

# ON TANGENT CONES AT INFINITY OF ALGEBRAIC VARIETIES

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ABSTRACT. In this paper, we establish the following version at infinity of Whitney's theorem [7, 8]: *Geometric and algebraic tangent cones at infinity of complex algebraic varieties coincide.* The proof of this fact is based on a geometric characterization of geometric tangent cones at infinity and using the global Lojasiewicz inequality with explicit exponents for complex algebraic varieties. We also show that tangent cones at infinity of complex algebraic varieties can be computed using Gröbner bases.

## 1. INTRODUCTION

Let  $V \subset (\mathbb{C}^n, 0)$  be an analytic variety in some neighborhood of the origin  $0$  in  $\mathbb{C}^n$ . In his seminal papers [7, 8], Whitney showed (among other results) that geometric and algebraic tangent cones of  $V$  at  $0$  coincide. The aim of this paper is to give a version at infinity of this result for complex algebraic varieties.

In order to state our main result, we need some notation. Let  $V \subset \mathbb{C}^n$  be an algebraic variety. The *geometric tangent cone*  $C_{g,\infty}(V)$  of  $V$  at infinity is defined by the set of “tangent” vectors, in the sense that  $v \in C_{g,\infty}(V)$  if, and only if, there exist a sequence  $\{x_k\}_{k \in \mathbb{N}} \subset V$ ,  $\|x_k\| \rightarrow \infty$ , and a sequence of numbers  $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$  such that  $t_k x_k \rightarrow v$ . By the *algebraic tangent cone*  $C_{a,\infty}(V)$  of  $V$  at infinity we mean the set

$$C_{a,\infty}(V) := \{v \in \mathbb{C}^n \mid f^*(v) = 0 \text{ for all } f \in \mathbf{I}(V)\},$$

where  $\mathbf{I}(V)$  denotes the ideal defining  $V$ , and for each polynomial  $f \in \mathbf{I}(V)$ ,  $f^*$  denotes its homogeneous component of highest degree.

The main result of this paper can be formulated as follows.

**Theorem 1.1.** *Let  $V \subset \mathbb{C}^n$  be an algebraic variety. Then  $C_{a,\infty}(V) = C_{g,\infty}(V)$ .*

**Example 1.1.** Let  $V := \{(x, y) \in \mathbb{C}^2 \mid x^2 - y^3 = 0\}$ . We have

$$C_{a,\infty}(V) = C_{g,\infty}(V) = \{y = 0\}.$$

**Remark 1.1.** The proof for Theorem 1.1 is different from the one given in [8, Theorem 10.6]. In fact, the proof of the inclusion  $C_{g,\infty}(V) \subseteq C_{a,\infty}(V)$  is rather straightforward (Lemma 2.3), while the proof of the converse one will follow from a geometric characterization of geometric

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tangent cones at infinity (Lemma 2.4) and the global Łojasiewicz inequality with explicit exponents for algebraic varieties (Lemma 2.5).

The paper is organized as follows. The proof of Theorem 1.1 will be given in Section 2. Then in Section 3, using Gröbner bases, we compute tangent cones at infinity of algebraic varieties.

## 2. PROOF OF THEOREM 1.1

Let  $V \subset \mathbb{C}^n$  be an algebraic variety. By definition, we can verify that the sets  $C_{g,\infty}(V)$  and  $C_{a,\infty}(V)$  are closed *cones* in  $\mathbb{C}^n$  with the vertex at the origin  $0 \in \mathbb{C}^n$ , i.e. for every element  $v \in C_\infty(V)$  and for every  $\lambda \in \mathbb{C}$ , we have  $\lambda v \in C_\infty(V)$ , where  $C_\infty(V)$  denotes for both  $C_{g,\infty}(V)$  and  $C_{a,\infty}(V)$ . Moreover, the following properties of these cones are obtained easily from their definitions.

**Lemma 2.1.** *For any two algebraic varieties  $V$  and  $W$  in  $\mathbb{C}^n$ , the following properties hold:*

- (i)  $C_\infty(W) \subseteq C_\infty(V)$  if  $W \subseteq V$ ;
- (ii)  $C_\infty(V \cup W) = C_\infty(V) \cup C_\infty(W)$ ;
- (iii)  $C_\infty(V \cap W) \subseteq C_\infty(V) \cap C_\infty(W)$ .

**Remark 2.1.** In general we cannot replace “ $\subseteq$ ” by “ $=$ ” in (iii). Indeed, let us consider the algebraic varieties  $V$  and  $W$  in  $\mathbb{C}^2$  defined respectively by  $x = 0$  and  $y - x^2 = 0$ . Then  $C_\infty(V) = C_\infty(W) = \{x = 0\}$ , while  $C_\infty(V \cap W) = \{0\}$ . Hence in this case,  $C_\infty(V \cap W) \subsetneq C_\infty(V) \cap C_\infty(W)$ .

The following statement shows that tangent vectors to  $V$  at infinity may be defined by analytic curves (compare [6, Proposition 2] and [8, Theorem 11.8]).

**Lemma 2.2.** *Let  $V \subset \mathbb{C}^n$  be an algebraic variety. Then, for each nonzero vector  $v \in C_{g,\infty}(V)$ , there exists an analytic curve  $\varphi: (0, \epsilon) \rightarrow \mathbb{C}^n$  ( $\epsilon > 0$ ) with  $\varphi(s) \in V$  for all  $s \in (0, \epsilon)$ , such that*

$$\lim_{s \rightarrow 0^+} \|\varphi(s)\| = +\infty \quad \text{and} \quad \lim_{s \rightarrow 0^+} \frac{\varphi(s)}{\|\varphi(s)\|} = - \lim_{s \rightarrow 0^+} \frac{\varphi'(s)}{\|\varphi'(s)\|} = \frac{v}{\|v\|}.$$

*Proof.* Indeed, let  $v$  be a nonzero vector in  $C_{g,\infty}(V)$ . By definition, there exist a sequence  $\{x_k\}_{k \in \mathbb{N}} \subset V$ ,  $\|x_k\| \rightarrow \infty$ , and a sequence of numbers  $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$  such that  $t_k x_k \rightarrow v$ . In view of the Curve Selection Lemma at infinity (see [2, 5]), there exist analytic curves  $\psi := (\psi_1, \dots, \psi_n): (0, \epsilon) \rightarrow \mathbb{C}^n$  and  $t: (0, \epsilon) \rightarrow \mathbb{C}$ , for some  $\epsilon > 0$ , such that

- (a)  $\psi(s) \in V$  for all  $s \in (0, \epsilon)$ ;
- (b)  $\lim_{s \rightarrow 0^+} \|\psi(s)\| = +\infty$ ;
- (c)  $\lim_{s \rightarrow 0^+} t(s)\psi(s) = v$ .

Since  $v \neq 0$ , we have that  $t(s) \neq 0$ , and so we can write

$$t(s) = as^p + \text{higher terms in } s,$$

for some  $a \in \mathbb{C}$ , with  $a \neq 0$ , and  $p \in \mathbb{Z}$ . Let  $J := \{j \mid \psi_j(t) \neq 0\} \neq \emptyset$ . For each  $j \in J$  we can write

$$\psi_j(s) = b_j s^{q_j} + \text{higher terms in } s,$$

for some  $b_j \in \mathbb{C}$ , with  $b_j \neq 0$ , and  $q_j \in \mathbb{Z}$ . It follows from conditions (b) and (c) that

$$p = -\min_{j \in J} q_j > 0$$

and

$$v = \begin{cases} ab_j & \text{if } j \in J \text{ and } q_j = -p, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\lambda \in \mathbb{C}$  be such that  $\lambda^{-p} = a$  and we may regard  $\psi$  as an analytic curve from the disc  $\{s \in \mathbb{C} \mid |s| < \epsilon\}$  to  $\mathbb{C}^n$ . Define the curve  $\varphi: (0, \epsilon/|\lambda|) \rightarrow \mathbb{C}^n$  by  $\varphi(s) := \psi(\lambda s)$ . Then, by condition (a),  $\varphi(s) \in V$  for all  $s \in (0, \epsilon/|\lambda|)$ . Furthermore, we have

$$\|\varphi(s)\| = cs^{-p} + \text{higher terms in } s$$

for some real number  $c > 0$ . Finally, a direct computation shows that

$$\lim_{s \rightarrow 0^+} \frac{\varphi_j(s)}{\|\varphi(s)\|} = -\lim_{s \rightarrow 0^+} \frac{\varphi'_j(s)}{\|\varphi'(s)\|} = \begin{cases} \frac{ab_j}{c} & \text{if } j \in J \text{ and } q_j = -p, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. □

**Remark 2.2.** By Lemma 2.2, it will not alter the definition of the cone  $C_{g,\infty}(V)$  if we require that the  $t_k$  be real and positive. In particular,  $v \in C_{g,\infty}(V) \setminus \{0\}$  if, and only if, there exists a sequence  $\{x_k\}_{k \in \mathbb{N}} \subset V$  such that

$$\lim_{k \rightarrow \infty} \|x_k\| = +\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{x_k}{\|x_k\|} = \frac{v}{\|v\|}.$$

**Lemma 2.3.** *Let  $V \subset \mathbb{C}^n$  be an algebraic variety. Then  $C_{g,\infty}(V) \subseteq C_{a,\infty}(V)$ .*

*Proof.* We first consider the case where  $V$  is a hypersurface defined by a polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$ . We can write

$$f = f_d + f_{d-1} + \dots + f_0,$$

where  $d := \deg f$  and each  $f_i$  is homogeneous polynomial of degree  $i$ .

Take any non-zero vector  $v \in C_{g,\infty}(V)$ . By Lemma 2.2, there exists a sequence  $\{x_k\}_{k \in \mathbb{N}} \subset V$  such that

$$\lim_{k \rightarrow \infty} \|x_k\| = +\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{x_k}{\|x_k\|} = \frac{v}{\|v\|}.$$

For  $k$  sufficiently large we have  $\|x_k\| > 0$  and hence

$$\begin{aligned} 0 &= f(x_k) = f\left(\|x_k\| \frac{x_k}{\|x_k\|}\right) \\ &= \|x_k\|^d \left[ f_d\left(\frac{x_k}{\|x_k\|}\right) + \frac{1}{\|x_k\|} f_{d-1}\left(\frac{x_k}{\|x_k\|}\right) + \cdots + \frac{1}{\|x_k\|^d} f_0\left(\frac{x_k}{\|x_k\|}\right) \right], \end{aligned}$$

which implies that

$$0 = f_d\left(\frac{x_k}{\|x_k\|}\right) + \frac{1}{\|x_k\|} f_{d-1}\left(\frac{x_k}{\|x_k\|}\right) + \cdots + \frac{1}{\|x_k\|^d} f_0\left(\frac{x_k}{\|x_k\|}\right).$$

Letting  $k \rightarrow \infty$  yields  $f_d(v) = 0$ , i.e.  $v \in C_{a,\infty}(V)$ .

Now let  $V$  be any algebraic variety in  $\mathbb{C}^n$  and  $v \in C_{g,\infty}(V)$ . Take any  $f \in \mathbf{I}(V)$ . We have  $V \subseteq \mathbf{V}(f) := \{x \in \mathbb{C}^n \mid f(x) = 0\}$ , and so, by Lemma 2.1(i),  $C_{g,\infty}(V) \subseteq C_{g,\infty}(\mathbf{V}(f))$ . It follows from the hypersurface case considered above that  $v \in C_{a,\infty}(\mathbf{V}(f))$ . Note that  $C_{a,\infty}(\mathbf{V}(f)) = \{w \in \mathbb{C}^n \mid f_d(w) = 0\}$ . Therefore,  $f_d(v) = 0$ . Since  $f$  is arbitrary,  $v \in C_{a,\infty}(V)$ . The proof is complete.  $\square$

In order to show the reverse inclusion  $C_{g,\infty}(V) \supseteq C_{a,\infty}(V)$ , we need the following geometric characterization of the cone  $C_{g,\infty}(V)$ .

**Lemma 2.4.** *Let  $V \subset \mathbb{C}^n$  be an algebraic variety and let  $v$  be a non-zero vector in  $\mathbb{C}^n$ . Then  $v \in C_{g,\infty}(V)$  if and only if there exists a sequence  $\{t_k\}_{k \in \mathbb{N}}$  of positive real numbers such that*

$$\lim_{k \rightarrow \infty} t_k = +\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\text{dist}(t_k v, V)}{t_k} = 0,$$

where  $\text{dist}(x, V)$  denotes the usual Euclidean distance from a point  $x \in \mathbb{C}^n$  to the variety  $V$ .

*Proof.*  $\Rightarrow$  Assume that  $v \in C_{g,\infty}(V)$ . By Lemma 2.2, there exists a sequence  $\{x_k\}_{k \in \mathbb{N}} \subset V$  such that

$$\lim_{k \rightarrow \infty} \|x_k\| = +\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{x_k}{\|x_k\|} = \frac{v}{\|v\|}.$$

Then we may assume that  $t_k := \frac{\|x_k\|}{\|v\|} > 0$  for all  $k \in \mathbb{N}$ . We have  $\lim_{k \rightarrow \infty} t_k = +\infty$  and

$$\frac{\text{dist}(t_k v, V)}{t_k} \leq \frac{\|t_k v - x_k\|}{t_k} = \left\| v - \frac{\|v\|}{\|x_k\|} x_k \right\| = \|v\| \left\| \frac{v}{\|v\|} - \frac{x_k}{\|x_k\|} \right\|.$$

Therefore,

$$\lim_{k \rightarrow \infty} \frac{\text{dist}(t_k v, V)}{t_k} = 0.$$

$\Leftarrow$  Conversely, assume there exists a sequence  $\{t_k\}_{k \in \mathbb{N}}$  of positive real numbers such that

$$\lim_{k \rightarrow \infty} t_k = +\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\text{dist}(t_k v, V)}{t_k} = 0.$$

For each  $k \in \mathbb{N}$ , let  $x_k$  be a point in  $V$  such that  $\text{dist}(t_k v, V) = \|t_k v - x_k\|$  (such a point does always exist because  $V$  is a closed set in the usual topology on  $\mathbb{C}^n$ ). Then

$$\lim_{k \rightarrow \infty} \left\| v - \frac{x_k}{t_k} \right\| = \lim_{k \rightarrow \infty} \frac{\|t_k v - x_k\|}{t_k} = \lim_{k \rightarrow \infty} \frac{\text{dist}(t_k v, V)}{t_k} = 0.$$

Hence  $\lim_{k \rightarrow \infty} \frac{x_k}{t_k} = v$  and so  $\lim_{k \rightarrow \infty} \|x_k\| = +\infty$  because  $\lim_{k \rightarrow \infty} t_k = +\infty$ . Furthermore we have

$$\lim_{k \rightarrow \infty} \frac{\|x_k\|}{t_k} = \|v\|.$$

Therefore

$$\lim_{k \rightarrow \infty} \frac{x_k}{\|x_k\|} = \lim_{k \rightarrow \infty} \left( \frac{x_k}{t_k} \times \frac{t_k}{\|x_k\|} \right) = \lim_{k \rightarrow \infty} \frac{x_k}{t_k} \times \lim_{k \rightarrow \infty} \frac{t_k}{\|x_k\|} = \frac{v}{\|v\|}.$$

The proof is complete.  $\square$

Let  $V \subset \mathbb{C}^n$  be an algebraic variety of dimension  $k$ . It is well-known that, for any generic  $(n-k)$ -dimensional linear subspace  $P$  in  $\mathbb{C}^n$ , the set  $V \cap P$  is finite, and its cardinality is called the *degree* of  $V$  and denoted by  $\text{deg}(V)$ . In particular, if  $V = \mathbf{V}(f) := \{x \in \mathbb{C}^n \mid f(x) = 0\}$  is a hypersurface defined by a complex polynomial  $f$  of degree  $d$ , then  $\text{deg}(V) = d$ . The following lemma will be useful in the sequel.

**Lemma 2.5** ([4, Lemma 8]). *Let  $V \subset \mathbb{C}^n$  be an irreducible algebraic variety of degree  $d$ . Then there are finitely many polynomials  $g_i, i = 1, \dots, s$ , of degree at most  $d$  vanishing on  $V$  and a constant  $c > 0$  such that*

$$\text{dist}(x, V)^d \leq c \max_{i=1, \dots, s} \{|g_i(x)|\} \quad \text{for all } x \in \mathbb{C}^n.$$

We are now ready to complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* By Lemma 2.3, we have  $C_{g, \infty}(V) \subseteq C_{a, \infty}(V)$ . Hence, it remains to show the reverse inclusion.

We first remark that it suffices to prove the inclusion  $C_{a, \infty}(V) \subseteq C_{g, \infty}(V)$  for the case where the variety  $V$  is irreducible. Indeed, if  $V_1, \dots, V_r$  are irreducible components of  $V$ , and assume that we have proved that  $C_{g, \infty}(V_i) = C_{a, \infty}(V_i)$  for every  $i$ , then by Lemma 2.1(ii) we have

$$C_{g, \infty}(V) = \bigcup_{i=1}^r C_{g, \infty}(V_i) = \bigcup_{i=1}^r C_{a, \infty}(V_i) = C_{a, \infty}(V).$$

Therefore, we may now assume that  $V$  is irreducible of degree  $d$ . By Lemma 2.5, there exist finitely many polynomials  $g_i, i = 1, \dots, s$ , of degree at most  $d$  vanishing on  $V$  and a constant  $c > 0$  such that for all  $x \in \mathbb{C}^n$  we have

$$\text{dist}(x, V)^d \leq c \max_{i=1, \dots, s} \{|g_i(x)|\}.$$

Let  $v$  be a non-zero vector in  $C_{a,\infty}(V)$ . Since  $V \subseteq \mathbf{V}(g_i)$  for every  $i = 1, \dots, s$ , by Lemma 2.1(i) we have  $C_{a,\infty}(V) \subseteq C_{a,\infty}(\mathbf{V}(g_i))$ , and so  $v \in C_{a,\infty}(\mathbf{V}(g_i))$ . Hence

$$(g_i)_{d_i}(v) = 0, \quad \text{for all } i = 1, \dots, s, \quad (1)$$

where  $(g_i)_{d_i}$  denotes the homogeneous component of  $g_i$  of degree  $d_i := \deg g_i$ .

Now let  $\{t_k\}_{k \in \mathbb{N}}$  be a sequence of positive real numbers such that  $\lim_{k \rightarrow \infty} t_k = +\infty$ . It follows from (1) that for every  $i = 1, \dots, s$ , there exists a positive constant  $c_i$  such that for all  $k$  sufficiently large,

$$|g_i(t_k v)| \leq c_i t_k^{d_i-1}.$$

Since  $d_i := \deg g_i \leq d$  for  $i = 1, \dots, s$ , we obtain for all  $k$  sufficiently large,

$$\max_{i=1,\dots,s} \{|g_i(t_k v)|\} \leq c' t_k^{d-1},$$

where  $c' := \max_{i=1,\dots,s} c_i > 0$ . Therefore

$$\frac{\text{dist}(t_k v, V)}{t_k} \leq \frac{(c \max_{i=1,\dots,s} \{|g_i(t_k v)|\})^{\frac{1}{d}}}{t_k} \leq (c'c)^{\frac{1}{d}} t_k^{\frac{d-1}{d}-1} = (c'c)^{\frac{1}{d}} t_k^{-\frac{1}{d}}.$$

By letting  $k \rightarrow \infty$ , we get

$$\lim_{k \rightarrow \infty} \frac{\text{dist}(t_k v, V)}{t_k} = 0.$$

Hence  $v \in C_{g,\infty}(V)$  by Lemma 2.4. The proof of the theorem is complete.  $\square$

### 3. COMPUTATIONS

In this section, given an algebraic variety  $V \subset \mathbb{C}^n$ , we shall compute the tangent cone at infinity  $C_\infty(V) := C_{a,\infty}(V) = C_{g,\infty}(V)$  of  $V$  by using Gröbner bases.

If  $\mathbf{I}(V) = \langle f \rangle \subset \mathbb{C}[x_1, \dots, x_n]$ , then it is easy to see that

$$C_\infty(V) = \mathbf{V}(f^*) := \{v \in \mathbb{C}^n \mid f^*(v) = 0\}.$$

(Recall that  $f^*$  stands for the homogeneous component of highest degree of  $f$ .) However, if  $\mathbf{I}(V) = \langle f_1, \dots, f_r \rangle$  has more generators, then it need *not* follow that

$$C_\infty(V) = \mathbf{V}(f_1^*, \dots, f_r^*) := \{v \in \mathbb{C}^n \mid f_1^*(v) = \dots = f_r^*(v) = 0\}.$$

For example, let  $V \subset \mathbb{C}^3$  be defined by the ideal generated by polynomials

$$f_1 := xy \quad \text{and} \quad f_2 := z(x^3 - y^2 + z^2).$$

We have  $f_1^* = xy$ ,  $f_2^* = x^3 z$ , and

$$\mathbf{V}(f_1^*, f_2^*) = \{x = 0\} \cup \{y = z = 0\}.$$

Consider the polynomial  $f \in \mathbb{C}[x, y, z]$  defined by

$$f(x, y, z) := yz(y^2 - z^2) = zx^2f_1 - yf_2.$$

We see that  $f^* = f$  vanishes on  $C_\infty(V)$ , however  $f^*$  does not vanish on  $\mathbf{V}(f_1^*, f_2^*)$ . It follows that  $C_\infty(V) \neq \mathbf{V}(f_1^*, f_2^*)$ .

We can overcome this difficulty by using an appropriate Gröbner basis for the ideal defining  $V$ . In order to compute the ideal defining the cone  $C_\infty(V)$  we need the following notation.

For a polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$  of degree  $d$ , its *homogenization*  $f^h$  is a polynomial in  $\mathbb{C}[x_0, x_1, \dots, x_n]$  defined by

$$f^h(x_0, x_1, \dots, x_n) := x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right),$$

where  $x_0$  is a new variable. For an ideal  $I$  in the ring  $\mathbb{C}[x_1, \dots, x_n]$ , its *homogenization* is the ideal

$$I^h := \langle f^h \mid f \in I \rangle \subset \mathbb{C}[x_0, x_1, \dots, x_n].$$

For any  $g \in \mathbb{C}[x_0, x_1, \dots, x_n]$ , let

$$g|_{x_0=0} := g(0, x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n].$$

Following [3] we define

$$I_\infty := \langle g|_{x_0=0} \mid g \in I^h \rangle \subset \mathbb{C}[x_1, \dots, x_n].$$

**Lemma 3.1.** *Let  $V \subset \mathbb{C}^n$  be an algebraic variety and let  $I := \mathbf{I}(V)$ . Then*

$$C_\infty(V) = \mathbf{V}(I_\infty).$$

*Proof.* Let us take any  $x = (x_1, \dots, x_n) \in C_\infty(V)$  and  $f \in I_\infty$ . Then there exists  $g \in I$  such that  $f = g^h|_{x_0=0}$ . Decomposing  $g$  into homogeneous components as

$$g = g_0 + g_1 + \dots + g_d,$$

where  $d$  is the degree of  $g$ , we have

$$g^h = g_0x_0^d + g_1x_0^{d-1} + \dots + g_{d-1}x_0 + g_d \in \mathbb{C}[x_0, x_1, \dots, x_n].$$

It follows that

$$f = g^h|_{x_0=0} = g_d = g^*.$$

By definition of  $C_\infty(V) = C_{a,\infty}(V)$ , we have  $f(x) = g^*(x) = 0$ , and so  $x \in \mathbf{V}(I_\infty)$ . Therefore,  $C_\infty(V) \subseteq \mathbf{V}(I_\infty)$ .

Conversely, let us take any  $x \in \mathbf{V}(I_\infty)$  and  $f \in I$ . Assume we decompose  $f$  into homogeneous components as

$$f = f_0 + f_1 + \dots + f_e,$$

where  $e$  is the degree of  $f$ . Then

$$f^h = f_0 x_0^e + f_1 x_0^{e-1} + \cdots + f_{e-1} x_0 + f_e \in I^h.$$

It follows that

$$f_e = f^h|_{x_0=0} \in I_\infty.$$

By definition of  $\mathbf{V}(I_\infty)$ , therefore  $f^*(x) = f_e(x) = 0$ . It follows that  $x \in C_{a,\infty}(V) = C_\infty(V)$ . The proof is complete.  $\square$

By [3, Lemma A.4.15], the variety  $\mathbf{V}(I_\infty) \subset \mathbb{C}^n$  equals to the so-called *part at infinity* of  $V$ . Hence, it follows from Lemma 3.1 and the remark after [3, Lemma A.4.15] the following computation for  $C_\infty(V) = \mathbf{V}(I_\infty)$ .

**Corollary 3.1.** *Let  $V \subset \mathbb{C}^n$  be an algebraic variety and let  $\{g_1, \dots, g_k\}$  be a Gröbner basis for  $\mathbf{I}(V)$  with respect to a degree ordering. Then*

$$C_\infty(V) = \mathbf{V}(g_1^h|_{x_0=0}, \dots, g_k^h|_{x_0=0}).$$

**Example 3.1.** Let us compute the tangent cone at infinity of the algebraic variety  $V \subset \mathbb{C}^3$  defined by the ideal  $I \subset \mathbb{C}[x, y, z]$  generated by polynomials  $f_1 := xy$  and  $f_2 := z(x^3 - y^2 + z^2)$  as considered at the beginning of this section.

It is easy to compute a Gröbner basis for  $I$  with respect to a degree ordering is

$$\{g_1 = xy, \quad g_2 = x^3z - y^2z + z^3, \quad g_3 = yz(y^2 - z^2)\}.$$

The homogenization of these polynomials by a new variable  $t$  in the polynomial ring  $\mathbb{C}[t, x, y, z]$ :

$$g_1^h = xy, \quad g_2^h = x^3z - y^2zt + z^3t, \quad g_3^h = yz(y^2 - z^2).$$

Substituting  $t = 0$ , using Corollary 3.1, we have

$$C_\infty(V) = \mathbf{V}(xy, x^3z, yz(y^2 - z^2)),$$

which is a union of 5 lines through the origin in  $\mathbb{C}^3$ .

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## REFERENCES

- [1] D. COX, J. LITTLE, AND D. O'SHEA, *Ideals, varieties, and algorithms*, Springer-Verlag, New York, Third edition, 2007.
- [2] S. T. DINH, H. V. HÀ, AND T. S. PHẠM, *A Frank-Wolfe type theorem for nondegenerate polynomial programs*, Math. Program. Ser. A., **147** (1-2) (2014), 519-538.
- [3] G.-M. GREUEL AND G. PFISTER, *A singular introduction to commutative algebra*, 2nd extended edition, Springer, Berlin, 2008.
- [4] S. JI, J. KOLLÁR AND B. SHIFFMAN, *A global Lojasiewicz inequality for algebraic varieties*, Trans. AMS., **329** (2) (1992), 813–818.
- [5] J. MILNOR, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies **61**, Princeton University Press, 1968.
- [6] D. O'SHEA AND L.S. WILSON, *Limits of tangent spaces to real surfaces*, Amer. J. Math., **126** (2004), 951–980.
- [7] H. WHITNEY, *Local properties of analytic varieties*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), 1965, pp. 205–244 Princeton Univ. Press, Princeton, N. J.
- [8] H. WHITNEY, *Tangents to an analytic variety*, Ann. of Math., **81** (1965), 496–549.

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