Two Lectures on Regular Singular Stratified Bundles

Lars Kindler
Universität Duisburg-Essen
Web: `http://www.esaga.uni-due.de/lars.kindler/`
Email: `lars.kindler@uni-due.de`

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Abstract

These are the notes for two 90 minute lectures given at the Vietnam Institute for Advanced Studies in Mathematics in August 2012. The author thanks VIASM and in particular Prof. Dr. Phung Ho Hai for the kind invitation and hospitality.

Contents

1 Lecture 1 - Differential Operators

1.1 Overview

The goal of these lectures is to give an elementary introduction into the theory of stratified bundles in positive characteristic, with a special emphasis on regular singularity as defined in [Gie75], and, more generally, the authors thesis [Kin12]. One of the main theorems of loc. cit. is

1.1 Theorem. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth, connected, separated, finite type k-scheme and E a stratified bundle on X, i.e. an $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}$-module. Then the following are equivalent:*

(a) *E is regular singular with finite monodromy group.*

(b) *E is trivialized on a finite, tame covering of X.* □

The notion of tameness used here will be defined in the lectures, and it is studied in [KS10]. The proof of Theorem 1.1 splits into two parts: An argument using more or less standard facts from the theory of Tannakian categories, and the following Main Lemma:

1.2 Theorem (Main Lemma). *Let $f : Y \to X$ be a finite galois étale morphism. Then $f_*\mathcal{O}_Y$ is a stratified bundle on X, i.e. an $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}$-module, and f is tame if and only if $f_*\mathcal{O}_Y$ is regular singular.* □

In the course of the lectures, we will define the terms used in the statement of the Main Lemma, and develop the theory far enough to give a proof. We try to avoid Tannakian and log-geometric language.

1.2 Differential operators

We give a brief summary of the relevant facts from [EGA4, §.16] and [BO78, Ch. 2], without proofs.

1.2.1 Definitions

Let $f : X \to S$ be a separated (for simplicity) morphism of schemes and $\mathrm{diag}_{X/S} : X \to X \times_S X$ the diagnoal. This is a closed immersion by assumption. Let $I_{X/S} \subseteq \mathcal{O}_{X\times_S X}$ be the associated sheaf of ideals and $\Delta^n_{X/S}$ the closed subscheme defined by $I^{n+1}_{X/S}$, which is called the *n-th infinitesimal neighborhood of X*. Note that the closed immersion $i_n : X \hookrightarrow \Delta^n_{X/S}$ is a homeomorphism on topological spaces.

1.3 Definition. We define

$$\mathcal{P}^n_{X/S} := i_n^{-1}\mathcal{O}_{\Delta^n_{X/S}} = \mathrm{diag}^{-1}_{X/S}\,\mathcal{O}_{X\times_S X}/I^{n+1}_{X/S}$$

and call it *the sheaf of n-th principal parts.* □

Note that the two projections $X \times_S X \to X$ induce maps $p^n_0, p^n_1 : \Delta^n_{X/S} \to X$, and accordingly, the sheaf of rings $\mathcal{P}^n_{X/S}$ carries a left- and a right-$\mathcal{O}_X$-structure via the corresponding ring morphisms $d^n_0, d^n_1 : \mathcal{O}_X \to \mathcal{P}^n_{X/S}$. In fact, for $n > 0$, one always has $(\mathcal{P}^n_{X/S} \cong \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X)/J^{n+1}$, where J is the kernel of the multiplication map $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X \to \mathcal{O}_X$, and with this identification we have $d_0^n(a) = a \otimes 1$, $d_1^n(a) = 1 \otimes a$.

1.4 Definition. Let E, F be two $\mathcal{O}_X$-modules. A *differential operator* $E \xrightarrow{h} F$ *of order* $\leq n$ is a $f^{-1}\mathcal{O}_S$-linear morphism which factors as

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "d_0"'] & F \\ \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E \arrow[ur, "\bar{h}"'] & \end{tikzcd}$$

with $\bar{h}$ being $\mathcal{O}_X$-linear. Here the tensor product $\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E$ is constructed with respect to the right-$\mathcal{O}_X$-structure of $\mathcal{P}^n_{X/S}$ and considered as an $\mathcal{O}_X$-module via the left-$\mathcal{O}_X$-structure of $\mathcal{P}^n_{X/S}$. ◻

In some sense a differential operator is hence a $f^{-1}\mathcal{O}_S$-linear map, which is *almost* $\mathcal{O}_X$-linear. Such a factorization is unique: If $\overline{a \otimes b} \otimes e \in \mathcal{P}^n_{X/S} \otimes E$, then

$$\bar{h}(\overline{a \otimes b} \otimes e) = a\bar{h}(\overline{1 \otimes b} \otimes e) = a\bar{h}(\overline{1 \otimes 1} \otimes be) = ah(be).$$

Thus we see that the set of differential operators $E \to F$ of order $\leq n$ is in fact

$$\mathrm{Diff}^{\leq n}_{X/S}(E, F) := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E, F),$$

which is an $\mathcal{O}_X(X)$-bimodule, and the *sheaf of differential operators* $E \to F$ of order $\leq n$ is

$$\mathcal{D}\mathrm{iff}^{\leq n}_{X/S}(E, F) = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} E, F),$$

which is an $\mathcal{O}_X$-bimodule.

1.5 Example. A differential operator of degree 0 is just multiplication with an element of $\mathcal{O}_X(X)$. ◻

Note that there are canonical surjections

$$\ldots \twoheadrightarrow \mathcal{P}^n_{X/S} \twoheadrightarrow \mathcal{P}^{n-1}_{X/S} \twoheadrightarrow \ldots,$$

and that they induce morphisms $\mathcal{D}\mathrm{iff}^{\leq n}_{X/S}(E, F) \to \mathcal{D}\mathrm{iff}^{\leq n+1}_{X/S}(E, F)$, so it makes sense to define

1.6 Definition. If E, F are $\mathcal{O}_X$ modules, then

$$\mathcal{D}\mathrm{iff}_{X/S}(E, F) := \varinjlim_n \mathcal{D}\mathrm{iff}^{\leq n}_{X/S}(E, F).$$

If $E = F = \mathcal{O}_X$, we also write $\mathscr{D}_{X/S} := \mathcal{D}\mathrm{iff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$. ◻

1.2.2 Composition

Now we define composition of differential operators. Consider the following diagram:

$$
\begin{tikzcd}
 & \Delta^{n+m}_{X/S} \arrow[dr, hook] & \\
X \arrow[ur, hook] \arrow[dr, hook] & & X \times_S X \\
 & \Delta^{n}_{X/S} \times_X \Delta^{n+m}_{X/S} \arrow[ur, hook] \arrow[uu, dotted, "\exists!"'] &
\end{tikzcd}
$$

One checks that the dotted arrow exists. This gives morphisms $\delta^{n,m} : \mathcal{P}^{n+m}_{X/S} \to \mathcal{P}^{n}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{m}_{X/S}$. It is induced by $\overline{a \otimes b} \mapsto \overline{a \otimes 1} \otimes \overline{1 \otimes b}$. Finally, if $E \xrightarrow{\phi} F$, $F \xrightarrow{\psi} G$ are two differential operators, then we construct:

$$
\begin{tikzcd}
F \arrow[r, "\phi"] \arrow[d, "d_0^n"'] \arrow[dd, bend right=60, "d_0^{n+m}"'] & G \arrow[r, "\psi"] \arrow[d, "d_0"'] & H \\
\mathcal{P}^{n}_{X/S} \otimes_{\mathcal{O}_X} F \arrow[ur, "\bar{\phi}"'] & \mathcal{P}^{m}_{X/S} \otimes_{\mathcal{O}_X} G \arrow[ur, "\bar{\psi}"'] & \\
\mathcal{P}^{n+m}_{X/S} \otimes_{\mathcal{O}_X} F \arrow[r, "\delta^{m,n}"] & \mathcal{P}^{m}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n}_{X/S} \otimes_{\mathcal{O}_X} F \arrow[u, "\mathrm{id} \otimes \bar{\phi}"'] &
\end{tikzcd}
$$

We see that $\psi \circ \phi \in \mathrm{Diff}^{\leq n+m}_{X/S}(F, H)$. Accoringly, $\mathcal{D}\mathrm{iff}_{X/S}(F, F)$ becomes a sheaf of bi-$\mathcal{O}_X$-algebras. Note that this sheaf is noncommutative in general: If $a \in \mathcal{O}_X(X)$ and $\psi : F \to F$ a differential operator, then $a\psi \neq \psi a$.

1.2.3 Functoriality

Now let $g : Y \to X$ be an S-morphism of S-schemes.

1.7 Proposition. *The following statements are true:*

- $g^*\mathscr{D}_{X/S} := \mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_X} g^{-1}\mathscr{D}_{X/S}$ *is a* $(\mathscr{D}_{Y/S}, g^{-1}\mathscr{D}_{X/S})$*-bialgebra.*
- *There exists a canonical left-*$\mathscr{D}_{Y/S}$*, right-*$g^{-1}\mathscr{D}_{X/S}$ *map*

$$\mathscr{D}_{Y/S} \to g^*\mathscr{D}_{Y/S}.$$

- *If* E *is a left-*$\mathscr{D}_{X/S}$*-module, then* g^*E *is isomorphic to* $\mathscr{D}_{X/S} \otimes_{g^{-1}\mathscr{D}_{X/S}} g^{-1}E$ *and carries a canonical left-*$\mathscr{D}_{X/S}$*-structure.* □

1.2.4 The smooth case

Assume there are global functions $x_1, \ldots, x_n \in \mathcal{O}_X(X)$ such that $dx_1, \ldots, dx_n$ are a basis for $\Omega^1_{X/S}$, i.e. such that the define a factorisation

$$\begin{tikzcd} X \arrow[r, "\text{étale}"] \arrow[dr, "f"'] & \mathbb{A}^n_S \arrow[d] \\ & S. \end{tikzcd}$$

1.8 Proposition. *In this situation $\mathscr{D}^{\leq m}_{X/S}$ is free (as left- and as right-$\mathcal{O}_X$-module) on generators of the form*

$$\partial^{(a_1)}_{x_1} \cdot \ldots \cdot \partial^{(a_m)}_{x_n}, \quad \sum a_i \leq m.$$

Here $\partial^{(a)}_{x_i}$ acts on $\mathcal{O}_X$ via $\partial^{(a)}_{x_i}(x^r_j) = \binom{r}{a} x_i^{r-a}$ if $i = j$ and $= 0$ otherwise. The following composition rules hold true:

- $\partial^{(a)}_{x_i}$ *and* $\partial^{(b)}_{x_j}$ *commute for all* i, j, a, b.
- $\partial^{(a)}_{x_i} \partial^{(b)}_{x_i} = \binom{a+b}{a} \partial^{(a+b)}_{x_i}$.
- $\partial^{(a)}_{x_i} \cdot f = \sum_{\substack{r+s=a \\ r,s \geq 0}} \partial^{(r)}_{x_i}(f) \cdot \partial^{(s)}_{x_i}$ *for* $f \in \mathcal{O}_X(X)$. □

1.9 Corollary.

- *If $f : Y \to X$ is an étale morphism of smooth S-schemes, then the morphism $\mathscr{D}_{Y/S} \to f^*\mathscr{D}_{X/S}$ from Proposition 1.7 is an isomorphism.*
- *If S is defined over $\mathbb{Q}$, then $\partial^{(a)}_{x_i} = \frac{1}{a!} \partial^a / \partial x^a_i$.*
- *If S is defined over $\mathbb{F}_p$, then $\partial^{(a)}_{x_i}$ cannot be written as product of lower order operators, but it nontheless* behaves *like $\frac{1}{a!} \partial^a / \partial x^a_i$, so we will use this notation, without it making literal sense. ("Evaluate in characteristic 0, notice that the coefficient is divisible by a sufficiently high power of p, cancel the p-power, and take the result mod p.")*
- *If S is defined over $\mathbb{F}_p$, we compute*

$$(\partial^{(a)}_{x_i})^p = \prod_{r=1}^{p} \binom{ra}{a} \partial^{(ap)}_{x_i} = 0$$

because $\binom{pa}{a} = 0$. □

To prove the last statement, and for later reference, we state the following:

1.10 Lemma. *Let p be a prime number.*

(a) *This is called* Lucas' Theorem: *For $a_0, \ldots, a_n, b_0, \ldots, b_n$ integers in $[0, p-1]$, $a := a_0 + a_1 p + \ldots + a_n p^n$, $b := b_0 + b_1 p + \ldots + b_n p^n$ we have*

$$\binom{a}{b} \equiv \prod_k \binom{a_k}{b_k} \mod p.$$

(b) *If $N, n, m, k \geq 0$ are integers such that $p^k > n$, then*

$$\binom{N + mp^k}{n} \equiv \binom{N}{n} \mod p.$$

Hence the term $\binom{b}{a} \in \mathbb{Z}/p\mathbb{Z}$ is well-defined for $a, b \in \mathbb{Z}/p^k\mathbb{Z}$, and it can be computed by identifying the set $\mathbb{Z}/p^k\mathbb{Z}$ with the set of integers $\{0, 1, \ldots, p^k - 1\}$. Similarly, $\binom{b}{a}$ is well-defined for $b \in \mathbb{Z}_p$ and $a \in \mathbb{Z}/p^k\mathbb{Z}$.

(c) *If $\alpha \in \mathbb{Z}_p$, then*

$$\alpha = \sum_{n=0}^{\infty} \overline{\binom{\alpha}{p^n}} p^n,$$

where $\overline{a}$ means the unique integer in $[0, \ldots, p-1]$ congruent to a.

(d) *If $\alpha, \beta \in \mathbb{Z}_p$,$d \geq 0$, then*

$$\binom{\alpha\beta}{p^d} \equiv \sum_{\substack{a+b=d \\ a,b \geq 0}} \binom{\alpha}{p^a} \binom{\beta}{p^b} \mod p.$$

□

Proof. Everything follows from (a), which is easily proven by computing the coefficient of x^b of

$$\sum_{k=0}^{a} \binom{a}{k} x^k = (1+x)^a \equiv \prod_{k=0}^{n} (1 + x^{p^k})^{a_k} \mod p.$$

■

Just like in characteristic 0 one proves:

1.11 Proposition. *Let $S = \operatorname{Spec} k$ for k a field. If X is smooth, separated, finite type over k, then a $\mathscr{D}_{X/k}$-module which is coherent as $\mathcal{O}_X$-module, is a locally free $\mathcal{O}_X$-module.* □

Proof. The characteristic 0 proof goes through mutatis mutandis: Without loss of generality we may assume that k is algebraically closed. Then it suffices to check that $\widehat{E} := E \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}_{X,x}}$ is free for all *closed points* $x \in X$, because X is of finite type over k. After choosing local coordinates, we can write $\widehat{\mathcal{O}_{X,x}} \cong k[\![x_1, \ldots, x_n]\!]$. Let $e_1, \ldots, e_r$ be a minimal system of generators of $\widehat{E}$ and assume that there is a relation $\sum_{i=1}^{r} a_i e_i = 0$ with $a_i \in k[\![x_1, \ldots, x_n]\!]$. Then all the a_i lie in the maximal ideal $\mathfrak{m} := (x_1, \ldots, x_n)$, by the minimality of the system of generators. Thus there exists some $N > 0$, such that all the a_i lie in $\mathfrak{m}^N$, but (after renumbering) $a_1 \notin \mathfrak{m}^{N+1}$. Then $a_1 = \lambda x_1^{\ell_1} \cdot \ldots x_n^{\ell_n} + \mathfrak{m}^{N+1}$, with $\lambda \in k^\times$, and at least one $\ell_i \neq 0$. We apply the operator

$$D^{\underline{\ell}} := \partial_{x_1}^{(\ell_1)} \cdot \ldots \partial_{x_n}^{(\ell_n)}$$

to the given relation and obtain

$$0 = D^{\underline{\ell}}(a_1)e_1 + D^{\underline{\ell}}(a_2)e_2 + \ldots + D^{\underline{\ell}}(a_n)e_n + \mathfrak{m}\widehat{E}.$$

But by construction $D^{\underline{\ell}}(a_1) \in k[\![x_1, \ldots, x_n]\!]^\times$, which is a contradiction due to Nakayama's lemma. ■

1.12 Remark. In fact $\widehat{E}$ is even trivial as a $\widehat{\mathscr{D}}_{k[\![x_1,\ldots,x_n]\!]/k}$-module, but we do not need this fact. □

1.3 Logarithmic differential operators

1.3.1 Definitions

From now on we set $S := \operatorname{Spec} k$ with k an algebraically closed field of characteristic $p > 0$. Let $\overline{X}$ be a smooth, connected, finite type, separated k-scheme and $X \subseteq \overline{X}$ an open dense subscheme, such that $D := \overline{X} \setminus X$ is a *strict normal crossings divisor*. Recall what this means:

1.13 Definition. D is a *strict normal crossings divisor* if X can be covered by open sets U, such that there exist $x_1, \dots, x_n \in \Gamma(U, \mathcal{O}_U)$ such that $\Omega^1_{U/k}$ is free with basis $dx_1, \dots, dx_n$, and such that $D \cap U = (x_1 \cdot \ldots \cdot x_r)$, with $1 \leq r \leq n$. □

In this situation, we define:

1.14 Definition ([Gie75]). $\mathscr{D}_{\overline{X}/k}(\log D)$ is defined to be the subsheaf of algebras of $\mathscr{D}_{\overline{X}/k}$, generated by all differential operators which locally fix all powers of the ideal of D. □

If $\overline{X}$ admits coordinates $x_1, \dots, x_n$, such that $D = (x_1 \cdot \ldots \cdot x_r)$, then $\mathscr{D}_{\overline{X}/k}(\log D)$ is generated by

$$\delta^{(a)}_{x_i} := x_i^a \partial^{(a)}_{x_i} \text{ for } 1 \leq i \leq r, a \geq 0 \text{ and } \partial^{(a)}_{x_i}, \text{ for } i > r, a \geq 0.$$

1.15 Remark.

- This sheaf of rings can be constructed more generally in a suitable category of schemes with logarithmic structures. As in the classical case, one constructs it from (an appropriate notion of) thickenings of the diagonal in this category, see [Kin12, Ch. 2].
- In Proposition 1.11 we saw that an $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}$-module is locally free. For $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-modules E, the situation is more complicated: E is not necessarily torsion free, and even if it is torsion free, then it is not necessarily locally free. We will see a partial remedy later on, once we have the notion of exponents at our disposal. □

1.3.2 Exponents

1.16 Proposition ([Gie75, Lemma 3.8]). *Assume that $\overline{X} \setminus X$ is a smooth divisor and that E is an $\mathcal{O}_{\overline{X}}$-locally free module of finite rank carrying a $\mathscr{D}_{\overline{X}/k}(\log D)$-structure. Then there exists a unique decomposition*

$$E|_D = \bigoplus_{\alpha \in \mathbb{Z}_p} E_\alpha$$

such that locally, after a choice of coordinates $x_1, \dots, x_n$ with $D = (x_1)$, if $e + x_1 E \in E_\alpha$, then $\delta^{(m)}_{x_1} := x_1^m \partial^{(m)}_{x_i}(e) = \binom{\alpha}{m} e + x_1 E$ for all $m \geq 0$. □

Proof. Gieseker gives a tricky coordinate free description of this decomposition, but we only do the local construction, without proving that it glues.

First some preliminaries: For $d \geq 1$, write $C_d := \operatorname{Maps}(\mathbb{Z}/p^d\mathbb{Z}, \mathbb{F}_p)$ for the set of maps of sets $\mathbb{Z}/p^d \to \mathbb{F}_p$. This is an $\mathbb{F}_p$-algebra of dimension p^{d+1}. For

$a \in \mathbb{Z}/p^d\mathbb{Z}$ define $h_a \in C_d$ by $h_a(b) = \binom{b}{a}$. Recall that this is well-defined by Lemma 1.10. Also define χ_a as the characteristic function of a. Then it is easy to see that $\{h_a | a \in \mathbb{Z}/p^d\mathbb{Z}\}$ and $\{\chi_a | a \in \mathbb{Z}/p^d\mathbb{Z}\}$ are two $\mathbb{F}_p$-bases for C_d, and that the χ_a are commuting, orthogonal idempotents.

Now back to geometry: We may assume that $\overline{X} = \operatorname{Spec} A$ with global coordinates $x_1, \dots, x_n$ and that $D = (x_1)$. Then $X = \operatorname{Spec} A[x_1^{-1}]$. We may also assume that E is free on A, because E is torsion free by assumption. Then the operators $\delta_{x_1}^{(a)}$ act linearly on $E/x_1E = E|_D$ by definition. Thus we get a map

$$\phi_d : C_d \to \operatorname{End}_{A/(x_1)}(E/x_1E)$$

by defining $\phi_d(h_a) = \delta_{x_1}^{(a)}$. This is in fact a map of *rings*, since the $\delta_{x_1}^{(a)}(x_1^r) = \binom{r}{a}x_1^r$.

But then the elements $\phi_d(\chi_a)$ (or a subset of the set of these elements) are orthogonal, commuting idempotents in $\operatorname{End}_{A/x_1}(E/x_1E)$, which means that we get a decomposition

$$E/x_1E = \bigoplus_{a \in \mathbb{Z}/p^d} E_a$$

such that χ_a acts trivially on E_b with $b \neq a$ and identically on E_a. Since $h_a = \sum_{b \in \mathbb{Z}/p^d} \binom{b}{a}\chi_b \in C_d$, we see that h_a acts as $\binom{b}{a}$ on E_a.

Finally we have to let d vary. Let $\rho : C_d \to C_{d+1}$ be the map coming from the projection $\mathbb{Z}/p^{d+1}\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^d\mathbb{Z}$. For $a \in \mathbb{Z}/p^d\mathbb{Z}$ we have

$$\rho(\chi_a) = \sum_{\substack{b \in \mathbb{Z}/p^{d+1} \\ b \equiv a \mod p^d}} \chi_b.$$

So the decomposition for C_{d+1} refines the decomposition for C_d, and this process has to stop since E/x_1E is finitely generated. The $\delta_{x_1}^{(n)}$-action can be computed for any d with $p^d > n$, and then by Lemma 1.10 $\binom{\alpha}{n} = \binom{\alpha \mod p^d}{n}$. ∎

1.17 Definition. In the situation of Proposition 1.16, write

$$\operatorname{Exp}_D(E) = \{\alpha \in \mathbb{Z}_p | \operatorname{rank} E_\alpha > 0\},$$

and call it *the set of exponents of E along D*.

Now let $\overline{X} \setminus X$ be an arbitrary strict normal crossings divisor with $D = \sum D_i$, and D_i smooth. If E is a torsion free $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module, then we define $\operatorname{Exp}_{D_i}(E)$ to as $\operatorname{Exp}_{D_i}(E|_{U_i})$ where $U_i \subseteq \overline{X}$ is an open set on which E is locally free, and which intersects D_j if and only if $i = j$. One easily checks that this definition does not depend on the choice of U_i. □

We can now remedy the problem with the local freeness of $\mathscr{D}_{\overline{X}/k}(\log D)$-modules:

1.18 Theorem (reformulation of [Gie75, Thm. 3.5]). *If E is an $\mathcal{O}_X$-coherent and $\mathcal{O}_{\overline{X}}$-torsion free $\mathscr{D}_{\overline{X}/k}(\log D)$-module such that $\operatorname{Exp}_{D_i}(E)$ maps injectively to $\mathbb{Z}_p/\mathbb{Z}$ for all i, i.e. such that the exponents along D_i do not differ by integers, then E is locally free if and only if E is reflexive.* □

The proof of the theorem is a rather complicated application of local cohomology, which we will skip, since we will not use the result. In fact, for the our purposes, we may always remove closed subsets of codimension ≥ 2 from $\overline{X}$, so we can always assume that E is actually locally free.

1.19 Corollary. *Assume we have open immersions $X \subseteq \overline{U}$ and $j : \overline{U} \subseteq \overline{X}$, such that $\overline{U} \cap D_i \neq \emptyset$ for every i. If E is a locally free $\mathscr{O}_{\overline{U}}$-module with $\mathscr{D}_{\overline{U}/k}(\log \overline{U} \cap D)$-action such that $\operatorname{Exp}_{D_i \cap \overline{U}}(E) \hookrightarrow \mathbb{Z}_p/\mathbb{Z}$, then j_*E is a* locally free *$\mathscr{D}_{\overline{X}/k}(\log D)$-module.* $\square$

Proof. It suffices to note that j_*E is reflexive. $\blacksquare$

1.3.3 Functoriality

We continue to denote by k an algebraically closed field of positive characteristic p.

In this section we consider the following situation: Let $X \subseteq \overline{X}$ and $Y \subseteq \overline{Y}$ be open immersions of smooth, finite type, separated, connected k-schemes, such that $D_Y := \overline{Y} \setminus Y$ and $D_X := \overline{X} \setminus X$ are strict normal crossings divisors. Furthermore let $\bar{f}$ and f be morphisms, fitting in the diagram

$$\begin{tikzcd} \overline{Y} \arrow[r, "\bar{f}"] & \overline{X} \\ Y \arrow[r, "f"] \arrow[u, hook] & X, \arrow[u, hook] \end{tikzcd}$$

i.e. such that $f = \bar{f}|_X$.

1.20 Remark. The readers familiar with logarithmic structures ([Kat89]) notice that in this situation $\bar{f}$ induces a morphism of the log-schemes associated with $X \subseteq \overline{X}$ and $Y \subseteq \overline{Y}$. $\square$

We have the similar functoriality results as in Proposition 1.7:

1.21 Proposition.

(a)

$$\bar{f}^*\mathscr{D}_{\overline{X}/k}(\log D_X) := \mathscr{O}_{\overline{Y}} \otimes_{\bar{f}^{-1}\mathscr{O}_{\overline{X}}} \bar{f}^{-1}\mathscr{D}_{\overline{X}/k}(\log D_X)$$

is a $(\mathscr{D}_{\overline{Y}/k}(\log D_Y), \bar{f}^{-1}\mathscr{D}_{\overline{X}/k}(\log D_X))$-bialgebra.

(b) *There exists a canonical morphism*

$$\mathscr{D}_{\overline{Y}/k}(\log D_Y) \xrightarrow{\bar{f}^\sharp} \bar{f}^*\mathscr{D}_{\overline{X}/k}(\log D_X)$$

fitting in the commutative diagram

$$\begin{tikzcd} \mathscr{D}_{\overline{Y}/k}(\log D_Y) \arrow[r] \arrow[d, hook] & \bar{f}^*\mathscr{D}_{\overline{X}/k}(\log D_X) \arrow[d, hook] \\ \mathscr{D}_{\overline{Y}/k} \arrow[r] & \bar{f}^*\mathscr{D}_{\overline{X}/k} \end{tikzcd}$$

where the lower horizontal morphism is the one from Proposition 1.7.

Now assume that $\bar{f}$ is finite, and f étale. Then $\bar{f}$ is faithfully flat. Moreover,

(c) *$\bar{f}^\sharp$ is an isomorphism if $\bar{f}$ is tamely ramified with respect to the strict normal crossings divisor D_X.*

(d) *Let $D_{Y,i}$ be a component of D_Y mapping to the component $D_{X,i}$ of D_X. If E is an $\mathcal{O}_{\overline{X}}$-coherent, $\mathcal{O}_{\overline{X}}$-torsion free $\mathscr{D}_{\overline{X}/k}(\log D_X)$-module, then the exponents of $\bar{f}^*E$ along $D_{Y,i}$ are the exponents of E along D_{X_i}, multiplied by the ramification index of the extension of discrete valuation rings associated with $D_{Y,i}$ and $D_{X,i}$.* □

Proof. We only give an explicit proof for the case that $\bar{f}$ is finite and f étale. Then all of this is essentially a question about finite extensions of discrete valuation rings. Let $A \hookrightarrow B$ be such an extension, $x \in A$ and $y \in B$ uniformizers. Then $x = uy^e$ for some $e \geq 1$ and $u \in B^\times$. We know that $K(B) \otimes \mathscr{D}_{B/k}(\log y) \xrightarrow{\cong} K(B) \otimes_A \mathscr{D}_{A/k}(\log x)$ is an isomorphism, and we claim that

$$\delta_y^{(p^m)} = \sum_{\substack{d+c=m\\ c,d\geq 0}} \binom{e}{p^c} \delta_x^{(p^d)} + y(B \otimes_A \mathscr{D}_{A/k}(\log x)) \tag{1}$$

Lets assume the truth of (1) for a minute. Then (a) follows directly, and (d) is also easy to derive: Let E be an A-module with $\mathscr{D}_{A/k}(\log x)$-action. If $a \in E \otimes_A B$ is such that $\delta_x^{(p^m)}(a) = \binom{\alpha}{p^m} a + x(E \otimes B)$ for some $\alpha \in \mathbb{Z}_p$, then

$$\delta_y^{(p^m)}(a) = \sum_{\substack{c+d=m\\ c,d\geq 0}} \binom{e}{p^c}\binom{\alpha}{p^d} a + y(E \otimes B) = \binom{e\alpha}{p^m} a + y(E \otimes B)$$

which proves (d).

Lets now prove that (1) holds. We compute:

$$\begin{aligned}
\delta_y^{(p^m)}(x^r) &= \delta_y^{(p^m)}(u^r y^{er}) \\
&= \sum_{\substack{a+b=p^m\\ a,b\geq 0}} \delta_y^{(a)}(u^r)\delta_y^{(b)}(y^{er}) \\
&= \binom{er}{p^m} x^r + \sum_{\substack{a+b=p^m\\ a>0,b\geq 0}} \delta_y^{(a)}(u^r) \binom{er}{b} y^{er} \\
&= \binom{er}{p^m} x^r + x^r \underbrace{\sum_{\substack{a+b=p^m\\ a>0,b\geq 0}} \binom{er}{b} \frac{\delta_y^{(a)}(u^r)}{u^r}}_{\in (y)}
\end{aligned} \tag{2}$$

This shows that $\delta_y^{(p^m)} - \sum_{c+d=m} \binom{e}{p^c} \delta_x^{(p^d)} \in y(B \otimes \mathscr{D}_{A/k}(\log x)$ as claimed, and to finish, we note that $\binom{er}{p^m} x^r = \sum_{c+d=m} \binom{e}{p^c} \delta_x^{(p^d)}(x^r)$.

For (d) assume that $A \hookrightarrow B$ is tamely ramified. It suffices to show that $\delta_x^{(p^m)}$ is in the image of $\bar{f}^\sharp$ for every $m \geq 0$. Consider the completions of A and

B: $\widehat{A} \hookrightarrow \widehat{B}$. Replacing $\widehat{B}$ by an étale extension does not change differential operators, so we may assume that $u = v^e$ in $\widehat{B}$. Indeed, by Hensel's Lemma, u has an e-th root in $\widehat{B}$, if and only if it has an e-th root modulo y, and since e is prime to p by assumption, the extension of the residue fields obtained by adjoining an e-th root is separable. Replacing y by vy, we may assume that $x = y^e$. Then (2) shows

$$\delta_y^{(p^m)} = \sum_{\substack{c+d=m \\ c,d\geq 0}} \binom{e}{p^c} \delta_x^{(p^d)}$$

In particular, $\delta_y^{(1)} = e\delta_x^{(1)}$, so $\delta_x^{(1)}$ is in the image of $\bar{f}^\sharp$. We proceed inductively:

$$\delta_y^{(p^m)} = e\delta_x^{(p^m)} + \underbrace{\sum_{\substack{c+d=m \\ c>1,d\geq 0}} \binom{e}{p^c} \delta_x^{(p^d)}}_{\operatorname{im} \bar{f}^\sharp}$$

which completes the proof. ■

1.22 Corollary. *Let f be étale and $\bar{f}$ finite and tamely ramified with respect to D_X, and let E be an $\mathcal{O}_{\overline{Y}}$-coherent, $\mathcal{O}_{\overline{Y}}$-torsion free $\mathscr{D}_{\overline{Y}/k}(\log D_Y)$-module. Then $\bar{f}_*E$ is an $\mathcal{O}_{\overline{X}}$-coherent, $\mathcal{O}_{\overline{X}}$-torsion free $\mathscr{D}_{\overline{X}/k}(\log D_X)$-module.* □

2 Lecture 2 - Stratified Bundles

2.1 Stratified bundles

2.1.1 Definitions and first properties

We continue to denote by k an algebraically closed field of positive characteristic p. Again let X be smooth, finite type, connected and separated over k.

2.1 Definition. A *stratified bundle* is an $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}$-module. A morphism of stratified bundles is a $\mathscr{D}_{X/k}$-linear morphism of $\mathcal{O}_X$-modules. We write $\mathrm{Strat}(X)$ for the category of stratified bundles. A stratified bundle is called *trivial*, if it is isomorphic to $\mathcal{O}_X^n$ in $\mathrm{Strat}(X)$. □

2.2 Remark. We make two remarks about the name "stratified bundle":

- Recall that by Proposition 1.11 a stratified bundle is in particular a locally free $\mathcal{O}_X$-module, so the word "bundle" is justified.
- Grothendieck defines in [Gro68] the term "stratification" as an "infinitesimal descent datum". In our situation with X a *smooth* k-scheme, the datum of a stratification on an $\mathcal{O}_X$-module is equivalent to the datum of a $\mathscr{D}_{X/k}$-action on an $\mathcal{O}_X$-module. For details see [BO78, Ch. 2]. □

2.3 Example. Let $f : Y \to X$ be a finite étale morphism, then the coherent $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ carries a canonical $\mathscr{D}_{X/k}$-structure, because $\mathscr{D}_{Y/k} \xrightarrow{\cong} f^*\mathscr{D}_{X/k}$. Even more, the covering is trivial if and only if the stratified bundle $f_*\mathcal{O}_Y$ is trivial.

Indeed, if the covering is trivial, then clearly the associated $\mathscr{D}_{X/k}$-structure is trivial. Conversely, assume that $f_*\mathcal{O}_Y \cong \mathcal{O}_X^n$ as $\mathscr{D}_{X/k}$-modules. We have to show that this is actually an isomorphism of $\mathcal{O}_X$-algebras. Let $e_1, \dots, e_r$ be a horizontal basis, i.e. a basis, such that $\psi(e_i) = 0$ for all differential operators $\psi \in \mathscr{D}_{X/k}(U)$ with $\psi(1) = 0$ for $U \subseteq X$ open. Let $U \subseteq X$ be an open with global coordinates $x_1, \dots, x_n$. By the construction of the $\mathscr{D}_{X/k}$-action on $f_*\mathcal{O}_Y$, we have the equation

$$\partial_{x_\ell}^{(m)}(e_i \cdot e_j) = \sum_{\substack{a+b=m \\ a,b \geq 0}} \partial_{x_\ell}^{(a)}(e_i)\partial_{x_\ell}^{(b)}(e_j) = 0$$

over U, for any m, ℓ, i, j. If $e_i e_j = \sum_{t=1}^r \lambda_t e_t$, with $\lambda_t \in \mathcal{O}_X(U)$, then

$$0 = \sum_{t=1}^r \partial_{x_\ell}^{(m)}(\lambda_t) e_t$$

for any $m > 0$. This shows that $\lambda_t \in k$ for all t, so the étale $\mathcal{O}_X$-algebra $f_*\mathcal{O}_Y$ comes via base-change from an étale k-algebra. But since k is algebraically closed, any such algebra is trivial. This completes the proof. $\square$

We give some functorial properties of the category $\mathrm{Strat}(X)$:

2.4 Proposition. *If $f : Y \to X$ is a morphism of smooth, separated, finite type k-schemes, then:*

(a) *Pull-back of $\mathcal{O}_X$-modules along f induces a functor*

$$f^* : \mathrm{Strat}(X) \to \mathrm{Strat}(Y).$$

(b) *If f is finite and étale, then push-forward of $\mathcal{O}_Y$-modules along f induces a functor*

$$f_* : \mathrm{Strat}(Y) \to \mathrm{Strat}(X)$$

which is right adjoint to f^*.

(c) *If E, F are stratified bundles, then $E \otimes_{\mathcal{O}_X} F$ and $\mathcal{H}om_{\mathcal{O}_X}(E, F)$ are stratified bundles in a bifunctorial way.* $\square$

PROOF. The first two statements follow from Proposition 1.7 and Corollary 1.9. The last statement is entirely analog to characteristic 0. ■

In fact we know more:

2.5 Theorem ([SR72]). *After the choice of a fiber functor $\mathrm{Strat}(X) \to \mathrm{Vectf}_k$ ($\mathrm{Vectf}_k =$ the category of finite dimension k-vector spaces), the category $\mathrm{Strat}(X)$ is neutral tannakian over k.* $\square$

We avoid Tannaka theory in these lectures, but we need the following definition:

2.6 Definition. Let E be a stratified bundle. Then we denote by $\langle E \rangle_\otimes$ the full subcategory of $\mathrm{Strat}(X)$ with objects all subquotients of stratified bundles of the form $P(E, E^\vee)$, for $P(x, y) \in \mathbb{N}[x, y]$. Here, if $P(x, y) = mx^a y^b$, then

$$P(E, E^\vee) := \bigoplus_{i=1}^m \underbrace{E \otimes \ldots \otimes E}_{a \text{ times}} \otimes \underbrace{E^\vee \otimes \ldots \otimes E^\vee}_{b \text{ times}}.$$

$\square$

2.1.2 Restriction to an open subset

In this section we analyze how the category $\mathrm{Strat}(X)$ is related to the category $\mathrm{Strat}(U)$ for $U \subseteq X$ an open dense subset.

2.7 Proposition. *Let $U \subseteq X$ be an open dense subset and $\rho : \mathrm{Strat}(X) \to \mathrm{Strat}(U)$ the restriction functor. Then the following statements are true:*

(a) *The restriction functor $\rho : \mathrm{Strat}(X) \to \mathrm{Strat}(U)$ is fully faithful.*

(b) *If $E \in \mathrm{Strat}(X)$, then the restriction functor $\rho_E : \langle E \rangle_\otimes \to \langle E|_U \rangle_\otimes$ is an equivalence.*

(c) *If $\operatorname{codim}_X X \setminus U \geq 2$, then the restriction functor $\rho : \mathrm{Strat}(X) \to \mathrm{Strat}(U)$ is an equivalence.* □

Proof. We sketch the proof and begin with (c), i.e. we assume that $\operatorname{codim}_X(X \setminus U) \geq 2$. Then j_*E is an $\mathcal{O}_X$-coherent $\mathscr{D}_{X/k}$-module restricting to E, so ρ is essentially surjective. Note that j_*E, it is locally free. If $\overline{E}$ is any other locally free extension of E, then there is a short exact sequence

$$0 \to \overline{E} \to j_*E \to G \to O$$

of $\mathcal{O}_X$-modules, and G is supported on $X \setminus U$. By the assumption on the codimension, $\mathcal{H}om_{\mathcal{O}_X}(G, \mathcal{O}_X) = \mathcal{E}xt^1_{\mathcal{O}_X}(G, \mathcal{O}_X) = 0$. Thus $\overline{E} \cong (j_*E)^{\vee\vee} \cong j_*E$. Finally, the $\mathscr{D}_{X/k}$-action on j_*E is uniquely determined by the $\mathscr{D}_{U/k}$-action on E, so the bijection $\mathrm{Hom}_{\mathcal{O}_X}(j_*E, j_*F) = \mathrm{Hom}_{\mathcal{O}_U}(E, F)$ induces a bijection $\mathrm{Hom}_{\mathrm{Strat}(X)}(j_*E, j_*F) = \mathrm{Hom}_{\mathrm{Strat}(U)}(E, F)$, so ρ is an equivalence.

Now for (a), by what we just proved, we may assume that $X \setminus U$ is a disjoint union of codimension 1 closed subsets, and then that $X \setminus U$ is a smooth divisor. We may also shrink X around the generic point of this divisor, to assume that $X = \operatorname{Spec} A$ has global coordinates $x_1, \ldots, x_n$ such that $X \setminus U = (x_1)$. Then $U = \operatorname{Spec} A[x_1^{-1}]$. It suffices to show that $\mathrm{Hom}_{\mathrm{Strat}(X)}(\mathcal{O}_X, E) \to \mathrm{Hom}_{\mathrm{Strat}(U)}(\mathcal{O}_U, E|_U)$ is bijective for every $E \in \mathrm{Strat}(X)$. Shrinking X further around η, we may assume that E is free with basis $e_1, \ldots, e_r$. Assume that $\phi : \mathcal{O}_U \to E|_U$ is given by $\phi(1) = \sum \phi_i e_i$, with $\phi_i \in A[x_1^{-1}]$. Then

$$0 = \partial^{(1)}_{x_1}(\phi(1)) = \sum \partial^{(1)}_{x_1}(\phi_i)e_i + \phi_i \partial^{(1)}_{x_1}(e_i).$$

This shows that $\partial^{(1)}_{x_1}(\phi_i) \in \phi_i \cdot A \subseteq A[x_1^{-1}]$. By induction we see that in fact $\partial^{(m)}_{x_1}\phi_i \in \phi_i A$ for all $m \geq 0$. But this means that the pole order of ϕ_i along x_1 is the same as the pole order of $\partial^{(m)}_{x_i}$ for all $m > 0$, so $\phi_i \in A$ and $\mathrm{Hom}_{\mathrm{Strat}(X)}(\mathcal{O}_X, E) \to \mathrm{Hom}_{\mathrm{Strat}(Y)}(\mathcal{O}_U, E|_U)$ is surjective.

It is also injective, since if $\psi_1, \psi_2 \in \mathrm{Hom}_{\mathrm{Strat}(X)}(\mathcal{O}_X, E)$, then $(\psi_1 - \psi_2) \otimes A[x_1^{-1}] = 0$ means that $\psi_1 = \psi_2$.

Finally, lets prove (b). By what we have seen, we only need to prove that $\rho : \langle E \rangle_\otimes \to \langle E|_U \rangle_\otimes$ is essentially surjective. First note that for all $P(x, y) \in \mathbb{N}[x, y]$, $P(E|_U, E|_U^\vee)$ are in the essential image of the restriction functor $\rho : \langle E \rangle_\otimes \to \langle E|_U \rangle_\otimes$, so we just have to check that for $F \in \mathrm{Strat}(X)$, all subquotients of $F|_U$ in $\mathrm{Strat}(U)$ lift to X. Let $F' \subseteq F|_U$ be a subobject. Then $j_*F' \subseteq j_*F$ and $F \subseteq j_*F$ are sub-$\mathscr{D}_{X/k}$-modules, and $j_*F' \cap F$ is a stratified bundle extending F'. Now if G is a quotient of $F|_U$, and F' the kernel of $F|_U \twoheadrightarrow G$, then $F/(j_*F' \cap F)$ is a lift of G. ■

2.2 $(X, \overline{X})$-regular singular stratified bundles

Now we fix $X \subseteq \overline{X}$ two smooth, connected, separated, finite type k-schemes, with $X \subseteq \overline{X}$ open dense and such that $D := \overline{X} \setminus X$ is a strict normal crossings divisor.

2.8 Definition. A stratified bundle $E \in \mathrm{Strat}(X)$ is called $(X, \overline{X})$-*regular singular* if there exists an $\mathcal{O}_{\overline{X}}$-coherent, $\mathcal{O}_{\overline{X}}$-torsion free $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{E}$ extending the $\mathscr{D}_{X/k}$-module E. We write $\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$ for the full subcategory of $\mathrm{Strat}(X)$ with objects the $(X, \overline{X})$-regular singular stratified bundles. □

2.9 Remark. The extension $\overline{E}$ is far from unique. □

2.10 Proposition. *For $E \in \mathrm{Strat}(X)$, the following are equivalent:*

(a) *E is $(X, \overline{X})$-regular singular.*

(b) *There is some open dense subset $\overline{U} \subseteq \overline{X}$ containing all generic points of $\overline{X} \setminus X$, such that $E|_{X \cap \overline{U}}$ is $(X \cap \overline{U}, \overline{U})$-regular singular.* □

Proof. "$\Rightarrow$" is clear. Conversely, Let $\overline{E}$ be an $\mathcal{O}_{\overline{U}}$-coherent, torsion free $\mathscr{D}_{\overline{U}/k}(\log \overline{U} \setminus X)$-module extending $E|_{\overline{U} \cap X}$. Then we can glue it to E over X to see that E is $(X, \overline{X} \cup \overline{U})$-regular singular, say with extension $\overline{E}'$. Finally, if $j : \overline{U} \hookrightarrow \overline{X}$ is the open immersion, then $j_*\overline{E}'$ is an $\mathcal{O}_{\overline{X}}$-coherent, torsion free $\mathscr{D}_{\overline{X}/k}(\log D)$-module, extending E, by the assumption on the codimension. ■

2.2.1 Exponents

For an $(X, \overline{X})$-regular singular stratified bundle E there are many extensions to an $\mathcal{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module. In the first lecture, we defined the notion of exponents of such an $\mathscr{D}_{\overline{X}/k}(\log D)$-module. Luckily, we understand very well how these exponents for different extensions of E are related:

2.11 Proposition. *Let E be an $(X, \overline{X})$-regular singular stratified bundle, and $\overline{E}$, $\overline{E}'$ two $\mathcal{O}_{\overline{X}}$-coherent, $\mathcal{O}_{\overline{X}}$-torsion free $\mathscr{D}_{\overline{X}/k}(\log D)$-extensions of E. If D_i is a smooth component of D, then the set of exponents of $\overline{E}$ and $\overline{E}'$ along D_i has the same image in $\mathbb{Z}_p/\mathbb{Z}$.* □

Proof. We clearly may assume that $D = \overline{X} \setminus X$ is a smooth divisor, say with generic point η, and we may shrink $\overline{X}$ around η, so that we can assume that $\overline{X} = \operatorname{Spec} A$ is affine, $\overline{E}, \overline{E}'$ are free, and that there are global coordinates $x_1, \ldots, x_n$, such that $D = (x_1)$. Write $j : X \hookrightarrow \overline{X}$. Then $\overline{E}, \overline{E}' \subseteq j_*E$ are $\mathscr{D}_{\overline{X}/k}(\log D)$-submodules, and $\overline{E} \cap \overline{E}'$ is also an $\mathcal{O}_{\overline{X}}$-coherent, torsion free $\mathscr{D}_{\overline{X}/k}(\log D)$-module extending E. Thus, to prove the proposition, we may assume that $\overline{E} \subseteq \overline{E}'$ is a $\mathscr{D}_{\overline{X}/k}(\log D)$-submodule.

We may now consider the situation over $\mathcal{O}_{\overline{X},\eta}$, which is a discrete valuation ring. Hence $\overline{E}'_\eta/\overline{E}_\eta$ is a torsion-module so there exists a minimal $N \geq 0$, such that $x_1^N \overline{E}_\eta \subseteq \overline{E}_\eta$. If $e \in \overline{E}'_\eta$ is an element such that $\delta_{x_1}^{(m)}(e) = \binom{\alpha}{m}e + x_1\overline{E}'_\eta$, then

$$\delta_{x_1}^{(m)}(x_1^N e) = \sum_{\substack{a+b=m\\a,b\geq 0}} \binom{N}{a}\binom{\alpha}{e} x_1^N e + x_1^{N+1}\overline{E}'_\eta = \binom{N+\alpha}{m} x_1^N e + x_1 \overline{E}_\eta$$

Thus $N+\alpha$ is an exponent of $\overline{E}$ along D. Since $\overline{E}$ and $\overline{E}'$ have the same number of exponents, the claim follows. ∎

2.12 Definition. The exponents of $E \in \mathrm{Strat}^{\mathrm{rs}}((X,\overline{X}))$ are the exponents of an $\mathcal{O}_{\overline{X}}$-coherent, torsion free $\mathscr{D}_{\overline{X}/k}(\log D)$-extension of E modulo $\mathbb{Z}$. Hence the exponents of E lie in $\mathbb{Z}_p/\mathbb{Z}$. □

2.2.2 Canonical extensions

2.13 Definition. Let τ be a (set theoretic) section of the projection $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p/\mathbb{Z}$ and $E \in \mathrm{Strat}^{\mathrm{rs}}((X,\overline{X}))$. A *$\tau$-extension of E* is an *$\mathcal{O}_{\overline{X}}$-locally free* finite rank $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{E}$ such that the exponents of $\overline{E}$ lie in the image of τ. □

While the extension to $\overline{X}$ of an $(X,\overline{X})$-regular singular stratified bundle is not unique, there always exists a unique τ-extension. This is completely parallel to the situation in characteristic 0.

2.14 Theorem ([Kin12, Sec. 3.2.2]). *Let E be an $(X,\overline{X})$-regular singular stratified bundle. Then a τ-extension exists and it is unique up to isomorphisms restricting to the identity on E.* □

Proof. We sketch the construction of a τ-extension, and do not discuss the unicity. The proof is an extension of methods of Gieseker. As usual we may reduce to the following situation (using Corollary 1.19): $\overline{X} = \operatorname{Spec} A$ with global coordinates $x_1,\dots,x_n$, such that $D = (x_1)$ and $X = \operatorname{Spec} A[x_1^{-1}]$. Let $\overline{E}$ be any $\mathcal{O}_{\overline{X}}$-coherent, torsion free extension of E to a $\mathscr{D}_{\overline{X}/k}(\log D)$-module. By shrinking $\overline{X}$ around the generic point of D, we may assume $\overline{E}$ is free. Write $\operatorname{Exp}(\overline{E}) = \{\alpha_1,\dots,\alpha_r\} \subseteq \mathbb{Z}_p$ for the set of exponents of $\overline{E}$ along D. Let $b := (b_1,\dots,b_r) \in \mathbb{Z}^r$ be an r-tuple, such that $b_i = b_j$ whenever $\alpha_i = \alpha_j$. From $\overline{E}$ we construct an $\mathcal{O}_{\overline{X}}$-coherent, torsion free $\mathscr{D}_{\overline{X}/k}(\log D)$-extension $\overline{E}^{(b)}$ of E, such that the set of exponents of $\overline{E}^{(b)}$ is $\{\alpha_1+b_1,\alpha_2+b_2,\dots,\alpha_r+b_r\}$. This would prove the theorem, we “move” the exponents of $\overline{E}$ into the image of the section τ.

The construction is done in two steps:

(a) If $a \in \mathbb{N}$, then $\overline{E}(aD)$ has exponents $\{\alpha_1 - a,\dots,\alpha_r - a\}$.

(b) For $i \in [1,r]$ we construct $\overline{E}_i$ such that $\operatorname{Exp}(E_i) = \{\alpha_j | j \neq i\} \cup \{\alpha_i + 1\}$. In fact, let $i = 1$. Then $\overline{E}_1$ is defined as the submodule of $\overline{E}$ generated by $x_1 E$ and the images of $\delta_{x_1}^{(p^j)} - \binom{\alpha_1}{j}\,\mathrm{id}$. This is a $\mathscr{D}_{\overline{X}/k}(\log D)$-submodule of $\overline{E}$, because $\partial_{x_j}^{(m)}$ commutes with $\delta_{x_1}^{(p^j)}$ for all m,i,n,j.

Now assume that $\alpha_1 = \alpha_2 = \ldots = \alpha_\ell$, and $\alpha_1 \neq \alpha_j$ for $j > \ell$. Let $s_1, \ldots, s_r$ be a basis of $\overline{E}$ such that

$$\delta_{x_1}^{(m)}(s_i) = \binom{\alpha_i}{m} s_i + x_1\overline{E}, \text{ for all } i, m$$

Then define $s_i' := x_1 s_i$ for $i \in [1, \ell]$ and $s_i' = s_i$ for $i > \ell$. Then the s_i' are a basis of $\overline{E}_1$ which finishes the proof. ∎

2.15 Corollary. *The essential image of the restriction functor*

$$\mathrm{Strat}(X) \to \mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$$

is the full subcategory of $(X, \overline{X})$-regular singular bundles with exponents equal to $0 \in \mathbb{Z}_p/\mathbb{Z}$. □

Proof. By the τ-extension theorem, we have to show that a locally free $\mathcal{O}_{\overline{X}}$-coherent, $\mathcal{O}_{\overline{X}}$-torsion free $\mathscr{D}_{\overline{X}/k}(\log D)$-module with exponents 0 has a canonical $\mathscr{D}_{\overline{X}/k}$-action. This is enough by Proposition 2.7. For this we may assume that $\overline{X}$ is affine with local coordinates $x_1, \ldots, x_n$ such that $D = (x_1)$ and $\overline{E}$ free. Then having exponents 0 means that for every $e \in \overline{E}$,

$$\delta_{x_1}^{(1)}(e) = 0 \cdot e + x_1\overline{E}.$$

Thus we can define $\partial_{x_1}^{(1)}(e) := \frac{\delta_{x_1}^{(1)}(e)}{x_1}$. In particular, the $\mathscr{D}_{\overline{X}/k}(\log D)$-action defines an honest flat connection with p-curvature 0 on $\overline{E}$. Then, by Cartier's Theorem ([Kat70]), if $(-)^{(1)}$ denotes Frobenius twist, then $\overline{E} = F_{X/k}^*\overline{E}_1$, where $\overline{E}_1$ is the $\mathscr{D}_{\overline{X}^{(1)}/k}$-module obtained as the sheaf of sections s of $\overline{E}$ such that $\partial_{x_i}^{(1)}(s) = 0$ for all i. Moreover, $\overline{E}_1$ also has exponents 0, and $\delta_{x_1}^{(p)}$ acts as $\delta_{x_1^{(1)}}^{(1)}$ on $\overline{E}_1$. We reapply the argument, to give meaning to the action of $\partial_{x_1^{(1)}}^{(1)} = \partial_{x_1}^{(p)}$ on $\overline{E}$. Then we apply Cartier's Theorem again, etc. ∎

2.2.3 Proof of the Main Lemma with respect to $(X, \overline{X})$

We now have all the tools we need to prove the Main Lemma Theorem 1.2 with respect to a fixed partial compactification $X \subseteq \overline{X}$.

2.16 Theorem ($(X, \overline{X})$-Main Lemma). *Let $X \subseteq \overline{X}$ be an open immersion of smooth, finite type, separated, connected k-schemes, such that $D_X := \overline{X} \setminus X$ is a strict normal crossings divisor. Let $f : Y \to X$ be a finite étale galois covering. Then f is tamely ramified with respect to D_X if and only if $f_*\mathcal{O}_Y$ is $(X, \overline{X})$-regular singular.* □

Proof. By taking the normalization $\bar{f} : \overline{Y} \to \overline{X}$ of $\overline{X}$ in $k(Y)$, and by removing codimension ≥ 2 subsets of $\overline{X}$ if necessary, we arrive at the following situation: $D_Y := \overline{Y} \setminus Y$ has strict normal crossings, the diagram

$$\begin{tikzcd} \overline{Y} \arrow[r, "\bar{f}"] & \overline{X} \\ Y \arrow[u, hook] \arrow[r, "f"] & X \arrow[u, hook] \end{tikzcd}$$

commutes, and $\bar{f}$ is finite. This implies that $\bar{f}$ is flat since $\overline{X}$ and $\overline{Y}$ are both smooth, and the dimensions of the fiber of $\bar{f}$ is constantly 0. Then it follows that $\bar{f}$ is surjective, as it is open and closed, and $\overline{X}$ is connected.

The direction "⇒" is easier: Corollary 1.22 shows that $\bar{f}_*\mathcal{O}_{\overline{Y}}$ is an $\mathcal{O}_{\overline{X}}$-coherent, $\mathcal{O}_{\overline{X}}$-torsion free $\mathscr{D}_{\overline{X}/k}(\log D_X)$-module extending the stratified bundle $f_*\mathcal{O}_Y$, so $f_*\mathcal{O}_Y$ is $(X, \overline{X})$-regular singular.

The converse direction is more involved: We assume $f_*\mathcal{O}_Y$ to be $(X, \overline{X})$-regular singular, and we construct a finite étale morphism $h : Z \to X$, which is tamely ramified with respect to $(X, \overline{X})$, such that $Y \times_X Z \to Z$ is the trivial covering $\coprod Z \to Z$. Then h dominates f, so f is tame with respect to $(X, \overline{X})$.

Again we may assume without loss of generality that $\overline{X} \setminus X$ is a smooth divisor with generic point η, and in the construction we may shrink $\overline{X}$ around η. We proceed in five steps:

(a) Note that the exponents of $f_*\mathcal{O}_Y$ are torsion in $\mathbb{Z}_p/\mathbb{Z}$, because by Proposition 1.21 pulling back an $\mathscr{D}_{\overline{X}/k}(\log D_X)$-extension of $f_*\mathcal{O}_Y$ along $\bar{f}$ multiplies the exponents by the ramification indices of $\bar{f}$ along D_X, and clearly $f^*f_*\mathcal{O}_Y = \mathcal{O}_Y^{\deg f}$ is trivial.

(b) By Theorem 2.14 we find an $\mathcal{O}_{\overline{X}}$-coherent, torsion free $\mathscr{D}_{\overline{X}/k}(\log D_X)$-extension $\overline{E}$ of $f_*\mathcal{O}_Y$ with exponents in $\mathbb{Z} \cap \mathbb{Q}$; say $\frac{a_1}{b}, \dots, \frac{a_r}{b}$ with $(b, p) = 1$.

(c) Shrinking $\overline{X}$ around η, if necessary, we may assume that $\overline{X} = \operatorname{Spec} A$, with local coordinates $x_1, \dots, x_n$ such that $D_X = (x_1)$. Then define $\overline{Z}_1 := \operatorname{Spec} A[x_1^{1/b}]$, and let $\bar{h} : \overline{Z}_1 \to \overline{X}$ be the associated covering. Let $Z_1 := \bar{h}^{-1}(X)$ and $h = \bar{h}|_{Z_1}$. Then h is étale and $\bar{h}$ finite and tamely ramified with respect to $\overline{X} \setminus X$. Then $h^*f_*\mathcal{O}_Y$ has exponents equal to 0 in $\mathbb{Z}_p/\mathbb{Z}$, which means by Corollary 2.15 there exists a stratified bundle $\overline{E}_1 \in \operatorname{Strat}(\overline{Z}_1)$ extending $h^*f_*\mathcal{O}_Y$.

(d) Now we claim that there exists a finite étale covering $\bar{g} : \overline{Z} \to \overline{Z}_1$ such that $\bar{g}^*\overline{E}_1$ is trivial. Indeed, this is true for $\overline{E}_1|_{Z_1}$ because it is true for $f_*\mathcal{O}_Y$, and Proposition 2.7 shows that the restriction functor $\langle \overline{E}_1 \rangle_\otimes \to \langle h^*f_*\mathcal{O}_Y \rangle_\otimes$ is an equivalence. This uses a tiny bit of Tannakian category. In other words: The Picard-Vessiot torsor associated with $\overline{E}_1$ restricts to the Picard-Vessiot torsor of $\overline{E}_1|_X = h^*f_*\mathcal{O}_Y$, which is finite.

(e) We can finish up: Write $Z := \bar{g}^{-1}(Z_1)$, $g = \bar{g}|_Z$ and $h = h_1 g$. Then we have the following diagram

$$\begin{tikzcd}
Y \times_X Z \arrow[r, "f_Z"] \arrow[dd, "h_Y"'] & Z \arrow[d, "g"] \arrow[dd, bend left=60, "h"] \\
 & Z_1 \arrow[d, "h_1"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

and h is tamely ramified with respect to $\overline{X} \setminus X$ by construction. But also by construction $h^*f_*\mathcal{O}_Y$ is trivial, and since $h^*f_*\mathcal{O}_Y = f_{Z,*}h_Y^*\mathcal{O}_Y$, Example 2.3 shows that f_Z is the trivial covering, so the proof is complete. ∎

2.3 Regular singular stratified bundles in general

This section was not covered in the lectures.

2.17 Definition. (a) Let X be a smooth, connected, separated, finite type k-scheme. A pair $(X, \overline{X}))$ is called *good partial compactification* if

- $\overline{X}$ is smooth, separated, and of finite type over k,
- $X \subseteq \overline{X}$ is a dense open subset.
- $\overline{X} \setminus X$ is a strict normal crossings divisor.

(b) A finite étale covering $f : Y \to X$ is called *tame*, if it is tamely ramified with respect to every good partial compactifications $(X, \overline{X})$ of X.

(c) A stratified bundle $E \in \mathrm{Strat}(X)$ is called *regular singular*, if and only if it is $(X, \overline{X})$-regular singular for every good partial compactification $(X, \overline{X})$ of X. □

2.18 Remark.

- Note that $\overline{X}$ is not required to be proper. Hence the name *partial* good compactification.
- The notion of tameness from the above definition was intensively studied in [KS10] under the name "divisor tameness". □

Finally, Theorem 2.16 immediately implies the Main Lemma Theorem 1.2

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