COMPLETE IDEALS IN 2-DIMENSIONAL REGULAR LOCAL RINGS

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1. Lecture 1 : Introduction and overview

The objective of these notes is to present a few important results about complete ideals in 2–dimensional regular local rings. The fundamental theorems about such ideals are due to Zariski found in appendix 5 of [26]. These results were proved by Zariski in [27] for 2- dimensional polynomial rings over an algebraically closed field of characteristic zero and rings of holomorphic functions. Zariski states in [27],

"It is the main purpose of the present investigation to develop an arithmetic theory parallel to the geometric theory of infinitely near points (in plane or on a surface without singularities.)"

Incidently [27] was Zariski's first paper in commutative algebra. In order to state Zariski's results, we recall the notion of integral closure of an ideal.

Definition 1.1. *Let I be an ideal of a commutative ring R. An element $x \in R$ is called integral over I, if*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

for some elements $a_i \in I^i$ for $i = 1, 2, \ldots, n$.

The set $\overline{I}$ of elements of R which are integral over I is an ideal called the **integral closure** of I. The ideal I is called a **complete ideal** if $\overline{I} = I$. An ideal is called a **simple ideal** if it cannot be written as a product of proper ideals of R.

We now state the two main theorems of Zariski.

These notes are based on a course offered by C. Huneke in 1987 at Purdue, the Purdue thesis of V. Kodiyalam and the recent book of Huneke and Swanson. These lectures were delivered at the Instiute of Mathematics, Hanoi and Vietnam Institute of Advanced Study in Mathematics, Hanoi during 29 August-14 September 2012.

Theorem 1.2 (**Zariski's Product Theorem**). *Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring. Then product of complete ideals in R is complete.*

Theorem 1.3 (**Zariski's Unique Factorization Theorem**). *Every complete ideal in a 2- dimensional regular local ring factors uniquely, upto order, as a product of simple complete ideals.*

The above two theorems are not valid in higher dimensions. Huneke constructed $\mathfrak{m}$-pimary ideals in a 3-dimensional regular local ring whose product is not complete.

Theorem 1.4 (**Huneke, 1986**). *Let I, J be complete $\mathfrak{m}$-primary ideals in $R = k[x,y]_{(x,y)}$ such that $I + J$ is not integrally closed. Let $I_1 = (I, z), J_1 = (J, z)$ in $S = R[z]_{(\mathfrak{m},z)}$. Then $I_1 J_1$ is not integrally closed, but I_1 and J_1 are integrally closed.*

Example 1.5. *Take $I = (\mathfrak{m}^4, x^2 + y^3)$ and $J = (\mathfrak{m}^4, y^3)$ where $\mathfrak{m} = (x, y)$. Let z be an indeterminate over R. Then the product of (I, z) and (J, z) is not complete although they are complete in $R[z]_{(m,z)}$. Jockusch and Swanson showed that for $I = (x^2, y^3, z^7) \subset k[x, y, z]$, where k is a field, $\overline{I}^2$ is not complete.*

Most of the positive results about complete ideals have been obtained in rings of dimension 2. The property that product of any two complete ideals is complete is essentially equivalent to R having a rational singlarity. We recall this important notion from singularity theory which was first introduced by M. Artin.

Definition 1.6. *A point x of a scheme X is* **regular** *if $\mathcal{O}_{X,x}$ is a regular local ring. A scheme X is called* **regular** *if all of its points are regular. A regular scheme X is called a desingularization of a scheme Y if there is a proper birational map $f : X \to Y$. A normal local ring domain $(R, \mathfrak{m})$ of dimension 2 is said to have a* **rational singularity** *if there exists a desingularization X of Spec R and such that $H^1(X, \mathcal{O}_X) = 0$.*

Theorem 1.7 (**Lipman, 1969**). *Product of complete ideals is complete in a 2-dimensional local ring having a rational singularity.*

S. D. Cutkosky investigated many aspects of complete ideals in a series of papers. A remarkable converse to the above theorem of Lipman was obtained by him in 1990 using deep results from algebraic geometry.

Theorem 1.8 (**Cutkosky,** [4])**.** *Let $(R, \mathfrak{m})$ be a 2-dimensional excellent normal local domain with algebraically closed residue field $k = R/\mathfrak{m}$. Then the following are equivalent:*
(1) *R has a rational singularity.*
(2) *Product of complete ideals in R is complete.*
(3) *Product of complete $\mathfrak{m}$-primary ideals is complete.*
(4) *If I is a complete $\mathfrak{m}$-primary ideal then I^2 is complete.*

The assumption in the above theorem on the residue field k of R is crucial. The theorem is not valid when k is not algebraically closed.

Theorem 1.9 (**Cutkosky,** [4])**.** *Let k be a field of characteristic not equal to 3. Consider the local ring: $R(k) = k[[x, y, z]]/(x^3 + 3y^3 + 9z^3)$. Then*
(1) *$R(k)$ is a normal local domain without a rational singularity.*
(2) *Product of complete ideals is complete in $R(\mathbb{Q})$.*
(3) *There exists a complete $\mathfrak{m}$-primary ideal whose square is not complete if k has positive characteristic or if k is algebraically closed.*

Definition 1.10 (Lipman, 1978)**.** *A Noetherian local ring $(R, \mathfrak{m})$ of dimension 2 is called* **pseudo-rational** *if it is normal, analytically unramified and for every birational proper map $W \to$ Spec R where W is normal, we have $H^1(W, \mathcal{O}_W) = 0$.*

Lipman showed that a 2-dimensional local domain having a rational singularity is pseudo-rational. Rees proved that product of complete ideals is complete in pseudo-ratioinal local rings. He approached this problem via the notion of normal Hilbert polynomials.

Normal Hilbert Polynomials

For any $\mathfrak{m}$-primary ideal I in an analytically unramified local ring $(R, \mathfrak{m})$ of dimension d, the **normal Hilbert function** $\overline{H}(I, n) = \lambda(R/\overline{I^n})$ for large n, is given by the **normal Hilbert polynomial** $\overline{P}(I, x)$:

$$\overline{P}(I, x) = \overline{e}_0(I)\binom{x + d - 1}{d} - \overline{e}_1(I)\binom{x + d - 2}{d - 1} + \cdots + (-1)^d \overline{e}_d(I),$$

for some integers $\overline{e}_0(I), \overline{e}_1(I), \ldots, \overline{e}_d(I)$ called the **normal Hilbert coefficients** of I.

Lipman and Rees determined the normal Hilbert polynomial for all $\mathfrak{m}$-primary ideals in 2-dimensional local domains having a rational singularity and pseudo-rational singularity respectively.

Theorem 1.11 (Lipman, Rees). *Let I be an $\mathfrak{m}$-primary complete ideal of a 2-dimensional pseudo-rational local domain $(R, \mathfrak{m})$. Then*
(1) *$H(I, n) = P(I, n)$ for all $n \geq 1$.*
(2) *$P(I, x) = e_0(I)\binom{x+1}{2} - (e_0(I) - \lambda(R/I))x$.*
(3) *If $R/\mathfrak{m}$ is infinite then for any minimal reduction J of I, $JI = I^2$.*

If $(R, \mathfrak{m})$ is pseudo-rational of dimension 2 then for any normal scheme W with a proper birational map $W \longrightarrow$ Spec R, $H^1(W, \mathcal{O}_W) = 0$. Take $W =$ Proj $\bigoplus_{n=0}^{\infty} \overline{I^n} t^n$. It can be shown that $\overline{e_2}(I) = \lambda(H^1(W, \mathcal{O}_W))$. Hence $\overline{e_2}(I) = 0$ for all $\mathfrak{m}$-primary ideals I. Huneke found necessary and sufficient conditions for the vanishing of $\overline{e_2}(I)$.

Theorem 1.12 (Huneke, 1987). *Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay analytically unramified local ring and let I be an ideal generated by system of parameters. Then $\overline{e}_2(I) = 0$ if and only if $I\overline{I^n} = \overline{I^{n+1}}$ for all $n \geq 1$.*

The above two theorems of Rees-Lipman and Huneke are unified by Rees in his work on pseudo-rational local rings. In order to state his theorem we define the concept of joint reductions.

Definition 1.13. *We say (a, b) is a* **joint reduction** *of the filtration $\{\overline{I^r J^s}\}$ if $a \in I, b \in J$ and*

$$\overline{I^r J^s} = a\overline{I^{r-1} J^s} + b\overline{I^r J^{s-1}} \text{ for } r, s \gg 0.$$

Rees proved existence of joint reductions for the filtration $\{I^r J^s\}$ if residue field of R is infinite.

Definition 1.14. *The ideal (a, b) is called a* **good joint reduction** *of $\{\overline{I^r J^s}\}$ if (a, b) is a joint reduction of $\{\overline{I^r J^s}\}$ so that*

$$\begin{aligned} (a) \cap \overline{I^r J^s} &= a\overline{I^{r-1} J^s} \text{ for all } r > 0, s \geq 0 \text{ and} \\ (b) \cap \overline{I^r J^s} &= b\overline{I^r J^{s-1}} \text{ for all } r \geq 0, s > 0. \end{aligned}$$

Lemma 1.15. *Let $(R, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension 2 with infinite residue field and I, J be $\mathfrak{m}$-primary ideals. Then there exists a good joint reduction (a, b) of $\{\overline{I^r J^s}\}$.*

Theorem 1.16 (Rees, 1981). *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension 2 with infinite residue field. Let I and J be $\mathfrak{m}$-primary ideals. Then following are equivalent.*

(1) $\overline{e}_2(IJ) = \overline{e}_2(I) + \overline{e}_2(J)$;
(2) *for all* $r, s > 0$, $\overline{I^r J^s} = a\overline{I^{r-1} J^s} + b\overline{I^r J^{s-1}}$.

where (a, b) *is a good joint reduction of the filtration* $\{\overline{I^r J^s} \mid r, s \geq 0\}$.

Theorem 1.17. *Product of complete* $\mathfrak{m}$*-primary ideals is complete in a* 2*-dimensional pseudo-rational local ring.*

Proof. We may assume that residue field of R is infinite. Let I, J be $\mathfrak{m}$-primary ideals. Let (a, b) is a good joint reduction of $\{\overline{I^r J^s}\}$. Then

$$\overline{I^r J^s} = a\overline{I^{r-1} J^s} + b\overline{I^r J^{s-1}} \text{ for } r, s > 0,$$

In particular, $\overline{IJ} = a\overline{J} + b\overline{I} \subseteq \overline{I}\ \overline{J}$. Since $\overline{I}\ \overline{J} \subseteq \overline{IJ}$ we get, $\overline{IJ} = \overline{I}\ \overline{J}$. □

Put $I = J$ in Rees' Theorem to deduce Huneke criterion for the vanishing of $\overline{e}_2(I)$.

Unique factorization of complete ideals

S. Cutkosky [2] showed that unique factorization fails in dimension 3. Huneke and Lipman [17] constructed the following explicit example in $k[[x, y, z]]$:

$$(x, y, z)(x^3, y^3, z^3, xy, yz, zx) = (x^2, y, z)(x, y^2, z)(x, y, z^2).$$

Let $m(R)$ denote the semigroup of complete $\mathfrak{m}$-primary ideals of a complete normal local domain of dimension two. The product operation in $m(R)$ is given by $I * J = \overline{(IJ)}$. Lipman [17] showed that if R is a UFD and $R/\mathfrak{m}$ is algebraically closed then $m(R)$ has unique factorization. Cutkosky [4] proved that if $m(R)$ has unique factorization then R is a UFD. He also constructed an example where the converse fails.

Quadratic transforms of a 2-dimensional regular local ring

In order to prove his main theorems about complete ideals, Zariski used the notion of a quadratic transform of R. Let R be a 2-dimensional regular local ring and $\mathfrak{m} = (x, y)$. Let K and k denote the fraction field of R and the residue field of R respectively. Consider the subring $S = R[y/x]$ of K. Then $\mathfrak{m}S = xS$ and $R_x = S_x$. As $S = R[t]/(xt - y)$, we have $S/\mathfrak{m}S \simeq k[t]$. Thus x is a prime element of S. As $S_x = R_x$, by Nagata's theorem, S is a UFD as R_x is a UFD. Let N be any height two prime ideal of S containing $\mathfrak{m}S$. Then S_N is a 2-dimensinal regular local ring. We say that S is a **quadratic transform** of R and S_N is a **first local quadratic transform of** R.

The unique maximal ideal of a local ring A will also be denoted by $\mathfrak{m}_A$.

Definition 1.18. *Let $R \subseteq S$ be 2-dimensional regular local rings We say that S dominates R birationally, written as $R \prec S$, if they have equal fraction fields and $\mathfrak{m}_S \cap R = \mathfrak{m}_R$.*

The follwoing result due to Abhyankar describes the structure of all 2-dimensional regular local rings birationally dominating a 2-dimensional regular local ring.

Theorem 1.19 (Abhyankar, [1]). *Let $R \subset S$ be 2-dimensional regular local rings where S is birationally dominating R. Then there is a unique sequence of 2-dimensional regular local rings*

$$R = R_0 \subset R_1 \subset R_2 \subset \ldots \subset R_n = S$$

such that R_i is a local quadratic transform of R_{i-1} for all $i = 1, 2, \ldots, n$.

Definition 1.20. *Let $(R, \mathfrak{m})$ be a regular local ring. The* **$\mathfrak{m}$-adic order** *$o(a)$ of a nonzero element $a \in R$ is the largest power r of $\mathfrak{m}$ so that $a \in \mathfrak{m}^r$. Similarly the $\mathfrak{m}$-adic order $o(I)$ of an ideal I of R is the largest power r of $\mathfrak{m}$ so that $I \subseteq \mathfrak{m}^r$.*

Finally we define the transform of an ideal.

Definition 1.21. *Let I be an $\mathfrak{m}$-primary ideal of a 2-dimensional regular local ring $(R, \mathfrak{m})$ and $S = R[y/x]$ where $\mathfrak{m} = (x, y)$. Then $IS = x^r J$ where $r = o(I)$ and $J = R$ or it is a height two ideal of S. Let N be a maximal ideal of S so that $\mathfrak{m}S \subset N$. Then J_N is called the* **transform of** *I in $T = S_N$ and it is denoted by I^T.*

We are now in a position to state an important formula called the Hoskin-Deligne formula in the literature [17].

Theorem 1.22 (Hoskin-Deligne formula). *Let I be an $\mathfrak{m}$-primary ideal of a 2-dimensional regular local ring $(R, \mathfrak{m})$. Then*

$$\lambda(R/\overline{I}) = \sum_{R \prec S} \binom{o(I^S) + 1}{2} [S/\mathfrak{m}_S : R/\mathfrak{m}].$$

The Hoskin-Deligne formula implies several important properties of complete ideals in a 2-dimensional regular local ring $(R, \mathfrak{m})$ with infinite residue field:

(1) Product of complete ideals is complete.

(2) We will determine the Hilbert-Samuel and Bhattacharya polynomials of complete $\mathfrak{m}$-primary ideals.
(3) For any minimal reduction J of an $\mathfrak{m}$-primary complete ideal $I^2 = JI$.
(4) For any complete m-primary ideals I and J of R, there exist $a \in I$ and $b \in J$ such that for all $r, s \geq 1$,

$$a^r J^s + b^s I^r = I^r J^s.$$

(5) The Rees algebra $R[It]$, the form ring $G(I)$, the fiber cone $F(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$, and the bigraded Rees algebra $R[Iu, Jv]$ are Cohen-Macaulay for any $\mathfrak{m}$-primary ideals I and J.

2. Lecture 2 : Reductions and integral closures of ideals

In this section we present some basic properties of integral closures and reductions of ideals. Zariski defined complete ideals in terms of valuation rings. We will present Lipman's theorem [17] that connects the two definitions.

We begin by setting up the notation for Hilbert polynomial of an ideal.

If I is an $\mathfrak{m}$-primary ideal of a local ring $(R, \mathfrak{m})$ of dimension d,then the Hilbert function of I is defined as $H(I, n) = \lambda(R/I^n)$. There is a polynomial $P(I, x)$ of degree d with rational coefficients so that $P(I, n) = H(I, n)$ for all large n. We write $P(I, x)$ as:

$$P(I, x) = e_0(I)\binom{x+d-1}{d} - e_1(I)\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

The coefficients $e_0(I), e_1(I), \ldots, e_d(I) \in \mathbb{Z}$ are called the Hilbert coefficients of I.

D.G. Northcott and D. Rees [21] introduced the concept of reduction of an ideal. This concept has turned out to be very useful concept in many questions in commutative algebra. An ideal J contained in an ideal I of a commutative ring R is called a **reduction of** I if $JI^n = I^{n+1}$ for some $n \in \mathbb{N}$. This relationship is preserved under ring homomorphisms and ring extensions. If I is a zero dimensional ideal of a local ring then the reduction process simplifies I without changing its multiplicity. A reduction J of I is called a minimal reduction of I if no ideal properly contained in J is a reduction of I.

Proposition 2.1. *Let $J \subseteq I$ be $\mathfrak{m}$-primary ideals of a local ring $(R, \mathfrak{m})$.*
(1) *If J is a reduction of I then $e_0(I) = e_0(J)$.*

(2) *If K is a reduction of J and J is a reduction of I then K is a reduction of I.*
(3) *An ideal J is a reduction of I if and only if $J + I\mathfrak{m}$ is a reduction of I.*

Proof. (1) If $JI^n = I^{n+1}$, then for all m,

$$\lambda(R/I^{n+m}) \geq \lambda(R/J^m) \geq \lambda(R/I^m).$$

Hence $P(I, n+m) \geq P(J, m) \geq P(I, m)$. Hence $P(I, x)$ and $P(J, x)$ have equal degrees and leading coefficients.
(2) Let $KJ^m = J^{m+1}$ and $JI^n = I^{n+1}$. Then $KI^{m+n} = KJ^mI^n = I^{m+n+1}$.
(3) Let $JI^n = I^{n+1}$. Then $JI^n + \mathfrak{m}I^{n+1} = I^{n+1}$, hence $(J + \mathfrak{m}I)I^n = I^{n+1}$. Conversely let $(J + I\mathfrak{m})I^n = I^{n+1}$. By Nakayama's lemma, $JI^n = I^{n+1}$. □

Definition 2.2. *For an ideal I of a local ring $(R, \mathfrak{m})$, the fiber cone of I is the graded ring $F(I) = \bigoplus_{n=0}^{\infty} I^n/I^n\mathfrak{m}$. The Krull dimension of $F(I)$ denoted by $s(I)$ is called the* **analytic spread of** *I.*

Proposition 2.3. *Let I be an ideal of a local ring $(R, \mathfrak{m})$ with residue field k. For $a \in I$, let a^* be the residue class of a in $I/\mathfrak{m}I$. Let $a_1, a_2, \ldots, a_s \in I$. Then $(a_1^*, a_2^*, \ldots, a_s^*)$ is a zero-dimensional ideal of $F(I)$ if and only if $J = (a_1, \ldots, a_s)$ is a reduction of I.*

Proof. The n^{th} homogeneous component of $K := (a_1^*, \ldots, a_s^*)$ is $(JI^{n-1} + mI^n)/mI^n$. Thus K is zero dimensional if and only if for all n large, $JI^{n-1} + \mathfrak{m}I^n = I^n$. This holds if and only if J is a reduction of I. □

Corollary 2.4. *Every reduction J of I contains a minimal reduction of I. Let $a_1, a_2, \ldots,$ a_s be chosen from J such that*
(a) *$a_1^*, \ldots, a_s^*$ are k-linearly independent,*
(b) $\dim F(I)/(a_1^*, \ldots, a_s^*) = 0$ *and*
(c) *The integer s in* (b) *is minimal with respect to* (b) *.*
Then $a_1, a_2, \ldots, a_s$ is a minimal basis of a minimal reduction of I contained in J.

Proof. Put $K = (a_1, a_2, \ldots, a_s)$. Observe that $K \cap \mathfrak{m}I = \mathfrak{m}K$ if and only if $\ker(K/\mathfrak{m}K \longrightarrow I/\mathfrak{m}I) = 0$. This is a consequence of (a). The assumption in b implies that K is a reduction of I. Suppose that $K' \subset K$ is a reduction of I. Then $K' + \mathfrak{m}I = K + \mathfrak{m}I$ by (c). Hence

$$K \subset (K' + \mathfrak{m}I) \cap K = K' + \mathfrak{m}I \cap K = K' + \mathfrak{m}K.$$

By Nakayama's lemma $K = K'$. It is clear that $a_1, \ldots, a_s$ minimally generate K. In fact $a_1, \ldots, a_s$ are part of a minimal basis of I. □

Proposition 2.5. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field k. Let $a_1, \ldots, a_s \in I$, an ideal of R. Then $a_1^*, \ldots, a_s^*$ form a homogeneous system of parameters of $F(I)$ if and only if $J = (a_1, \ldots, a_s)$ is a minimal reduction of I. In particular, every minimal reduction of I is minimally generated by $s(I)$ elements.*

Proof. If $a_1^*, \ldots, a_s^*$ form a homogeneous system of parameters of $F(I)$ then $s = \dim F(I)$ and $F(I)/(a_1^*, \ldots, a_s^*)$ is zero-dimensional. Hence $a_1, \ldots, a_s$ generate a minimal reduction of I. Conversely if $J = (a_1, \ldots, a_s)$ is a minimal reduction of I then $\dim F(I)/(a_1^*, \ldots, a_s^*) = 0$ and s is minimal with respect to this property. Hence $a_1^*, \ldots, a_s^*$ constitute a homogeneous system of parameters.
Since k is infinite, it is possible to choose a homogeneous system of parameters of $F(I)$ from the degree one component of $F(I)$. Hence every minimal reduction of I is minimally generated by $\dim F(I) = s(I)$ elements. □

For an ideal I of a local ring $(R, \mathfrak{m})$, we set $\mu(I) = \dim I/\mathfrak{m}I$. The number $\mu(I)$ is the minimal number of generators of I.

Proposition 2.6. *For ideal I of a local ring $(R, \mathfrak{m})$ we have*

$$\operatorname{alt} I := \sup\{ht\, p : p \text{ is a minimal prime of } I\} \leq s(I) \leq \mu(I).$$

Proof. We may assume that $R/\mathfrak{m}$ is infinite. Let J be a minimal reduction of I. Since $JI^n = I^{n+1}$ for some n, $V(I) = V(J)$. Therefore by the Krull's altitude theorem $\operatorname{alt} I = \operatorname{alt} J \leq \mu(J) = s(I)$. Since $\dim F(I) \leq \dim I/I\mathfrak{m}$, we get $s(I) \leq \mu(I)$. □

Proposition 2.7. *Let I be an ideal of a commutative ring R. Then the integral closure $\bar{I}$ of I is an ideal of R.*

Proof. Consider the Rees algebra $R(I) = \bigoplus_{n=0}^{\infty} I^n t^n$ of I, where t is an indeterminate. Let $x \in \bar{I}$ satisfy the equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, for some $a_i \in I^i,\ i = 1, 2, \ldots, n$. Then

$$(xt)^n + (a_1 t)(xt)^{n-1} + \cdots + (a_i t^i)(xt)^{n-i} + \cdots + a_n t^n = 0.$$

Hence xt is integral over $R(I)$. If $x, y \in \bar{I}$ then xt, yt are integral over $R(I)$. Thus $xt + yt$ is integral over $R(I)$. Let $u \in R$ and ut be integral over $R(I)$. Then there exist $b_1, b_2, \ldots, b_n \in R(I)$ such that $(ut)^n + b_1(ut)^{n-1} + \cdots + b_n = 0$. Equating coefficient of t^n we obtain $u^n + b_{1n}u^{n-1} + \cdots + b_{nn} = 0$ where b_{ij} are defined by $b_i = \sum b_{ij} t^j$ where $b_{ij} \in I^j$ for $i = 1, 2, \ldots, n$. This shows that $u \in \bar{I}$. In particular $x + y \in \bar{I}$. If $x \in \bar{I}$ and $c \in R$, it is easy to see that $cx \in \bar{I}$. Hence $\bar{I}$ is an ideal. □

Proposition 2.8. *Let I be an ideal of a commutative ring R. Then $x \in \bar{I}$ if and only if I is a reduction of (I, x).*

Proof. Suppose $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $a_i \in I^i$, $i = 1, 2, \ldots, n$. Then $x^n \in I(I, x)^{n-1}$ which yields $I(I, x)^{n-1} = (I, x)^n$. Conversely suppose that I is a reduction of (I, x) and $I(I, x)^{n-1} = (I, x)^n$. Then $x^n = \sum_{i=1}^m a_i b_i$ where $a_i \in I$ and $b_i \in (I, x)^{n-1}$. Thus $b_i = \sum_{j=0}^{n-1} a_{ij} x^{n-1-j}$ for some $a_{ij} \in I^j, j = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, m$. Hence $x^n - \sum_{i=1}^m \sum_{j=0}^{n-1} a_i a_{ij} x^{n-1-j} = 0$. Thus $x \in \bar{I}$. □

Proposition 2.9. *Let $I \subseteq J$ be ideals of a commutative ring R such that J is finitely generated. Then I is a reduction of J if and only if $J \subseteq \bar{I}$.*

Proof. Let $J = (I, x_1, x_2, \ldots, x_m)$. Let $J \subseteq \bar{I}$. Then x_1 is integral over I, hence I is a reduction of (I, x_1). Now apply induction on m to see that I is a reduction of J. Conversely let I be a reduction of J. Then for an indeterminate $t, (It)(Jt)^{n-1} = (Jt)^n$ for some n. Therefore $R[Jt]$ is a finite $R[It]$-module. Hence xt is integral over $R[It]$ for any $x \in J$. Therefore $x \in \bar{I}$. □

Proposition 2.10. *Let R be a commutative ring and let S be a multiplicatively closed subset of R. Let I be an ideal of R. Then $\overline{I} R_S = \overline{I R_S}$. In particular localization of a complete ideal is complete.*

Proof. Let $x \in \bar{I}$ and $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ be an equation of integral dependence where $a_i \in I^i, i = 1, 2, \ldots, n$. Then

$$(x/1)^n + (a_1/1)(x/1)^{n-1} + \cdots + a_n/1 = 0.$$

Hence $x/1 \in \overline{I R_S}$. Conversely, let $x/s \in \overline{I R_S}$. Then $x/1 \in \overline{I R_S}$. Hence there exist $b_i \in I^i$, for $i = 1, 2, \ldots, m$ and $t \in S$ such that

$$(x/1)^m + (b_1/t)(x/1)^{m-1} + \cdots + b_n/t = 0.$$

Multiply this equation by t^m to get

$$(tx)^m + (b_1)(tx)^{m-1} + (tb_2)(tx)^{m-2} + \cdots + b_m t^{m-1} = 0,$$

which implies $tx \in \overline{I}$. Thus $x \in \overline{I} R_S$.

□

Proposition 2.11. *Let I be an ideal in a Noetherian ring R. Suppose the associated graded ring $G(I) = \oplus_{n=0}^{\infty} I^n / I^{n+1}$ of I is reduced. Then $\overline{I^n} = I^n$ for all $n \geq 1$.*

Proof. Let there be an $n \geq 1$, such that $I^n \neq \overline{I^n}$ and pick an $r \in \overline{I^n} \setminus I^n$. Then there is a k and elements $a_i \in I^{ni}, i = 1, 2, \ldots, k$ such that

$$r^k + a_1 r^{k-1} + \cdots + a_k = 0. \tag{1}$$

We can find a $p \leq n-1$ such that $r \in I^p \setminus I^{p+1}$. Let $r^\star$ denote the initial form of r in the p^{th}-graded component of $G(I)$. Then the equation (1) gives $r^k \in I^{pk+1}$. Hence $r^\star$ is nilpotent. This is a contradiction. □

Corollary 2.12. *Let $(R, \mathfrak{m})$ be a regular local ring. Then $\mathfrak{m}^n = \overline{\mathfrak{m}^n}$ for all $n \geq 1$.*

Proposition 2.13. *Let I and J be ideals of a Noetherian ring R. Let M be a finitely generated R-module with nilpotent annihilator $\operatorname{ann}(M)$. Suppose $IM = JM$. Then $\overline{I} = \overline{J}$.*

Proof. Since $IM = JM$, we have $IM = (I+J)M$. Thus we may assume that $I \subseteq J$. We only need to show that $J \subset \overline{I}$. Let $b \in J$. Pick $u_1, u_2, \ldots, u_n \in M$, such that $M = Ru_1 + Ru_2 + \cdots + Ru_n$. Then for $i, j = 1, 2, \ldots, n$, there exist $a_{ij} \in I$ such that $bu_i = \sum_{j=1}^n a_{ij} u_j$. Put $A = (a_{ij})$. Then $(bI_n - A)u = 0$ where I denotes the $n \times n$ identity matrix and $u = (u_1, u_2, \ldots, u_n)^t$. Thus $\det(bI_n - A)u_i = 0$ for all $i = 1, 2, \ldots, n$. Hence there exists an r so that $(\det(bI_n - A))^r = 0$. This yields an equation of integral dependence over I for b. □

Complete ideals and discrete valuation rings

Zariski defined complete ideals in terms of valuations. The definition given in these notes refers to integral elements. We prove a theorem of Lipman which shows that these two definitions are in fact equivalent. We first show the existence of discrete valuation rings birationally dominating a given local domain.

Proposition 2.14. *Let $(R, \mathfrak{m})$ be a local domain of positive dimension. Then there is a discrete valuation ring $(V, \mathfrak{n})$ birationally dominating $(R, \mathfrak{m})$.*

Proof. We show that there exists an $x \in \mathfrak{m}$ such that $x^k \notin \mathfrak{m}^{k+1}$ for all $k \geq 1$. Let $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ and assume by way of contradiction that $x_i^k \in \mathfrak{m}^{k+1}$ for some k and for all $i = 1, 2, \ldots, n$. Since $\mathbf{x}^{[k]} := (x_1^k, x_2^k, \ldots, x_n^k)$ is a reduction of $\mathfrak{m}^k$, there exists an s such that $\mathbf{x}^{[k]}\mathfrak{m}^{ks} = \mathfrak{m}^{ks+k}$. Hence $\mathfrak{m}^{ks+k} \subset \mathfrak{m}^{sk+k+1}$ which yields $\mathfrak{m}^{ks+k} = 0$ This is a contradiction as $\dim R \geq 1$. Thus we may assume without loss of generality that for $x_1 = x$, $x^k \notin m^{k+1}$ for all k.

The ring $S = R[\mathfrak{m}/x] = R[x_2/x, x_3/x, \ldots, x_n/x]$ is called a *monoidal transform* of R. It is easy to see that $S = \{b/x^k : b \in \mathfrak{m}^k$ for some $k\}$. The ideal $xS = \mathfrak{m}S$ is a proper ideal. Indeed, if $1 \in xS$ then $1 = bx/x^d$ for some $d \geq 1$ and $b \in \mathfrak{m}^d$. Hence $x^d \in \mathfrak{m}^{d+1}$ contradicting the choice of x. Thus xS is a height one ideal of S. Let Q be a minimal prime of xS. By Krull-Akizuki theorem, the integral closure T of S_Q in its fraction field K is a one dimensional Noetherian domain. Let N be a maximal ideal of T contracting to the maximal ideal of S_Q. then $NT_N \cap R = \mathfrak{m}$. Hence T_N is the desired discrete valuation ring birationally dominating R. $\square$

Theorem 2.15 (Lipman's theorem). *Let S be a Noetherian domain with fraction field K and let I be a proper ideal of S. Then*

$$\overline{I} = \bigcap_V IV \cap S$$

where the intersection is over all discrete valuation rings V in K such that $V \supset R$.

Proof. Since principal ideals in integrally closed domains are complete and intersections of complete ideals are complete, the ideal J on the right hand side of the above equation is complete. Hence $\overline{I} \subseteq J$. Conversely let $x \notin \overline{I}$. Then we find a discrete valuation ring $V \supset S$ in K such that $x \notin IV$. Put $T = S[Ix^{-1}]$. Then $x^{-1}IT$ is a proper ideal of T. Indeed, if $x^{-1}IT = T$, then $1 = a_1/x + a_2/x^2 + \cdots + a_n/x^n$, where $a_i \in I^i$ for $i = 1, 2, \ldots, n$. Hence $x^n = a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ which shows that $x \in \overline{I}$. This is a contradiction. Pick a minimal prime Q of $x^{-1}IT$. By Proposition 2.14, there exists a discrete valuation ring $(V, \mathfrak{n})$ such that $V \geq T_Q$. Hence $x^{-1}IT \subset Q \subset QT_Q = \mathfrak{n} \cap T_Q$ and $x^{-1}IV \subseteq \mathfrak{n}$. Thus $x \notin IV$. $\square$

Theorem 2.16. *Let $(R, \mathfrak{m})$ be a local domain and let I be an $\mathfrak{m}$-primary ideal. Then there exist discrete valuation rings $V_1, V_2, \ldots, V_n$, birationally dominating R such that*

$$\overline{I} = \bigcap_{i=1}^{n} IV_i \cap R.$$

Proof. Let K be a fraction field of R. By Theorem (2.15) we have $\overline{I} = \cap(IV \cap R)$ where the intersection varies over discrete valuation rings in K containing R. If $IV = V$, then we may remove V from this intersection. Thus we may assume that $IV < V$ for all the discrete valuation rings appearing in (2.15). If I is $\mathfrak{m}$-primary then it follows that $R \prec V$. Since R/I is Artinian, the descending chain of ideals $\{\cap_{i=1}^r IV_i \cap R\}_{r \geq 1}$ terminates. $\square$

3. **Lecture 3 : Quadratic transforms and contracted ideals**

In this section we introduce (local) quadratic transform of a 2-dimensional regular local ring. We will prove that a local quadratic transform of a 2-dimensional regular local ring is again a 2- dimensional regular local ring. This construction facilitates inductive arguments for the proofs of main theorems about complete ideals.

Lemma 3.1. *Let S be a commutative ring and a, b be a regular sequence in S. Let x be an indeterminate. Then*

$$S[x]/(ax - b) \simeq S[b/a].$$

Proof. Consider the map $\phi : S[x] \to S[b/a]$ defined by $\phi(x) = b/a$. We show that the $\ker(\phi) = (ax - b)$. Let $f(x) = r_n x^n + \cdots + r_0 \in \ker(\phi)$. Apply induction on the degree $\deg(f(x))$ of $f(x)$. Since $r_n(b/a)^n + \cdots + r_0 = 0$, $r_n b^n + \cdots + r_0 a^n = 0$. Hence $r_n b^n \in (a)$. Since a, b is a regular sequence, $r_n \in (a)$. Write $r_n = a s_n$ for some $s_n \in S$. Then

$$g(x) = f(x) - (ax - b) s_n x^{n-1} \in \ker(\phi)$$

and $\deg g(x) < \deg f(x)$. By induction $g(x) \in (ax - b)$, and hence so does $f(x)$.

□

Proposition 3.2. *Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring. Let $\mathfrak{m} = (x, y)$ and $S = R[y/x]$. Then*
(1) *The ideal $xS = \mathfrak{m}S$ is a prime ideal.*
(2) *The maximal ideals of S containing $\mathfrak{m}S$ are in one-to-one correspondence with irreducible polynomials of the polynomial ring $k[t]$ over $k = R/\mathfrak{m}$.*
(3) *If N is any maximal ideal of S containing $\mathfrak{m}S$ then S_N is a 2- dimensional regular local ring.*
(4) *We have $\operatorname{Spec}(S) = \operatorname{Spec}(R_x) \cup \operatorname{Spec}(k[t])$.*
(5) *The ring S is a unique factorization domain.*
(6) *The valuation ring of the $\mathfrak{m}$-adic order valuation is $S_{\mathfrak{m}S}$.*

Proof. (1) Since x, y is a regular sequence, $S/\mathfrak{m}S \simeq R[t]/(xt - y, \mathfrak{m}[t]) \simeq k[t]$. Hence $xS = \mathfrak{m}S$ is a prime ideal.
(2) The maximal ideals of S containing $\mathfrak{m}S$ are therefore in 1-1 correspondence with maximal ideals of $S/\mathfrak{m}S \simeq k[t]$. But the maximal ideals $k[t]$ are principal and generated by irreducible polynomials in $k[t]$.

(3) Let N be a maximal ideal of S containing xS. Then N/xS is generated by an irreducible polynomial $g(t) \in k[t]$. Let $g(y/x)$ be any lift of $g(t)$ to S. Then $N = (x, g(y/x))$. Thus $\mu(N) = 2$, and S_N is a 2-dimensional regular local ring.
(4) Notice that $R_x = S_x$. Hence prime ideals of S not containing x are in 1-1 correspondence with prime ideals of R not containing x. The remaining primes of S are in 1-1 correspondence with primes in $k[t]$.
(5) Since x is a prime element of S, by Nagata's theorem, it is enough to see that S_x is a UFD. But $R_x = S_x$ and R is a UFD, hence S_x is a UFD. So S is a UFD.
(6) Let V be the valuation ring of the $\mathfrak{m}$-adic order valuation. Since $o(y/x) = 0$ $S \subset V$. It is easy to see that $S_{\mathfrak{m}S} \subseteq V$. Since $S_{\mathfrak{m}S}$ is a discrete valuation ring it follows that $V = S_{\mathfrak{m}S}$. □

Definition 3.3. *The local ring S_N is called a* **first local quadratic transform of** *R.*

Let $\mathfrak{m} = (x, y)$. Any first local quadratic transform of R is a localization of either $R[\mathfrak{m}/x]$ or $R[\mathfrak{m}/y]$. Any first local quadratic transform of R is a 2-dimensional regular local ring birationally dominating R. The n^{th} local quadratic transform of R is defined to be the first local quadratic transform of an $(n-1)^{\text{st}}$ local quadratic transform of R.

Ideals contracted from quadratic transforms

An important step in the proofs of Zariski's theorems is the fact that any $\mathfrak{m}$-primary complete ideal of a 2-dimensional regular local ring is a contraction of an ideal from a local quadratic transform T of R and the transform of I in T is also complete.

We assume in the rest of the section that $(R, \mathfrak{m})$ is a 2-dimensional regular local ring with residue field k and $\mathfrak{m} = (x, y)$.

Definition 3.4. *An ideal I of R is called a* **contracted ideal** *if there is an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ and an ideal K of $S = R[\mathfrak{m}/x]$ such that $K \cap R = I$, equivalently $IS \cap R = I$.*

Proposition 3.5. *An ideal I of R is contracted from $S = R[\mathfrak{m}/x]$ if and only if $I : \mathfrak{m} = I : x$.*

Proof. Let $\mathfrak{m} = (x, y)$. Suppose I is contracted from S. It is clear that $I : \mathfrak{m} \subset I : x$. Let $r \in I : x$. Then $ry = rx(y/x) \in IS \cap R = I$. Hence $r \in I : y$. Thus $I : \mathfrak{m} = I : x$.

Conversely let $I : \mathfrak{m} = I : x$. Let $r \in IS \cap R$. We may write

$$r = a_0 + a_1(y/x) + \cdots + a_n(y/x)^n$$

where $a_i \in I^i$ for $i = 0, 1, \ldots, n$. We induct on n. If $n = 0$ then $r \in I$. Let $n \geq 1$. Then

$$x^n r = x^n a_0 + \cdots + a_{n-1} x y^{n-1} + a_n y^n.$$

Thus $x | a_n$. Write $a_n = x b_n$. for some $b_n \in R$. Hence

$$r x^{n-1} = x^{n-1} a_0 + \cdots + a_{n-1} y^{n-1} + b_n y^n.$$

Therefore

$$r = a_0 + a_1 (y/x) + \cdots + (y/x)^{n-1} (a_{n-1} + b_n y).$$

Since $x b_n \in I$, we have $y b_n \in I$. Thus $a_{n-1} + y b_n \in I$. By induction $r \in I$. □

Proposition 3.6. *An $\mathfrak{m}$-primary ideal I is contracted from $S = R[\mathfrak{m}/x]$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ if and only if there exists an $a \in I$ such that $\mathfrak{m} I = a\mathfrak{m} + xI$.*

Proof. Suppose that I is contracted from $S = R[\mathfrak{m}/x]$. Then $I : \mathfrak{m} = I : x$. Since $R/(x)$ is a discrete valuation ring , there is an $a \in I$ so that $(a, x)/(x) = (I, x)/(x)$. Hence $(I, x) = (a, x)$. Let $b \in I$. Then $b = ap + xq$ for some $p, q \in R$. Hence $q \in I : x = I : \mathfrak{m}$. Hence $I \subseteq (a) + x(I : \mathfrak{m})$ and thus $\mathfrak{m} I = a\mathfrak{m} + xI$. Conversely suppose that there is an $a \in I$ such that $\mathfrak{m} I = a\mathfrak{m} + xI$. Let $r \in I : x$. Then $rxy = ap + xq$, for some $p \in \mathfrak{m}$ and $q \in I$. Hence $x(ry - q) = ap$. Since a, x is a regular sequence, $ry - q = as$ for some $s \in R$. Hence $ry \in I$. Therefore $I : x = I : \mathfrak{m}$. □

Proposition 3.7. *The product of two $\mathfrak{m}$-primary ideals I and J contracted from $S = R[\mathfrak{m}/x]$ is also contracted from S.*

Proof. Let $a \in I$ and $b \in J$ such that $I\mathfrak{m} = a\mathfrak{m} + xI$ and $J\mathfrak{m} = b\mathfrak{m} + yJ$. Hence

$$IJ\mathfrak{m} = I(b\mathfrak{m} + xJ) = b(a\mathfrak{m} + xI) + xIJ = ab\mathfrak{m} + xIJ.$$

Therefore IJ is contracted from S. □

We will now prove a very useful numerical criterion due to Lipman and Rees for $\mathfrak{m}$-primary ideals that are contracted from some quadratic transform of R.

We will need the Hilbert-Burch theorem which identifies the structure of ideals of projective dimension one in regular local rings. We state the following special version useful to us.

Theorem 3.8 (Hilbert-Burch Theorem). *Let I be an $\mathfrak{m}$-primary ideal of a 2-dimensional regular local ring $(R, \mathfrak{m})$ with $\mu(I) = n$. Then there is an $(n-1)\times n$ matrix A with entries from $\mathfrak{m}$ such that I is generated by the maximal minors of A. Furthermore, there is an exact sequence*

$$0 \longrightarrow R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \longrightarrow R/I \longrightarrow 0,$$

where the maps ϕ_1 and ϕ_2 are defined as follows: Let $\Delta_i = (-1)^{i+1}\det A_i$ where A_i is the submatrix of A obtained by deleting the i^{th} column of A. The map ϕ_2 is the matrix multiplication by A and and the map ϕ_1 is the matrix multiplication by $(\Delta_1, \Delta_2, \ldots, \Delta_n)^t$.

Example 3.9. Let $I = (x^2, xy, y^3)$. Then I is generated by 2×2-minors of the matrix

$$A = \begin{bmatrix} y & -x & 0 \\ 0 & y^2 & -x \end{bmatrix}$$

and we have the following minimal resolution of R/I :

$$0 \longrightarrow R^2 \xrightarrow{A} R^3 \xrightarrow{B} R \longrightarrow R/I \longrightarrow 0.$$

where $B = (x^2, xy, y^3)^t$.

Example 3.10. Let $I = (y^5, y^4x, y^3x^3, x^6)$. Then I is generated by the maximal minors of the matrix

$$A = \begin{bmatrix} x & -y & 0 & 0 \\ 0 & x^2 & -y & 0 \\ 0 & 0 & -x^3 & y^3 \end{bmatrix}.$$

Put $B = (y^5, y^4x, y^3x^3, x^6)^t$. Then we have the following minimal resolution of R/I :

$$0 \longrightarrow R^3 \xrightarrow{A} R^4 \xrightarrow{B} R \longrightarrow R/I \longrightarrow 0.$$

Lemma 3.11. *Let I be an $\mathfrak{m}$-primary ideal of R. Then $\lambda\left(\frac{I:\mathfrak{m}}{I}\right) = \mu(I) - 1$.*

Proof. Let $\mu(I) = n$. We compute $\operatorname{Tor}_2^R(R/I, k)$ in two ways. By Hilbert-Burch theorem, we have the following minimal resolution of R/I,

$$0 \longrightarrow R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \longrightarrow R/I \longrightarrow 0.$$

tensor with k to get the complex:

$$0 \longrightarrow k^{n-1} \xrightarrow{\overline{\phi_2}} k^n \xrightarrow{\overline{\phi_1}} k \longrightarrow 0.$$

Since the maps in the above complex are zero maps, $\operatorname{Tor}_2^R(R/I, k) = \ker \overline{\phi_2} = k^{n-1}$. Hence $\lambda\left(\operatorname{Tor}_2^R(R/I, k)\right) = \mu(I) - 1$. We can calculate $\operatorname{Tor}_2^R(R/I, k)$ from the Koszul complex resolving k as an R-module. Tensor the Koszul complex with R/I to get the complex:

$$0 \longrightarrow R/I \xrightarrow{\alpha} R/I \oplus R/I \xrightarrow{\beta} R/I \longrightarrow 0$$

where $\alpha = (\bar{y}, -\bar{x})$ and $\beta = (\bar{x}, \bar{y})^t$. Hence $\operatorname{Tor}_2^R(R/I, k) \simeq \{\bar{r} \in R/I : ry, rx \in I\} = \frac{I:\mathfrak{m}}{I}$. □

Theorem 3.12 (Lipman, Rees). *Let $(R, \mathfrak{m})$ be a 2–dimensional regular local ring with infinite residue field k. Then an $\mathfrak{m}$-primary ideal is contracted from $S = R[\mathfrak{m}/x]$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ if and only if $\mu(I) = o(I) + 1$.*

Proof. Set $o(I) = r$ and $L_r(I) = I + \mathfrak{m}^{r+1}/\mathfrak{m}^{r+1} \subseteq G(\mathfrak{m}) = k[x^*, y^*]$ where $x^* = x + \mathfrak{m}^2, y^* = y + \mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2$. Since k is infinite, we can choose a linear form in x^*, y^* so that it does not divide the gcd of elements in $L_r(I)$. Let x be a lift of this linear form. Then $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. We claim that $\lambda((I : x)/I) = o(I)$. Indeed, consider the exact sequence

$$0 \longrightarrow I : x/I \longrightarrow R/I \xrightarrow{x} R/I \longrightarrow R/(I, x) \longrightarrow 0.$$

Hence $\lambda\left(I : x/I\right) = \lambda\left(R/(I, x)\right)$. The choice of x shows that $(I, x)/(x) \nsubseteq (\mathfrak{m}^{r+1}, x)/(x)$. Therefore $o(I) = \lambda(R/(I, x)) = \lambda\left((I : x)/I\right)$. Hence

I is contracted from $R[y/x] \Leftrightarrow I : \mathfrak{m} = I : x \Leftrightarrow \lambda\left(\frac{I : \mathfrak{m}}{I}\right) = \lambda\left(\frac{I : x}{I}\right) \Leftrightarrow \mu(I) - 1 = o(I)$.

□

4. Lecture 4 : The characteristic form and transforms of ideals

The characteristic form of an ideal

Let I be an $\mathfrak{m}$-primary ideal of a 2-dimensional regular local ring $(R, \mathfrak{m})$ with residue field k and $\mathfrak{m} = (x, y)$ and $o(I) = r$. It is natural to ask: what power of maximal ideal is a factor of I where I is an $\mathfrak{m}$-primary ideal of R? To decide this, we introduce the characteristic form of an ideal which is a homogeneous polynomial in $k[u, v]$ where $G(\mathfrak{m}) := \bigoplus_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1} \simeq k[u, v]$ The subspace $L_r(I) = I + \mathfrak{m}^{r+1}/\mathfrak{m}^{r+1}$ of $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ is generated by certain forms of degree r in u, v. The greatest common divisor of these forms is called the **characteristic form of** I and it is denoted by $c(I)$.

Example 4.1. Let $I = (x^2, xy, y^3)$. Then $o(I) = 2$. The space $L_2(I)$ is generated by u^2, uv. Hence $c(I) = u$.

Proposition 4.2. *Let I, J be $\mathfrak{m}$-primary ideals. Then $c(IJ) = c(I)c(J)$.*

Proof. For an $\mathfrak{m}$-primary ideal I of order r, let I^* denote the ideal of $G(\mathfrak{m})$ generated by $I + \mathfrak{m}^{r+1}/\mathfrak{m}^{r+1}$. Then $I^* = c(I)I'$, $J^* = c(J)J'$ where I', J' are either unit ideals or (u, v)-primary. Since $o(IJ) = o(I) + o(J)$,

$$(IJ)^* = I^*J^* = c(I)c(J)I'J' = c(IJ)(IJ)'.$$

It follows that $c(IJ) = c(I)c(J)$. □

Proposition 4.3. *Let I be an $\mathfrak{m}$-primary contracted ideal with $r = o(I)$. Then*
(1) *$I = \mathfrak{m}J$ for some ideal J of R, if and only if* $\deg c(I) < r$.
(2) *If $r > \deg c(I)$ and $s = r - \deg c(I)$ then $I = \mathfrak{m}^s(I : \mathfrak{m}^s)$.*
(3) *$c(I) = 1$ if and only if $\overline{I} = \mathfrak{m}^r$.*
(4) *If I, J is contracted then $\mathfrak{m} \mid IJ$ if and only if $\mathfrak{m} \mid I$ or $\mathfrak{m} \mid J$.*

Proof. (1) First we note that if I is contracted from $S = R[y/x]$ then for any ideal K of R, $I : K$ is also contracted from S. Indeed,

$$(I : K) : x = (I : x) : K = (I : \mathfrak{m}) : K = (I : K) : \mathfrak{m}.$$

Let $I = \mathfrak{m}J$ for some ideal J of R. Then $J \subseteq I : \mathfrak{m}$. Hence $I = \mathfrak{m}J \subseteq \mathfrak{m}(I : \mathfrak{m}) \subseteq I$. Thus $I = \mathfrak{m}(I : \mathfrak{m})$. We may therefore assume that if $\mathfrak{m} \mid I$ then $I = \mathfrak{m}(I : \mathfrak{m})$. Hence we assume that $J = (I : \mathfrak{m})$. Now consider the exact sequence for any contracted $\mathfrak{m}$-primary ideal I where we have put $J = I : \mathfrak{m}$,

$$0 \longrightarrow I/\mathfrak{m}J \longrightarrow J/\mathfrak{m}J \longrightarrow J/I \longrightarrow 0.$$

Hence

$$\dim I/\mathfrak{m}J = \dim J/\mathfrak{m}J - \dim(I : \mathfrak{m})/I = \mu(J) - \mu(I) + 1 = o(J) - o(I) + 1 \geq 0.$$

Case 1: If $o(I) = o(J)$ then $\dim I/\mathfrak{m}J = 1$. As $\mathfrak{m}J \subseteq \mathfrak{m}^{r+1}$, we have a surjection $I/\mathfrak{m}J \longrightarrow I + \mathfrak{m}^{r+1}/\mathfrak{m}^{r+1}$. Hence $L_r(I)$ is a one-dimensional k-vector space. Therefore $o(I) = \deg c(I)$.
Case 2: If $o(I) = o(J) + 1$ then $I = \mathfrak{m}J$ by the above sequence. But then $c(I) = c(\mathfrak{m}J) = c(\mathfrak{m})c(J) = c(J)$. Hence

$$\deg c(I) = \deg c(J) \leq o(J) < o(I).$$

(2) Apply induction on $s(I) = o(I) - \deg c(I)$. The $s = 0$ case is already proved in (1). Now let $s > 0$. Then $\mathfrak{m} \mid I$ and so $I = \mathfrak{m}(I : \mathfrak{m})$. Then $c(I) = c(I : \mathfrak{m})$ and

$$s(I : \mathfrak{m}) = O(I : \mathfrak{m}) - \deg c(I : \mathfrak{m}) = o(I) - 1 - \deg c(I) = s(I) - 1.$$

By induction we conclude that $I = \mathfrak{m}^{s(I)}(I : \mathfrak{m}^{s(I)})$.
(3) If $\overline{I} = \mathfrak{m}^n$ then $c(I) = c(\overline{I}) = c(\mathfrak{m}^n) = 1$. Conversely let I be contracted and $c(I) = 1$. Let $s(\overline{I}) = s(I) = o(I) - \deg c(I) = o(I) = n > 0$. Then $\overline{I} = \mathfrak{m}^n(\overline{I} : \mathfrak{m}^n)$. Thus $o(\overline{I} : \mathfrak{m}^n) = 0$. Therefore $\overline{I} : \mathfrak{m}^n = R$ which means $\overline{I} = \mathfrak{m}^n$.
(4) Let J be contracted and $\mathfrak{m} \mid IJ$. Then

$$s(IJ) = o(IJ) - \deg c(IJ) = o(I) + o(J) - \deg c(I) - \deg c(J) > 0.$$

Therefore either $s(I) > 0$ or $s(J) > 0$. Hence either $\mathfrak{m} \mid I$ or $\mathfrak{m} \mid J$. □

Proposition 4.4. *Let I be an $\mathfrak{m}$-primary and $\mu(I) = o(I) + 1$. Then I is contracted from $S = R[y/x]$ where $\mathfrak{m} = (x, y)$ if and only if the initial form $x^* \in G(\mathfrak{m})$ does not divide $c(I)$.*

Proof. Let I be contracted from $S = R[y/x]$ where $\mathfrak{m} = (x, y)$. Then $I : x = I : \mathfrak{m}$. We know that $\lambda(R/(I, x)) = \lambda(I : x/I) = \lambda(I : \mathfrak{m}/I) = r$ where $r = o(I)$. If $x^* \mid c(I)$ then $(I, x) \subseteq (x, \mathfrak{m}^{r+1})$. Hence $\lambda(R/(I, x)) \geq \lambda(R/(x, m^{r+1})) = r + 1$. This is a contradiction. Therefore $x^* \nmid c(I)$. Conversely suppose that $x^* \nmid c(I)$. Then

$$\lambda(R/(I, x)) = \lambda(I : x/I) = r = \mu(I) - 1 = \lambda(I : \mathfrak{m})/I.$$

This shows that $I : \mathfrak{m} = I : x$. Hence I is contracted from $R[y/x]$.

□

Example 4.5. A contracted ideal may not be complete. The ideal $I = (x^4, x^2y, xy^4, y^5)$ is contracted as $\mu(I) = 4 = o(I) + 1$. Also $(xy^3)^2 = (x^2y)y^5 \in I^2$. Hence $xy^3 \in \overline{I} \setminus I$.

Transform of an ideal

Proposition 4.6. *Let Let $o(I) = r$. Then $IS = x^r J$ where J is either a height two ideal of S or $J = S$.*

Proof. Since $I \subseteq \mathfrak{m}^r$, $IS \subseteq x^r S$. If $IS \subseteq x^{r+1}S$ then $I \subseteq x^{r+1}S \cap R = \mathfrak{m}^{r+1}$. This is a contradiction. Hence $IS = x^r J$ where $J \nsubseteq xS$. If $J \subseteq P$ where P is a height one prime of S, then $x \notin P$. As $R_x = S_x$, there is a height one prime Q of R so that $Q_x = P_x$. In fact $Q = P \cap R$. Therefore $I \subseteq Q$. This is a contradiction since ht$I = 2$ and ht$Q = 1$.

□

Definition 4.7. *The uniquely determined ideal J, denoted by I^S, is called the* **transform of I in S.** *If N is a height two maximal ideal of S then $J_N \subseteq T = S_N$ denoted by I^T is called* **the transform of I in T.**

Example 4.8. Let $I = (y^2 - x^3, x^2y, x^4)$. As $\mu(I) = 3 = o(I) + 1$, I is a contracted ideal. In fact it is contracted from $S = R[y/x]$. Put $y/x = y_1$ and $y = xy_1$. Then

$$IS = (x^2y_1^2 - x^3, x^3y_1, x^4) = x^2(y_1^2 - x, xy_1, x^2) = x^2(y_1^2 - x, xy_1).$$

Thus $I^S = (y_1^2 - x, xy_1)$. The only height two maximal ideal containing I^S is $N = (x, y_1)$. Moreover I^S is complete.

5. Lecture 5 : Zariski's Theorems

Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring with infinite residue field throughout this section. The goal in this section is to prove the main theorems of Zariski about complete ideals. This requires several steps:

(1) The transform of a complete $\mathfrak{m}$-primary ideal is complete.
(2) Complete $\mathfrak{m}$-primary ideals are contracted.
(3) If I is $\mathfrak{m}$-primary then $\lambda(R/I) < \lambda(T/I^T)$ for any local quadratic transform T of R.

We need a crucial preparatory result about valuation rings.

Proposition 5.1. *Let $(S, \mathfrak{n})$ be a local domain with infinite residue field L. Let $a_1, a_2, \dots, a_r$ be a minimal set of generators of of a proper ideal I. If $x = (x_1, x_2, \dots, x_r) \in S^r$ then put $x' = (x'_1, x'_2, \dots, x'_n) \in L^r$ where $'$ denotes residue class in L.*
(1) *Let $(V, \mathfrak{m}_V)$ be a DVR birationally dominating S. Then the set*

$$W(V) = \{(x'_1, x'_2, \dots, x'_r) \in L^r \mid (x_1a_1 + x_2a_2 + \dots + x_ra_r)V < IV\}$$

is a proper subspace of L^r.
(2) *Let $V_1, V_2, \dots, V_g$ be DVRs birationally dominating R. Then there exists $d \in I$ such that $dV_i = IV_i$ for all $i = 1, 2, \dots, g$.*

Proof. (1) First note that $W(V)$ is well-defined. Indeed, let v be the valuation defined by V. Let $x = (x_1, x_2, \ldots, x_r) \in S^r$. Put $a = (a_1, a_2, \ldots, a_r)$ and $x.a = x_1a_1 + \cdots + x_ra_r$. Note that $x' \in W(V)$ if and only if $v(x.a) > v(I)$. Here $v(I)$ denotes the smallest value among $v(a)$ where $a \in I$. Let $y \in S^r$ and $x' = y'$. Then

$$c := x.a - y.a = (x - y).a \in I\mathfrak{n} \subseteq I\mathfrak{m}_V.$$

Hence $v(y.a) = v(c - x.a) \geq \min\,(v(c), v(x.a)) > v(I)$. Thus $W(V)$ is well-defined. Now we show that W is a subspace of L^r. Let $x', y' \in W(V)$. Then

$$v(x.a + y.a) = v((x + y).a)) \geq \min\,(v(x.a), v(y.a)) > v(I).$$

Hence $x' + y' \in W(V)$. Let $b' \in L^{\times}$ and $x' \in W(V)$. Then $v(bx.a) = v(x.a) > v(I)$. Therefore $b'x' \in W(V)$. As $IV = a_iV$ for some $i = 1, 2, \ldots, r$, it follows that $W(V)$ is a proper subspace of L^r.

(2) By (1), $W_i = W(V_i)$ for $i = 1, 2, \ldots, g$. are proper subspaces of L^r. As L is infinite, these subspaces cannot cover L^r. Let

$$z' = (z'_1, z'_2, \ldots, z'_r) \in L^r \setminus \{W_1 \cup W_2 \cup \ldots \cup W_g\}.$$

Then $d := z.a$ satisfies $dV_i = IV_i$ for all $i = 1, 2, \ldots, r$. $\square$

Proposition 5.2. *A complete $\mathfrak{m}$-primary ideal of a 2-dimensional regular local ring with infinite residue field is a contracted ideal.*

Proof. Let $V_1, V_2, \ldots, V_n$ be finitely many DVRs which dominate R birationally such that $I = \cap_{i=1}^n IV_i \cap R$. Let $x \in \mathfrak{m}$ such that $\mathfrak{m}V_i = xV_i$ for all $i = 1, 2, \ldots, n$. We show that I is contracted from $R[y/x]$. For this it is enough to show that $I : \mathfrak{m} = I : x$. Let $r \in R$ and $rx \in I$. Then $r\mathfrak{m}V_i = rxV_i \subseteq IV_i$ for all $i = 1, 2, \ldots, n$. Therefore $r\mathfrak{m} \subseteq \cap_{i=1}^n IV_i \cap R = I$ and so $r \in I : \mathfrak{m}$. Hence $I : \mathfrak{m} = I : x$ which implies that I is contracted from $R[y/x]$.

$\square$

Proposition 5.3. *Let I be an $\mathfrak{m}$-primary complete ideal. Then $I\mathfrak{m}^i$ is complete for all $i \geq 1$.*

Proof. Let $I = IV_1 \cap IV_2 \cap \cdots \cap IVg \cap R$ where $V_1, V_2, \ldots, V_g$ are certain discrete valuations domains birationally dominating R. Choose an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $xV_i = \mathfrak{m}V_i$ for all $i = 1, 2, \ldots, g$. It is enough to show that $I\mathfrak{m}$ is complete. Put $J = \overline{\mathfrak{m}I}$. Since R/xR is a discrete valuation ring $(J, x) = (\mathfrak{m}I, x)$. Hence $J = \mathfrak{m}I + (J : x)x$. We show that

$J : x = I$ to finish the proof. If $rx \in J$ then $rxV_i \subseteq JV_i = I\mathfrak{m}V_i = xIV_i$ for all i. Hence $r \in \cap_{i=1}^{g} IV_i \cap R = I$. $\square$

Proposition 5.4. *Let I be an $\mathfrak{m}$-primary ideal contracted from $S = R[\mathfrak{m}/x]$. Then I^S is complete if and only if I is complete.*

Proof. Let I be complete. Since x is a nonzerodivisor, $x^r I^S = IS$ is complete if and only if I^S is complete. Hence it is enough to prove that IS is complete. Let $s \in \overline{IS}$ and

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

where $a_i \in I^i S$ for $i = 1, 2, \ldots, n$. Since $I^i S = \cup_{n \geq 0} I^i \mathfrak{m}^n / x^n$, we may write $s = t/x^l$ and $a_i = b_i/x^l$ where $l \geq 1$ and $b_i \in I^i \mathfrak{m}^l$ for $i = 1, 2, \ldots, n$. Substitute these in the above equation and multiply by x^{ln} to get

$$t^n + b_1 t^{n-1} + b_2 x^l t^{n-2} + \cdots + b_n x^{ln-l} = 0.$$

Hence $t \in \overline{I\mathfrak{m}^l} = I\mathfrak{m}^l$ by Proposition 5.3. Therefore $s = t/x^l \in IS$ and hence IS is complete.

Conversely let I^S be complete. Then $IS = x^r I^S$ is also complete. As I is contracted from S, $I = IS \cap R$ is also complete. $\square$

Example 5.5. Put $I = (x^2 + y^3, x^3, x^2 y)$. Observe that I is contracted from $S = R[x/y]$. Moreover $o(I) = 2$ and $\mu(I) = 1 + o(I) = 3$. Put $x_1 = x/y$. Then $IS = y^2 I^S$ where $I^S = (x_1^2 + y, yx_1^2)$. The only maximal ideal of S containing I^S is $N = (x_1, y)$. Now consider the ideal $I^S S_N$ in the regular local ring S_N. Then $I^S S_N$ is contracted from $S_N[y/x_1]$. Moreover $I^S S_N$ is a complete ideal. Since $I^S = I^S S_N \cap S$, we see that I^S and hence I is a complete ideal.

Proposition 5.6. *Let I be an $\mathfrak{m}$-primary ideal of order r. Let x be a minimal generator of $\mathfrak{m}$ and $S = R[\mathfrak{m}/x]$. Suppose that I is contracted from S. Then the natural map*

$$\phi : \mathfrak{m}^r / I \longrightarrow \mathfrak{m}^r S / IS$$

is an R-module isomorphism.

In particular, if $T = S_N$ for a height two maximal ideal N of S containing I^S, then

$$\lambda(R/I) > \lambda(T/I^T).$$

Proof. First we note that $\mathfrak{m}^r = I + x\mathfrak{m}^{r-1}$. By the exact sequence

$$0 \longrightarrow (I : x)/I \longrightarrow R/I \xrightarrow{x} R/I \longrightarrow R/(I, x) \longrightarrow 0,$$

we get

$$\lambda\left(R/(I, x)\right) = \lambda\left((I : x)/I\right) = \lambda\left((I : \mathfrak{m})/I\right) = \mu(I) - 1 = r.$$

Since $\lambda\left(R/(x, \mathfrak{m}^r)\right) = r = \lambda\left(R/(I, x)\right)$, we have $(I, x) = (\mathfrak{m}^r, x)$. Thus $\mathfrak{m}^r \subseteq (I, x)$, and hence $\mathfrak{m}^r = I + x(\mathfrak{m}^r : x) = I + x\mathfrak{m}^{r-1}$. Therefore for all $n \geq 0$, by using induction on n we get $\mathfrak{m}^{r+n} = I\mathfrak{m}^n + x^n\mathfrak{m}^r$.
An element of $\mathfrak{m}^r S$ is of the form a/x^n where $a \in \mathfrak{m}^{r+n}$. Since $\mathfrak{m}^{r+n} = I\mathfrak{m}^n + x^n\mathfrak{m}^r$, we can write $a = b + cx^n$ where $b \in I\mathfrak{m}^n$ and $c \in \mathfrak{m}^r$. Thus $a/x^n = b/x^n + c$. Since $b/x^n \in I\mathfrak{m}^n/x^n \subset IS$, it follows that c and a/x^n have the same image in $\mathfrak{m}^r S/IS$. Thus ϕ is surjective.
For injectivity of ϕ, note that since I is contracted from S and $I \subset \mathfrak{m}^r$,

$$IS \cap \mathfrak{m}^r = IS \cap R \cap \mathfrak{m}^r = I \cap \mathfrak{m}^r = I.$$

To prove the statement about length, note that

$$\lambda(S/I^S) \leq \lambda_R(S/I^S) = \lambda_R(\mathfrak{m}^r/I) < \lambda(R/I).$$

□

Zariski's Theorems

Theorem 5.7 (Zariski's Product Theorem). *The product of complete ideals ideals in a 2-dimensional regular local ring is complete.*

Proof. Let I and J be complete ideals in R. Since R is a UFD, it is enough to prove the theorem when I and J are $\mathfrak{m}$-primary and complete. We apply induction on $\ell = \lambda(R/I) + \lambda(R/J)$. If $\ell = 2$, then $I = J = \mathfrak{m}$. In this case since $G(\mathfrak{m})$ is a polynomial ring, hence $\mathfrak{m}^n = \overline{\mathfrak{m}^n}$ for all $n \geq 1$. Now suppose that $\ell \geq 3$. We can find a minimal generator x of $\mathfrak{m}$ such that both I and J are contracted from $S = R[\mathfrak{m}/x]$. Hence IJ is also contracted from S. Since $IJ = IJS \cap R$, it is enough to show that IJS is complete. Let $o(I) = r$ and $o(J) = s$. Then $IJS = x^{r+s}I^S J^S$. Hence it is enough to show that $I^S J^S$ is complete. Let $N_1, N_2, \ldots, N_g$ be all the maximal ideals containing $I^S J^S$. Then $I^S J^S = \cap_{i=1}^g I^S J^S S_{N_i} \cap R$. Thus it is enough to show that the product of the complete ideals $I^S S_{N_i}$ and $J^S S_{N_i}$ is complete. Since the co-lengths of these ideals is smaller than the co-lengths of I and J respectively, by induction hypothesis, we are done. □

Zariski's Unique Factorization Theorem

Now we embark on the proof of Zariski's unique factorization theorem. One of the ingredients of the proof is the fact that the transform of a simple complete ideal is again simple and complete. For this we need the concept of inverse transform of an ideal.

Inverse transform of an ideal

Let J be a height two ideal of S and $J = (s_1/x^{a_1}, s_2/x^{a_2}, \dots, s_n/x^{a_n})$, where $a_1, a_2, \dots, a_n \in \mathbb{N}$, $\;s_i \in \mathfrak{m}^{a_i}$ for $i = 1, 2, \dots, n$ and $a = \max\{a_1, a_2, \dots, a_n\}$. Then $x^a J$ is an extension of an ideal in R. Let a be the least integer such that with this property. Then the ideal $x^a J \cap R$ is called the **inverse transform of** I. Clearly the inverse transform of a height two ideal of S is a contracted ideal.

Lemma 5.8. *Let J be a height two ideal of $S = R[y/x]$ and let $I = x^a J \cap R$ be the inverse transform of J. Then* (1) $IS = x^a J$, (2) $o(I) = a$ *and* (3) $\mathfrak{m}$ *does not divide* I.

Proof. (1) and (2) Let e and c denote the extension and contraction operations on ideals. For an ideal L of R, $L^{ece} = L^e$. As $x^a J$ is an extended ideal from R, we have $IS = (x^a J \cap R)S = x^a J$. Let $o(I) = r$. Then $IS = x^r I' = x^a J$ where I' is a height two ideal of S. Since I' and J have height 2, it follows that $a = r$.

(3) Suppose $\mathfrak{m}$ divides I and write $\mathfrak{m}K = I$ for an ideal K of R. Then $\mathfrak{m}KS = xKS = IS = x^a J$. Hence $KS = x^{a-1} J$. This contradicts the choice of a. Hence $\mathfrak{m}$ does not divide I. $\square$

Proposition 5.9. *Let I and J be $\mathfrak{m}$-primary contracted ideals and $\mathfrak{m}I = \mathfrak{m}J$. Then $I = J$.*

Proof. As $R/\mathfrak{m}$ is infinite, we may pick $x, y \in \mathfrak{m}$ so that $\mathfrak{m} = (x, y)$ and I and J are contracted from $S = R[y/x]$. Assume for a moment that $I\mathfrak{m} : x = I$ and $J\mathfrak{m} : x = J$. Then $I\mathfrak{m} : x = I = \mathfrak{m}J : x = J$. To show that $I\mathfrak{m} : x = I$, pick an $a \in I$ so that $\mathfrak{m}I = xI + a\mathfrak{m}$. Let $z \in R$ and $zx \in \mathfrak{m}I$. Then $zx = xb + ap$ for some $b \in I$ and $p \in \mathfrak{m}$. Hence $x(z - b) = ap$. As x, a is a regular sequence, $z - b = aq$ for some $q \in R$. Thus $z = b + aq \in I$. Hence $I\mathfrak{m} : x = I$ and similarly $J\mathfrak{m} : x = J$. $\square$

Proposition 5.10. *Let $I \neq \mathfrak{m}$ be a simple $\mathfrak{m}$-primary ideal contracted from $S = R[y/x]$. Then the transform I^S is also simple. If I is simple and complete then I^S is contained in a unique maximal ideal N of S and I^T is simple and complete where $T = S_N$.*

Proof. Suppose that I^S is not simple. Then $I^S = J'K'$ where J', K' are proper ideals of S. Let $J = x^a J' \cap R$ and $K = x^b K' \cap R$ be the inverse transforms of J' and K' respectvely. Put $r = o(I)$. Then

$$\mathfrak{m}^{a+b} IS = x^{r+a+b} I^S = \mathfrak{m}^r JKS.$$

As $\mathfrak{m}, I, J$ and K are all contracted ideals from S, we get $\mathfrak{m}^{a+b} I = \mathfrak{m}^r JK$. We know that $\mathfrak{m}$ does not divide J and K. Therefore by simplicity of I we conclude that $I = JK$ which contradicts the simplicity of I. Hence I^S is simple.

Now suppose that $I \neq \mathfrak{m}$ is complete and simple. Let $N := N_1, N_2, \ldots, N_g$ be the height two maximal ideals of S which contain I^S. If $g \geq 2$ then I^S is product of its N_i-primary components for $i = 1, 2, \ldots, g$. As I^S is simple, $g = 1$. If I^S_N is product of two proper ideals of S_N then $I^S_N = J_N K_N$ for some ideals $J, K \subset S$ that are N-primary. Thus $I^S = JK$ which is a contradiction. Hence I^S_N is simple and complete.

□

Theorem 5.11 (Zariski's Unique Factorization Theorem). *A complete $\mathfrak{m}$-primary ideal of a 2-dimensional regular local ring R with $R/\mathfrak{m}$ infinite is product of uniquely determined simple complete ideals up to ordering of its factors.*

Proof. Any ideal I can be factored as a product of simple ideals using just the Noetherian property of R. Let $I = I_1 I_2 ... I_g$ where $I_1, I_2, \ldots, I_g$ are simple. Then $I = \overline{I_1 I_2} \ldots \overline{I_g}$. The factors $\overline{I_i}$ may also be factored as a product of complete ideals if they are not simple. Therefore, as R is Noetherian, this process stops. Hence every complete ideal of R is a product of simple complete ideals.

The monoid of complete ideals has cancellation property due the fact that if M is a finite faithful R-module and $IM = JM$ then, $\overline{I} = \overline{J}$.

Now we show that if I is a simple complete $\mathfrak{m}$-primary ideal and $I \mid J_1 J_2 \ldots J_n$ where $J_1, J_2, \ldots, J_n$ are simple and complete then $I = J_i$ for some i.

To prove this apply induction on $\lambda = \lambda(R/I)$. If $\lambda = 1$ then $I = \mathfrak{m}$. In this case we have already proved the cancellation property in the larger monoid of contracted $\mathfrak{m}$-primary ideals. Now let $\lambda > 1$. Pick an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $I, J_1, J_2, \ldots, J_n$ are contracted from $S = R[y/x]$. Let $'$ denote transform of an ideal in S. Put $IK = J_1 J_2 \ldots J_n$. where K is an ideal of R.

Put $o(I)=r, o(K)=s, o(J_i)=r_i$ for $i=1,2,\ldots,n$. Taking transforms, we have

$$x^{r+s}I'K' = x^{a_1+a_2+\cdots+a_n}J_1'J_2'\ldots J_n'.$$

Note that $r+s=o(IK)=o(J_1J_2\ldots J_n)=x^{a_1+a_2+\cdots+a_n}$. Hence $I'K'=J_1'J_2'\ldots J_n'$. Since I is simple and complete $\mathfrak{m}$-primary ideal, there is a unique maximal ideal N of S such that I' is N-primary. Hence $I_N'K_N'=(J_1')_N(J_2')_N\ldots(J_n')_N$. Since $\lambda(R/I)<\lambda(S_N/I_N')$, by induction, $I_N'=(J_i')_N$ for some i. Since I', J_i' are simple N-primary ideals, we have $I'=J_i$.

Therefore $x^{r+a_i}I'=\mathfrak{m}^{a_i}IS=x^rJ_iS=\mathfrak{m}^rJ_iS$. Contracting back to R we have $\mathfrak{m}^{a_i}I=\mathfrak{m}^rJ_i$. Due to cancellation of $\mathfrak{m}$ and simplicity of I and J_i we have $I=J_i$. □

6. Lecture 6 : The Hoskin-Deligne length formula

In this section we prove a formula originally due to Hoskin [9] and reproved several times thereafter by various authors (see [8],[20], [17] and [15]). for the co-length of an $\mathfrak{m}$-primary complete ideal I of a 2-dimensional regular local ring R. A number of fundamental properties of complete ideals follow from this formula.

If T is a quadratic transform of R, then the residue field of T is a finite algebraic extension of the residue field of R by Hilbert's Nullstellensatz. We denote this field degree by $[T/\mathfrak{m}_T : R/\mathfrak{m}]$. The order of I^T with respect to the maximal ideal of T will be denoted by $o(I^T)$. By the symbol $R \prec T$ we mean T is either R or a local quadratic transform of R.

Proposition 6.1. *Let I be an $\mathfrak{m}$-primary ideal of order r contracted from $S=R[\mathfrak{m}/x]$ where x is a minimal generator of $\mathfrak{m}$. Then*

$$\mathfrak{m}^r/I \simeq \bigoplus_T T/I^T,$$

where the direct sum extends over all the first local quadratic transforms of R.

Proof. We know that $\mathfrak{m}^r/I \simeq \mathfrak{m}^rS/IS$. Since $x^rI^S=IS$ we have $\mathfrak{m}^rS/IS \simeq S/I^S$. Since I^S is a height two ideal of S, there are finitely many maximal ideals $N_1, N_2, \ldots, N_g$ of S containing I^S. Put $T_i=S_{N_i}$ for $i=1,2,\ldots,g$. Thus by Chinese Remainder Theorem, we have the isomorphism

$$S/I^S \simeq \bigoplus_{i=1}^{g} T_i/I^ST_i = \bigoplus_{i=1}^{g} T_i/I^{T_i}.$$

It remains to show that if T is a first quadratic transform of R which is not a localization of S then $I^T = T$. By a result of Sally [24] $\mathfrak{m}T$ is a principal ideal. Let $\mathfrak{m}T = yT$ where y is a minimal generator of $\mathfrak{m}$. Then $x/y \in T$ and hence there exists a height two maximal ideal P of $S = R[\mathfrak{m}/y]$ such that $T = S_P$. Since $\mathfrak{m}^r = I + x\mathfrak{m}^{r-1}$, we get $y^r \in I + x\mathfrak{m}^{r-1}$. Therefore $1 \in y^{-r}IT + \mathfrak{m}T$. Since $\mathfrak{m}T \subset P$, and $y^{-r}IT = I^T$, we obtain $I^T = T$. $\square$

We will also need the following theorem due to Abhyankar [1].

Theorem 6.2 (Abhyankar). *Let $(R, \mathfrak{m}) \prec (T, \mathfrak{n})$ be 2-dimensional regular local rings with same field K of fractions. Then there is unique sequence of 2-dimensional regular local rings*

$$R = R_0 \prec R_1 \prec R_2 \prec \ldots \prec R_n = T$$

such that R_i is a first local quadratic transform of R_{i-1} for $i = 1, 2, \ldots, n$.

Theorem 6.3 (Hoskin-Deligne). *Let $(R, \mathfrak{m})$ be 2- dimensional regular local ring. Let I be a complete $\mathfrak{m}$-primary ideal of R. Then*

$$\lambda(R/I) = \sum_{R \preceq T} \binom{o(I^T)+1}{2} [T/\mathfrak{m}_T : R/\mathfrak{m}]$$

Proof. Apply induction on $l = \lambda(R/I)$. If $l = 1$ then $I = \mathfrak{m}$. In this case $\mathfrak{m}^T = T$ for all quadratic transforms of R other than R. Hence the formula is valid. Now let $l \geq 2$. Since I is complete, it is contracted from some quadratic transform $S = R[\mathfrak{m}/x]$ where x is a minimal generator of $\mathfrak{m}$. Let $o(I) = r$. Then $\mathfrak{m}^r/I \simeq \oplus T/I^T$ where the direct sum is over all the first quadratic transforms of R by the result proved above. Since $I \subseteq \mathfrak{m}^r$,

$$\lambda(R/I) = \lambda(R/\mathfrak{m}^r) + \lambda\left(\mathfrak{m}^r/I\right) = \binom{r+1}{2} + \sum_{R \prec T} \lambda_T\left(T/I^T\right)[T/\mathfrak{m}_T : R/\mathfrak{m}]. \qquad (2)$$

Hence $\lambda(T/I^T) < \lambda(R/I)$ for all $R \prec T$ in the above sum. Since I^T is complete for all T, by induction hypothesis, for all the first quadratic transforms T of R,

$$\lambda_T\left(T/I^T\right) = \sum_{T \prec U} \binom{o(I^U)+1}{2} [U/\mathfrak{m}_U : T/\mathfrak{m}_T]$$

Using $[U/\mathfrak{m}_U : R/\mathfrak{m}] = [U/\mathfrak{m}_U : T/\mathfrak{m}_T][T/\mathfrak{m}_T : R/\mathfrak{m}]$, we get the formula. Note that if $I^U \neq U$ and $R \prec U$ then there is a unique $T_i \prec U$ by Abhyankar's Theorem. $\square$

Some consequences of the Hoskin-Deligne formula

We have proved that $\lambda(R/I) > \lambda_T(T/I^T)$ where I is a contracted ideal in R and T is a first quadratic transform of R. This gives us an inductive tool needed to prove several results about complete ideals. We prove the Lipman-Rees formula for the Hilbert function of a complete $\mathfrak{m}$-primary ideal in a 2-dimensional regular local ring.

For any $\mathfrak{m}$-primary ideal in a local ring $(R, \mathfrak{m})$ of dimension d, the Hilbert function $H(I,n) = \lambda(R/I^n)$ is given by the Hilbert polynomial $P(I,n)$ for all large n. This polynomial is written in the form

$$P(I,n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I),$$

for some integers $e_0(I), e_1(I), \ldots, e_d(I)$ called the Hilbert coefficients of I. As a consequence of the HD formula, we derive a formula for $P(I,n)$ where I is a complete $\mathfrak{m}$-primary ideal of a 2-dimensional regular local ring.

Corollary 6.4. *Let I be an $\mathfrak{m}$-primary complete ideal of a 2-dimensional regular local ring $(R, \mathfrak{m})$. Then*
(1) *$H(I,n) = P(I,n)$ for all $n \geq 1$.*
(2) *$P(I,n) = e_0(I)\binom{n+1}{2} - (e_0(I) - \lambda(R/I))n$.*
(3) *If $R/\mathfrak{m}$ is infinite then for any minimal reduction J of I, $JI = I^2$.*

Proof. Since I^n is complete for all $n \geq 1$, and $o((I^n)^T) = n\, o(I^T)$, we have for all $n \geq 1$,

$$\lambda(R/I^n) = \sum_{R \leq T} \binom{n\, o(I^T)+1}{2} [T/\mathfrak{m}_T : R/\mathfrak{m}].$$

Hence $H(I,n) = P(I,n)$ for all $n \geq 1$. Writing this formula in the standard form we get

$$e_0(I) = \sum_{R \leq T} o(I^T)^2 [T/\mathfrak{m}_T : R/\mathfrak{m}] \ \text{ and } \ e_1(I) = \sum_{R \leq T} \binom{o(I^T)}{2} [T/\mathfrak{m}_T : R/\mathfrak{m}].$$

Hence $e_0(I) - e_1(I) = \lambda(R/I)$. It is well known that this condition implies (3), however we present a short proof. The H-D formula gives

$$e_0(I) = \lambda(R/I^2) - 2\lambda(R/I).$$

Let $J = (a,b)$ be a minimal reduction of I. Then we have $R/I \oplus R/I \simeq J/JI$. Hence $2\lambda(R/I) = \lambda(R/JI) - e_0(I)$. The last two formulae yield $I^2 = JI$. □

Example 6.5. Cutkosky [5] showed existence of a family of examples of normal ideals in a 3-dimensional regular local ring whose reduction number is not 2 and hence their Rees algebras are not Cohen-Macaulay. Huckaba and Huneke gave the following explicit example. Let k be a field of characteristic different from 3. Set $R = k[x, y, z]$ and $\mathfrak{m} = (x, y, z)$ and let

$$N = (x^4, x(y^3 + z^3), y(y^3 + z^3), z(y^3 + z^3)).$$

and put $I = N + \mathfrak{m}^5$. Then I is a height 3 normal ideal, $G(I^n)$ is not Cohen-Macaulay for any $n \in \mathbb{N}$ and $\overline{e}_3(I) \neq 0$.

Next we show that Zariski's Product Theorem can be deduced from the HD formula. Recall that for large r, s the normal Hilbert function of $\mathfrak{m}$ primary ideals I and J in a d-dimensional analytically unramified local ring $(R, \mathfrak{m})$ is defined [20] as $\overline{H}(r, s) = \lambda(R/\overline{I^r J^s})$. There is a polynomial $\overline{P}(x, y) \in \mathbb{Q}[x, y]$ of total degree d such that $\overline{P}(r, s) = \overline{H}(r, s)$ for all large r, s. For $d = 2$ the polynomial $\overline{P}(x, y)$ can be written as

$$\overline{P}(x, y) = e(I)\binom{x+1}{2} + xye(I \mid J) + e(J)\binom{y+1}{2} - ex - fy + g.$$

where $e, f, g \in \mathbb{Z}$. If $R/\mathfrak{m}$ is infinite then there exist $a \in I, b \in J$ such that $aJ + bI$ is a reduction of IJ and $e(I \mid J) = e(a, b)$. We say that (a, b) is a joint reduction of I, J.

Theorem 6.6. *Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring and let I, J be $\mathfrak{m}$-primary complete ideals. Then IJ is complete. Moroever $\overline{H}(r, s) = P(r, s)$ for all $r, s > 0$.*

Proof. We may assume without loss of generality that $R/\mathfrak{m}$ is infinite. Let a, b be a joint reduction of I, J. Then for all $r, s \geq 0$,

$$\overline{H}(r, s) = \sum_{R \preceq S} \binom{ro(I^S) + so(I^S) + 1}{2} d_S$$

where $d_S = [S/\mathfrak{m}_S : R/\mathfrak{m}]$. Rewriting the binomials in this formula in standard form we see that for all $r, s \geq 0$,

$$\overline{H}(r, s) = \lambda(R/I^r) + rse(I \mid J) + \lambda(R/J^s).$$

Hence $e(I \mid J) = \lambda(R/\overline{IJ}) - \lambda(R/I) - \lambda(R/J)$. The map

$$\phi : R/I \oplus R/J \longrightarrow \frac{(a, b)}{aJ + bI}$$

defined by $\phi(c', d') = (bc + ad)'$ for $c, d \in R$ is an R-module isomorphism. Therefore

$$\lambda(R/I) + \lambda(R/J) = \lambda(R/(aJ + bI)) - e(I \mid J) = \lambda(R/\overline{IJ}) - e(I \mid J).$$

It follows that $\overline{IJ} = aJ + bI$. As $aJ + bI \subseteq IJ$ we conclude that IJ is complete. $\square$

Generalizations of HD formula in higher dimensions

Since The HD formula has many consequences in dimension two regular local rings, it is natural to ask if there is a version in higher dimensions. The HD formula gives the multiplicity of an $\mathfrak{m}$-primary ideal I of a 2-dimensional regular local ring:

$$e(I) = \sum_{R \prec S} o(I^S)^2[S/\mathfrak{m}_S : R/\mathfrak{m}].$$

B. Johnston [14] proved an analogue of this formula for any finitely supported $\mathfrak{m}$-primary ideal I in regular local ring of dimension d :

$$e(I) = \sum_{R \prec S} o(I^S)^d[S/\mathfrak{m}_S : R/\mathfrak{m}].$$

Let us recall the notion of finitely supported ideal in a regular local ring which was introduced by Lipman in [17]. Let K be a field. Let the Greek letters $\alpha, \beta, \gamma, \ldots$ denote regular local rings of dimensions at least two with fraction field K. These will be called **points**. For any point α the unique maximal ideal and its order valuation will be denoted by of $\mathfrak{m}_\alpha$ and o_α. A **quadratic transform** of α is of the form $Q = \alpha[\mathfrak{m}_\alpha/x]_\mathfrak{p}$ where $x \in \mathfrak{m}_\alpha \setminus \mathfrak{m}_\alpha^2$ and $\mathfrak{p}$ is a prime ideal of $\alpha[\mathfrak{m}_\alpha/x]$ which contains $\mathfrak{m}_\alpha$. Then $\alpha \prec Q$.

Definition 6.7. *A point β is called* **infinitely near** α *if there is a sequence*

$$\alpha = \alpha_0 \prec \alpha_1 \prec \ldots \prec \alpha_n = \beta$$

of points in K so that each α_i is a quadratic transform of α_{i-1} for $i = 1, 2, \ldots, n$. If such a sequence exists, it is unique. We call it the quadratic sequence from α to β.

Abhyankar [1] proved that if α is 2-dimensional then such a sequence always exists. If $\dim \alpha \geq 3$ then one does not have such a structure theorem for points.

Definition 6.8. *Let α be a point and I be a nonzero ideal of α. A* **point basis** *of I is a family of nonnegative integers $\mathbf{B(I)} = \{\mathbf{o(I^\beta)}\}_{\alpha \prec \beta}\}$. We say that β is a* **base point** *of I if $o(I^\beta) > 0$. The ideal I is called* **finitely supported** *if it has finitely many base points.*

Clare D'Cruz [6] considered the problem of identification of finitely supported complete ideals in regular local rings whose product may be complete. This turns out to be connected to higher dimensional version of the HD formula. She showed [7] that in any

regular local ring $(R, \mathfrak{m})$ of of dimension atleast 2, and an $\mathfrak{m}$-primary complete ideal I, $\mu(I) \geq \binom{o(I)+d-1}{d-1}$. If equality holds then $I\mathfrak{m}^n$ is complete for all n. This in turn implies that the quadratic transforms of such complete ideals are again complete.
The following theorem of D'Cruz emphasized the condition about $\mu(I)$.

Theorem 6.9 (D'Cruz). *Let $(R, \mathfrak{m})$ be a regular local ring of dimension ≥ 3 with $R/\mathfrak{m}$ algebraically closed. Let I be a complete $\mathfrak{m}$-primary monomial ideal of R. Then $I\mathfrak{m}$ is complete if and only if $\mu(I) = \binom{o(I)+d-1}{d-1}$.*

D'Cruz proved the above result via the HD formula for finitely supported $\mathfrak{m}$-primary complete ideals. M. Lejeune Jalabert [16] also proved a version of HD formula in dimension 3. This generalization in higher dimensions involves lengths of right derived functors of direct images of certain sheaves. We refer the reader to [7] for further details. In case I is a complete finitely supported $\mathfrak{m}$-primary monomial ideal then we have

$$\lambda(R/I) = \sum_{R \prec T} \binom{o(I^T) + d - 1}{d} [T/\mathfrak{m}_T : R/\mathfrak{m}].$$

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