Z$ be a single-valued function between Banach spaces. It is said to be Lipschitz continuous around $\bar{x}$ if there is a neighborhood $U$ of $\bar{x}$ and a constant $\ell \geq 0$ such that
\[ \| f(x) - f(u) \| \leq \ell \| x - u \| \quad \text{for all } x, u \in U. \]
It is said to be strictly Lipschitz continuous at $\bar{x}$ if it is Lipschitz continuous around $\bar{x}$ and there is a neighborhood $U$ of the origin in $X$ such that any sequence $\{z_k\}$ defined by
\[ z_k = \frac{f(x_k + tkv) - f(x_k)}{tk}, \quad \forall k \in \mathbb{N}, \]
where $v \in U$, $x_k \to \bar{x}$ and $t_k \downarrow 0$, contains a norm convergent subsequence.
Obviously, if the image space $Z$ is a finite-dimensional space, then a Lipschitz continuous function is automatically strictly Lipschitz. Other properties of strictly Lipschitzian continuity can be found in, e.g., [16].

The following proposition presents several important properties of Lipschitzian functions.

**Proposition 6** Let $f : X \to Z$ be a Lipschitz continuous function at $\bar{x} \in \text{dom } f$. The following hold:

(i) $\partial^\infty f(\bar{x}) = \{0\}$.

(ii) $D_M f(\bar{x})(z^*) = \partial \langle z^*, f(\bar{x}) \rangle$ for all $z^* \in Z^*$.

(iii) $D_N f(\bar{x})(z^*) = \partial \langle z^*, f(\bar{x}) \rangle$ for all $z^* \in Z^*$ provided that $f$ is locally strictly Lipschitz continuous around $\bar{x}$.

In infinite-dimensional spaces, the validity of calculus and characterizations for these generalized differentiation requires certain additional “sequential normal compactness” properties of sets and mappings, which are automatic in finite dimensions, while being a crucial ingredient of variational analysis in infinite dimensions; see the books [16,17] for a comprehensive theory and numerous applications of various properties of this type. In this paper, for the sake of simplicity, we assume that the functions in questions enjoy the Lipschitzian behavior ensuring the SNC requirements of the involved calculus rules for generalized differentiation.

We conclude this section with the definition of indicator mappings. It is used to convert a constrained vector optimization problems to an unconstrained one.

**Definition 7 (indicator mappings).** The indicator mapping $\Delta : X \to Z$ of a set $\Omega \subset X$ between two Banach spaces is defined by

$$
\Delta(x; \Omega) := \begin{cases}
\{0\} \subset Z & \text{if } x \in \Omega, \\
\emptyset & \text{if } x \notin \Omega.
\end{cases}
$$

It can be viewed as an extended form of the classical indication $\delta(x; \Omega) : X \to \mathbb{R}$ equal to 0 if $x \in \Omega$ and $+\infty$ for the case of $Z = \mathbb{R}$ since we have $\text{gph } \Delta(\cdot; \Omega) = \text{gph } \delta(\cdot; \Omega) = \Omega \times \{0\}$. The Cartesian product structure of the graph of an indicator mapping surely yields

$$
D_N \Delta(\bar{x}; \Omega)(z^*) = D_M \Delta(\bar{x}, \Omega)(z^*) = N(\bar{x}, \Omega)
$$

for any $\bar{x} \in \Omega$ and $z^* \in Z^*$.

### 3 Main Result

Following the dual-space approach in [2,3,5] we establish new necessary optimality conditions for $\varepsilon$e-Pareto minimizers of the unconstrained and constrained vector optimization problems:

**$\Theta$-minimize** $f(x)$ subject to $x \in X$, \hspace{1cm} (3.7)

**$\Theta$-minimize** $f(x)$ subject to $x \in \Omega$, \hspace{1cm} (3.8)

where $f : X \to Z$ is a vector-valued function between Asplund spaces, $\Omega$ is a closed set in the domain space $X$, and $\Theta$ is an ordering set of the image space $Z$.

Note that a unconstrained problem can be considered as a constrained one by adding the implicit constraint $x \in \text{dom } f$ while a constrained problem can be equivalently described as an unconstrained one by adding the objective by the indicator mapping of the geometric constraint $\Omega$ to the image space $Z$, i.e., the new objective is $f_\Omega(x) := f(x) + \Delta(x; \Omega)$, where $\Delta(\cdot; \Omega)$ is defined in Definition 7, known as the restriction of $f$ over $\Omega$.

First let us recall some auxiliary results needed to establish our new necessary conditions.
Lemma 8 Let $f : X \to Z$ be a function between Banach spaces and let $\Theta \subset Z$ be an ordering set of the space $Z$ (i.e., $0 \in \Theta$). Associate with $f$ a set-valued mapping, namely the epigraphical multifunction of $f$ (with respect to $\Theta$), $\mathcal{E}_f : X \rightrightarrows Z$ with the images $\mathcal{E}_f(x) := f(x) + \Theta$. The epigraph of $f$ (with respect to $\Theta$) is defined via the graph of $\mathcal{E}_f$, i.e.,

$$\text{epi } f := \text{gph} \mathcal{E}_f = \{(x, z) \in X \times Z \mid z \in f(x) + \Theta\}.$$

Fix $\bar{x} \in \text{dom } f$ and $\bar{u} \in f(\bar{x}) + \Theta$ and then denote $\bar{z} := f(\bar{x})$. Then the following hold:

(a)

$$
\begin{align*}
\begin{cases}
\hat{N}((\bar{x}, \bar{u}), \text{epi } f) \subset \hat{N}((\bar{x}, \bar{z}), \text{epi } f) \subset \hat{N}((\bar{x}, \bar{z}), \text{gph } f), \\
(x^*, -z^*) \in \hat{N}((\bar{x}, \bar{u}), \text{epi } f) \Rightarrow -z^* \in \hat{N}(\bar{u} - \bar{z}, \Theta).
\end{cases}
\end{align*}
$$

(3.9)

If $\Theta$ is a convex cone, then $\hat{N}(\bar{u} - \bar{z}, \Theta) \subset \hat{N}(0, \Theta)$.

(b) Assume further that both $X$ and $Z$ are Asplund, that $f$ is continuous around $\bar{x}$, and that $\Theta$ is locally closed around $0 \in Z$. Then the relationships in (3.9) hold for the limiting normal cones as well.

$$
\begin{align*}
\begin{cases}
N((\bar{x}, \bar{u}), \text{epi } f) \subset N((\bar{x}, \bar{z}), \text{epi } f) \subset N((\bar{x}, \bar{z}), \text{gph } f), \\
(x^*, -z^*) \in N((\bar{x}, \bar{u}), \text{epi } f) \Rightarrow -z^* \in N(\bar{u} - \bar{z}, \Theta).
\end{cases}
\end{align*}
$$

Proof. First let us justify the first inclusion in the first line in (3.9) since the second one $\hat{N}((\bar{x}, \bar{z}), \text{epi } f) \subset \hat{N}((\bar{x}, \bar{z}), \text{gph } f)$ is trivial thanks to the decreasing monotonicity property of Fréchet normal cones.

Take arbitrarily $(x^*, -z^*) \in \hat{N}((\bar{x}, \bar{u}), \text{epi } f)$. By the definition of the Fréchet normal cone in (1), for any $\varepsilon > 0$ there is $\eta > 0$ such that

$$
\langle (x^*, -z^*), (x, z) - (\bar{x}, \bar{u}) \rangle \leq \varepsilon \| (x, z) - (\bar{x}, \bar{u}) \|, \quad \forall \,(x, u) \in \text{epi } f \text{ with } \|(x, u) - (\bar{x}, \bar{u})\| \leq \eta.
$$

(3.10)

Since $\text{epi } f = \text{gph } \mathcal{E}_f$, we can find $\hat{\theta} \in \Theta$ such that $\bar{u} = \bar{z} + \hat{\theta}$ (recalling that $\bar{z} = f(\bar{x})$). Taking now an arbitrary pair $(x, z) \in \text{epi } f$ with $\|(x, z) - (\bar{x}, \bar{z})\| \leq \eta$, we have

$$
\| (x, z + \hat{\theta}) - (\bar{x}, \bar{z} + \hat{\theta}) \| = \| (x, z) - (\bar{x}, \bar{z}) \| \leq \eta \text{ with } (x, z + \hat{\theta}) \in \text{epi } f,
$$

i.e., the pair $(x, z + \hat{\theta})$ satisfies (4.22). Substituting it to (4.22) we obtain

$$
\langle (x^*, -z^*), (x, z + \hat{\theta}) - (\bar{x}, \bar{z} + \hat{\theta}) \rangle \leq \varepsilon \| (x, z + \hat{\theta}) - (\bar{x}, \bar{z} + \hat{\theta}) \|
$$

$$
\Rightarrow \langle (x^*, -z^*), (x, z) - (\bar{x}, \bar{z}) \rangle \leq \varepsilon \| (x, z) - (\bar{x}, \bar{z}) \|
$$

justifying $(x^*, -z^*) \in \hat{N}((\bar{x}, f(\bar{x})), \text{epi } f)$ since $(x, z)$ and $\varepsilon$ were arbitrary.

Similarly, by fixing $x = \bar{x}$ in (4.22) we can find an estimate for the dual element $z^*$. Precisely, for any $\theta \in \Theta$ with $\| \theta - \hat{\theta} \| \leq \eta$, we have $(\bar{x}, \bar{z} + \theta) \in \text{epi } f$ and $\| (\bar{x}, \bar{z} + \theta) - (\bar{x}, \bar{u}) \| = \| \theta - (\bar{u} - \bar{z}) \| \leq \eta$. Plugging the pair $(\bar{x}, \bar{z} + \theta)$ into (4.22) we have

$$
\langle -z^*, \theta - (\bar{u} - \bar{z}) \rangle \leq \varepsilon \| \theta - (\bar{u} - \bar{z}) \|
$$

which ensures that $-z^* \in \hat{N}((\bar{u} - \bar{z}, \Theta)$. Assume furthermore that $\Theta$ is a convex cone. Then we have $-z^* \in \hat{N}(0; \Theta)$ since

$$
\hat{N}((\bar{u} - \bar{z}, \Theta) = \hat{N}(0, \Theta - \bar{u} + \bar{z}) = \hat{N}(0, \Theta - \theta) \subset \hat{N}(0, \Theta)
$$
since $\Theta \subset \Theta - \hat{\theta}$ for any convex cone $\Theta$ and $\hat{\theta} \in \Theta$ and since the Fréchet normal cones enjoys the reverse monotonicity property. Note finally that both the Fréchet and limiting normal cones to convex sets reduce to the normal cone of convex analysis. Thus, $-z^* \in N(0; \Theta) = \bar{N}(0; \Theta)$. We have proved (a).

To check the validity of (b), from the limiting normal vector $(x^*, -z^*) \in N((\bar{x}, \bar{u}), \text{gph} \mathcal{E}_f)$ we can find a sequence of Fréchet normal vectors $(x_k^*, z_k^*) \xrightarrow{\text{epi}} (x^*, z^*)$ as $k \to \infty$ satisfying

$$(x_k^*, -z_k^*) \in \bar{N}((x_k, u_k); \text{epi} f) \text{ for some sequence } (x_k, u_k) \xrightarrow{\text{epi}} (\bar{x}, \bar{u}).$$

By (a), we have $-z_k^* \in \bar{N}(u_k - f(x_k); \Theta)$ and

$$(x_k^*, -z_k^*) \in \bar{N}((x_k, f(x_k)); \text{epi} f) \subset \bar{N}((x_k, f(x_k)); \text{gph} f).$$

Passing these to limit as $k \to \infty$ and taking into account that $(x_k, f(x_k)) \to (\bar{x}, \bar{z})$ we arrive at $-z^* \in N(\bar{u} - f(\bar{x}); \Theta)$ and

$$(x^*, -z^*) \in N((\bar{x}, \bar{z}); \text{epi} f) \subset N((\bar{x}, \bar{z}); \text{gph} f).$$

The proof of (b) is complete. \qed

Note that Lemma 8 can be seen as a combination of [17, Proposition 5.23] and [2, Proposition 4.3]. In [17, Proposition 5.23] there is no estimate to the normal cone to the epigraph of $f$ or the subdifferential of $f$. In [2, Proposition 4.3] the relationships in (4.22) were formulated for a broader class of set-valued mappings enjoying the so-called order continuity property. This kind of continuity is automatics for continuous single-valued functions and lower semicontinuous extended-real-valued functions. We provide here a much simpler proof for the class of single-valued functions involved in this paper. The reader can find an alternative proof for set-valued mappings relying on the calculus rules for generalized differentiation in [6].

We are now ready to formulate new necessary optimality conditions for $\varepsilon\varepsilon$-Pareto optimal solutions of the unconstrained problem (3.7) and the constrained problem (3.8). To justify them we employ advanced tools and techniques of variational analysis and generalized differentiation. It is important to emphasize that the ordering set $\Theta$ is not assumed to be either convex or solid in general. However, if it happens to be convex, then the obtained results can be further improved.

**Theorem 9** Let $f : X \to Z$ be a continuous function between Asplund spaces and let $\Theta \subset Z$ is a closed ordering set of the image space $Z$, which might be nonconvex and nonsolid. Assume that $\bar{x}$ is an $\varepsilon\varepsilon$-optimal solution of problem (3.7) or simply of $f$. Then, for every positive number $\lambda > \varepsilon$, there exist $\bar{y} \in \text{dom } f$ with $\|\bar{y} - \bar{x}\| < \lambda$, $\hat{\theta} \in \Theta$, and a nonzero dual element $z^* \in -N(\hat{\theta}; \Theta)$ with $\|z^*\| = 1$ such that

$$\|y^*\| < \frac{\varepsilon}{\lambda - \varepsilon} \text{ for some } y^* \in \partial f(\bar{y})(z^*) \subset D_{\xi}^* f(\bar{y})(z^*). \tag{3.11}$$

**Proof.** By the definition of $\varepsilon\varepsilon$-optimality, we have

$$f(x) \notin (\bar{y} - \Theta - \varepsilon\varepsilon), \forall x \in \text{dom } f$$

since $\bar{x}$ is an $\varepsilon\varepsilon$-optimal solution of $f$. Denote $\bar{z} := f(\bar{x})$ and consider the set $\Xi$ in the product space $X \times Z$ given by

$$\Xi := \{(x, z) \in X \times Z \mid x \in \text{dom } f \text{ and } z \in f(x) + \varepsilon\varepsilon + \Theta\}.$$

Obviously, $\Xi$ is locally closed around $(\bar{x}, \bar{z} + \varepsilon\varepsilon e)$ due to the continuity of $f$ the closedness of $\Theta$. It is not difficult to check that $(\bar{x}, \bar{z}) \notin \Xi$. Arguing by contradiction, assume that $(\bar{x}, \bar{z}) \in \Xi$, i.e., $\bar{x} \in \text{dom } f$ and $\bar{z} \in f(\bar{x}) + \varepsilon\varepsilon e + \Theta$. The latter inclusion yields $-\varepsilon e \in \Theta$ contradicting to the definition of $\varepsilon\varepsilon$-optimality.

Considering now a scalar optimization problem: Find the distance from the point $(\bar{x}, \bar{z})$ to the set $\Xi$

$$\text{minimize} \quad \varphi(x, z) := \|z - \bar{z}\| \quad \text{subject to} \quad (x, z) \in \Xi, \tag{3.12}$$
we have \((\bar{x}, \bar{z} + \varepsilon \mathbf{e}) \in \Xi\) and
\[
\varphi(\bar{x}, \bar{z} + \varepsilon \mathbf{e}) = \|((\bar{z} + \varepsilon \mathbf{e}) - \bar{z})\| = \varepsilon \leq \inf_{(x,z) \in \Xi} \varphi(x, z) + \varepsilon.
\]
Thus, \((\bar{x}, \bar{z})\) is an \(\varepsilon\)-optimal solution of the auxiliary problem (3.12). In the other words, \((\bar{x}, \bar{z})\) is an \(\varepsilon\)-optimal solution to the function \(\varphi(x, z) + \delta((x, z); \Xi)\), where \(\delta(\cdot; \Xi)\) is the indicator function of \(\Xi\).

Employing the lower subdifferential variational principle from [16, Theorem 2.28] to the function \(\varphi + \delta(\cdot; \Xi)\), the \(\varepsilon\)-optimal solution \((\bar{x}, \bar{z} + \varepsilon \mathbf{e})\), and the number \(\lambda > 0\), we can find \((\bar{y}, \bar{w} + \varepsilon \mathbf{e}) \in \Xi\) with \(\bar{w} \in f(\bar{y}) + \Theta = E_{f}(\bar{y})\) and \((u^*, v^*) \in X^* \times Z^*\) satisfying
\[
\frac{\|((\bar{y}, \bar{w} + \varepsilon \mathbf{e}) - (\bar{x}, \bar{z} + \varepsilon \mathbf{e})\|}{\|u^*\|} \leq \lambda \quad \text{and} \quad \frac{\|(u^*, v^*)\|}{\|z\|} = \frac{\bar{z}}{\lambda}
\]
for some \((u^*, v^*) \in \partial(\varphi + \delta(\cdot; \Xi))(\bar{y}, \bar{w}) \subset \partial(\varphi + \delta(\cdot; \Xi))(\bar{y}, \bar{w})\).

Applying the limiting subdifferential sum rule for pairs of semi-Lipschitz functions in [16, Theorem 3.36] we have
\[
\partial(\varphi + \delta(\cdot; \Xi))(\bar{y}, \bar{w} + \varepsilon \mathbf{e}) \subset \partial \| - \bar{z}\|((\bar{y}, \bar{w} + \varepsilon \mathbf{e}) + \partial \delta((\bar{y}, \bar{w} + \varepsilon \mathbf{e}); \Xi))
\]
\[
= \{0\} \times \partial \|\bar{w} + \varepsilon \mathbf{e}\| - \bar{z}\| + N((\bar{y}, \bar{w} + \varepsilon \mathbf{e}); \Xi).
\]

Note that \(\bar{w} + \varepsilon \mathbf{e} \neq \bar{z}\). Arguing by contradiction, assume that \(\bar{w} + \varepsilon \mathbf{e} = \bar{z}\) and get from \(\bar{w} \in f(\bar{y}) + \Theta\) the existence of \(\theta \in \Theta\) such that \(\bar{z} = f(\bar{y}) + \varepsilon \mathbf{e} + \theta\). Thus, we have
\(\bar{y} \in \text{dom } f\) and \(f(\bar{y}) = \bar{z} - \varepsilon \mathbf{e} - \theta \in \bar{z} - \varepsilon \mathbf{e} - \Theta\)
contradicting to the \(\varepsilon\mathbf{e}\)-optimality of \(\bar{x}\). By the convexity of the norm and the nonzero of \(\bar{v} := \bar{w} + \varepsilon \mathbf{e} - \bar{z}\) we have
\(w^* \in \partial \|\bar{v}\| \Rightarrow \|w^*\| = 1\).

Inclusion (3.13) can be expressed in the form
\(\langle u^*, v^* \rangle - \langle 0, w^* \rangle \in N((\bar{y}, \bar{w} + \varepsilon \mathbf{e}); \Xi)\)
for some \(w^* \in \partial \|\bar{v}\|\) with \(\|w^*\| = 1\). Set
\[
(y^*, -z^*) := \left(\frac{u^*}{\|u^* - w^*\|}, \frac{v^* - w^*}{\|v^* - w^*\|}\right)
\]
we have
\((y^*, -z^*) \in N((\bar{y}, \bar{w} + \varepsilon \mathbf{e}); \Xi) = N((\bar{y}, \bar{w}); \text{epi } f)\);
the inequality can be checked by using the definition directly or by using the sum rule with \(f\) and the constant function \(g(x) \equiv \varepsilon \mathbf{e}\).

By Lemma 8 (b) we have \(z^* \in -N(\tilde{\Theta}; \Theta)\) with \(\tilde{\Theta} := \bar{w} - f(\bar{y})\) and
\[
(y^*, -z^*) \in N((\bar{y}, \bar{f}(\bar{y})); \text{epi } f) \subset N((\bar{y}, \bar{f}(\bar{y})); \text{gph } f).
\]
Taking into account the definitions of subdifferential and coderivative, we get
\(y^* \in \partial f(\bar{y})(z^*) \subset D_N^* f(\bar{y})(z^*)\).

It is not easy to check the estimates for the dual elements: \(\|z^*\| = 1\) and
\[
\|y^*\| \leq \frac{\|u^*\|}{\|v^* - w^*\|} \leq \frac{\bar{z}}{1 - \frac{\varepsilon}{\lambda}} = \frac{\varepsilon}{\lambda - \varepsilon}.
\]
The proof is complete.

Assume now that \(\bar{x}\) is an \(\varepsilon\mathbf{e}\)-optimal solution of \(f\) or problem (3.7) for every \(\varepsilon > 0\) sufficiently small; that is automatics for the conventional closed and convex ordering cones. Then, the necessary condition for \(\varepsilon\mathbf{e}\)-optimal solutions in Theorem 9 can be taken the following form.
Corollary 10 Assume in addition to the hypotheses in Theorem 9 that $\bar{x}$ is an $\varepsilon\textbf{e}$-optimal solution of $f$ for all $\varepsilon \leq \bar{\varepsilon}$. Then, for every $\lambda > 0$, there exist $\bar{y} \in \text{dom} f$ with $\|\bar{y} - \bar{x}\| \leq \lambda$, $\bar{\theta} \in \Theta$, and a nonzero dual element $z^* \in -N(\bar{\theta}; \Theta)$ with $\|z^*\| = 1$ such that
\begin{equation}
0 \in \partial f(\bar{y})(z^*) + \lambda \mathbb{B}^* \subset D^*_N f(\bar{y})(z^*) + \lambda \mathbb{B}^*.
\end{equation}

Proof. Observe that if the assertion in (3.14) holds for some $\bar{\lambda}$, then it is still valid for any $\lambda$ bigger than $\bar{\lambda}$. Therefore, it is sufficient to verify it for all $\lambda$ sufficiently small.

Take $\lambda \in (0, \sqrt{\bar{\varepsilon}})$ arbitrarily, then choose $\eta > 0$ such that $\eta < \lambda$ and $\frac{\eta}{1 - \eta} < \lambda$. Since $\eta^2 \leq \lambda^2 < \bar{\varepsilon}$, $\bar{x}$ is an $\eta^2\varepsilon$-optimal solution of $f$. Employing Theorem 9 to this solution, $\varepsilon = \eta^2$, and $\lambda = \eta$, we have $\bar{y} \in \text{dom} f$ with $\|\bar{y} - \bar{x}\| \leq \eta < \lambda$ and a nonzero dual element $z^* \in Z^*$ with $\|z^*\| = 1$ such that
\begin{equation}
0 \in \partial f(\bar{y})(z^*) + \frac{\eta}{1 - \eta} \mathbb{B}^* \subset \partial f(\bar{y})(z^*) + \lambda \mathbb{B}^*,
\end{equation}
where the latter inclusion holds due to the choice of $\eta$, and thus we also have
\begin{equation}
0 \in D^*_N f(\bar{y})(z^*) + \lambda \mathbb{B}^*
\end{equation}
due to Lemma 8. The proof is complete. \hfill \Box

The results in Theorem 9 and Corollary 10 are formulated for the unconstrained optimization problems. As mentioned before, a constrained optimization problem (3.8) is equivalent to the unconstrained one, where the objective function is the restriction of the cost $f$ over the constraint $\Omega$, denoted by $f_{\Omega} : X \rightarrow Z$ and defined by $f_{\Omega}(x) := f(x) + \triangle(x; \Omega)$. We have the following estimates for the coderivative and the subdifferential of $f_{\Omega}$:
\begin{align*}
\partial f_{\Omega}(\bar{y})(z^*) &\subset \partial f(\bar{y})(z^*) + N(\bar{y}; \Omega), \\
D^*_N f_{\Omega}(\bar{y})(z^*) &\subset D^*_N f(\bar{y})(z^*) + N(\bar{y}; \Omega).
\end{align*}

It is important to stress that by replacing such an upper estimate for the subdifferential/coderivative of the restriction $f_{\Omega}$ makes necessary results less efficient.

Theorem 11 Let $f : X \rightarrow Z$ be a Lipschitz continuous function between Asplund spaces, let $\Omega \subset X$ be a nonempty closed subset in $X$, and let $\Theta \subset Z$ be a closed ordering set $\Theta \subset Z$ which might be nonconvex and nonsolid. Suppose that $\bar{x}$ is an $\varepsilon, \textbf{e}$-optimal solution of problem (3.8). Then for every $\lambda > \varepsilon$, there exist $\bar{y} \in \Omega$ with $\|\bar{y} - \bar{x}\| \leq \lambda$ and $z^* \in -N(\bar{\theta}; \Theta)$ with $\|z^*\| = 1$ for some $\theta \in \Theta$ with $\|\theta\| \leq \lambda$ such that
\begin{equation}
0 \in D^*_N f_{\Omega}(\bar{y})(z^*) + N(\bar{y}; \Omega) + \frac{\varepsilon}{\lambda - \varepsilon} \mathbb{B}^*.
\end{equation}
Assume furthermore that $f$ is strictly Lipschitzian around $\bar{x}$. Then, condition (3.15) can be expressed in the form
\begin{equation}
0 \in (z^*, f)(\bar{y}) + N(\bar{y}; \Omega) + \frac{\varepsilon}{\lambda - \varepsilon} \mathbb{B}^*.
\end{equation}

Proof. By Proposition 6, we have the equivalent between inclusion (3.15) and inclusion (3.16) when $f$ is strictly Lipschitz around $\bar{x}$. Therefore, it is sufficient to prove condition (3.15) only.

Since $\bar{x}$ is an $\varepsilon, \textbf{e}$-optimal solution of $f_{\Omega}$, we apply Theorem 9 to the function $f_{\Omega}$ and the approximate solution $\bar{x}$. For every $\lambda > \varepsilon$, there exist $\bar{y} \in \Omega$ with $\|\bar{y} - \bar{x}\| \leq \lambda$ and $z^* \in -N(\bar{\theta}; \Theta)$ with $\|z^*\| = 1$ for some $\theta \in \Theta$ with $\|\theta\| \leq \lambda$ such that
\begin{equation}
0 \in D^*_N f_{\Omega}(\bar{y})(z^*) + \frac{\varepsilon}{\lambda - \varepsilon} \mathbb{B}^*.
\end{equation}
Note that the function $f_{\Omega}$ is, indeed, a sum of the Lipschitz continuous function $f$ and the lower semicontinuous mapping $\triangle(\cdot; \Omega)$. Applying the coderivative sum rule for Lipschitz-like mappings from [16, Corollary 3.11] to this sum, we get
\begin{equation}
D^*_N f_{\Omega}(\bar{y})(z^*) = D^*_N (f + \triangle(\cdot; \Omega))(\bar{y})(z^*) \\
\subset D^*_N f(\bar{y})(z^*) + D^* \triangle(\bar{y}; \Omega)(z^*) = D^* f(\bar{y})(z^*) + N(\bar{y}; \Omega).
\end{equation}
Substituting (3.18) into (3.17) we arrives at the desired inclusion in (3.15). The proof is complete.

Corollary 12 Let \( f, \Omega \) and \( \Theta \) be as in Theorem 11. Assume that \( \bar{x} \) is an \( \varepsilon \)-optimal solution of the constrained problem (3.8) for all \( \varepsilon \leq \bar{\varepsilon} \). Then for every \( \lambda > 0 \), there exist \( \bar{y} \in \Omega \) with \( \| \bar{y} - \bar{x} \| \leq \lambda \) and \( z^* \in \nabla \Theta (\bar{y}; \Theta) \) with \( \| z^* \| = 1 \) for some \( \bar{\theta} \in \Theta \) with \( \| \bar{\theta} \| \leq \lambda \) satisfying

\[
\mathbf{0} \in \partial (z^*, f)(\bar{y}) + N(\bar{y}; \Omega) + \lambda \Theta^*
\]

provided that \( f \) is strictly Lipschitz continuous at \( \bar{y} \).

Proof. It follows from the proof of Theorem 10 in which we replace Theorem 9 with Corollary 11.

4 Revised necessary conditions for \( \varepsilon \)-optimal solutions.

In this section we present refined necessary conditions for \( \varepsilon \)-optimal solutions of both the unconstrained problem (3.7) and the constrained one (3.8) with the usual ordering cone in vector optimization, i.e., \( \Theta \) is a closed and convex ordering cone in the image space \( Z \). In contrast to general ordering sets in Section 3, the additional properties of \( \Theta \) ensures the transitivity property of the ordering relation \( \leq \). We obtain more efficient optimality conditions in two regards: (1) we remove the restriction \( \lambda > \varepsilon \) in Theorem 9 and Theorem 11 and (2) we formulate necessary conditions in terms of Fréchet subdifferential objects which are smaller than the corresponding limiting ones.

Our techniques base on a version of the (subdifferential) variational principle for vector-valued functions; see [3, Theorem 3.8] for set-valued mappings: Let \( f : X \rightarrow Z \) be a vector-valued function between Asplund spaces, where \( Z \) is partially ordered by a proper, closed and convex cone \( \Theta \subset Z \) with \( \Theta \setminus (-\Theta) \neq \emptyset \), i.e., \( \Theta \) is not a linear subspace of \( Z \). Assume that \( f \) is quasibounded from below (i.e., there is a bounded set \( M \) in \( Z \) such that \( f(X) \subset M + \Theta \), epi-closed (i.e., the epigraph of \( f \) is closed in the product space \( X \times Z \)). Then for any \( \varepsilon > 0 \), \( \lambda > 0 \), \( e \in \Theta \setminus (-\Theta) \) with \( \| e \| = 1 \), and a strict approximate \( \varepsilon \)-optimal solution \( \bar{x} \) of the function \( f \) (i.e., there is a \( \varepsilon \in \varepsilon \) such that \( \bar{x} \) is a \( \varepsilon \)-optimal solution of \( f \)), there are \( \bar{u} \in \partial f(\bar{x}) \) with \( \| \bar{u} - \bar{x} \| < \lambda \) and \( v^* \in -\bar{\nabla}(0; \Theta) \) with \( \| v^* \| = 1 \) satisfying

\[
\| u^* \| < \frac{\varepsilon}{\lambda} \text{ for some } u^* \in \partial f(\bar{u})(v^*) \subset \bar{\partial} f(\bar{u})(v^*). \quad (4.19)
\]

Note that the requirement of strict approximate \( \varepsilon \)-optimality is essential in vector optimization, but not in scalar optimization. Note also that if \( \bar{\varepsilon} \) is a \( \varepsilon \)-optimal point to a set \( \Xi \), then it is a strict \( (\varepsilon + \delta) \)-optimal point to that set for any \( \delta > 0 \). But, this implication does not hold for \( \delta = 0 \). Indeed, consider the set \( \Xi \) in \( \mathbb{R}^2 \) with the conventional Pareto order, i.e., \( \Theta = \mathbb{R}^2_+ \) defined by

\[
\Xi := \{(x, y) \in \mathbb{R}^2 \mid x = 2 - \frac{1}{y} \text{ or } y = 2 - \frac{1}{x}\}.
\]

Set \( e := (1, 1) \in \mathbb{R}^2_+ \). It is not difficult to check that the point \( \bar{\varepsilon} := (1, 1) \) is a \( \sqrt{2} \)-optimal point to \( \Xi \), but it is not a strict \( \sqrt{2} \)-optimal point to \( \Xi \).

The next result presents a refined version of Theorem 11 as well as its Corollary 12.

Theorem 13 Let \( f : X \rightarrow Z \) be a Lipschitz continuous function between Asplund spaces, let \( \Omega \subset X \) be a nonempty closed subset in \( X \), and let \( \Theta \subset Z \) be a closed and convex ordering cone satisfying \( \Theta \setminus (-\Theta) \neq \emptyset \) (which might have an empty interior.) Suppose that \( \bar{x} \) is a strict \( \varepsilon \)-optimal solution of problem (3.8). Then, for every \( \lambda > 0 \), there exist \( \bar{y}_1 \in \partial f(\bar{x}) \) with \( \| \bar{y}_1 - \bar{x} \| < \varepsilon \), \( \bar{y}_2 \in \Omega \) with \( \| \bar{y}_2 - \bar{x} \| < \varepsilon \), and a nonzero dual element \( z^* \in -\bar{\nabla}(0; \Theta) \) with \( \| z^* \| = 1 \) such that

\[
\| x^* \| < \frac{\varepsilon}{\lambda} \text{ for some } x^* \in \hat{\partial} f(\bar{y}_1)(z^*) + \hat{\nabla}(\bar{y}_2; \Omega); \quad (4.20)
\]

and thus we also have

\[
\| x^* \| < \frac{\varepsilon}{\lambda} \text{ for some } x^* \in \hat{\partial} f(\bar{y}_1)(z^*) + \hat{\nabla}(\bar{y}_2; \Omega). \]
**Proof.** Let us first assume that $f$ is quasi-bounded from below over $\Omega$, i.e., there is a bounded set $M$ in $Z$ such that $f(\Omega) \subset M + \Theta$; in the other words, the restriction of $f$ over $\Omega$ denoted by $f_{|\Omega}$ is quasi-bounded from below. Employing the subdifferential variational principle from [3, Theorem 3.8] to the function $f_{|\Omega}$ under the assumptions made, there are $\bar{u} \in \Omega$ with $\|\bar{u} - \bar{x}\| \leq t_1 < \lambda$ and $\nu^* \in -\hat{N}(0; \Theta)$ with $\|\nu^*\| = 1$ such that

$$
\|u^*\| \leq t_2 < \frac{\epsilon}{\lambda} \text{ with } u^* \in \hat{D}f_{|\Omega}(\bar{u}, f(\bar{u}))(\nu^*) = \hat{D}^*\mathcal{E}_{f_{|\Omega}}(\bar{u}, f(\bar{u}))(\nu^*).
$$

We need to specify the numbers $t_1$ and $t_2$ here for estimates in what follows. By the definitions of subderivative and the implications in Lemma 8 we have

$$
u^* \in \hat{D}^*\mathcal{E}_{f_{|\Omega}}(\bar{u}, f(\bar{u}))(\nu^*) \iff (u^*, -\nu^*) \in \hat{N}((\bar{u}, \bar{v}); \epi f_{|\Omega}) \Rightarrow -\nu^* \in \hat{N}(0; \Theta).
$$

Define two sets $\Xi_1$ and $\Xi_2$ in the product space $T := X \times Z$ by

$$
\Xi_1 := \epi f = \{(x, z) \in T \mid z = f(x) + \Theta\},
$$

$$
\Xi_2 := \Omega \times Z = \{(x, z) \in T \mid x \in \Omega\}.
$$

(4.21)

It is easy to check that $\epi f_{|\Omega} = \Xi_1 \cap \Xi_2$.

Taking $\eta > 0$ such that $t_1 + \eta \leq \lambda$ and $\frac{t_1 + \eta}{\Lambda} \leq \frac{\xi}{\lambda}$, and then applying the fuzzy rule for set intersections from [16, Lemma 3.1] with the parameter $\eta$, we can find

$$
\left\{
(\bar{y}_1, w_1) \in \Xi_1 \text{ with } \|\bar{y}_1, w_1\| - (\bar{u}, \hat{v}) \| \leq \eta,
\right.

$$

$$
(\bar{y}_2, w_2) \in \Xi_2 \text{ with } \|\bar{y}_2, w_2\| - (\bar{u}, \hat{v}) \| \leq \eta,
$$

$$
(u_1^*, -v_1^*) \in \hat{N}((\bar{y}_1, w_1); \Xi_1),
$$

$$
(u_2^*, 0) \in \hat{N}((\bar{y}_2, w_2); \Xi_2) = \hat{N}(\bar{y}_2; \Omega) \times \{0\},
$$

$$
\|\|u_1^* - v_1^*\| + \|u_2^*, 0\| - k(\|u^*, -\nu^*\|) \leq \eta \text{ and max}\{\|u_1^*\|, \|v_1^*\|, k\} = 1.
$$

(4.22)

By Lemma 8 one has

$$
\left\{
\begin{array}{l}
(u_1^*, -v_1^*) \in \hat{N}((\bar{y}_1, w_1); \epi f) \Rightarrow v_1^* \in -\hat{N}(0; \Theta),
\end{array}
\right.
$$

$$
\left\{
\begin{array}{l}
u_1^* \in -\hat{N}((\bar{y}_1, w_1); \epi f) \subset \hat{N}((\bar{y}_1, f(\bar{y}_1)); \text{gph} f).
\end{array}
\right.
$$

The second line clearly gives $u_1^* \in \hat{D}^*f(\bar{y}_1)(v_1^*)$. Taking into account the estimate of coderivatives of Lipschitz functions in [16, Theorem 1.43], we have

$$
\|u_1^*\| \leq \ell\|v_1^*\|
$$

where $\ell$ is the Lipschitz constant of $f$. Without loss of generality, we may assume $\ell \leq 1$; otherwise, we consider the function defined by $\ell^{-1}f(\cdot)$ being Lipschitz continuous with the Lipschitz constant equal to 1.

By (4.22) we get $\|v_1^*\| \geq k\|\nu^*\| - \eta = k - \eta$ from $\|v_1^* + kv^*\| \leq \eta$. Thus we have

$$
1 = \max\{\|u_1^*\|, \|v_1^*\|, k\} = \max\{\ell(k - \eta), k - \eta, k\} = k.
$$

By scaling dual elements

$$
y_1^* := \frac{u_1^*}{\|v_1^*\|} \text{ and } y_2^* := \frac{u_2^*}{\|v_1^*\|}, \quad z^* := \frac{v_1^*}{\|v_1^*\|}
$$

we now have

$$
\|z^*\| = 1, \quad y_1^* \in \hat{D}f(\bar{y}_1) \subset \hat{D}^*f(\bar{y}_1) \text{ and } y_2^* \in \hat{N}(\bar{y}_2; \Omega)
$$

satisfying

$$
\|y_1^* + y_2^*\| = \frac{\|u_1^* + u_2^*\|}{\|v_1^*\|} \leq \frac{\eta + \|u^*\|}{\|v^*\| - \eta} \leq \frac{\eta + t_1}{1 - \eta} \leq \frac{\epsilon}{\lambda},
$$

$$
\|\bar{y}_1 - \bar{x}\| \leq \|\bar{y}_1 - \bar{u}\| + \|\bar{u} - \bar{x}\| \leq \eta + t_1 < \lambda \text{ and }
$$

$$
\|\bar{y}_2 - \bar{x}\| \leq \|\bar{y}_2 - \bar{u}\| + \|\bar{u} - \bar{x}\| \leq \eta + t_1 < \lambda.
$$

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due to the choices of $t_1$, $t_2$ and $\eta$. We have proved the necessary suboptimality condition (4.20) provided that $f$ is quasi-bounded from below.

It remains to prove it when $f$ is not quasi-bounded from below. Fix $r > \lambda > 0$ and define $$\widehat{\Omega} := \Omega \cap (\bar{x} + rB_X).$$

Since $f$ is locally Lipschitz with the Lipschitz constant $\ell$ over $\Omega$, so does it over the truncated set $\widehat{\Omega}$. Then, we have
\[ f(u) \in f(\bar{x}) + \ell\|u - \bar{x}\|B_Z \subset f(\bar{x}) + r\ell B_Z \]
surely verifying that $f$ is quasi-bounded from below over $\widehat{\Omega}$. By what had been proved above, for every $\lambda > 0$, there are $\bar{y}_1 \in \text{dom } f$ with $\|\bar{y}_1 - \bar{x}\| < \lambda$, $\bar{y}_2 \in \Omega \subset \Omega$ with $\|\bar{y}_2 - \bar{x}\| < \lambda$ and $z^* \in -\hat{N}(\bar{0}; \Theta)$ with $\|z^*\| = 1$ such that
\[ \|x^*\| < \frac{\varepsilon}{\lambda} \] for some $x^* \in \partial f(\bar{y}_1)(z^*) + \hat{N}(\bar{y}_2; \widehat{\Omega}) \subset \hat{D}^* f(\bar{0}) + \hat{N}(\bar{y}_2; \widehat{\Omega}).$

Since $\hat{N}(\bar{y}_2; \widehat{\Omega}) = \hat{N}(\bar{y}_2; \Omega)$ due to the structure of $\widehat{\Omega}$ and the estimate $\|\bar{y}_2 - \bar{x}\| < \lambda$, we complete the proof of the theorem.

As a direct consequence of Theorem 4 we recapture the fuzzy necessary conditions for Pareto optimal solutions to problem (3.8).

**Corollary 14** Let $f$, $\Omega$ and $\Theta$ be as in Theorem 13. Assume that $\bar{x}$ is a Pareto optimal solution of the constrained optimization problem (3.8). Then, for every $\varepsilon > 0$, there exist $\bar{y}_1 \in \text{dom } f$ with $\|\bar{y}_1 - \bar{x}\| < \varepsilon$, $\bar{y}_2 \in \Omega$ with $\|\bar{y}_2 - \bar{x}\| < \varepsilon$, and $z^* \in -\hat{N}(\bar{0}; \Theta)$ with $\|z^*\| = 1$ such that
\[ \|x^*\| < \varepsilon \] for some $x^* \in \hat{D}^* f(\bar{y}_1)(z^*) + \hat{N}(\bar{y}_2; \Omega). \quad (4.23)$

**Proof.** Fix $\varepsilon > 0$ and take any $\varepsilon \in \Theta \setminus (-\Theta)$ with $\|\varepsilon\| = 1$. Since $\bar{x}$ is a Pareto minimal solution of problem (3.8), it is a strict $\varepsilon^2 \varepsilon$-minimal solution for this problem. Employing Theorem 13 for the strict $\varepsilon^2 \varepsilon$-minimal solution $\bar{x}$ and the parameter $\lambda = \varepsilon$, we arrive at the desired condition (4.23).

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**References**


