STRUCTURAL STABILITY OF AUTONOMOUS SEMILINEAR NONLOCAL EVOLUTION EQUATIONS AND THE RELATED SEMI-DYNAMICAL SYSTEMS

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Abstract. Our work investigates firstly the unique existence and the continuous dependence (on the singular kernel and initial data) of solutions to nonlocal evolution equations on Hilbert spaces. Secondly, we prove the well-definedness of a related semi-dynamical system consisting of Lipschitz continuous mappings in the space of continuous functions by constructing a metric utilizing the kernel of nonlocal derivative. Our results extend and generalize the existing studies on Caputo fractional differential equations, namely the stability and structural stability results in Diethelm & Ford (J. Math. Anal. Appl. 265, no. 2, 229–248, 2002), the related semi-dynamical systems in Son & Kloeden (Vietnam J. Math. 2021), to the case of nonlocal differential equations.

1. Introduction

Let H be a separable Hilbert space. We consider in this work the following initial value problem of the semilinear nonlocal evolution equation

$$k * u'(t) + Au(t) = f(u(t)), t > 0, u(0+) = u_0, \tag{1.1}$$

where $k * u'(t) := \int_0^t k(t-\tau)u'(\tau)d\tau$ stands for the nonlocal derivative in time, the kernel k belongs to a class of completely positive functions, A is a self-adjoint operator and f is a global Lipschitz continuous function.

Nonlocal evolution equations have been an active topic of research because of their important appearance in many mathematical models in processes in materials with memories, see e.g. [1, 9, 16]. In a particular setting, when $H = L^2(\Omega), \Omega \subset \mathbb{R}^N$ and $A = -\Delta$ is the Laplace operator associated with a Dirichlet or Neumann boundary condition, Equation (1.1) with different kernel functions k is employed to describe anomalous diffusion phenomena including slow and ultraslow diffusions, see e.g. [20].

In the case of special kernel $k(t) = g_{1-\alpha}(t) := t^{-\alpha}/\Gamma(1-\alpha), \alpha \in (0,1)$, the term $\frac{d}{dt}[k * (u - u_0)]$ represents the Caputo fractional derivative of order α, the corresponding equation

$$^C D_0^\alpha u(t) + Au(t) = f(u(t)), t > 0, u(0+) = u_0, \tag{1.2}$$

has been studied extensively. We refer to [17, 3] for fractional differential equations on finite dimensional case and [10] for results on existence, uniqueness, certain asymptotic behavior of the solutions to nonlocal differential equations on Hilbert

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spaces. Moreover, in typical applications, the parameters of the equation, namely the order and the initial data, depend on the material constants, and are known to a moderate accuracy, see [5]. Therefore, the continuous dependence of the solutions on the initial condition and the order α, in other words, the stability and structural stability of the solutions, are also of great interest. These types of stability were testified in [6, 7] and were applied to present numerical schemes for the solutions of certain simple or concrete nonlinearities f. Last but not least, the knowledge about the asymptotic behavior of solutions to Equation (1.2) would possibly attain by using the theory of attractors if this equation could generate an autonomous dynamical system. In [2], it was proved that such equations in general case cannot generate a dynamical system, however they can associate with a certain semi-dynamical system as shown in [4].

Our research originates from the existing results for equations involving the Caputo fractional derivative mentioned above. In the present work, we aim at extending and generalizing the stability, structural stability results and the well-definedness of related semi-dynamical system to nonlocal differential equation (1.1). Our main results are Theorem 3.5 on continuous dependence of the solution on the initial condition and the kernel; Theorem 3.6 on Lipschitz continuous of the solution on the initial data; and Theorem 4.2 on the semi-dynamical system generated by Equation (1.1).

We set the following standing assumptions

- (A) *The operator $A : D(A) \to H$ is densely defined and self-adjoint with compact resolvent.*
- (K) *The kernel function $k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative and nonincreasing, and there exists a function $l \in L^1_{loc}(\mathbb{R}^+)$ such that $k * l = 1$ on $(0, \infty)$.*
- (F) *The nonlinearity f satisfies $f(0) = 0$ and is global Lipschitz continuous on H, that is, there exists a positive number L_f such that*

$$\|f(u) - f(v)\| \leq L_f \|u - v\|, \ \forall u, v \in H. \tag{1.3}$$

The condition (K) covers a lot of interesting kernels. For example, the case $k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $\alpha \in (0, 1)$, satisfies (K) with $l(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ thanks to the identity

$$\int_0^t (t - \tau)^{-\alpha} \tau^{\alpha-1} d\tau = \Gamma(\alpha)\Gamma(1 - \alpha).$$

The well-known ultraslow process (more details see [11, 20, 10, 16]) corresponds to the distributed-order derivative, where the kernel $k(t) = \int_0^1 g_{1-\alpha}(t) d\alpha$. This kernel also fulfills the condition (K) with $l(t) = \int_0^\infty \frac{e^{-pt}}{1+p} dp, t > 0$. For further discussion on the generation of various kernels of ultraslow type satisfying condition (K) we refer to the recent paper [13] for more details.

Based on the resolvent families of Prüss, by taking the convolution with L^1-function l, one can transform Equation (1.1) into a Volterra integral equation of the following form

$$u(t) + l * Au(t) = \underbrace{u_0 + l * N_f(u)(t)}_{F(t)}, t > 0. \tag{1.4}$$

This representation has some disadvantages. Because the equation takes place in an infinite dimensional Hilbert space and the operator A is unbounded in H, the solution $u(t)$ in search must take values in the domain of A, which narrows the suitable functional space. Let $D(A) = \{u \in H : Au \in H\}$ furnishes with the graph norm $\|u\|_{D(A)} = (\|u\|^2 + \|Au\|^2)^{1/2}$. By the assumption (A), $D(A)$ is a Banach subspace of H. Consequently, it is harder to verify property of operators acting on that narrower functional space $C([0,T]; D(A))$ in proving existence result. The standard idea to overcome these disadvantages is to establish first the wellposedness of the linear problem and then using the solution operators and perturbation technique to formulate an expression of mild solution to the nonlinear problem. Following this approach, let us denote $S(t,A)u_0$ the unique solution of the linear Volterra equation

$$w(t) = u_0 - A\left(l * w(t)\right), t \geq 0. \tag{1.5}$$

and $R(t,A)$ is the solution operator, which is determined from the expression

$$R(t,A) = l(t)I - A\left(\int_0^t l(t-\tau)R(\tau,A)d\tau\right), t > 0. \tag{1.6}$$

Then, a mild solution of the linear problem and the nonlinear equation (1.1) can be defined as in (3.2) and (3.3). The existence can be shown by a fixed point argument under the global Lipschitz continuity of the nonlinearity f.

Our present work is organized as follows: in Section 2, we summarize some facts about the generalized Mittag–Leffler functions and the resolvent families; in Section 3, we state and prove the main results on the existence and continuous dependence on the kernel of the solutions to semilinear equations; in Section 4 we construct the related semi-dynamical system to Equation (1.1).

2. Preliminaries

2.1. **Generalized Mittag–Leffler functions.** Let k be a singular kernel which satisfies the hypothesis (K). Due to the fact that $l \in L^1_{loc}(\mathbb{R}^+)$, as mentioned in [12], see also [8, Theorem 3.1] the following scalar Volterra equation

$$r(t,\mu) + \mu l * r(t,\mu) = l(t), t > 0, \tag{2.1}$$

has a unique $L^1_{loc}(\mathbb{R}^+)$ solution, for an arbitrary $\mu \in \mathbb{R}$. As a consequence of [1, Theorem 2.2], condition (K) implies that l is completely positive, hence $r(t,\mu) > 0$ for all $\mu > 0$. On the other hand, $l(t) > 0$ for almost all $t > 0$. Combining this with the representation

$$r(t,\mu) = l(t) + \sum_{j=1}^{\infty} (-\mu)^j \underbrace{l * l * \ldots * l}_{j \text{ times } l}(t), t > 0, \tag{2.2}$$

consequently, we get $r(t,\mu) > 0$ for $\mu \leq 0$. Thus, $r(t,\mu) > 0$ for all $\mu \in \mathbb{R}$. We denote by $s(t,\mu)$ the unique solution of the following relaxation equation

$$s(t,\mu) + \mu \int_0^t l(t-\tau)s(\tau,\mu)d\tau = 1, t > 0.$$

Then the functions $s(t,\mu)$ and $r(t,\mu)$ satisfy the relation

$$s(t,\mu) = \int_0^t k(t-\tau)r(\tau,\mu)d\tau, \quad \frac{ds(t,\mu)}{dt} = -\mu r(t,\mu), t > 0. \tag{2.3}$$

Therefore, $s(t,\mu) > 0$ for arbitrary $\mu \in \mathbb{R}$.

We now recall some further properties of these functions, which play an important role in our analysis.

Proposition 2.1. [10, 21] *Let the hypothesis* (K) *hold. Then for every* $\mu > 0$, $s(\cdot,\mu), r(\cdot,\mu) \in L^1_{loc}(\mathbb{R}^+)$. *In addition, we have:*

(1) *The function* $s(\cdot,\mu)$ *is nonnegative and nonincreasing if* $\lambda > 0$, *nonnegative and nondecreasing if* $\lambda < 0$. *Moreover,*

$$s(t,\mu)\left[1+\mu\int_0^t l(\tau)d\tau\right] \le 1,\ \forall t \ge 0,$$

hence if $l \notin L^1(\mathbb{R}^+)$ *then* $\lim_{t\to\infty} s(t,\mu) = 0$ *for every* $\mu > 0$.

(2) *For* $\mu > 0$, *the function* $r(\cdot,\mu)$ *is nonnegative and one has*

$$s(t,\mu) = 1 - \mu\int_0^t r(\tau,\mu)d\tau = k * r(\cdot,\mu)(t),\ t \ge 0,$$

so $\int_0^t r(\tau,\mu)d\tau \le \mu^{-1},\ \forall t > 0$.

If $\lim_{t\to+\infty} 1 * l(t) = +\infty$ *then*

$$\lim_{t\to+\infty} s(t,\mu) = 0 \text{ and } \int_0^\infty r(\tau,\mu)d\tau = \mu^{-1}.$$

(3) *For each* $t > 0$, *the functions* $\mu \mapsto s(t,\mu)$ *and* $\mu \mapsto r(t,\mu)$ *are nonincreasing.*

(4) *Let* $v(t) = s(t,\mu)v_0 + (r(\cdot,\mu) * g)(t)$, *here* $g \in L^\infty(\mathbb{R}^+)$. *Then* v *solves the problem*

$$\frac{d}{dt}[k * (v - v_0)](t) + \mu v(t) = g(t),\ v(0) = v_0.$$

We also refer to the following Gronwall type inequality, which was shown in [10, Proposition 2.2].

Lemma 2.2. *Let* v *be a nonnegative function satisfying*

$$v(t) \le s(t,\mu)v_0 + \int_0^t r(t-\tau,\mu)[\alpha v(\tau) + \beta(\tau)]d\tau, t \ge 0,$$

for $\mu > 0, v_0 \ge 0, \alpha > 0$ *and* $\beta \in L^1_{loc}(\mathbb{R}^+)$. *Then*

$$v(t) \le s(t,\mu-\alpha)v_0 + \int_0^t r(t-\tau,\mu-\alpha)\beta(\tau)d\tau.$$

Particularly, if β *is a constant then*

$$v(t) \le s(t,\mu-\alpha)v_0 + \frac{\beta}{\mu-\alpha}(1 - s(t,\mu-\alpha)).$$

2.2. **Resolvent families.**

2.2.1. *Solution operators.* It is easy to see that equation (1.1) is equivalent to the following Volterra equation (Prüss form):

$$u(t) + l * Au(t) = u_0 + l * f(t). \tag{2.4}$$

Note that this right-hand side $l * f$ is different from the well-known one in Prüss formulation. Via the Laplace transformation, the solution operator is given by

$$\widehat{u}(\lambda)u_0 = \widehat{k}(\lambda)R(\lambda\widehat{k}(\lambda), -A)u_0 + R(\lambda\widehat{k}(\lambda), -A)\widehat{f}(\lambda),$$

where $R(z,-A) = (I + zA)^{-1}$.

Motivated by the work in [16] for general Volterra integral equations, we use the followings:

Definition 2.1. *A family $\{S(t) = S(t, A)\}_{t\geq 0}$ of bounded linear operators in H is called a Caputo resolvent for* (1.1) *if the following conditions are satisfied:*

- (S1) *$S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$.*
- (S2) *$S(t, A)$ is commuted with A in the sense that $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax, \forall x \in D(A)$ and $t \geq 0$;*
- (S3) *The resolvent equation holds*

$$S(t)x + \int_0^t l(t-s)AS(s)xds = x \text{ for all } x \in D(A), t \geq 0. \tag{2.5}$$

Definition 2.2. *A family $\{R(t) = R(t, A)\}_{t>0}$ of bounded linear operators in H is called an integral resolvent generated by A if the following conditions are satisfied:*

- (R1) *$R(t)$ is strongly continuous on $t > 0$.*
- (R2) *$R(t)$ is commuted with A in the sense that $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax, \forall x \in D(A)$ and $t > 0$;*
- (R3) *The resolvent equation holds*

$$R(t)x + \int_0^t l(t-s)AR(s)xds = l(t)x \text{ for all } x \in D(A), t > 0. \tag{2.6}$$

2.2.2. *Known results.* When A is a (bounded) normal operator in a Hilbert space H then the necessary and sufficient conditions for the well-posedness of Equation (1.1) is formulated in terms of spectral functions $s(t, \mu), r(t, \mu)$, the solutions of the following scalar Volterra equations:

$$\begin{aligned} s(t,\mu) + \mu \int_0^t l(t-\tau)s(\tau,\mu)d\tau &= 1, \qquad t > 0, \\ r(t,\mu) + \mu \int_0^t l(t-\tau)r(\tau,\mu)d\tau &= l(t), \quad t > 0. \end{aligned} \tag{2.7}$$

See [16, Prop. 1.5] for more details.

Under the assumption (A), there exists a normal basis of H which consists of eigenvectors $\{e_n\}_{n\geq 1}$ corresponding to the nondecreasing sequence of eigenvalues $\{\lambda_n\}_{n\geq 1}$ of the operator A. In this case, the solution operators $S(t), R(t)$ can be represented as Fourier series as follow:

$$\begin{aligned} S(t)v &:= \sum_{n=1}^{\infty} s(t,\lambda_n)\langle v, e_n\rangle e_n, \\ R(t)v &:= \sum_{n=1}^{\infty} r(t,\lambda_n)\langle v, e_n\rangle e_n, \; v \in H. \end{aligned}$$

In case A has positive spectrum, that means $\lambda_1 > 0$, the well-definedness and qualitative properties of $S(t), R(t)$ are considered in detail by the authors in [10]. Note that it easily extends to the case $\lambda_1 < 0$.

Lemma 2.3. [10] *Let $\{S(t)\}_{t\geq 0}$ and $\{R(t)\}_{t>0}$, be the families of linear operators defined by* (2.5) *and* (2.6)*, respectively. Then*

(1) For each $v \in H$ and $T > 0$, $S(\cdot)v \in C([0,T];H)$ and $AS(\cdot)v \in C((0,T];H)$. Moreover,

$$\|S(t)v\| \le s(t,\lambda_1)\|v\|,\ t \in [0,T],$$

$$\|AS(t)v\| \le \frac{\|v\|}{(1*l)(t)},\ t \in (0,T].$$

(2) Let $v \in H, T > 0$ and $g \in C([0,T];H)$. Then $R(\cdot)v \in C((0,T];H)$, $R*g \in C([0,T];H)$ and $A(R*g) \in C([0,T];V_{-\frac{1}{2}})$. Furthermore,

$$\|R(t)v\| \le r(t,\lambda_1)\|v\|,\ t \in (0,T],$$

$$\|(R*g)(t)\| \le \int_0^t r(t-\tau,\lambda_1)\|g(\tau)\|d\tau,\ t \in [0,T],$$

$$\|A(R*g)(t)\|_{-\frac{1}{2}} \le \left(\int_0^t r(t-\tau,\lambda_1)\|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}},\ t \in [0,T].$$

3. Solvability and structural stability

3.1. **Solvability of semilinear initial value problems.** We consider the linear Cauchy problem of differential form

$$k * u'(t) + Au(t) = g(t), u(0) = u_0, \tag{3.1}$$

where g is a bounded continuous function on $\mathbb{R}^+$. It is proved in [10] that the equation (3.1) possesses a mild solution (and actually a weak solution) $u : \mathbb{R}^+ \to H$, which can be expressed as follows

$$u(t) = S(t)u_0 + \int_0^t R(t-\tau)g(\tau)d\tau,\ t \ge 0. \tag{3.2}$$

Motivated by formula (3.2), a function u is called a mild solution to the original problem (1.1) if

$$u(t) = S(t)u_0 + \int_0^t R(t-\tau)N_f(u)(\tau)d\tau, t > 0, \tag{3.3}$$

where $N_f(u)(t) = f(u(t))$ stands for the Nemytskii operator induced by the nonlinear map f. Based on this point of view, we are now examining the semilinear Volterra equation of the form

$$u(t) = \varphi(t) + \int_0^t R(t-\tau)N_f(u)(\tau)d\tau, t > 0. \tag{3.4}$$

where φ is a given function. Mild solutions for initial value problem (1.1) is a special case when $\varphi(t) = S(t,A)u_0$.

Utilizing perturbation technique and working in a Banach space furnished by a suitable Bielecki weighted norm, we obtain the solvability result of nonlinear problem.

Theorem 3.1. *Assume that (A), (K) and (F) are fulfilled. Then for an arbitrary $T > 0$ and given continuous function $\varphi \in C([0,T];H)$, Problem (3.4) has a unique mild solution $u(t,\varphi)$ on $[0,T]$. Furthermore, one has*

$$\|u(t)\| \le M_\varphi s(t,\lambda_1 - L_f),\ \text{where } M_\varphi = \sup_{t\in[0,T]} \frac{\|\varphi(t)\|}{s(t,\lambda_1)}. \tag{3.5}$$

Proof. Fix a suitable constant γ which will be specified later. Let $C(\gamma)$ denote the Banach space of all functions $w \in C([0,T];H)$ endowed with the following Bielecki weighted norm

$$\|w\|_{C(\gamma)} = \sup_{t\in[0,T]} \frac{\|w(t)\|}{s(t,\gamma)}. \tag{3.6}$$

We define an operator Φ on the space $C([0,T];H)$ by

$$\Phi(u)(t) = \varphi(t) + \int_0^t R(t-\tau) f(u(\tau)) d\tau.$$

By properties of the operators $R(t)$ in Lemma 2.3(2), one concludes that Φ acts on $C([0,T];H)$.

Let u, v be in $C([0,T];H)$. Because $f(\cdot)$ is Lipschitz continuous, we have following estimate

$$\begin{aligned}
\|\Phi(u)(t) - \Phi(v)(t)\| &= \left\| \int_0^t R(t-\tau)\big[f(u(\tau)) - f(v(\tau))\big] d\tau \right\| \\
&\le L_f \int_0^t \|R(t-\tau)\| \|u(\tau) - v(\tau)\| d\tau \\
&\le L_f \int_0^t r(t-\tau,\lambda_1) s(\tau,\gamma) \frac{\|u(\tau)-v(\tau)\|}{s(\tau,\gamma)} d\tau \\
&\le L_f \sup_{\tau\in[0,T]} \frac{\|u(\tau)-v(\tau)\|}{s(\tau,\gamma)} \int_0^t r(t-\tau,\lambda_1) s(\tau,\gamma) d\tau. \qquad (3.7)
\end{aligned}$$

We compute the last integral by using the fundamental properties of generalized Mittag-Lefller functions

$$s(t,a) - s(t,b) = (b-a) \int_0^t r(t-\tau,a) s(\tau,b) d\tau, a, b \in \mathbb{R}.$$

Indeed, direct computation from the defining integral equations of $s(t,a)$ and $s(t,b)$ we have

$$s(t,a) - s(t,b) + al * (s(t,a) - s(t,b)) = (b-a) l * s(t,b).$$

That means $w = s(t,a) - s(t,b)$ is a solution to the equation

$$w + al * w = l * (b-a) s(t,b).$$

Moreover, $r(t,a)$ is the unique solution to the equation $r(t,a) + al * r(t,a) = l$, we obtain

$$s(t,a) - s(t,b) = w = r(t,a) * (b-a) s(t,b) = (b-a) \int_0^t r(t-\tau,a) s(\tau,b) d\tau.$$

Therefore, the estimate (3.7) induces

$$\begin{aligned}
\|\Phi(u)(t) - \Phi(v)(t)\| &\le L_f . \|u-v\|_{C(\gamma)} \frac{s(t,\gamma) - s(t,\lambda_1)}{\lambda_1 - \gamma} \\
&\le \frac{L_f}{\lambda_1 - \gamma} \|u-v\|_{C(\gamma)} s(t,\gamma).
\end{aligned}$$

Consequently,

$$\sup_{t\in[0,T]} \frac{\|\Phi(u)(t) - \Phi(v)(t)\|}{s(t,\gamma)} \le \frac{L_f}{\lambda_1 - \gamma} \|u-v\|_{C(\gamma)}.$$

It yields that Φ is a contraction mapping in $C(\gamma)$ provided that $\gamma < \lambda_1 - L_f$. Hence, by Banach fixed point theorem, Φ has a unique fixed point $u \in C([0,T]; H)$.

Moreover, by the definition of $\Phi(u)$, we can rewrite

$$u(t) = \varphi(t) + \int_0^t R(t-\tau)\left(f(u(\tau)) - f(0)\right) d\tau,$$

which together with Lemma 2.1 imply that

$$\begin{aligned}\|u(t)\| \leq & \|\varphi(t)\| + L_f \int_0^t \|R(t-\tau)\|.\|u(\tau)\| d\tau \\ \leq & M_\varphi s(t,\lambda_1) + L_f \int_0^t r(t-\tau,\lambda_1)\|u(\tau)\| d\tau,\end{aligned}$$

Appyling Lemma 2.2, we obtain

$$\|u(t)\| \leq M_\varphi s(t, \lambda_1 - L_f). \tag{3.8}$$

□

In particular, when $\varphi(t) = S(t,A)u_0$, it holds for a mild solution to Equation (1.1)

$$\|u(t,\varphi)\| \leq s(t, \lambda_1 - L_f)\|u_0\|, t \geq 0.$$

3.2. **Continuous dependence of solutions on the kernel.** In this section, we denote $s(t,\mu) = s(t,\mu;l)$, $r(t,\mu) = r(t,\mu;l)$ to emphasize the dependence of s and r on the kernel l.

Lemma 3.2. *For a given number $\mu \in \mathbb{R}$, the mapping $l \mapsto s(t,\mu;l)$ is continuous and bounded from $L^1(0,T)$ to $C([0,T];\mathbb{R})$.*

Proof. The function $s(t,\mu;l)$ is the unique solution of the scalar Volterra equation

$$w + \mu l * w = 1.$$

Therefore, $s(t,\mu;l)$ possesses the following representation

$$s(t,\mu;l) = 1 - \mu 1 * l + \mu^2 1 * l * l - \ldots + (-\mu)^m 1 * \underbrace{l * \ldots * l}_{m \text{ times } l} + \ldots \tag{3.9}$$

Take two kernels $l, \bar{l} \in L^1(0,T)$, and denote the corresponding solutions by $s(t,\mu;l)$ and $\bar{s}(t,\mu;\bar{l})$.

Set $\mathcal{E}(t) = |s(t,\mu;l) - \bar{s}(t,\mu;\bar{l})|$. From the scalar defining integral equation of s, we have

$$\begin{aligned}\mathcal{E}(t) &= \left|\mu \int_0^t l(t-\tau)\left(s(\tau,\mu;l) - \bar{s}(\tau,\mu;\bar{l})\right) d\tau - \mu \int_0^t (\bar{l} - l)(t-\tau)\bar{s}(\tau,\mu;\bar{l}) d\tau\right| \\ &\leq |\mu| l * |s - \bar{s}|(t) + |\mu| 1 * |l - \bar{l}|(t).\end{aligned}$$

By Lemma 2.2, one has

$$\mathcal{E}(t) \leq |\mu| s(t, -|\mu|)\|l - \bar{l}\|_{L^1(0,T)} \leq |\mu| s(T, -|\mu|; l)\|l - \bar{l}\|_{L^1(0,T)}.$$

Hence, $\|s(\cdot,\mu;l) - \bar{s}(\cdot,\mu;\bar{l})\|_{C[0,T]} \leq |\mu| s(T, -|\mu|; l)\|l - \bar{l}\|_{L^1(0,T)}$, which finishes the proof. □

Let CP_0 be the set of all function $l \in L^1(0,T)$ such that there exists a nonnegative, noninceasing function $k \in L^1(0,T)$ such that $k * l = 1$ on $(0,T)$.

Lemma 3.3. *For $\mu \geq 0$, the mapping $l \mapsto r(t, \mu; l)$ is continuous from $L^1(0, T) \cap CP_0$ to $L^1(0, T) \cap CP_0$.*

Proof. If $k*l = 1$ then by direct computation, the function $k_\mu(t) = k(t) + \mu$ satisfies

$$\int_0^t k_\mu(t-\tau) r(\tau, \mu) d\tau = k_\mu * r = 1, \text{ for all } t > 0.$$

Therefore, when $\mu \geq 0$, the mapping $l \mapsto r(t, \mu; l)$ maps $L^1(0, T) \cap CP_0$ to itself.

Let $r(t, \mu; l), \bar{r}(t, \mu; \bar{l})$ be solutions corresponding to the kernels l and $\bar{l}$, respectively. Set $\tilde{\mathcal{E}}(t) = r(t, \mu; l) - \bar{r}(t, \mu; \bar{l})$. From the defining equations of r and $\bar{r}$, one reduces

$$\tilde{\mathcal{E}}(t) \leq \mu l * \tilde{\mathcal{E}}(t) + |l - \bar{l}|(t) + \mu \bar{r}(\cdot, \mu) * |l - \bar{l}|(t). \tag{3.10}$$

Choosing a positive constant γ such that

$$\mu \int_0^T l(t) e^{-\gamma t} dt < 1/2 \tag{3.11}$$

thanks to the fact that $\lim_{\gamma \to +\infty} \int_0^T l(t) e^{-\gamma t} dt = 0$. Multiplying both sides of (3.10) by the function $e^{-\gamma t}$ and taking the L^1-norm, one gets

$$\begin{aligned} \|e^{-\gamma t} \tilde{\mathcal{E}}\|_{L^1(0,T)} \leq & \mu \int_0^T l(t) e^{-\gamma t} dt \int_0^T \tilde{\mathcal{E}}(t) e^{-\gamma t} dt \\ & + \|l - \bar{l}\|_{L^1(0,T)} (1 + \|\mu \bar{r}(\cdot, \mu)\|_{L^1(0,T)}). \end{aligned} \tag{3.12}$$

By Lemma 2.1 (2), $\|\mu \bar{r}(\cdot, \mu)\|_{L^1(0,T)} \leq \|\mu \bar{r}(\cdot, \mu)\|_{L^1(0,\infty)} \leq 1$. Therefore, one obtains

$$e^{-\gamma T} \|\tilde{\mathcal{E}}\|_{L^1(0,T)} \leq \|e^{-\gamma t} \tilde{\mathcal{E}}\|_{L^1(0,T)} \leq 2 \frac{\|l - \bar{l}\|_{L^1(0,T)}}{1 - \mu \int_0^T e^{-\gamma t} l(t) dt}. \tag{3.13}$$

Finally, one concludes that

$$\|\tilde{\mathcal{E}}\|_{L^1(0,T)} \leq 4 e^{\gamma T} \|l - \bar{l}\|_{L^1(0,T)},$$

where the number γ depends continuously on μ and $\|l\|_{L^1(0,T)}$. □

Remark 3.1. *If $\mu < 0$ then the mapping $l \mapsto r(t, \mu; l)$ is continuous from $L^1(0, T)$ to $L^1(0, T)$ by an analogous argument.*

As we can choose a common γ for all μ varying in a compact set, the family of functions $r(t, \mu; l)$ is uniformly continuous. The continuous dependence of $r(t, \mu; l)$ on the kernel l, for a given μ can be obtained from [8, Theorem 3.1, p.42]. *Now we are in a position now to extend this continuous dependence to the infinite dimensional case as follow.*

Proposition 3.4. *For any $\epsilon > 0$, there exists $\delta > 0$ such that for all completely positive kernels $l, \bar{l}$ satisfying $\|l - \bar{l}\| \leq \delta$, it holds*

$$\int_0^T \|R(t) - \bar{R}(t)\| dt < \epsilon,$$

where $R, \bar{R}$ are solution operators with respect to the kernels $l, \bar{l}$, respectively.

Proof. By the representation formula of solution operator, one has

$$[R-\bar{R}](t)v = \sum_{i=1}^{\infty}(r(t,\lambda_i;l) - r(t,\lambda_i;\bar{l}))\langle v, e_i\rangle e_i, \tag{3.14}$$

For a given μ, decompose $R-\bar{R}$ into three parts

$$\begin{aligned}[R-\bar{R}](t)v =& \sum_{i:\lambda_i\le\mu}(r(t,\lambda_i;l) - \bar{r}(t,\lambda_i;\bar{l}))\langle v, e_i\rangle e_i + \sum_{i:\lambda_i>\mu} r(t,\lambda_i;l)\langle v, e_i\rangle e_i \\ &- \sum_{i:\lambda_i>\mu} \bar{r}(t,\lambda_i;\bar{l})\langle v, e_i\rangle e_i \\ =& E_1(t)v + E_2(t)v - E_3(t)v.\end{aligned}$$

By the decreasing monotonicity of the map $\lambda \mapsto r(t,\lambda;l)$, the last two terms can be easily estimated

$$\|E_2(t)\| = r(t,\lambda(\mu);l), \|E_3(t)\| = \bar{r}(t,\lambda(\mu);\bar{l}) \tag{3.15}$$

where $\lambda(\mu) = \min\{\lambda_i : \lambda_i > \mu\}$. Clearly,

$$\|E_2\|_{L^1(0,T)} = \frac{1-s(T,\lambda(\mu);l)}{\lambda(\mu)} \le \frac{1}{\lambda(\mu)},$$
$$\|E_3\|_{L^1(0,T)} = \frac{1-\bar{s}(T,\lambda(\mu);\bar{l})}{\lambda(\mu)} \le \frac{1}{\lambda(\mu)}.$$

Combining with the fact that $\lim_{\mu\to+\infty}\lambda(\mu) = +\infty$, for an arbitrary given $\epsilon > 0$, one can fix a number μ such that

$$\|E_2\|_{L^1(0,T)} + \|E_3\|_{L^1(0,T)} \le \frac{2}{\lambda(\mu)} \le \epsilon/2. \tag{3.16}$$

We now consider the first term $E_1(t)$. Obviously, definition of E_1 yields

$$\|E_1(t)\| \le \max_{\lambda_i\le\mu}|r(t,\lambda_i;l) - \bar{r}(t,\lambda_i;\bar{l})| \le \sum_{\lambda_i\le\mu}|r(t,\lambda_i;l) - \bar{r}(t,\lambda_i;\bar{l})|.$$

Denote by $n(\mu)$ the number of eigenvalues of A which are less than or equal to μ. By Lemma 3.3, one can choose a $\delta > 0$ such that

$$\|r(\cdot,\lambda_i;l) - \bar{r}(\cdot,\lambda_i;\bar{l})\|_{L^1(0,T)} \le \frac{\epsilon}{2n(\mu)+1}, \forall\lambda_i < \mu, \tag{3.17}$$

provided that $\|l-\bar{l}\|_{L^1(0,T)} < \delta$. Hence,

$$\|E_1\|_{L^1(0,T)} \le \sum_{\lambda_i\le\mu}\|r(\cdot,\lambda_i;l) - \bar{r}(\cdot,\lambda_i;\bar{l})\|_{L^1(0,T)} \le n(\mu)\frac{\epsilon}{2n(\mu)+1} < \epsilon/2. \tag{3.18}$$

Combining (3.16) and (3.18), it follows

$$\|R-\bar{R}\|_{L^1(0,T)} \le \|E_1\|_{L^1(0,T)} + \|E_2\|_{L^1(0,T)} + \|E_3\|_{L^1(0,T)} < \epsilon,$$

provided that $\|\bar{l}-l\|_{L^1(0,T)} < \delta$. This completes the proof. □

Theorem 3.5. *Let $u_l(t,\varphi)$ denote the unique solution to Equation* (3.4) *corresponding to the kernel l and the initial condition φ. Then the solution mapping $(\varphi, l) \mapsto u_l(t,\varphi)$ is continuous from $C([0,T];H)\times L^1(0,T)$ into $C([0,T];H)$.*

Proof. We first observe that: combining the estimate (3.7) and Lemma 3.2, the solution mapping is the composition of bounded maps, so it is also bounded.

Fix a point (φ, l) and consider $(\bar{\varphi}, \bar{l})$ belonging to a neighbourhood of (φ, l). Let $u_l(t,\varphi), \bar{u}_{\bar{l}}(t,\bar{\varphi})$ be the unique mild solutions to Equation (3.4) with the kernels l and $\bar{l}$ and the initial conditions $\varphi(t)$ and $\bar{\varphi}(t)$, respectively.

Let denote $\bar{S}$ and $\bar{R}$ the solution operators corresponding to the kernel $\bar{k}$. By the formulae determining a mild solution to (3.4), we have

$$\begin{aligned}
\|u_l(t,\varphi) - \bar{u}_{\bar{l}}(t,\bar{\varphi})\| &= \left\| \varphi(t) - \bar{\varphi}(t) + \int_0^t \left[R(t-\tau) f(u(\tau)) - \bar{R}(t-\tau) f(\bar{u}(\tau)) \right] d\tau \right\| \\
&\le \|\varphi(t) - \bar{\varphi}(t)\| + \int_0^t \|R(t-\tau)\| \|f(u(\tau)) - f(\bar{u}(\tau))\| d\tau + \\
&\qquad + \int_0^t \|R(t-\tau) - \bar{R}(t-\tau)\| \|f(\bar{u}(\tau))\| d\tau \\
&\le \|\varphi - \bar{\varphi}\|_{C[0,T]} + \int_0^t r(t-\tau, \lambda_1; l) L_f \|u_l(\tau,\varphi) - \bar{u}_{\bar{l}}(\tau,\bar{\varphi})\| d\tau \\
&\qquad + \int_0^T \|R(t) - \bar{R}(t)\| dt \cdot \sup_{[0,T]} \|N_f(\bar{u}_{\bar{l}})(\tau)\|.
\end{aligned}$$

Fix a positive number δ_0. Because the solution map is bounded, there exists a constant $M = M(\delta_0)$ such that whenever $\|\varphi - \bar{\varphi}\| \le \delta_0, \|l - \bar{l}\|_{L^1(0,T)} \le \delta_0$ then

$$\sup_{[0,T]} \|N_f(\bar{u}_{\bar{l}})(\tau)\| \le \|f(0)\| + L_f \sup_{[0,T]} \|\bar{u}_{\bar{l}}(\tau)\| \le M. \tag{3.19}$$

Let ϵ be an arbitrary positive number. Thanks to Proposition 3.3, one can choose a $\delta \in (0,\delta_0) \cap (0, s(T,\lambda_1; l)\epsilon/2)$ such that

$$\|l - \bar{l}\|_{L^1(0,T)} \le \delta \text{ yields } \int_0^T \|R(t) - \bar{R}(t)\| dt \le \frac{s(T,\lambda_1;l)\epsilon}{2M+1}.$$

Set $e(t) = \|u_l(t,\varphi) - \bar{u}_{\bar{l}}(t,\bar{\varphi})\|$. Plugging this estimate above, one gets

$$\begin{aligned}
e(t) &\le \epsilon s(T,\lambda_1; l) + L_f \int_0^t r(t-\tau,\lambda_1; l) e(\tau) d\tau && (3.20)\\
&\le \epsilon s(t,\lambda_1; l) + L_f \int_0^t r(t-\tau,\lambda_1; l) e(\tau) d\tau, && (3.21)
\end{aligned}$$

due to the decreasing monotonicity of the function $t \mapsto s(t,\lambda_1; l)$. Applying Lemma 2.2, one finally gains

$$e(t) \le s(t, \lambda_1 - L_f; l)\epsilon.$$

This means that

$$\|u_l(\cdot,\varphi) - u_{\bar{l}}(\cdot,\bar{\varphi})\|_{C([0,T];H)} = \sup_{[0,T]} e(t) \le \epsilon \|s(\cdot, \lambda_1 - L_f; l)\|_{C[0,T]}.$$

This finishes the proof. □

The kernel k naturally appears in Equation (1.1), while its related kernel l presents in the Prüss form Equation (1.4). If CP_0 is the set of all function $l \in L^1(0,T)$ such that there exists a nonnegative, nonincreasing function $k \in L^1(0,T)$ such that $k * l = 1$ on $(0,T)$, then the mapping $k \mapsto l$ is an injection. One can use either k or l to present the pair of kernels (k,l). One may furnish the set of kernels k with the induced topology from this injective mapping and then obtain the continuous dependence of solutions on the kernel k directly. However, since the original kernel k also requires the nondecreasing monotonicity, which is difficult to be characterised from topological point of view, hence, we decide to formulate our continuous dependence results on the induced kernel l. In some special cases, when the mapping $k \mapsto l$ is continuous, then one concludes that the solution continuously depends on the kernel k too. Let us consider the following two examples.

(1) The subclass of fractional kernels, $k(t) = g_{1-\alpha}(t), \alpha \in (0,1)$. Clearly, if $k_n(t) = g_{1-\alpha_n}(t), \{\alpha_n\}$ converges to a fractional order $\alpha \in (0,1)$ then the corresponding kernel $l_n = g_{\alpha_n}$ tends to g_α in $L^1(0,T)$. Therefore, one gains that the solution of fractional order continuously depends on their order.

(2) The subclass of ultraslow diffusion kernels $k(t) = \int_0^1 g_{1-\alpha}(t)d\mu(\alpha)$ with μ is a nonnegative continuous function on $[0,1]$ as in [11]. If $\{k_n\}_{n\ge 1}$ is a sequence of kernels related to a sequence of weight functions $\{\mu_n(\alpha)\}_{n\ge 1}$ then by the formula stated in [11, page 261], the corresponding kernels $\{l_n\}_{n\ge 1}$ is given by

$$l_n(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\tau} \frac{\int_0^1 \tau^\alpha \sin(\alpha\pi)\mu_n(\alpha)d\alpha}{[\int_0^1 \tau^\alpha \sin(\alpha\pi)\mu_n(\alpha)d\alpha]^2 + [\int_0^1 \tau^\alpha \cos(\alpha\pi)\mu_n(\alpha)d\alpha]^2} d\tau.$$

Therefore, if $\mu_n \to \mu$ in $C^3([0,1])$ then $l_n \to l$ as $n \to \infty$. Thus one obtains the continuous dependence of solutions on the weight function $\mu \in C^3([0,1])$.

3.3. **Continuous dependence of solutions on the initial function.**

Theorem 3.6. *For a given kernel k, the solution mapping $\varphi \mapsto u(t,\varphi)$ corresponding to Equation* (3.4) *is Lipschitz continuous on $C([0,T];H)$.*

Proof. Assume $\varphi, \bar\varphi \in C([0,T];H)$. Let $er(t) = \|u(t,\varphi) - u(t,\bar\varphi)\|, t \in [0,T]$. Using the similar argument as in the proof of Theorem 3.5, one has

$$er(t) \le \|\varphi(t) - \bar\varphi(t)\| + \int_0^t r(t-\tau,\lambda_1)L_f er(\tau)d\tau \tag{3.22}$$

$$\le \|\varphi - \bar\varphi\|_{C[0,T]} + L_f \int_0^t r(t-\tau,\lambda_1)er(\tau)d\tau. \tag{3.23}$$

Denote $\Psi \in C[0,T]$ is the unique solution of the following Volterra equation

$$\Psi(t) = 1 + L_f \int_0^t r(t-\tau,\lambda_1)\Psi(\tau)d\tau, t > 0.$$

Then one concludes that

$$er(t) \le \Psi(t)\|\varphi - \bar\varphi\|_{C([0,T];H)},$$

which finally implies that

$$\|u(\cdot,\varphi) - u(\cdot,\bar\varphi)\|_{C([0,T];H)} \le \|\Psi\|_{C[0,T]} \cdot \|\varphi - \bar\varphi\|_{C([0,T];H)}.$$

The proof is completed. □

4. The related semi-group construction

In the case of finite dimensional without nonlinearity, that is, $H = \mathbb{R}^d$, A is a given matrix and $f = 0$, the nonlocal equation (1.1) can be transformed into a linear Volterra equation, which has been investigated thoroughly in [8, Chapter 8] via the initial and forcing function semigroups. Motivated by this approach, we now to consider the infinite dimensional case with Lipschitzian nonlinearity.

Lemma 4.1. *Assume that l is Laplace transformable on the right half plane, that is, for all $\epsilon > 0$,*

$$\hat{l}(\epsilon) = \int_0^{+\infty} e^{-\epsilon t} l(t)dt < +\infty. \tag{4.1}$$

For a given $\mu \in \mathbb{R}_+$ and a the kernels k satisfying (K), there exists a constant $\omega = \omega(\mu, l) > 0$ such that $s(t, -\mu; l) \leq 2e^{\omega t}$ for all $t \geq 0$.

Proof. The function $s(t, -\mu)$ is the unique solution of scalar Volterra equation $s(t, -\mu) = 1 + \mu l * s(t, -\mu)$, which possesses the following representation (see (3.9))

$$s(t, -\mu) = 1 + \mu 1 * l(t) + \mu^2 1 * l * l(t) + \ldots + \mu^j 1 * \underbrace{l * \ldots * l}_{j \text{ times } l}(t) + \ldots$$

Multiplying both sides by the function $e^{-\omega t}$ and direct computation implies

$$s(t, -\mu)e^{-\omega t} = e^{-\omega t} + e^{-\omega t} * m + e^{-\omega t} * m * m + \ldots$$

where $m(t) = \mu e^{-\omega t} l(t)$. Thanks to (4.1), we choose $\omega > 0$ such that

$$\mu \int_0^{+\infty} e^{-\omega t} l(t)dt < 1/2.$$

Because $m(t) \geq 0$, $\omega > 0$, we have

$$\begin{aligned} e^{-\omega t} * \underbrace{m * \ldots * m}_{j \text{ times } m}(t) &\leq 1 * \underbrace{m * \ldots * m}_{j \text{ times } m}(t) \\ &\leq [1 * m(t)]^j \text{ (by induction on } j) \\ &\leq \left(\int_0^{+\infty} m(t)dt \right)^j < 2^{-j}. \end{aligned}$$

Therefore, one concludes that

$$e^{-\omega t} s(t, -\mu) \leq \sum_{j=0}^{+\infty} 2^{-j} = 2. \tag{4.2}$$

Thus, $s(t, -\mu) \leq 2e^{\omega t}$ for all $t \geq 0$. $\square$

We denote by $\mathcal{C}([0, +\infty); H)$ the space of continuous functions $\varphi\colon \mathbb{R}^+ \to H$, with the topology of uniform convergence on compact subsets. This topology is metrizable, whose metric is given by

$$\rho(\varphi, \phi) = \sum_{n=1}^{\infty} \frac{1}{q^n} \rho_n(\varphi, \phi), \tag{4.3}$$

where

$$\rho_n(\varphi, \phi) = \frac{\sup_{t \in [0,n]} \|\varphi(t) - \phi(t)\|}{1 + \sup_{t \in [0,n]} \|\varphi(t) - \phi(t)\|},$$

and $q > q_*$ with $q_* := \begin{cases} 1, \text{ if } \lambda_1 > 0 \\ e^{\omega}, \text{ if } \lambda_1 \leq 0 \end{cases}$. Here we design the metric inducing the topology of uniform convergence on compact intervals by specifying the value of q such that the following semi-dynamical family consists of Lipschitz continuous transformations. We now in a position to follow closely the idea in Son et. al. in [4] (see also [19, Chapter XI]).

For a given Lipschitzian nonlinearity f, we define the following family $\{T_t\}_{t\geq 0}$ of transformation on $\mathcal{C}([0,+\infty);H)$

$$T_t\varphi(\theta) = \varphi(t+\theta) + \int_0^t R(t+\theta-\tau)N_f(u)(\tau)d\tau, \text{ for } \theta \geq 0,\ \varphi \in \mathcal{C}([0,+\infty);H), \quad (4.4)$$

where $u : \mathbb{R}^+ \to H$ is the unique solution of the initial value problem (3.4), namely

$$u(t) = \varphi(t) + \int_0^t R(t-\tau)N_f(u)(\tau)d\tau.$$

Theorem 4.2. *Suppose that the assumptions (A), (K) and (F) hold. Then the family of operators $\{T_t\}_{t\geq 0}$ associated to Problem (3.4) forms a semigroup of Lipschitzian mapping in $\mathcal{C}([0,+\infty);H)$ with respect to the metric ρ.*

Proof. The proof is divided into two parts.

Step 1. The operator $T_t\colon \mathcal{C}([0,+\infty);H) \to \mathcal{C}([0,+\infty);H)$ is Lipschitz continuous. For a given t, let $\lceil t \rceil = \min\{m \in \mathbb{N} : m \geq t\}$. One has for $\varphi, \phi \in \mathcal{C}([0,+\infty);H)$:

$$\begin{aligned}
\|T_t\varphi(\theta) - T_t\phi(\theta)\| &\leq \|\varphi(t+\theta) - \phi(t+\theta)\| + \\
&\qquad + L_f \sup_{\tau\in[0,t]} \|u(\tau,\varphi) - u(\tau,\phi)\| \int_0^t \|R(t+\theta-\tau)\| d\tau \\
&\leq \|\varphi(t+\theta) - \phi(t+\theta)\| + \\
&\qquad + L_f \sup_{\tau\in[0,t]} \|u(\tau,\varphi) - u(\tau,\phi)\| \int_0^t r(t-\tau+\theta,\lambda_1) d\tau.
\end{aligned}$$

1. Case 1: $\lambda_1 \leq 0$. By the decreasing monotonicity with respect to λ of $r(t,\lambda)$, one sees that $r(\tau,\lambda) \leq r(\tau,-\mu)$, where $\mu = |\lambda_1| + 1 > 0$. Hence,

$$\int_0^t r(t-\tau+\theta,\lambda_1)d\tau \leq \frac{s(t+\theta,-\mu) - s(\theta,-\mu)}{\mu} \leq \frac{s(t+\theta,-\mu)}{\mu}. \quad (4.5)$$

Applying Lemma 4.1 for this μ, it follows that

$$\begin{aligned}
\sup_{[0,n]} \|T_t\varphi(\theta) - T_t\phi(\theta)\| &\leq \sup_{[0,n]} \|\varphi(\theta+t) - \phi(\theta+t)\| + \\
&\qquad + \sup_{[0,n]} s(t+\theta,-\mu)\frac{L_f}{\mu}\|u(\theta,\varphi) - u(\theta,\phi)\|,
\end{aligned}$$

which yields

$$\rho_n(T_t\varphi, T_t\phi) \leq \rho_{n+\lceil t \rceil}(\varphi,\phi) + \frac{L_f}{\mu} 2e^{\omega(n+\lceil t \rceil)} \sup_{\theta\in[0,t]} \|u(\theta,\varphi) - u(\theta,\phi)\|.$$

It reduces to

$$\rho(T_t\varphi, T_t\phi) \leq q^{\lceil t \rceil}\rho(\varphi,\phi) + \sum_{n\geq 1} \frac{2L_f}{\mu q^n} e^{\omega(n+\lceil t \rceil)} \sup_{\theta\in[0,t]} \|u(\theta,\varphi) - u(\theta,\phi)\|.$$

This means that

$$\rho(T_t\varphi, T_t\phi) \le q^{\lceil t\rceil}\rho(\varphi,\phi) + \frac{2L_f e^{\omega(\lceil t\rceil)+1}}{\mu(q-e^{\omega})}\sup_{\theta\in[0,t]}\|u(\theta,\varphi)-u(\theta,\phi)\|. \tag{4.6}$$

2. Case 2: $\lambda_1 > 0$. We have

$$\int_0^t r(t-\tau+\theta,\lambda_1)d\tau = \int_\theta^{t+\theta} r(\xi,\lambda_1)d\xi \le \int_0^{+\infty} r(\xi,\lambda_1)d\xi = 1/\lambda_1,$$

Hence, one concludes that

$$\|T_t\varphi(\theta)-T_t\phi(\theta)\| \le \|\varphi(t+\theta)-\phi(t+\theta)\| + \frac{L_f}{\lambda_1}\sup_{\tau\in[0,t]}\|u(\tau,\varphi)-u(\tau,\phi)\|.$$

Similar to case 1, one obtains

$$\begin{aligned}\sup_{[0,n]}\|T_t\varphi(\theta)-T_t\phi(\theta)\| \le \sup_{[0,n]}\|\varphi(\theta+t)-\phi(\theta+t)\|+ \\ + \frac{L_f}{\lambda_1}\sup_{\tau\in[0,t]}\|u(\tau,\varphi)-u(\tau,\phi)\|.\end{aligned}$$

This estimate yields that

$$\rho_n(T_t\varphi,T_t\phi) \le \rho_{n+\lceil t\rceil}(\varphi,\phi) + \frac{L_f}{\lambda_1}\sup_{\tau\in[0,t]}\|u(\tau,\varphi)-u(\tau,\phi)\|,$$

and consequently,

$$\begin{aligned}\rho(T_t\varphi,T_t\phi) &\le q^{\lceil t\rceil}\rho(\varphi,\phi) + \sum_{n\ge1}\frac{1}{q^n}\frac{L_f}{\lambda_1}\sup_{\tau\in[0,t]}\|u(\tau,\varphi)-u(\tau,\phi)\| \\ &\le q^{\lceil t\rceil}\rho(\varphi,\phi) + \frac{1}{q-1}\frac{L_f}{\lambda_1}\sup_{\tau\in[0,t]}\|u(\tau,\varphi)-u(\tau,\phi)\|. \qquad (4.7)\end{aligned}$$

As a consequence of Theorem 3.6, there exists a constant C which is independent of φ,ϕ such that

$$\|u(\cdot,\varphi)-u(\cdot,\phi)\|_{C([0,t];H)} \le C\|\varphi-\phi\|_{C([0,t];H)} \le \widetilde{C}\rho(\varphi,\phi).$$

Combining this fact and the estimates (4.6) and (4.7) in each case mentioned above, we conclude that T_t is a Lipschitz continuous mapping on $\mathcal{C}([0,+\infty);H)$.

Step 2. The family $\{T_t, t\ge 0\}$ satisfies the law of semigroup. Namely, take $\sigma,\tilde\sigma\in\mathbb{R}^+$, $\varphi\in\mathcal{C}([0,+\infty);H)$ and denote $u(\theta)=u_\varphi(\theta)$ the solution of Equation (3.4). By this equation, we have

$$\begin{aligned}u(\sigma+\theta) &= \varphi(\sigma+\theta) + \int_0^{\sigma+\theta} R(t+\theta-\tau)f(u(\tau))d\tau \\ &= \varphi(\sigma+\theta) + \int_0^{\sigma} R(\sigma+\theta-\tau)f(u(\tau))d\tau + \int_\sigma^{\sigma+\theta} R(\sigma+\theta-\tau)f(u(\tau))d\tau \\ &= T_\sigma\varphi(\theta) + \int_0^\theta R(\theta-r)f(u(\sigma+r))dr, (\text{we change } \tau=\sigma+r).\end{aligned}$$

It means that $\psi(\theta):=u(\sigma+\theta)$ is the unique solution to the following equation

$$\psi(\theta) = T_\sigma\varphi(\theta) + \int_0^\theta R(\theta-r)f(\psi(r))dr.$$

$T_\sigma \varphi$ plays the same role as φ in Equation (3.4), therefore, it holds

$$\begin{aligned}T_{\tilde{\sigma}}\left(T_\sigma \varphi\right)(\theta) &= T_\sigma \varphi(\tilde{\sigma}+\theta) + \int_0^{\tilde{\sigma}} R(\tilde{\sigma}+\theta-\tau) f(\psi(\tau)) d\tau \\ &= \varphi(\sigma+\tilde{\sigma}+\theta) + \int_0^{\sigma} R(\sigma+\tilde{\sigma}+\theta-\tau) f(u(\tau)) d\tau + \\ &\quad + \int_0^{\tilde{\sigma}} R(\tilde{\sigma}+\theta-\tau) f(\psi(\tau)) d\tau.\end{aligned}$$

We change $\tau := r - \sigma$ in the last integral, and substitute $\psi(r-\sigma) = u(r)$, which yields

$$\int_0^{\tilde{\sigma}} R(\tilde{\sigma}+\theta-\tau) f(u(\sigma+\tau)) d\tau = \int_{\sigma}^{\sigma+\tilde{\sigma}} R(\sigma+\tilde{\sigma}+\theta-r) f(u(r) dr.$$

We obtain that

$$\begin{aligned}T_{\tilde{\sigma}}\left(T_\sigma \varphi\right)(\theta) &= \varphi(\sigma+\tilde{\sigma}+\theta) + \int_0^{\sigma+\tilde{\sigma}} R(\sigma+\tilde{\sigma}+\theta-\tau) f(u(\tau)) d\tau \\ &= T_{\sigma+\tilde{\sigma}} \varphi(\theta) \text{ for all } \theta \geq 0.\end{aligned}$$

φ is arbitrary, therefore, we conclude that $T_{\tilde{\sigma}} \circ T_\sigma = T_{\tilde{\sigma}+\sigma}$, $\{T_t\}$ satisfies the semigroup law. □

In the present work, we consider derivatives of Caputo type, hence, some results are stated in the class of continuous functions. For other types of derivatives, for example the nonlocal derivatives of Riemann-Liouville type ${}^{RL}D^{(k)}u(t) = \frac{d}{dt}[k * u(t)], t > 0$, one must work on a larger class of functional spaces, such as $L^p((0,T); H)$. As one can see from the proofs above, most analogous results still hold in this larger setting with a minor modification.

As mentioned in Prüss monograph [16, Chapter 13], the semigroup approach can be utilized to investigate the solvability of linear singular Volterra integral equations via many different ways, such as the forcing function approach or the history function approach,... For linear evolution equations, although it is very hard to obtain *optimal* solvability results for nonlocal equations in terms of semigroup theory, the embedding of solution operators into a semi-dynamical setting is still meaningful, as it allows us to use powerful tool and very rich concepts of dynamical systems theory to gain asymptotic properties of solutions. The related semi-dynamical system constructed in this paper can be seen as our first attempt to study asymptotic behavior of solutions to nonlocal evolution equations via dynamical approach. The pullback of the objects from dynamical theory might induce some lights on the study of stability theory for nonlocal evolution equations.

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