International Workshop on Probability Theory and Related Fields

Time: February 26th – March 1st, 2019
Venue: Vietnam Institute for Advanced Studies in Mathematics
7th floor, Ta Quang Buu Library, No. 1 Dai Co Viet, Hanoi

Program

26 Feb (Tue)	27 Feb (Wed)	28 Feb (Thu)	1 Mar (Fri)
9:00 – 9:50 Registration/Coffee	9:30 – 10:20 Suito	9:30 – 10:20 Uchitane	9:30 – 10:20 Long
9:50 – 10:00 Welcoming	10:20 – 10:50 Hong	10:20 – 11:10 Viet	10:20 – 10:50 Yang
10:00 – 10:50 Nakano	10:50 – 11:10 Coffee break	Free discussions (afternoon)	10:50 – 11:10 Coffee break
10:50 – 11:40 Linh	11:10 – 11:40 Huong		11:10 – 12:00 Ton
11:40 – 13:30 Photo & Lunch	11:40 – 13:30 Lunch		12:00 – Closing
13:30 – 14:20 Anh	13:30 – 14:20 Yagi		
14:20 – 15:10 Duy	14:20 – 14:50 Tuong		
15:10 – 15:40 Coffee Break	14:50 – 15:20 Coffee Break		
15:40 – 16:30 Hung	15:20 – 16:10 Dieu		
	18:00 – 20:00 Banquet		

Organizers: Nguyen Huu Du, Trinh Khanh Duy, Le Minh Ha,
Nguyen Trong Hieu, Ngo Hoang Long, Ta Viet Ton
Contact: T.K.D.: trinh(@)m.tohoku.ac.jp; N.T.H.: hieunguyentrong(@)gmail.com
N.H.L.: ngolong(@)hnue.edu.vn; T.V.T. tavietton(@)agr.kyushu-u.ac.jp

Program

Tueday, 26 Feb.

9:00 – 9:50	Registration/Coffee
10:00 – 10:50	Fumihiko Nakano (Gakushuin University, Japan) Asymptotic behavior of eigenfunctions for 1D random Schrödinger operators
10:50 – 11:40	Linh Thi Hoai Nguyen (Kyushu University, Japan) Bounded Confidence Gossip Algorithms for Opinion Formation and Application to Multidimensional Data Clustering
13:30 – 14:20	Cung The Anh (Hanoi National University of Education, Vietnam) Continuous data assimilation for Navier-Stokes-Voigt equations with stochastically noisy data
14:20 – 15:10	Khanh Duy Trinh (Tohoku University, Japan) Betti numbers of random geometric complexes in the thermodynamic regime
15:40 – 16:30	Pham Viet Hung (Institute of Mathematics, Vietnam) Distribution of the maximum of stationary Gaussian fields

Wednesday, 27 Feb.

9:30 – 10:20	Hiroshi Suito (Tohoku University, Japan) Geometrical characteristics study for thoracic diseases
10:20 – 10:50	Phan Thanh Hong (Thang Long University, Vietnam) Asymptotic stability for stochastic dissipative systems with a Hölder noise
11:10 – 11:40	Le Dieu Huong (Pharos Digital Books Company) Spatial meta-population dynamics as a point pattern dynamics
13:30 – 14:20	Atsushi Yagi (Osaka University, Osaka, Japan) Maximal Regularity of Parabolic Evolution Equations and Stochastic Problems
14:20 – 14:50	Tran Dinh Tuong (Ho Chi Minh University of Transport, Vietnam) On the asymptotic behavior of a stochastic model with Ivlev's function response and jumps
15:20 – 16:10	Nguyen Thanh Dieu (Vinh University, Vietnam) Long-time behaviour of epidemic models in random environment

Thusday, 26 Feb.

9:30 – 10:20	Takeshi Uchitane (Kobe University, Japan) Evolutionary Design of Experiments to Find Significant Factors for Tsunami Evacuation Simulation
10:20 – 11:10	Quang Huy Viet Huynh (Tohoku University, Japan) Multi-GPU accelerated matrix solver based on the BiCGSafe algorithm and its application to simulation of blood flows in the aorta

Friday, 1 Mar.

9:30 – 10:20	Hoang-Long Ngo (Hanoi National University of Education, Vietnam) Parameter estimation for a discretely observed non-smooth threshold diffusion model
10:20 – 10:50	Jian Yang (Osaka University, Japan) Segregation Patterns of Tree-grass Competition System
11:10 – 12:00	Ton Viet Ta (Kyushu University, Japan) Mathematical Models for Swarm Behavior
12:00 –	Closing

Asymptotic behavior of eigenfunctions for 1D random Schrödinger operators

Fumihiko NAKANO
Gakushuin University, Japan

We consider the 1D random Schrödinger operator of the following type:

$$H_L := -\frac{d^2}{dt^2} + L^{-\alpha}F(X_t) \quad \text{on } L^2(0,L) \quad \text{with Dirichet b.c., } \alpha > 0$$

where $F \in C^\infty(M)$, M : torus such that $\int_M F(x)dx = 0$, and $\{B_t\}$ is the Brownian motion of M. Since the localization length of H_L is in the order of $L^{2\alpha}$, we expect the transition between the extended and localized states at $\alpha = \frac{1}{2}$. Level statistics problem, that is, to study the local distribution of eigenvalues of H_L, has been carried out in [N14, Kotani-N17]. In this talk, we discuss the scaling limit of the corresponding eigenfunctions.

Bounded Confidence Gossip Algorithms for Opinion Formation and Application to Multidimensional Data Clustering

Linh Thi Hoai NGUYEN
Kyushu University, Japan

In this talk, a bounded confidence gossip algorithm for describing the process of opinion forming over a communication network is presented. Each agent in the network keeps a time-varying opinion vector (or state) which represents its opinion about a set of matters. A common confidence threshold is set for all of the agents.

The states of agents in the network will be updated time by time according to an iterative procedure: At each time, (i) one agent is chosen randomly, it then chooses one of its neighbor on the communication graph to contact with, (ii)they exchange their states, and (iii) if they have different states and the distance between their states is strictly smaller than the confidence threshold, they update their states as the average of the two.

This algorithm converges almost surely to some equilibrium point such that any two adjacent agents either have the same state or have distinct states which differ at least the confidence threshold from each other. An increasing confidence threshold algorithm, which repeats the constant confidence threshold algorithm several times with increasing confidence threshold, is also proposed. The algorithm is also convergent almost surely to some equilibrium point. It performs as a hierarchical clustering method.

Since the proposed algorithms reveal clustering phenomenon of opinion formation, they have application to multi-dimensional data clustering. Two clustering algorithms are proposed: Clustering and Consensus Clustering. The first one is based directly on the constant confidence threshold algorithm. The latter considers the robustness of derived clustering by performing multiple runs of the Clustering algorithm. The application of the proposed algorithms is presented through some numerical examples.

Continuous data assimilation for Navier-Stokes-Voigt equations with stochastically noisy data

Cung The ANH
Department of Mathematics, Hanoi National University of Education, Vietnam
Email: anhctmath@hnue.edu.vn

We consider the continuous data assimilation for three-dimensional Navier-Stokes-Voigt equations, where observational measurements are given by finite volume elements or nodal points of the velocity field and measurement errors are represented by stochastic noise. First, we show that the stochastic data assimilation equations are well-posed. Then we provide explicit conditions on the observation density (resolution) which guarantee explicit asymptotic bounds, as the time tends to infinity, on the error between the approximate solution and the actual solution which is corresponding to these measurements, in terms of the variance of the noise in the measurements.

Betti numbers of random geometric complexes in the thermodynamic regime

Khanh Duy TRINH
Tohoku University, Japan

Given points $\mathcal{X}_n = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^d$ and a radius $r > 0$, the Čech complex $\mathcal{C}(\mathcal{X}_n, r)$ and the Rips complex $\mathcal{R}(\mathcal{X}_n, r)$ are defined to be abstract simplicial complexes

$$\mathcal{C}(\mathcal{X}_n, r) = \left\{ \emptyset \neq \sigma \subset \mathcal{X}_n : \bigcap_{x \in \sigma} B_r(x) \neq \emptyset \right\},$$
$$\mathcal{R}(\mathcal{X}_n, r) = \left\{ \emptyset \neq \sigma \subset \mathcal{X}_n : B_r(x) \cap B_r(y) \neq \emptyset \text{ for all } x, y \in \sigma \right\}.$$

Here $B_r(x) = \{y \in \mathbb{R}^d : \|x - y\| \leq r\}$ denotes the closed ball of radius r centered at x with respect to the Euclidean norm $\|\cdot\|$. They are typical examples of geometric complexes. When the points $\mathcal{X}_n$ are random, one may be interested in studying the limiting behavior of the random Čech complex $\mathcal{C}(\mathcal{X}_n, r_n)$ (or the Rips complex $\mathcal{R}(\mathcal{X}_n, r_n)$) through their Betti numbers, where the radius $r_n = r(n)$ is a deterministic sequence. The randomness may be taken from an iid sequence of $\mathbb{R}^d$-valued random variables having density with respect to the Lebesgue measure on $\mathbb{R}^d$. In this case, according to the limit of $\Lambda_n = n^{1/d} r_n$, zero, finite or infinite, random geometric complexes behave quite differently. This talk aims to give a brief survey on this topic with emphasizing on the middle regime, called the thermodynamic regime or the critical regime.

Distribution of the maximum of stationary Gaussian fields

Pham Viet HUNG
Institute of Mathematics, Vietnam Academy of Science and Technology, Vietnam

In this talk, we present the relation between the asymptotic formula of the tail distribution of the maximum of Gaussian fields and the geometric functionals of the domain such as: Euler-Poincare characteristic, volume, area surface,.... We recall the results given by Piterbarg, Sun, Adler and Taylor, Takemura and Kuriki, Azais and Wschebor and also discuss some recent results. Some extensions for the conjunction probability is also presented.

Geometrical characteristics study for thoracic diseases

Hiroshi SUITO and Quang Huy Viet HUYNH
Advanced Institute for Materials Research, Tohoku University, Japan

In this talk, we present computational analyses for blood flows in the human aorta. Aortic aneurysm and aortic dissections persist as life-threatening hazards. In recent years, patient-specific simulations are common in biomedical engineering. Although they are extremely useful for grasping the flow/stress distributions and for patient-specific treatment planning, they remain insufficient to elucidate the general mechanisms of a targeted disease. We introduce a geometrical characterization of blood vessels, which vary widely among individuals. Differences in the vessel morphology can produce different flow characteristics, stress distributions, and ultimately different outcomes. Therefore, the characterization of the morphologies of these vessels poses an important clinical question. Through close collaboration between mathematical science and clinical medicine, these analyses yield greater understanding leading to better risk assessments.

References

[1] T. Ueda, H. Suito, H. Ota and K. Takase, "Computational fluid dynamics modeling in aortic diseases", Cardiovascular Imaging Asia, Vol. 2, No. 2, pp. 58–64, 2018.

[2] H. Suito, K. Takizawa, V. Huynh, D. Sze and T. Ueda, "FSI analysis of the blood flow and geometrical characteristics in the thoracic aorta", Computational mechanics, Vol. 54, No. 4, pp. 1035–1045, 2014.

Asymptotic stability for stochastic dissipative systems with a Hölder noise

Phan Thanh HONG
Thang Long University, Vietnam

We prove the exponential stability of the zero solution of a stochastic differential equation with a Hölder noise, under the strong dissipativity assumption. As a result, we also prove that there exists a random pullback attractor for a stochastic system under a multiplicative fractional Brownian noise.
(Joint work with Nguyen Dinh Cong and Luu Hoang Duc)

Spatial meta-population dynamics as a point pattern dynamics

Le Dieu HUONG
Pharos Digital Books Company
(Join work with Nguyen Hong NHUNG, Nguyen Trong HIEU and Fugo TAKASU)

We are interested in the meta-population model as a stochastic point pattern dynamics with special focus on its spatial distribution at equilibrium, each point on a continuous space is either "empty" or "occupied". The colonization occurs with a colonization kernel as a function of distance from "occupied" to "empty". This stochastic process has been mathematically described by the hierarchical dynamics of the probabilities that a point, a pair made by two points, and a triplet made by three points, etc., is in a specific configuration of status. The talk will show the advantage of point pattern approach to model spatial meta-population dynamics when they have global and local interactions among individuals likely depend on density and distance between them.

Maximal Regularity of Parabolic Evolution Equations and Stochastic Problems

Atsushi YAGI
Osaka University, Osaka, Japan
Email: yagi-atsushi-ch@alumni.osaka-u.ac.jp

Keywords: parabolic evolution equation, maximal regularity, stochastic problems

Consider the Cauchy problem for an abstract linear evolution equation

$$\frac{du}{dt} + A(t)u = f(t), \quad 0 < t \le T; \qquad u(0) = u_0$$

in a Banach space X. Here, $A(t)\,(0 \le t \le T)$ are sectorial operators of X of angles $< \frac{\pi}{2}$ having bounded inverses.

We are concerned with the strict solutions, i.e., $u \in \mathcal{C}([0,T];X) \cap \mathcal{C}^1((0,T];X)$, and interested in maximal regularity of the correspondence $[u_0, f(t)] \mapsto \frac{du}{dt}(t)$. Assume that the operators $A(t)$ satisfy one of the regularity conditions: (I) $\|A(t)^\nu[A(t)^{-1} - A(s)^{-1}\|_{\mathcal{L}(X)} \le C|t-s|^\mu$ with some exponents $0 < \mu,\ \nu \le 1$ such that $\mu + \nu > 1$; or (II) $\left\|A(t)^\nu \frac{dA(t)^{-1}}{dt}\right\|_{\mathcal{L}(X)} \le C$ with some exponent $0 < \nu \le 1$. Then, for exponents $0 < \sigma < \beta \le 1$, the mapping

$$S : [u_0, f(t)] \mapsto \tfrac{du}{dt}(t), \quad S : \mathcal{D}(A(0)^\beta) \times \mathcal{F}^{\beta,\sigma}((0,T];X) \longrightarrow \mathcal{F}^{\beta,\sigma}((0,T];X)$$

is a bounded multilinear operator. Here, $\mathcal{F}^{\beta,\sigma}((0,T];X)$ is a specific space of weighted Hölder continuous functions on $(0,T]$ with values in X which is set up in view of the properties that the functions $A(0)e^{-\tau A(0)}u_0\,(0 < \tau \le T)$, where $u_0 \in \mathcal{D}(A(0)^\beta)$, enjoy, for the details see [1, Chapter 3] and [2].

These facts in turn indicate that the class of strict solutions is one of the optimal solution spaces in order to handle the nonlinear parabolic evolution equation

$$\frac{du}{dt} + A(t,u)u = f(t,u), \quad 0 < t \le T; \qquad u(0) = u_0.$$

This talk will be addressed for reporting some recent results on stochastic parabolic evolution equations ([3]) and for presenting some nonlinear problems arising in Mathematical Biology.

References

[1] A. Y: *Abstract parabolic evolution equations and their applications*, Springer, 2010.

[2] A. Y: Maximal regularity of Hölder type for abstract parabolic evolution equations, *Funkcial. Ekvac.*, Vol. 55(2012), 405–428.

[3] T. V. Ta, A. Y., Y. Yamamoto: Strict solutions to stochastic parabolic evolution equations in M-Type 2 Banach spaces, *Funkcial. Ekvac.*, Vol. 61(2018), 191–217.

On the asymptotic behavior of a stochastic model with Ivlev's function response and jumps

Tran Dinh TUONG
Ho Chi Minh University of Transport, Vietnam
(Joint work with Nguyen Huu DU and Nguyen Thanh DIEU)

This work is concerned with long-time behavior of a stochastic predator-prey model with Ivlev effected by both white noise and Lévy jumps. We firstly focus on conditions for the existence of the ergodic stationary distribution to the logistic equation. Next, by introducing a threshold λ, sufficient and almost necessary conditions for the permanence as well as ergodicity and extinction are investigated: when $\lambda < 0$, the species converge to the population on the boundary in exponential rate. If $\lambda > 0$, there exists unique invariant probability measure concentrated on an appropriate space. Finally, a numerical solution to illustrate our results is presented.

Keywords: Ivlev-type; extinction; permanence; stationary distribution; ergodicity; jumps

Long-time behaviour of epidemic models in random environment

Nguyen Thanh DIEU
Department of Mathematics, Vinh University, 182 Le Duan, Vinh, Nghe An, Vietnam
Email: dieunguyen2008@gmail.com

In this talk, we focus on asymptotic behavior of a stochastic SIR epidemic model represented by a system of stochastic differential equations perturbed by white noise, markov switching and Lévy jump. The talk provides sufficient conditions that are very close to necessary for the permanence. In addition, our work develops ergodicity of the underlying system. We also characterize the support of a unique invariant probability measure and prove the convergence in total variation norm of transition probability to the invariant measures. In special case, rates of convergence are also ascertained. It is shown that the rate is not too far from exponential in that the convergence speed is of the form of a polynomial of any degree.

Evolutionary Design of Experiments to Find Significant Factors for Tsunami Evacuation Simulation

Takeshi UCHITANE
Kobe University, Japan

Gathering and leaning social data get match attention to realize smart and resilient societies. Such data includes electric power, human mobility and economic money flows which are surely associated with daily human life. Under some special events like a natural disaster, on the other hand, such data is not enough to estimate the result of the events, because such data can be obtained only in the usual time. Therefore, a numerical simulation with a dynamical social model is also required to estimate the result of special events.

Such a numerical simulation has various and many input variables associated not only with physical phenomena but also with human?s decision making. Moreover, it is required to consider uncontrollable and probabilistic many events in the physical and social phenomena. Thus, it is very difficult to run simulations and to find better and worse conditions because the number of conditions which should be concerned exponentially increase as the number of input variables and considerable events get larger. Therefore, such a comprehensive search is a target of super computer applications, although the estimation results are strongly required for each local government and for each citizen.

A practical goal of the social problem like an evacuation simulation from earth quake trigged Tsunami is to find more sensitive factors which strongly affect the evacuation time. It may be possible to find such factors within a smaller number of simulation-runs when change values of specific input variables less affect the simulation results. Therefore, a probabilistic search algorithm named "Evolutionary Design of Experiments" was built so as to find all significant input variables as if the algorithm was an evolutionary computation algorithm to find a optimal solution.

In the workshop, 1) a problem of Tsunami evacuation simulation to find significant factors, 2) the basic mechanism of evolutionary computation as an optimization algorithm and 3) the stochastic search algorithm to reduce simulation executions and to find almost risks of Tsunami disaster, are introduced.

Multi-GPU accelerated matrix solver based on the BiCGSafe algorithm and its application to simulation of blood flows in the aorta

Quang Huy Viet HUYNH* Koki OTERA**
Hiroshi SUITO *
*Advanced Institute for Materials Research, Tohoku University, Japan
**Graduate School of Environmental and Life Science, Okayama University, Japan

Recently, GPU (graphics processing unit) computing has been recognized as a powerful platform to achieve high performance in numerical simulation. GPU computing is the use of GPUs in association with the use of CPU to speed up computations. Numerical simulations by using GPUs to achieve high performance are receiving much attention at present. Simulation of blood flow by using numerical methods has become an emerging research area. The understanding of flow patterns of blood flow is useful in the prevention, diagnosis, and treatment of cardiovascular disease. On numerical simulation of blood flow by finite element methods, it is necessary to solve large scale systems of linear equations. Conventionally, these large scale linear systems can be solved by using iterative methods such as BiCGStab or GPBiCG algorithms. In recent times, a new BiCGSafe algorithm has been proposed, and its convergence behavior has been shown to be superior to that of the GPBiCG algorithm. In this talk, we present an effective implementation of the BiCGSafe algorithm, in which single or multiple GPUs are used in order to achieve high speedups of execution time of large scale linear system solvers, and its application to simulation of blood flows in the aorta.

Keywords: Simulation of Blood Flow, Finite Element Method,
High Performance Computing, GPU Computing.

Parameter estimation for a discretely observed non-smooth threshold diffusion model

Arturo Kohatsu-Higa, Hoang-Long NGO and Ngoc Khue TRAN
Ritsumiekan University, Japan
Hanoi National University of Education, Vietnam
Pham Van Dong University, Vietnam

Threshold diffusions are regime switching continuous time models where the model switches between two different behaviors as the state variable crosses a threshold value. We consider the statistical estimation of this threshold value for a discretely observed one-dimensional diffusion process. Specifically, in our situation, the drift coefficient of the diffusion process is discontinuous in the threshold parameter which introduces technical problems. We present the asymptotic properties of a moment estimator and a quasi-likelihood estimator. The second estimator performs better than the first but suffers from lack of regularity. In order to solve this problem, we propose a regularization and its asymptotic properties are studied.

Segregation Patterns of Tree-grass Competition System

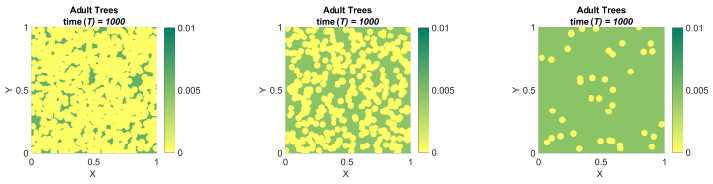
Jian YANG
Osaka University, Japan

We proposed a tree-grass competition model for the kinematics of forest together with grassland by modifying the classical age-structured continuous model which was by Kuznetsov-Antonovsky-Biktashev-Aponina [1] in 1994. We simplified our model to equations shown by (1),

$$\begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta w - \varphi(v)u - fu & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = d_w \Delta w - \beta w + \alpha v & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & \text{in} \quad \Omega, \end{cases} \tag{1}$$

in which $\varphi(v)$ is just a square function of the form $\varphi(v) = a'(v-b')^2 + c'$.

This simplified model coincides with the classical model. As for mathematical analysis on (1), we have some known analytical results on the classical model[2]. We conducted numerical experiments on (1) [3]. With adjusting h, which stands for the mortality of adult trees, we found three different kinds of tree-grass segregation patterns as inhomogeneous stabilized stationary solutions to (1), namely, low-connectivity forests, intermediate-connectivity forests and high-connectivity forests, as shown by Figure 1.



(a) (b) (c)

Figure 1: Three tree-grass segregation patterns.

The three tree-grass segregation patterns differ in two aspects, namely, forest connectivity level and tree-area ratio. We demonstrated that forest connectivity level and tree-area ratio are two individual factors which are influencing the tree-grass dynamics. In addition, we found a clear correlation between the three different segregation patterns and the values of the mortality of adult trees, as Figure 2 shows.

Segregation patterns \ Value of h	0.0019	0.018	0.0335
High-connectivity forests			O
Intermediate-connectivity forests		O	
Low-connectivity forests	O		

Figure 2: A correlation between h and the segregation patterns.

As widely observed, forests in the real world are unavoidably heavily influenced by stochastic disturbances. Considering this, turning the mortality, h, from a fixed constant to a variable perturbed by stochastic disturbances remains to be an important work for the future development on our tree-grass competition system. Referring to the work by T. V. Ta,

L. T. H. Nguyen and A. Yagi [4], in which a stochastic forest model is proposed and mathematically studied, we want to do the same substitution on h in (1), which is shown by (2),

$$h \to h - \sigma dw_t, \tag{2}$$

where $\{\omega_t, t \geq 0\}$ is a one-dimensional Brownian motion, and σdw_t stands for gaussian white noises.

References

[1] Yu A. Kuznetsov, M. Ya. Antonovsky, V. N. Biktashev and A. Aponina, *A cross-diffusion model of forest boundary dynamics*, J. Math. Biol. **32** (1994), 219–232.

[2] L. H. Chuan, T. Tsujikawa and A. Yagi, *Stationary solutions to forest kinematic model*, Glasg. Math. J. **51**(2009), 1–17.

[3] J. Yang and A. Yagi, *Tree-grass segregation patterns*, accepted for publication.

[4] T. V. Ta, L. T. H. Nguyen, A. Yagi, *A sustainability condition for stochastic forest model*, Comm. Pu. Appl. An. **16**(2017), 699–718.

Mathematical Models for Swarm Behavior

Ton Viet TA
Center for Promotion of International Education and Research, Kyushu University, Japan

Swarm behavior consisting of a large number of individuals often surprises us. They move coherently, matching their velocity without collision and maintaining a constant scale of school, even though they have only moderate ability of information processing and of execution of programming.

Several mathematical models for swarming have been presented on the basis of experimental results concerning interactions between nearby mates which are rather simple. In this talk, I will present our recent work on swarm behavior including

a) Mathematical models of stochastic (partial) differential equations using local rules of individuals in swarm (e.g., repulsion, attraction, alignment, and reaction to the environment)

b) The effects of noise on swarm behavior

c) Numerical simulations

Credit: Atsushi Yagi