Variational Sets of Perturbation Maps and Applications to Sensitivity Analysis for Constrained Vector Optimization

N. L. H. Anh · P. Q. Khanh

Abstract We consider sensitivity analysis in terms of variational sets for nonsmooth vector optimization. First, relations between variational sets, or their minima/weak minima, of a set-valued map and that of its profile map are obtained. Second, given an objective map, relationships between the above sets of this objective map and that of the perturbation map and weak perturbation map are established. Finally, applications to constrained vector optimization are given. Many examples are provided to illustrate the essentialness of the imposed assumptions and some advantages of our results.

Keywords Nonsmooth vector optimization · Sensitivity analysis · Perturbation maps · Weak perturbation maps · Variational sets · Singular variational sets

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1 Introduction

Stability and sensitivity analysis are of great importance for optimization, as well as for other fields of applied mathematics, from both the theoretical and practical view points. As usual, stability is understood as a qualitative analysis, which concerns mainly studies of various continuity (or semicontinuity) properties of solution maps and optimal-value maps. Sensitivity means a quantitative analysis, i.e., studies of kinds of derivatives of the mentioned maps. For sensitivity results in nonlinear programming using classical derivatives, see, e.g., [1]. However, practical optimization problems are often nonsmooth. To cope this crucial difficulty, most of approaches to studies of optimality conditions and sensitivity analysis are based on generalized derivatives. Nowadays, set-valued maps (or multimaps) are involved frequently in optimization-related models. In particular, for vector optimization, both perturbation and solution maps are set-valued. One of the first and most important derivatives of a multimap is the contingent derivative. In [2-8], behaviors of perturbation maps for vector optimization were investigated quantitatively by making use of contingent derivatives. Higher-order sensitivity analysis was studied in [9, 13], applying kinds contingent derivatives. To the best of our knowledge, no other kinds of generalized derivatives have been used in contributions to this topic, while so many notions of generalzed differentiability have been introduced and applied effectively in investigations of optimality conditions, see

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excellent books [14-18]. We mention in more detail only several recent papers on generalized derivatives of set-valued maps and optimality conditions. In [19, 20], optimality conditions for set-valued vector optimization were discussed in terms of contingent derivatives. Variants of contingent epidderivatives were developed with applications in optimality conditions for several types of solutions in [21-25]. In [26, 27], lower and upper Dini derivatives were used for a similar purpose. In [28, 29], radial epiderivatives were the generalized differentiability for getting optimality conditions. Variants of higher-order radial derivatives for establishing higher-order conditions were proposed in [31, 32]. In [33] the higher-order Neustadt derivative was employed to extend the classical Dubovitski-Milyutin scheme. The higher-order lower Hadamard directional derivative was the tool for set-valued vector optimization in [34]. In [10, 11], higher-order variational sets of a multimap were proposed in dealing with optimality conditions for set-valued optimization. Calculus rules for variational sets were established in [12] to ensure the applicability of variational sets. We can expect that many generalized derivatives, besides the contingent ones, can be employed effectively in sensitivity analysis. However, only few generalized derivatives admit extentions to higher orders. Here we choose variational sets for higher-order considerations of perturbation maps, since some advantages of this generalized differentiability were shown in [11-13], e.g., almost no assumptions are required for variational sets to exist (to be nonempty); direct calculating these sets is simply a computation of a set limit; extentions to higher orders are direct; they are bigger than corresponding sets of most derivatives (this property is decisively advantageous in establishing necessary optimality conditions by separation techniques), etc.

The aim of this paper is to study properties of perturbation maps, in terms of higher-order variational sets. Regarding solutions of vector optimization, we restrict ourselves to the two basic notions of (Pareto) minima and weak minima. Correspondingly, our concern is to deal with perturbation maps and weak perturbation maps. We employ variational sets in both assumptions and conclusions of our results. We also show cases where our results can be employed but some existing results cannot. Examples are provided to ensure the essentialness of each imposed assumption.

The plan of the paper is as follows. Some preliminary facts are given in Sect. 2 for our later use. In Sect. 3, we prove relations between a variational set of a multifunction or the minima/weak minima of this set and that of the corresponding profile multifunction. The obtained results are employed in Sect. 4 to get relationships between the variational sets of a perturbation map or weak perturbation map or the minima/weak minima of these sets and the corresponding ones of the feasible-set map to the objective space. Sect. 5 is devoted to applications of the above sensitivity analysis to a set-constrained vector optimization problem. In this case we go further to have an analysis including variational sets of the constrained-set map into the decision space. Concluding observations are included in Sect. 6.

2 Preliminaries

Let X and Y be normed spaces, $A \subseteq Y$, B_Y the closed unit ball in Y, and $S = \{y \in Y : ||y|| = 1\}$. By int$A$ and cl$A$ we denote the interior and closure of A. $\mathbb{N}$, $\mathbb{R}^k$, and $\mathbb{R}^k_+$ stand for the set of natural numbers, the k-dimensional space and its nonnegative orthant, respectively (resp). We often use the following notations: cone$A = \{\lambda a : \ \lambda \geq 0, \ a \in A\}$, cone$_+A = \{\lambda a : \ \lambda > 0, \ a \in A\}$. A nonempty subset Q of a cone K is called a base of K iff $K = \text{cone}Q$ and $0 \notin \text{cl}Q$. It is easy to see that if K has a convex base, then K is convex and pointed. For $A \subseteq Y$, $\hat{y} \in A$ is said to be a (Pareto) minimum of A (with respect to, shortly wrt, the ordering cone K) iff $(A - \hat{y}) \cap (-K) \subseteq K$. When int$K \neq \emptyset$, $\hat{y} \in A$ is said to be a weak minimum of A iff $(A - \hat{y}) \cap (-\text{int}K) = \emptyset$. Let $\text{Min}_K A$ ($\text{WMin}_K A$) stand for the set of all minima (weak minima) of A.

A subset A of Y is said to have the domination property iff $A \subseteq \text{Min}_K A + K$. Similarly, when int$K \neq \emptyset$, we say that A has the weak domination property iff $A \subseteq \text{WMin}_K A + (\text{int}K \cup \{0\})$.

For a set-valued map $H : X \rightrightarrows Y$, the domain, graph and epigraph of H are denoted by dom$H$, gr$H$, and epi$H$, resp. $H + K$ is called the profile mapping of H. H is said to be calm around $x_0 \in \text{dom}H$ iff $\exists V$ (neighborhood of x_0), $\exists M > 0$, $\forall x \in V$, $H(x) \subseteq H(x_0) + M||x - x_0||B_Y$. The Kuratowski-Painlevé upper limit (lower limit, resp) of H at x_0 is defined by

$$\operatorname*{Limsup}_{x \xrightarrow{H} x_0} H(x) = \{y \in Y :\ \exists x_n \in \mathrm{dom}H :\ x_n \to x_0, \exists y_n \in H(x_n),\ y_n \to y\}$$

$$(\operatorname*{Liminf}_{x \xrightarrow{H} x_0} H(x) = \{y \in Y :\ \forall x_n \in \mathrm{dom}H :\ x_n \to x_0, \exists y_n \in H(x_n),\ y_n \to y\}),$$

where $x \xrightarrow{H} x_0$ means that $x \in \mathrm{dom}H$ and $x \to x_0$. If we have $\operatorname{Limsup}_{x \xrightarrow{H} x_0} H(x) = \operatorname{Liminf}_{x \xrightarrow{H} x_0} H(x)$, then this value is called the (Kuratowski-Painlevé) limit of H at x_0 and denoted by $\operatorname{Lim}_{x \xrightarrow{H} x_0} H(x)$.

3 Variational Sets of Set-Valued Maps

In this section, let X and Y be normed spaces, $K \subseteq Y$ a closed convex cone and $F : X \rightrightarrows Y$. We recall first the concept of variational sets of set-valued maps and establish some results on the relationship between variational sets of $F + K$ and F.

Definition 3.1 Let $(x_0, y_0) \in \mathrm{gr}F$, $v_1, \cdots, v_{m-1} \in Y$, and $m \in \mathbb{N}$.

(i) ([10, 11]) The mth-order variational set of type 1 (type 2, resp) of F at (x_0, y_0) (relative to $v_1, \cdots, v_{m-1}$) is

$$V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) = \operatorname*{Limsup}_{x \xrightarrow{F} x_0,\ t \to 0^+} \frac{1}{t^m}(F(x) - y_0 - tv_1 - \cdots - t^{m-1}v_{m-1})$$

$$(W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) = \operatorname*{Limsup}_{x \xrightarrow{F} x_0\ t \to 0^+} \frac{1}{t^{m-1}}(\mathrm{cone}_+(F(x) - y_0) - v_1 - \cdots - t^{m-2}v_{m-1})).$$

(ii) If the upper limit in (i) is equal to the lower one, then this limit is called a proto-variational set of order m of type 1 of F at (x_0, y_0). Similar terminology is defined for type 2.

We can easily prove the following formulas for V^m and W^m:

$$\begin{aligned} V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) &= \{y \in Y | \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists v_n \to y, \forall n, \\ &\quad y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_n)\}, \\ W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) &= \{y \in Y | \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists v_n \to y, \forall n, \\ &\quad v_1 + ... + t_n^{m-2} v_{m-1} + t_n^{m-1} v_n \in \mathrm{cone}_+(F(x_n) - y_0)\}. \end{aligned}$$

Remark 3.1 Recall that the mth-order contingent derivative of F at (x_0, y_0) (relative to $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$) is the map $D^m F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1}) : X \rightrightarrows Y$ defined by

$$D^m F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(u) = \operatorname*{Limsup}_{u' \to u, t \to 0^+} \frac{1}{t^m}(F(x_0 + tu_1 + \cdots + t^{m-1}u_{m-1} + t^m u') - y_0 - tv_1 - \cdots - t^{m-1}v_{m-1})$$

We can say roughly that the contingent derivative is a directional variant of variational set V^m. Similarly, most of generalized derivatives (e.g., the (upper) Dini derivative, Hadamard derivative, adjacent derivative, etc) are also based on directional rates, while for the variational sets we allow the flexibility $x_n \xrightarrow{F} x_0$. That is why these sets are big:

$$D^m F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})X \subseteq V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}).$$

The first simple result is about a relation between the variational sets of the two maps F and $F + K$.

Proposition 3.1

(i) $V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + K \subseteq V^m(F + K, x_0, y_0, v_1, \cdots, v_{m-1})$;

(ii) $W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + K \subseteq W^m(F + K, x_0, y_0, v_1, \cdots, v_{m-1})$.

Proof By the similarity we present only a proof for (ii). Let $y \in W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + K$, i.e., there exist $v \in W^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$ and $k \in K$ such that $y = v + k$. Then, there are $t_n \to 0^+$ and $x_n \xrightarrow{F} x_0$ such that $h_n(v_1 + \cdots + t_n^{m-2} v_{m-1} + t_n^{m-1}(v_n + k)) \in F(x_n) + K - y_0$.
So, $v + k \in W^m(F + K, x_0, y_0, v_1, \cdots, v_{m-1})$ and the proof is complete. □

The inclusions opposite to those in Proposition 3.1 may not hold as the following example shows.

Example 3.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $(x_0, y_0) = (0, (0, 0))$, and $F(x)$ is equal to $\{(0, 0)\}$ if $x = 0$, and to $\{(-1, -1)\}$ if $x \neq 0$. Then, we have $V^1(F, x_0, y_0) = \{(0, 0)\}$ and $V^1(F + K, x_0, y_0) = \mathbb{R}^2$. Thus, $V^1(F+K, x_0, y_0) \not\subseteq V^1(F, x_0, y_0)+K$. Let $v_1 = (0, 1) \in V^1(F+K, x_0, y_0)$. Then, $V^2(F+K, x_0, y_0, v_1) \neq \emptyset$ and $V^2(F, x_0, y_0, v_1) = \emptyset$. Consequently, $V^2(F + K, x_0, y_0, v_1) \not\subseteq V^2(F, x_0, y_0, v_1) + K$.

For variational sets of type 2 one has $W^1(F, x_0, y_0) + K = \mathbb{R}^2 = W^1(F + K, x_0, y_0)$, and $v_1 \in W^1(F + K, x_0, y_0)$. But, $W^2(F + K, x_0, y_0, v_1) = \mathbb{R}^2$, $W^2(F, x_0, y_0, v_1) = \emptyset$. Hence, $W^2(F + K, x_0, y_0, v_1) \not\subseteq W^2(F, x_0, y_0, v_1) + K$.

Proposition 3.2 *Suppose K have a compact convex base. Then,*

(i) $\mathrm{Min}_K V^m(F + K, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq V^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$;

(ii) $\mathrm{Min}_K W^m(F + K, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$.

Proof We prove only (i). Let $v \in \mathrm{Min}_K V^m(F + K, x_0, y_0, v_1, \cdots, v_{m-1})$. Then, there exist $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$, $y_n \in F(x_n)$, and $k_n \in K$ such that $\overline{v_n} := \frac{1}{t_n^m}(y_n + k_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}) \to v$. Then,

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \overline{v_n} - k_n \in F(x_n). \tag{1}$$

We claim that $\frac{k_n}{t_n^m} \to 0$ (for a subsequence). For a convex compact base Q of K, one has $k_n = \alpha_n b_n$ for some $\alpha_n \geq 0$ and $b_n \in Q$ and all n. If $\alpha_n = 0$ for infinitely many $n \in \mathbb{N}$, we are done. Hence, let $\alpha_n > 0$ for all n. We may assume that $b_n \to b \in Q$. Then, $\frac{k_n}{t_n^m} = \frac{\alpha_n b_n}{t_n^m} \to 0$ if and only if $\frac{\alpha_n}{t_n^m} \to 0$. Suppose $\frac{\alpha_n}{t_n^m}$ does not converge to 0. Then, nothing is lost by assuming that $\frac{\alpha_n}{t_n^m} \geq \epsilon$ for some $\epsilon > 0$ and all n. Let $\overline{k_n} := (\epsilon \frac{t_n^m}{\alpha_n}) k_n$. Then, $\overline{k_n} - k_n \in -K$ and $y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \overline{v_n} - \overline{k_n} \in F(x_n) + K$. Since $\frac{\overline{k_n}}{t_n^m} \to \epsilon b \neq 0$, one has $\overline{v_n} - \frac{\overline{k_n}}{t_n^m} \to v - \epsilon b$, and hence $v - \epsilon b \in V^m(F + K, x_0, y_0, v_1, \cdots, v_{m-1})$. Thus,

$$-\epsilon b \in (V^m(F + K, x_0, y_0, v_1, \cdots, v_{m-1}) - v) \cap (-K).$$

Because K has a convex base, $-\epsilon b \notin K$, contradicting the minimality of v. Therefore, $\frac{k_n}{t_n^m} \to 0$. It follows from (1) that $y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m (\overline{v_n} - \frac{k_n}{t_n^m}) \in F(x_n)$, and $\overline{v_n} - \frac{k_n}{t_n^m} \to v$. Therefore, $v \in V^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$. □

For weak minima we do not have a similar result, as indicated by the following example.

Example 3.2 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $(x_0, y_0) = (0, (0, 0))$, and $F(x)$is equal to $\{(0, 0)\}$ if $x = 0$, to $\{(0, -1)\}$ if $x = \frac{1}{n}$, to $\{(\frac{1}{n}, \frac{-1}{n})\}$ if $x = \sin \frac{1}{n}$ for $n \in \mathbb{N}$, and to $\emptyset$ otherwise. Then, we have

$$V^1(F, x_0, y_0) = \{(x, y) \in Y : y = -x, x \geq 0\},$$

$$W^1(F, x_0, y_0) = \{(0, y) \in Y : y \leq 0\} \cup \{(x, y) \in Y : y = -x, x \geq 0\},$$
$$V^1(F + K, x_0, y_0) = W^1(F + K, x_0, y_0) = \mathbb{R}_+ \times \mathbb{R}.$$

Consequently, $\mathrm{WMin}_K V^1(F + K, x_0, y_0) = \mathrm{WMin}_K W^1(F + K, x_0, y_0) = \{0\} \times \mathbb{R}$. Therefore, $\mathrm{WMin}_K V^1(F + K, x_0, y_0) \not\subseteq V^1(F, x_0, y_0)$, $\mathrm{WMin}_K W^1(F + K, x_0, y_0) \not\subseteq W^1(F, x_0, y_0)$.

If $\mathrm{int}K \neq \emptyset$, for weak minima, we have the following analogous properties.

Proposition 3.3 *Suppose $\hat{K} \subseteq \mathrm{int}K \cup \{0\}$ be a closed convex cone with a compact convex base. Then,*

(i) $\mathrm{WMin}_K V^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq V^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$;

(ii) $\mathrm{WMin}_K W^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$.

Proof Consider (ii). Since $\hat{K} \subseteq \mathrm{int}K \cup \{0\}$, any $v \in \mathrm{WMin}_K W^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1})$ satisfies

$$v \in W^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1}) \cap \mathrm{Min}_{\hat{K}} W^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1}). \tag{2}$$

Hence, there exist $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$, $v_n \to v$, $k_n \in \hat{K}$, and $h_n > 0$ such that, for all n,

$$\frac{v_1 + \cdots + t_n^{m-2} v_{m-1} + t_n^{m-1} v_n}{h_n} - k_n \in F(x_n) - y_0. \tag{3}$$

For a compact base $\hat{Q}$ of $\hat{K}$, there exist $\alpha_n \geq 0$ and $q_n \in \hat{Q}$ such that $k_n = \alpha_n q_n$. We may assume that $q_n \to q \in \hat{Q}$. We claim that $\frac{h_n \alpha_n}{t_n^{m-1}} \to 0$ (for a subsequence). This is true if $\alpha_n = 0$ for infinitely many $n \in \mathbb{N}$. Now, suppose to the contrary that $\alpha_n > 0$, for all n, and $\frac{h_n \alpha_n}{t_n^{m-1}} \not\to 0$. Then, we may assume that $\frac{h_n \alpha_n}{t_n^{m-1}} \geq \epsilon$ for some $\epsilon > 0$ and all n. Let $\overline{k_n} := \frac{\epsilon t_n^{m-1}}{h_n \alpha_n} k_n \in \hat{K}$. Then, we have $k_n - \overline{k_n} \in \hat{K}$. By (3), we obtain $\frac{v_1 + \cdots + t_n^{m-2} v_{m-1} + t_n^{m-1} v_n}{h_n} - \overline{k_n} \in F(x_n) + \hat{K} - y_0$. As $\frac{h_n \overline{k_n}}{t_n^{m-1}} \to \epsilon q \neq 0$, this implies that $v - \epsilon q \in W^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1})$. Therefore, $-\epsilon q \in (W^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1}) - v) \cap (-\hat{K})$, and $-\epsilon q \notin \hat{K}$, which contradicts (2). Hence, $\frac{h_n \alpha_n}{t_n^{m-1}} \to 0$ and $v_n - \frac{h_n k_n}{t_n^{m-1}} \to v$. It follows from (3) that $v \in W^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$ and the proof is complete. □

To get the equalities in Proposition 3.1, we need the following new notion.

Definition 3.2 Let $(x_0, y_0) \in \mathrm{gr}F$, $v_1, \cdots, v_{m-1} \in Y$, and $m \in \mathbb{N}$. The mth-order singular variational set of type 1 (type 2, resp) of F at (x_0, y_0) is defined by

$$\begin{aligned} V^{\infty(m)}(F, x_0, y_0, v_1, \cdots, v_{m-1}) &= \{y \in Y \,|\, \exists x_n \xrightarrow{F} x_0, \exists t_n \to 0^+, \exists \lambda_n \to 0^+, \\ &\quad \exists y_n \in \frac{F(x_n) - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}}{t_n^m}, \lambda_n y_n \to y\} \end{aligned}$$

$$\begin{aligned} (W^{\infty(m)}(F, x_0, y_0, v_1, \cdots, v_{m-1}) &= \{y \in Y \,|\, \exists x_n \xrightarrow{F} x_0, \exists t_n \to 0^+, \exists \lambda_n \to 0^+, \\ &\quad \exists y_n \in \frac{\mathrm{cone}_+(F(x_n) - y_0) - v_1 - \cdots - t_n^{m-2} v_{m-1}}{t_n^{m-1}}, \lambda_n y_n \to y\}). \end{aligned}$$

Proposition 3.4 *Let K have a compact convex base.*

(i) *Let either of the following conditions hold:*

(i$_1$) *$V^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1})$ has the domination property;*

(i$_2$) *$V^{\infty(m)}(F, x_0, y_0, v_1, \cdots, v_{m-1}) \cap (-K) = \{0\}$.*

Then,

$$V^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1}) = V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + K, \tag{4}$$

$$\mathrm{Min}_K V^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_K V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}). \tag{5}$$

(ii) *Let either of the following two conditions hold:*

(ii$_1$) *$W^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1})$ has the domination property;*

(ii$_2$) *$W^{\infty(m)}(F, x_0, y_0, v_1, \cdots, v_{m-1}) \cap (-K) = \{0\}$.*

Then,

$$W^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1}) = W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + K,$$

$$\mathrm{Min}_K W^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_K W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}).$$

Proof We prove only (i). First, we check (4). By Proposition 3.1(i), we need simply to verify that

$$V^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + K.$$

If (i_1) holds, then $V^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq \text{Min}_K V^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1}) + K$. Hence, (4) is satisfied, since we have (by Proposition 3.2)

$$\text{Min}_K V^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1}) + K \subseteq V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + K.$$

If (i_2) holds and $v \in V^m(F+K, x_0, y_0, v_1, \cdots, v_{m-1})$, then there exist $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$, $y_n \in F(x_n)$, and $k_n \in K$ such that $\frac{1}{t_n^m}(y_n + k_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}) \to v$. If one has n_0 such that $k_n = 0$ for all $n \geq n_0$, then, $v \in V^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$. If there is a subsequence, denoted again by $\{k_n\}$ with $k_n \neq 0$, we claim that $\{\frac{||k_n||}{t_n^m}\}$ be bounded. Indeed, otherwise we may assume that $\frac{||k_n||}{t_n^m} \to \infty$ and $\frac{k_n}{||k_n||} \to \overline{k} \in K \setminus \{0\}$. Setting

$$v_n := \frac{y_n + k_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}}{t_n^m}, \quad \lambda_n := \frac{t_n^m}{||k_n||},$$

we get

$$\lambda_n \frac{y_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1} - t_n^m v_n}{t_n^m} \to -\overline{k} \in -K \setminus \{0\}.$$

As $\lambda_n \to 0^+$, this means $-\overline{k} \in V^{\infty(m)}(F, x_0, y_0, v_1, \cdots, v_{m-1}) \cap -K \setminus \{0\}$, contradicting ($i_2$). So, $\{\frac{||k_n||}{t_n^m}\}$ is bounded and $\frac{||k_n||}{t_n^m} \to a \geq 0$. With $\overline{v_n} := \frac{1}{||k_n||}(y_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1} - t_n^m v_n)$, one has

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m (v_n + \frac{||k_n|| \overline{v_n}}{t_n^m}) = y_n \in F(x_n).$$

It easy to see that $v_n + \frac{||k_n||\overline{v_n}}{t_n^m} \to v - a\overline{k}$. Thus, $v - a\overline{k} \in V^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$ and (4) is satisfied. (5) is implied directly from (4) and Proposition 2.2. □

The following result for weak minima can be proved similarly as Proposition 4.3.

Proposition 3.5 *Let $\hat{K} \subseteq \text{int}K \cup \{0\}$ be a closed convex cone with a compact convex base.*

(i) *Impose either of the following two conditions:*

(i_1) $V^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1})$ *has the weak domination property;*

(i_2) $V^{\infty(m)}(F, x_0, y_0, v_1, \cdots, v_{m-1}) \cap (-\hat{K}) = \{0\}$.

Then,

$$\text{WMin}_K V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) = \text{WMin}_K V^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1}).$$

(ii) *Let either of the following conditions hold:*

(ii_1) $W^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1})$ *possesses the weak domination property;*

(ii_2) $W^{\infty(m)}(F, x_0, y_0, v_1, \cdots, v_{m-1}) \cap (-\hat{K}) = \{0\}$.

Then,

$$\text{WMin}_K W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) = \text{WMin}_K W^m(F+\hat{K}, x_0, y_0, v_1, \cdots, v_{m-1}).$$

4 Variational Sets of Perturbation Maps

In this section, we apply the results of Sect. 3 to set-valued optimization. Let U be a normed space of perturbation parameters, Y be the objective (normed) space ordered partially by a closed convex cone K, and $F : X \rightrightarrows Y$. One aims at finding minima or weak minima of $F(u)$ for a given parameter value u. Hence, we define set-valued maps G and S from U to Y by, for $u \in U$,

$$G(u) = \text{Min}_K F(u), \; S(u) = \text{WMin}_K F(u).$$

As it is well-known, G and S are called the perturbation map and weak perturbation map, resp. The purpose of this section is to investigate relationships between the variational sets of F and that of G and S, including relations between minima or weak minima of these variational sets.

F has the domination property around u_0 iff there exists a neighborhood V of u_0 such that $F(u)$ has the domination property for all $u \in V$. F has the weak domination property around u_0 with respect to (wrt) $\hat{K}$ iff there exists a neighborhood V of u_0 such that $F(u) \subseteq \mathrm{WMin}_K F(u) + \hat{K}$ for all $u \in V$.

Remark 4.1 (i) Suppose $y_0 \in G(u_0)$ and F have the domination property around u_0. Then,

$$V^m(G+K, u_0, y_0, v_1, \cdots, v_{m-1}) = V^m(F+K, u_0, y_0, v_1, \cdots, v_{m-1}),$$

$$W^m(G+K, u_0, y_0, v_1, \cdots, v_{m-1}) = W^m(F+K, u_0, y_0, v_1, \cdots, v_{m-1}).$$

(ii) If $y_0 \in S(u_0)$ and F has the weak domination property around u_0 wrt $\hat{K}$, then

$$V^m(S+\hat{K}, u_0, y_0, v_1, \cdots, v_{m-1}) = V^m(F+\hat{K}, u_0, y_0, v_1, \cdots, v_{m-1}),$$

$$W^m(S+\hat{K}, u_0, y_0, v_1, \cdots, v_{m-1}) = W^m(F+\hat{K}, u_0, y_0, v_1, \cdots, v_{m-1}).$$

Theorem 4.1 *Let $(u_0, y_0) \in \mathrm{gr}G$ and $v_1, \cdots, v_{m-1} \in Y$. Let F have the domination property around u_0, and K have a compact convex base.*

(i) *Assume further either of the following two conditions:*

(i$_1$) *$V^m(F+K, u_0, y_0, v_1, \cdots, v_{m-1})$ has the domination property;*

(i$_2$) *$V^{\infty(m)}(F, u_0, y_0, v_1, \cdots, v_{m-1}) \cap (-K) = \{0\}$.*

Then,

$$\mathrm{Min}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_K V^m(G, u_0, y_0, v_1, \cdots, v_{m-1}).$$

(ii) *Impose either of the following conditions:*

(ii$_1$) *$W^m(F+K, u_0, y_0, v_1, \cdots, v_{m-1})$ has the domination property;*

(ii$_2$) *$W^{\infty(m)}(F, u_0, y_0, v_1, \cdots, v_{m-1}) \cap (-K) = \{0\}$.*

Then,

$$\mathrm{Min}_K W^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_K W^m(G, u_0, y_0, v_1, \cdots, v_{m-1}).$$

Proof We prove only Assertion (i). Remark 4.1(i) yields that $V^m(G+K, u_0, y_0, v_1, \cdots, v_{m-1})$ also has the domination property. Because either (i$_1$) or (i$_2$) holds, from Proposition 3.4 we get

$$\mathrm{Min}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_K V^m(F+K, u_0, y_0, v_1, \cdots, v_{m-1})$$

$$= \mathrm{Min}_K V^m(G+K, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_K V^m(G, u_0, y_0, y_1, \cdots, y_{m-1}).$$

□

The following example illustrates Theorem 4.1.

Example 4.1 Let $U = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $u_0 = 0$, $y_0 = (0,0)$, and $F(u) = \{(y_1, y_2) \in Y : y_1 = u,\ y_2 \geq |y_1|\}$ for $u \in U$. Then, $G(u) = \{(y_1, y_2) \in Y : y_1 = u,\ y_2 = |y_1|\}$. Let $v_i = (-1, 1)$ for $i = 1, \cdots, m-1$. Direct calculations give

$$V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = W^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \begin{cases} \{(y_1, y_2) \in Y : y_2 \geq |y_1|\}, & \text{if } m = 1, \\ \{(y_1, y_2) \in Y : y_1 + y_2 \geq 0\}, & \text{if } m > 1. \end{cases}$$

$$V^m(G, u_0, y_0, v_1, \cdots, v_{m-1}) = W^m(G, u_0, y_0, v_1, \cdots, v_{m-1}) = \begin{cases} \{(y_1, y_2) \in Y : y_2 = |y_1|\}, & \text{if } m = 1, \\ \{(y_1, y_2) \in Y : y_1 + y_2 = 0\}, & \text{if } m > 1. \end{cases}$$

We can check that the assumptions of Theorem 4.1 are satisfied for all m. Direct checking yields

$$\mathrm{Min}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_K V^m(G, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_K W^m(G, u_0, y_0, v_1, \cdots, v_{m-1})$$

$$= \begin{cases} \{(y_1, y_2) \in Y : y_1 \leq 0,\ y_2 = |y_1|\}, & \text{if } m = 1, \\ \{(y_1, y_2) \in Y : y_1 + y_2 = 0\}, & \text{if } m > 1. \end{cases}$$

Similarly, by Remark 4.1(ii) and Proposition 3.5, we have the following for weak minima.

Theorem 4.2 *Let $(u_0, y_0) \in \mathrm{gr}S$ and $v_1, \cdots, v_{m-1} \in Y$. Let F have the weak domination property around u_0 wrt $\hat{K}$, where $\hat{K} \subseteq \mathrm{int}K \cup \{0\}$ is a closed convex cone having a compact convex base.*

(i) *Let either of the following two conditions hold:*

(i$_1$) $V^m(F + \hat{K}, u_0, y_0, v_1 \cdots, v_{m-1})$ *has the weak domination property;*

(i$_2$) $V^{\infty(m)}(F, u_0, y_0, v_1, \cdots, v_{m-1}) \cap (-\hat{K}) = \{0\}$.

Then,

$$\mathrm{WMin}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{WMin}_K V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}).$$

(ii) *Impose one of the following two conditions:*

(ii$_1$) $W^m(F + \hat{K}, u_0, y_0, v_1, \cdots, v_{m-1})$ *has the weak domination property;*

(ii$_2$) $W^{\infty(m)}(F, u_0, y_0, v_1, \cdots, v_{m-1}) \cap (-\hat{K}) = \{0\}$.

Then,

$$\mathrm{WMin}_K W^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{WMin}_K W^m(S, u_0, y_0, v_1, \cdots, v_{m-1}).$$

Example 4.2 Let $U = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $u_0 = 0$, $y_0 = (0, 0)$, and $F(u)$ is equal to $\{(y_1, y_2) \in Y : y_1 = u,\ y_2 \geq -y_1\}$ if $u \leq 0$, and to $\{(y_1, y_2) \in Y : 0 \leq y_1 \leq u,\ y_2 \geq 0\}$ if $u > 0$. Then, $S(u)$ is equal to $\{(y_1, y_2) \in Y : y_1 = u,\ y_2 \geq -y_1\}$ if $u \leq 0$, and to $\{(y_1, y_2) \in Y : 0 \leq y_1 \leq u,\ y_2 = 0\}$ if $u > 0$. Let $v_i = (1, 0)$ for $i = 1, \cdots, m-1$. Direct computations yield that $V^m(F, u_0, y_0, v_1, \cdots, v_{m-1})$ is equal to $\mathbb{R}^2_+ \cup \{(y_1, y_2) \in Y : y_1 \leq 0,\ y_2 \geq -y_1\}$ if $m = 1$, and to $\mathbb{R} \times \mathbb{R}_+$ if $m > 1$, and $V^m(S, u_0, y_0, v_1, \cdots, v_{m-1})$ is equal to $\{(y_1, y_2) \in Y : y_1 \leq 0,\ y_2 \geq -y_1\} \cup (\mathbb{R}_+ \times \{0\})$ if $m = 1$, and to $\mathbb{R} \times \{0\}$ if $m > 1$.

For each of F and S, the variational sets of the two types coincide for all $m \geq 1$. We can check that the assumptions of Theorem 4.2 are fulfilled for all m (for an arbitrary closed convex cone $\hat{K}$ such that $\hat{K} \in \mathrm{int}\mathbb{R}^2_+ \cup \{(0, 0)\}$). Direct verifying gives

$$\mathrm{WMin}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{WMin}_K V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{WMin}_K W^m(S, u_0, y_0, v_1, \cdots, v_{m-1})$$

$$= \begin{cases} \{(y_1, y_2) \in Y : y_1 \leq 0,\ y_2 = -y_1\} \cup (\mathbb{R}_+ \times \{0\}), & \text{if } m = 1, \\ \mathbb{R} \times \{0\}, & \text{if } m > 1. \end{cases}$$

Note that the set of (Pareto) minima is much smaller than that of weak minima: $G(u)$ is equal to $\{(y_1, y_2) \in Y : y_1 = u,\ y_2 = -y_1\}$ if $u \leq 0$, and to $\{(0, 0)\}$ if $u > 0$. For $v_i = (1, 0)$, $i = 1, \cdots, m-1$, we have $V^1(G, u_0, y_0) = W^m(G, u_0, y_0) = \{(y_1, y_2) \in Y : y_1 \leq 0,\ y_2 = -y_1\}$, and they are empty for $m > 1$. We can check that the assumptions of Theorem 4.1 are satisfied for $m = 1$ and

$$\mathrm{Min}_K V^1(F, u_0, y_0) = \mathrm{Min}_K V^1(G, u_0, y_0) = \mathrm{Min}_K W^1(G, u_0, y_0) = \{(y_1, y_2) \in Y : y_1 \leq 0,\ y_2 = -y_1)\}.$$

In the following case, Theorems 4.1 and 4.2 can be used, but some recent existing results cannot.

Example 4.3 Let $U = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $u_0 = 0$, $y_0 = (0, 0)$, and $F(u)$ is equal to $\{(0, 0)\}$ if $u = 0$, and to $\{(0, 0); (\frac{1}{n^3}, \frac{-1}{n^3}); (\frac{-1}{n^3}, \frac{1}{n^3})\}$ if $u = \frac{1}{n}$ for $n \in \mathbb{N}$, and is empty otherwise.

Then, $S(u) = G(u) = F(u)$. For $v_1 = (1, -1)$, $v_2 = (-1, 1)$. Calculations give

$$V^1(F, u_0, y_0) = W^1(F, u_0, y_0) = \{(y_1, y_2) \in Y : y_1 + y_2 = 0\},$$

$$V^2(F, u_0, y_0, v_1) = W^2(F, u_0, y_0, v_1) = \{(y_1, y_2) \in Y : y_1 + y_2 = 0\},$$

$$V^3(F, u_0, y_0, v_1, v_2) = W^3(F, u_0, y_0, v_1, v_2) = \{(y_1, y_2) \in Y : y_1 + y_2 = 0\}.$$

We can check that the assumptions of Theorems 4.1 and 4.2 are satisfied. Calculating the lower Studniarski derivative of F at (u_0, y_0) (see [13] for the definition), we have $\underline{d}^m F(u_0, y_0)(u) = \emptyset$ for all $u \in \mathbb{R}$. Hence, Theorems 4.1-4.3 and Corollaries 4.1-4.3 of [13] cannot be in use.

Theorems 4.3, 4.7, and 4.10 of [9] in terms of second-order contingent derivatives cannot be applied either, since $D^2 F(u_0, y_0, u_1, v_1)(u) = \emptyset$, for all $u \in \mathbb{R}$.

Proposition 4.1 *Let* $(u_0, y_0) \in \text{gr}S$ *and* $v_1, \cdots, v_{m-1} \in Y$*. Let* F *have a proto-variational set of order* m *of type 1 at* (u_0, y_0)*, and* $\text{int}K \neq \emptyset$*. Then,*

$$V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}) \subseteq \text{WMin}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}).$$

Proof Let $y \in V^m(S, u_0, y_0, v_1, \cdots, v_{m-1})$, i.e., there exist $t_n \to 0^+$, $u_n \overset{S}{\to} u_0$, and $y_n \to y$ such that

$$y_0 + t_n v_1 + \cdots + t_n^m y_n \in S(u_n) \subseteq F(u_n), \tag{6}$$

and $y \in V^m(F, u_0, y_0, v_1, \cdots, v_{m-1})$. Suppose $y \notin \text{WMin}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1})$, i.e., there exists $y' \in V^m(F, u_0, y_0, v_1, \cdots, v_{m-1})$ such that $y - y' \in \text{int}K$. For the above sequences t_n and u_n, there exists $y'_n \to y'$ such that $y_0 + t_n v_1 + \cdots + t_n^m y'_n \in F(u_n)$, and $y_n - y'_n \in \text{int}K$ for large n. Consequently,

$$(y_0 + t_n v_1 + \cdots + t_n^m y_n) - (y_0 + t_n v_1 + \cdots + t_n^m y'_n) = t_n^m (y_n - y'_n) \in \text{int}K,$$

i.e., $y_0 + t_n v_1 + \cdots + t_n^m y_n \notin \text{WMin}_K F(u_n) = S(u_n)$, which contradicts (6). □

Unfortunately, the similar result is not true for W^m, as indicated by the next example.

Example 4.4 Let $U = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, and $F(u)$ is equal to $(\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup \{(x, y) : x^2 + y^2 = 1\}$ $u = 0$, and empty if $u \neq 0$. Then, $S(u) \equiv \text{WMin}_K F(u)$ and $S(u)$ is equal to $((-\infty, -1) \times \{0\}) \cup (\{0\} \times (-\infty, -1)) \cup \{(x, y) : x^2 + y^2 = 1, x \leq 0, y \leq 0\}$ if $u = 0$, and empty if $u \neq 0$. F has a proto-variational set at $(0, (-1, 0))$ and $W^1(F, 0, (-1, 0)) = (\mathbb{R}_+ \times \mathbb{R}) \cup (\mathbb{R}_- \times \{0\})$. However, we have

$$\text{WMin}_K W^1(F, 0, (-1, 0)) = (\{0\} \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \{0\}),$$

$$W^1(S, 0, (-1, 0)) = (\mathbb{R}_- \times \{0\}) \cup \{(x, y) : y \leq -x, x \geq 0\},$$

and hence $W^1(S, 0, (-1, 0)) \not\subseteq \text{WMin}_K W^1(F, 0, (-1, 0))$.

Replacing "S" in Proposition 4.1 by "G" is impossible as shown now.

Example 4.5 Let $U = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, and $F(u)$ is equal to $\{(x, y) : x > 0, y < -x\} \cup \{(x, y) : y = -x\}$ if $u = 0$, and empty if $u \neq 0$. Then, $G(u) \equiv \text{Min}_K F(u)$ is defined by $G(0) = \{(x, y) : y = -x, x \leq 0\}$ and $G(u) = \emptyset$ for any $u \neq 0$. We see that F has the following proto-variational set $V^1(F, 0, (0, 0)) = \{(x, y) : x \geq 0, y < -x\} \cup \{(x, y) : y = -x\}$. Since $\text{Min}_K V^1(F, 0, (0, 0)) = \{(x, y) : y = -x, x < 0\}$, $V^1(G, 0, (0, 0)) = \{(x, y) : y = -x, x \leq 0\}$, one has $V^1(G, 0, (0, 0)) \not\subseteq \text{Min}_K V^1(F, 0, (0, 0))$.

Theorem 4.3 *Let* $(u_0, y_0) \in \text{gr}S$ *and* $v_1, \cdots, v_{m-1} \in Y$*. Let* $\hat{K}$ *be a closed convex cone contained in* $\text{int}K \cup \{0\}$ *and have a compact convex base. Let the following conditions be satisfied:*

(i) *either of the following holds*

(i$_1$) $V^m(F + \hat{K}, u_0, y_0, v_1, \cdots, v_{m-1})$ *has the weak domination property;*

(i$_2$) $V^{\infty(m)}(F, u_0, y_0, v_1, \cdots, v_{m-1}) \cap (-\hat{K}) = \{0\}$*;*

(ii) *F has the weak domination property around u_0 wrt $\hat{K}$;*

(iii) *F has a proto-variational set of order m of type 1 at (u_0, y_0).*

Then,

$$V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}) = \text{WMin}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}).$$

Proof Obviously, by Proposition 4.1, we need to prove only that

$$V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}) \supseteq \text{WMin}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}).$$

Propositions 3.3, 3.5, and Remark 4.1(ii) together imply that

$$\text{WMin}_K V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \text{WMin}_K V^m(F + \hat{K}, u_0, y_0, v_1, \cdots, v_{m-1})$$

$$= \text{WMin}_K V^m(S + \hat{K}, u_0, y_0, v_1, \cdots, v_{m-1}) \subseteq V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}).$$

□

5 Applications to Constrained Vector Optimization

In this section, we consider the following two constrained vector optimization problems, where both the objective map and the constraint set depend on a perturbation parameter,

$$\mathrm{Min}_K\ F(x,u),\ \text{ subject to } x \in X(u), \tag{7}$$

$$\mathrm{WMin}_K\ F(x,u),\ \text{ subject to } x \in X(u). \tag{8}$$

Here, as before U, W, Y are normed spaces, K is a nonempty closed convex ordering cone in Y, which is now assumed additionally to be pointed, F is a set-valued objective map from $W \times U$ to Y, and X is a set-valued map from U to W. We define a set-valued map H from U to Y by

$$H(u) = F(X(u),u) = \{y \in Y : y \in F(x,u), x \in X(u)\}.$$

So, $H(u)$ is the parameterized feasible set in the objective space. In problems (7) and (8), we aim to obtain minima and weak minima of $H(u)$, resp. The solution sets in Y to problems (7) and (8) are denoted by $\mathrm{Min}_K H(u)$ and $\mathrm{WMin}_K H(u)$, resp. Like in Sect. 4, we define

$$G(u) = \mathrm{Min}_K H(u),\ \ S(u) = \mathrm{WMin}_K H(u).$$

We need the following new definition.

Definition 5.1 Let W, U, Y be normed spaces, $F : W \times U \rightrightarrows Y$, $((x_0,u_0),y_0) \in \mathrm{gr}F$, $x \in W$, $(w_i, v_i) \in W \times Y$ for $i = 1, \cdots, m-1$.

(i) The mth-order upper (lower, resp) variation of F at $((x_0,u_0),y_0)$ wrt x is

$$V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1}) = \{v : \exists t_n \to 0^+, \exists h_n \to 0^+, \exists x_n \to x, \exists u_n \to u_0, \exists v_n \to v, \forall n,$$

$$y_0 + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m v_n \in F(x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n)\}$$

$$(\underline{V}_q^m(F,(x_0[x],u_0),y_0,w_1,v_1 ..., w_{m-1},v_{m-1}) = \{v : \forall t_n \to 0^+, \forall x_n \to x, \forall u_n \to u_0, \exists v_n \to v, \forall n,$$

$$y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n w_1 + ... + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n)\}).$$

(ii) F is said to have a mth-order proto variation of F at $((x_0,u_0),y_0)$ iff, for all x,

$$V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1}) = \underline{V}_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1}).$$

We now investigate connections of a proto variation of F and a variational set of X to the corresponding variational set of H.

Proposition 5.1 *Let $u_0 \in U$, $x_0 \in X(u_0)$, and $y_0 \in F(x_0,u_0)$. If F has a mth-order proto variation at $((x_0,u_0),y_0)$, then*

$$\left(\bigcup_{x \in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})} V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})\right) \subseteq V^m(H,u_0,y_0,v_1,\cdots,v_{m-1}). \tag{9}$$

Moreover, if W is finite dimensional, $\widetilde{X}(u,y) := \{x \in \mathbb{R}^n : x \in X(u), y \in F(x,u)\}$ is calm around (u_0,y_0), $\widetilde{X}(u_0,y_0) = \{x_0\}$, and $V_q^1(\widetilde{X},(u_0,y_0[0]),x_0) = \{0\}$, then, the inclusion opposite to (9) is valid.

Proof Let there exist $x \in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})$ such that $v \in V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})$. $x \in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})$ means the existence of $t_n \to 0^+$, $u_n \to u_0$, $x_n \xrightarrow{X} x$ such that, for all n, $x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m x_n \in X(u_n)$. Then,

$$F(x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n) \subseteq H(u_n). \tag{10}$$

Because $v \in V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})$, with the above t_n, u_n, x_n, there exists $y_n \in F(x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n)$ such that $\frac{1}{t_n^m}(y_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}) \to v$. So, we have

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \left(\frac{y_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}}{t_n^m}\right)$$

$$= y_n \in F(x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n).$$

It follows from (10) that $v \in V^m(H, u_0, y_0, v_1, \cdots, v_{m-1})$.

Next, we prove the inclusion reverse to (9). Let $v \in V^m(H, u_0, y_0, v_1, \cdots, v_{m-1})$, i.e., there exist $t_n \to 0^+$, $u_n \to u_0$, and $v_n \overset{H}{\to} v$ such that $y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n \in H(u_n)$ for all n. Then, there exists $x_n \in X(u_n)$ such that $y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_n, u_n)$. Hence, $x_n \in \widetilde{X}(u_n, y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n)$. The calmness of $\widetilde{X}$ yields $M > 0$ such that

$$||x_n - x_0|| \le M||(u_n, y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n) - (u_0, y_0)||.$$

Then, $x_n \to x_0$ and hence $(x_n - x_0 - t_n w_1 - \cdots - t_n^{m-1} w_{m-1}) \to 0$. We claim that $\{\frac{1}{t_n^m}(x_n - x_0 - t_n w_1 - \cdots - t_n^{m-1} w_{m-1})\}$ is bounded. Indeed, we have

$$x_0 + ||x_n - x_0|| \frac{(x_n - x_0)}{||x_n - x_0||} = x_n \in \widetilde{X}(u_n, y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n). \qquad (11)$$

We may assume that $a_n := \dfrac{(x_n - x_0)}{||x_n - x_0||} \to a$ with norm one. Setting $r_n^m = ||x_n - x_0 - t_n w_1 - \cdots - t_n^{m-1} w_{m-1}||$, we have $r_n^m \to 0^+$ and

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n = y_0 + r_n \frac{t_n}{r_n} v_1 + \cdots + r_n^{m-1} \frac{t_n^{m-1}}{r_n^{m-1}} v_{m-1} + r_n^m \frac{t_n^m}{r_n^m} v_n$$

$$= y_0 + r_n\left(\frac{t_n}{r_n} v_1 + \cdots + r_n^{m-2} \frac{t_n^{m-1}}{r_n^{m-1}} v_{m-1} + r_n^{m-1} \frac{t_n^m}{r_n^m} v_n\right) := y_0 + r_n q_n.$$

For $h_n = ||x_n - x_0||$, (11) is written equivalently as $x_0 + h_n a_n \in \widetilde{X}(u_n, y_0 + r_n q_n)$. If $\dfrac{t_n^m}{r_n^m} \to 0^+$, then $q_n \to 0$ and $a \in V_q^1(\widetilde{X}, (u_0, y_0[0]), x_0)$, impossible. Thus, $\{\frac{1}{t_n^m}(x_n - x_0 - t_n w_1 - \cdots - t_n^{m-1} w_{m-1})\}$ is bounded and $\overline{x_n} := \frac{1}{t_n^m}(x_n - x_0 - t_n w_1 - \cdots - t_n^{m-1} w_{m-1})$ converges to some $\overline{x} \in \mathbb{R}^n$. Since $x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m \overline{x_n} \in X(u_n)$, one has $y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m \overline{x_n}, u_n)$. Therefore, $\overline{x} \in V^m(X, u_0, x_0, w_1, \cdots, w_{m-1})$ and $v \in V_q^m(F, (x_0[\overline{x}], u_0), y_0, w_1, v_1, \cdots, w_{m-1}, v_{m-1})$. □

The following four examples ensure the essentialness of each assumption of Proposition 5.1.

Example 5.1 ($\widetilde{X}(u_0, y_0) = \{x_0\}$ is needed) Let $U = W = Y = \mathbb{R}$, $F(x, u) = \{x(x-1)\}$, $u_0 = 0$, $x_0 = 1$, $y_0 = 0 \in F(x_0, u_0)$, and $X(u)$ is equal to $\{x : 0 \le x \le 1\}$ if $u = 0$, to $\{x : -u \le x \le 1\}$ if $u = \frac{1}{n}$, $n \in \mathbb{N}$, and empty otherwise. Then,

$$\widetilde{X}(u, y) = \begin{cases} \{\frac{1 - \sqrt{1+4y}}{2}, \frac{1 + \sqrt{1+4y}}{2}\}, & \text{if } u \in \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\},\ \frac{-1}{4} \le y \le 0, \\ \{\frac{1 - \sqrt{1+4y}}{2}\}, & \text{if } u \in \{\frac{1}{n} : n \in \mathbb{N}\},\ 0 < y \le u(u+1), \\ \emptyset, & \text{otherwise}, \end{cases}$$

and $H(u)$ is equal to $\{y : \frac{-1}{4} \le y \le 0\}$ if $u = 0$, to $\{y : \frac{-1}{4} \le y \le u(u+1)\}$ if $u = \frac{1}{n}$, $n \in \mathbb{N}$, and empty otherwise. $\widetilde{X}$ is clearly calm around (u_0, y_0) and we can obtain by direct calculations that

$$V_q^1(\widetilde{X}, (u_0, y_0[0]), x_0) = \{0\},\ V^1(X, u_0, x_0) = -\mathbb{R}_+,$$

$$V_q^1(F, (x_0[x], u_0), y_0) = \{x\},\ V^1(H, u_0, y_0) = \mathbb{R}.$$

So,

$$\bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0) = -\mathbb{R}_+.$$

Thus, since $\widetilde{X}(u_0, y_0) = \{0, 1\} \neq \{x_0\}$, we have

$$V^1(H, u_0, y_0) \not\subseteq \bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0).$$

Example 5.2 (the calmness around (u_0, y_0) cannot be dropped) Let $U = W = Y = \mathbb{R}$, $F(x, u) = \{x(x-1)\}$, $u_0 = 0$, $x_0 = 1$, $y_0 = 0 \in F(x_0, u_0)$, and $X(u)$ is equal to $\{x : 0 < x \leq 1\}$ if $u = 0$, to $\{x : -u < x \leq 1\}$ if $u = \frac{1}{n}$, $n \in \mathbb{N}$, and empty otherwise. Then,

$$\widetilde{X}(u, y) = \begin{cases} \{1\}, & \text{if } u \in \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\},\ y = 0, \\ \{\frac{1 - \sqrt{1 + 4y}}{2}, \frac{1 + \sqrt{1 + 4y}}{2}\}, & \text{if } u \in \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\},\ \frac{-1}{4} \leq y < 0, \\ \{\frac{1 - \sqrt{1 + 4y}}{2}\}, & \text{if } u \in \{\frac{1}{n} : n \in \mathbb{N}\},\ 0 < y < u(u+1), \\ \emptyset, & \text{otherwise,} \end{cases}$$

and $H(u)$ is as in Example 5.1 with only "$y \leq u(u+1)$" replaced by the strict inequality. Hence,

$$\widetilde{X}(u_0, y_0) = \{x_0\},\ V_q^1(\widetilde{X}, (u_0, y_0[0]), x_0) = \{0\},$$

$$V^1(X, u_0, x_0) = -\mathbb{R}_+,\ V_q^1(F, (x_0[x], u_0), y_0) = \{x\},\ V^1(H, u_0, y_0) = \mathbb{R}.$$

Consequently, because $\widetilde{X}$ is not calm around (u_0, y_0), we really have

$$\bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0) = -\mathbb{R}_+,$$

$$V^1(H, u_0, y_0) \not\subseteq \bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0).$$

Example 5.3 ($V_q(\widetilde{X}, (u_0, y_0[0]), x_0) = \{0\}$ is essential) Let $U = Y = \mathbb{R}$, $W = \mathbb{R}^2$, $X(u) = \{x \in \mathbb{R}^2 : x_1 = u,\ x_2 = 0\}$, $F(x, u) = \{x_1^2(x_1 - 1)\}$, $u_0 = 0$, $x_0 = (0, 0)$, and $y_0 = f(x_0, u_0) = 0$. Then, $\widetilde{X}(u, y)$ is equal to $\{(u, 0)\}$ if $u \in \mathbb{R}$, $y = u^2(u-1)$, and is empty otherwise, and $H(u) = \{u^2(u-1)\}$. Hence, $\widetilde{X}(u_0, y_0) = \{x_0\}$ and $\widetilde{X}$ is calm around (u_0, y_0). Direct calculations give $V^1(X, u_0, x_0) = \mathbb{R} \times \{0\}$, $V_q^1(F, (x_0[x], u_0), y_0) = \{0\}$. Therefore,

$$\bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0) = \{0\}.$$

By taking $t_n = \frac{1}{n}$, $u_n = \frac{1}{\sqrt{n}}$, $x_n = (u_n, 0) \in X(u_n)$, $v_n = \frac{1}{\sqrt{n}} - 1 \to -1$, we can check that $y_0 + t_n v_n \in H(u_n)$. Thus, $-1 \in V^1(H, u_0, y_0)$. Consequently,

$$V^1(H, u_0, y_0) \not\subseteq \bigcup_{x \in V^1(X, u_0, x_0)} V_q^1(F, (x_0[x], u_0), y_0).$$

To see the reason, let $t_n = \frac{1}{n}$, $u_n = \frac{1}{n}$, $y_n = \frac{1}{n}(\frac{1}{n} - 1) \to 0$, $x_n = (1, 0)$ to have that $x_0 + t_n x_n \in \widetilde{X}(u_n, y_0 + t_n y_n)$, and so $(1, 0) \in V_q^1(\widetilde{X}, (u_0, y_0[0]), x_0)$.

Example 5.4 (W needs be finite dimensional) Let $U = Y = \mathbb{R}$ and $W = l_1$, the space of all real sequences $x = (x^i)_{i \in \mathbb{N}}$ with $\sum_{i=1}^{\infty} |x^i| < \infty$. Let $X(u)$ be $\{0\}$ if $u = 0$, $\{x = (x^i)_{i \in \mathbb{N}} \in X : x^i = u \text{ if } i = n;\ x^i = 0, \text{ if } i \neq n\}$ if $u = \frac{1}{n}$, $n \in \mathbb{N}$, and empty otherwise, $F(x, u) = \{||x||(||x|| - 1)\}$, $u_0 = 0$, $x_0 = 0 \in X(u_0)$, and

$y_0 = 0 \in F(x_0, u_0)$. Then, $\widetilde{X}(u,y)$ is equal to $\{0\}$ if $u=0$, to $\{x=(x^i)_{i\in\mathbb{N}} \in X : x^i = u$ if $i=n$, $x^i=0$, if $i \neq n\}$ if $u = \dfrac{1}{n}$, $n \in \mathbb{N}$, $y = |u|(|u|-1)$, and empty otherwise, and $H(u)$ is $\{0\}$ if $u=0$, $\{|u|(|u|-1)\}$ if $u = \dfrac{1}{n}$, $n \in \mathbb{N}$, and empty otherwise. Hence, $\widetilde{X}$ is calm around (u_0, y_0). We can compute directly that

$$\widetilde{X}(u_0,y_0) = \{x_0\},\ V_q^1(\widetilde{X},(u_0,y_0[0]),x_0) = \{0\},$$

$$V_q^1(F,(x_0[x],u_0),y_0) = \{-||x||\},\ V^1(X,u_0,x_0) = \{0\}.$$

Therefore,

$$\bigcup_{x\in V^1(X,u_0,x_0)} V_q^1(F,(x_0[x],u_0),y_0) = \{0\}.$$

By taking $t_n = \dfrac{1}{n}$, $u_n = \dfrac{1}{n}$, $x_n = (x_n^i)_{i\in\mathbb{N}} \in X(u_n)$ satisfying $x_n^i = u_n$ if $i=n$ and $x_n^i = 0$ if $i \neq n$, and $v_n = \dfrac{1}{n} - 1 \to -1$, we can check that $y_0 + t_n v_n \in H(u_n)$. Hence, $-1 \in V^1(H,u_0,y_0)$. Thus,

$$V^1(H,u_0,y_0) \not\subseteq \bigcup_{x\in V^1(X,u_0,x_0)} V_q^1(F,(x_0[x],u_0),y_0).$$

Finally, invoking to Proposition 5.1 and results of Sect. 4, we easily establish relations between minima and weak minima of the mentioned variational sets stated in the following three theorems.

Theorem 5.1 *Let $(u_0,y_0) \in \mathrm{gr}G$, $x_0 \in X(u_0)$, $y_0 \in F(x_0,u_0)$, W be finite dimensional, and K have a compact convex base. Suppose*

(i) *H has the domination property around u_0;*

(ii) *either of the following two conditions holds:*

(ii$_1$) *$V^m(H+K,u_0,y_0,v_1,\cdots,v_{m-1})$ has the domination property;*

(ii$_2$) *$V^{\infty(m)}(H,x_0,y_0,v_1,\cdots,v_{m-1}) \cap (-K) = \{0\}$;*

(iii) *F has a mth-order proto variation at $((x_0,u_0),y_0)$;*

(iv) *$\widetilde{X}$ is calm around (u_0,y_0);*

(v) *$\widetilde{X}(u_0,y_0) = \{x_0\}$ and $V_q^1(\widetilde{X},(u_0,y_0[0]),x_0) = \{0\}$.*

Then,

$$\mathrm{Min}_K\left(\bigcup_{x\in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})} V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})\right)$$
$$= \mathrm{Min}_K V^m(G,x_0,y_0,v_1,\cdots,v_{m-1}).$$

Proof This follows from Theorem 4.1(i) and Proposition 5.1. □

Theorem 5.2 *Let $(u_0,y_0) \in \mathrm{gr}S$, $x_0 \in X(u_0)$, $y_0 \in F(x_0,u_0)$, W be finite dimensional, and $\hat{K}$ be a closed convex cone contained in* $\mathrm{int}K \cup \{0\}$ *and have a compact convex base. Suppose*

(i) *Y has the weak domination property around u_0 wrt $\hat{K}$;*

(ii) *either of the following two conditions is satisfied:*

(ii$_1$) *$V^m(H+\hat{K},u_0,y_0,v_1,\cdots,v_{m-1})$ has the weak domination property;*

(ii$_2$) *$V^{\infty(m)}(H,x_0,y_0,v_1,\cdots,v_{m-1}) \cap (-\hat{K}) = \{0\}$;*

(iii) *F has a mth-order proto variation at $((x_0,u_0),y_0)$;*

(iv) *$\widetilde{X}$ is calm around $((u_0,y_0),x_0)$;*

(v) *$\widetilde{X}(u_0,y_0) = \{x_0\}$ and $V_q^1(\widetilde{X},(u_0,y_0[0]),x_0) = \{0\}$.*

Then,

$$\mathrm{WMin}_K\left(\bigcup_{x\in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})} V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})\right)$$

$$= \mathrm{WMin}_K V^m(G, x_0, y_0, v_1, \cdots, v_{m-1}).$$

Proof Theorem 4.2(i) and Proposition 5.1 together imply this theorem. □

Theorem 5.3 *Let the assumptions of Theorem 5.2 be satisfied and H have a proto-variational set of order m of type 1 at (u_0, y_0). Then*

$$V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{WMin}_K \left(\bigcup_{x \in V^m(X, u_0, x_0, w_1, \cdots, w_{m-1})} V_q^m(F, (x_0[x], u_0), y_0, w_1, v_1, \cdots, w_{m-1}, v_{m-1}) \right).$$

Proof Applying Theorem 4.3 and Proposition 5.1, we are done. □

Remark 5.1 Though there have been several contributions to analysis of perturbation map G and weak perturbation map S for unconstrained feasible map F (defined in Sect. 4), we see only Tanino [2] dealing with this topic for a map F in a set-constrained smooth single-valued problem. That paper was limited to first-order results in terms of gradients of F. The present paper is the first attempt of higher-order considerations of F for a set-constrained nonsmooth multivalued problem. The extension has been performed in several aspects. Furthermore, we have extended successfully almost directly Theorem 4.1 of [2]. However, a drawback here is that the results are technically complicated. We hope that, excluding inevitable complexity, e.g. with higher-order derivatives (at least because of long expressions) and a high level of nonsmoothness, improvements can be obtained in future. In this paper, we restrict ourselves to making sure that the relatively complicated assumptions imposed in the results cannot be avoided by showing (in examples) their essentialness.

6 Conclusion

Since quantitative properties of perturbation maps of nonsmooth vector optimization is of high importance, but there have been only considerations in terms of contingent derivatives, we discuss higher-order analysis of such maps in terms of variational sets, a kind of generalized derivatives which is suitable for a high level of nonsmoothness and relatively easy to compute. We establish relations between variational sets of a perturbation map or weak perturbation map or the minima/weak minima of these sets and the corresponding ones of the feasible-set map to the objective space. These results are applied to sensitivity analysis for set-constrained vector optimization. As some results look complicated, we have tried to confirm the essentialness of each imposed assumption as well as to illustrate advantages of our results by a number of examples, which indicate also that computing variational sets is not a hard work.

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