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A novel method for finding minimum-norm solutions to pseudomonotone variational inequalities

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Abstract In this paper, we introduce a novel iterative method for finding the minimum-norm solution to a *pseudomonotone* variational inequality problem in Hilbert spaces. We establish strong convergence of the proposed method and its linear convergence under some suitable assumptions. Some numerical experiments are given to illustrate the performance of our method. Our result improves and extends some existing results in the literature.

Keywords Subgradient extragradient method · variational inequality problem · pseudomonotone operator · strong convergence · convergence rate.

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1 Introduction

Throughout this paper, assume that C is a nonempty, convex and closed subset of the real Hilbert space H with the inner product $\langle , \rangle$ and the norm $\|.\|$. Let $F : H \to H$ be a Lipschitz continuous operator. The object of our investigation is the following variational inequality problem (shortly, $VI(C,F)$):
Find $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0 \ \ \forall x \in C. \tag{1}$$

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We denote the solution set of $VI(C,F)$ by $Sol(C,F)$.

Many problems in various fields such as physic, economics, engineering, optimization theory can be led to variational inequalities. Iterative methods for solving these problems have been proposed and analyzed (see, for example, [8,9,10] and references therein). One of the most famous methods for solving $VI(C,F)$ is the extragradient method introduced by Korpelevich [11]. In this method, one needs to calculate two projections onto C at each iteration. This may affect the efficiency of the method when finding a projection onto a closed and convex set C is not an easy problem.

In recent years, many authors are interested in the extragradient method and improved it in various ways, see, e.g. [1,3,4,5,16,17,19,20,22,23] and references therein. The subgradient extragradient method, proposed by Censor et al. [2] for solving $VI(C,F)$ in real Hilbert spaces is one of these modifications.

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda F x_n), \\ T_n = \{x \in H : \langle x_n - \lambda F x_n - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda F y_n), \end{cases} \tag{2}$$

where $\lambda \in (0, \frac{1}{L})$, and L is a Lipschitz constant of F. This method replaces two projections onto C by one projection onto C and one onto a half-space. The sequence $\{x_n\}$ generated by (2) converges weakly to an element of $Sol(C,F)$ provided that $Sol(C,F)$ is nonempty.

Kraikaew and Seajung [12] used the subgradient extragradient method and Halpern method to introduce an algorithm for solving $VI(C,F)$ as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda F x_n), \\ T_n = \{x \in H : \langle x_n - \lambda F x_n - y_n, x - y_n\rangle \leq 0\}, \\ z_n = P_{T_n}(x_n - \lambda F y_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) z_n, \end{cases} \tag{3}$$

where $\lambda \in (0, \frac{1}{L}), \{\alpha_n\} \subset (0,1), \alpha_n \to 0, \sum_{n=1}^{\infty} \alpha_n = +\infty$. They proved that the sequence $\{x_n\}$ generated by (3) converges strongly to $P_{Sol(C,F)}x_0$ if F is monotone and L-Lipschitz continuous. The main disadvantage of algorithms (2), (3) is a requirement to know the Lipschitz constant of F or at least to know some its estimation.

Very recently, Yang [23] proposed a modification of subgradient extragradient method with step size rule using the inertial-type method as follows:
Given $\lambda_0 > 0, \mu < \mu_0 \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$

$$y_n = P_C(w_n - \lambda_n F w_n),$$

$$T_n := \{x \in H : \langle w_n - \lambda_n F w_n - y_n, x - y_n\rangle \leq 0\},$$

$$x_{n+1} = P_{T_n}(w_n - \lambda_n F y_n),$$

$$\lambda_{n+1} = \begin{cases} \min\{\mu \dfrac{\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2}{2\langle F w_n - F y_n, x_{n+1} - y_n\rangle}, \lambda_n\} & \text{if } \langle F w_n - F y_n, z_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Under the pseudomonotonicity and sequentially weak continuity of the mapping, the convergence of the algorithm was established without the knowledge of the Lipschitz constant of the mapping.

Motivated and inspired by the above mentioned works, and by the ongoing research in these directions, in this paper, we suggest a new iterative scheme for finding the minimum-norm solution to $VI(C,F)$ (1). It is worth pointing out that the proposed algorithm does not require the prior knowledge of the Lipschitz-type constant of the variational inequality mapping and only requires to compute one projection onto a feasible set per iteration as well as without the assumption on the weakly sequential continuity of the mapping. Moreover, the convergence rate is obtained under strong pseudomonotonicity and Lipschitz continuity assumptions of the variational inequality mapping.

The paper is organized as follows. In Section 2, we recall some basic definitions and results. In Section 3, we present and analyze the convergence of the proposed algorithms. Finally in Section 4, we present some numerical experiments to illustrate the performance of the proposed method.

2 Preliminaries

Lemma 2.1 ([6, Lemma 2.1]) Consider the problem $VI(C,F)$ with C being a nonempty, closed, convex subset of a real Hilbert space H and $F : C \to H$ being pseudo-monotone and continuous. Then, x^* is a solution of $VI(C,F)$ if and only if

$$\langle Fx, x - x^* \rangle \geq 0 \ \ \forall x \in C.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following results hold:*
i) $\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \ \forall x,y \in H$;
ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \ \forall x,y \in H$.

Definition 2.1 Let $T : C \to H$ be an operator, where C is a closed and convex subset of a real Hilbert space H. Then

- T is called L-Lipschitz continuous with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x-y\| \quad \forall x,y \in C.$$

- T is called monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x,y \in C.$$

- T is said to be *pseudo-monotone* if

$$\langle Tx, y - x\rangle \geq 0 \Longrightarrow \langle Ty, y - x\rangle \geq 0.$$

  It is called δ-strongly pseudo-monotone if there is $\delta > 0$ such that

$$\langle Tx, y - x\rangle \geq 0 \Longrightarrow \langle Ty, y - x\rangle \geq \delta\|y-x\|^2.$$

- T is said to be *weakly sequentially continuous* if, for each sequence $\{x_n\}$ in C, $\{x_n\}$ converges weakly to a point $x \in C$, then $\{Tx_n\}$ converges weakly to Tx.
- T is called weakly closed on C if for any $\{x_n\} \subset C$, $x_n \rightharpoonup x$, and $T(x_n) \rightharpoonup y$, then $T(x) = y$.

– T is said to have $*$-property on C, if the function $\|T(x)\|$ is weakly lower-semicontinuous (w.l.s.c.) on C, i.e., for any $\{x_n\} \subset C$, $x_n \rightharpoonup x$,

$$\|T(x)\| \leq \liminf_{n\to\infty} \|T(x_n)\|.$$

A relation between the weakly sequential continuity, weak closedness and $*$-property are revealed in the following simple statement.

Lemma 2.3 *i. Any weakly sequentially continuous operator is weakly closed and have the $*$-property.*
ii. A weakly closed operator, mapping bounded subsets into bounded subsets, is weakly sequentially continuous.
iii. An operator having the $*$-property and mapping bounded subsets into bounded subsets is not necessarily weakly sequentially continuous, and hence is not necessarily weakly closed.

Proof i. Suppose T is weakly sequentially continuous on C. Then it is weakly closed by definition. Further, let $C \ni x_n \rightharpoonup x$, then $T(x_n) \rightharpoonup T(x)$, and due to the weak lower continuity of the norm, one gets $\|T(x)\| \leq \liminf_{n\to\infty} \|T(x_n)\|$, which means the $*$-property of T.
ii. Assume that T is weakly closed and maps bounded subsets into bounded subsets. Let $x_n \rightharpoonup x$, than the sequence $\{x_n\}$ is bounded, hence, the set $\{T(x_n)\}$ is also bounded. Let ζ be a weak cluster point of $\{T(x_n)\}$. There exists a weakly convergent subsequence $T(x_{n_k}) \rightharpoonup \zeta$. Since $x_{n_k} \rightharpoonup x$, by the weak closedness of T, one gets $\zeta = T(x)$. Thus, $T(x_n) \rightharpoonup T(x)$.
iii. Let H be a real Hilbert space with an orthonormal basis $\{e_n\}$ and C be a closed ball centered at 0 with radius $r := \sqrt{2}$. Define the operator $T : C \to H$ by $T(x) := \|x\|x$. Obviously, T maps bounded subsets into bounded subsets. Further, T has the $*$-property. Indeed, let $x_n \rightharpoonup x$, then $\|T(x)\| = \|x\|^2 \leq (\liminf_{n\to\infty} \|x_n\|)^2 \leq \liminf_{n\to\infty} \|x_n\|^2 = \liminf_{n\to\infty} \|T(x_n)\|$. On the other hand, T is not weakly sequentially continuous. Indeed, let $x_n = e_n + e_1$. Then $x_n \rightharpoonup e_1$, and for $n \geq 2$, $T(x_n) = \sqrt{2}(e_n + e_1) \rightharpoonup \sqrt{2}e_1 \neq T(e_1) = 2e_1$.

Lemma 2.4 ([15]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$ then $\lim_{n\to\infty} a_n = 0$.

Definition 2.2 ([14]) Let $\{x_n\}$ be a sequence in H.
i) $\{x_n\}$ is said to converge R-linearly to x^* with rate $\rho \in [0,1)$ if there is a constant $c > 0$ such that

$$\|x_n - x^*\| \leq c\rho^n \quad \forall n \in \mathbb{N}.$$

ii) $\{x_n\}$ is said to converge Q-linearly to x^* with rate $\rho \in [0,1)$ if

$$\|x_{n+1} - x^*\| \leq \rho \|x_n - x^*\| \quad \forall n \in \mathbb{N}.$$

3 Main results

In this work, we assume the following conditions:

Condition 1 *The feasible set C is nonempty, closed, and convex.*

Condition 2 *The mapping $F : H \to H$ is L-Lipschitz continuous, pseudomonotone on H. However, the information of L is not necessary to be known.*

Condition 3 *The solution set $Sol(C,F)$ is nonempty.*

The proposed algorithm is of the form:

Algorithm 3.1

Initialization: *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying $\sum_{n=1}^{\infty} \alpha_n < +\infty$. Let $\theta > 0$, $\tau_1 > 0$, $\mu \in (0,1)$ and $x_0, x_1 \in H$ be arbitrary. We assume that $\{\theta_n\}$, $\{\varepsilon_n\}$ and $\{\gamma_n\}$ are three positive sequences such that $\{\theta_n\} \subset [0,\theta)$ and $\varepsilon_n = o(\gamma_n)$, i.e., $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\gamma_n} = 0$, where $\{\gamma_n\} \subset (0,1)$ satisfies the following conditions:*

$$\lim_{n\to\infty} \gamma_n = 0, \quad \sum_{n=1}^{\infty} \gamma_n = \infty.$$

Iterative Steps: *Calculate x_{n+1} as follows:*

Step 1. *Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise}. \end{cases} \tag{4}$$

Step 2. *Set $u_n = (1-\gamma_n)(x_n + \theta_n(x_n - x_{n-1}))$ and compute*

$$q_n = P_C(u_n - \tau_n F u_n).$$

Step 3. *Compute*

$$x_{n+1} = P_{T_n}(u_n - \tau_n F q_n),$$

where $T_n := \{x \in H | \langle u_n - \tau_n F u_n - q_n, x - q_n\rangle \leq 0\}$.
Update

$$\tau_{n+1} := \begin{cases} \min\{\mu \dfrac{\|u_n - q_n\|^2 + \|x_{n+1} - q_n\|^2}{2\langle F u_n - F q_n, x_{n+1} - q_n\rangle}, \tau_n + \alpha_n\} & \text{if } \langle F u_n - F q_n, x_{n+1} - q_n\rangle > 0, \\ \tau_n + \alpha_n & \text{otherwise.} \end{cases} \tag{5}$$

*Set $n := n+1$ and go to **Step 1**.*

Remark 3.1 As noted in [13], the sequence generated by (5) is allowed to increase from iteration to iteration. Hence, our results in this work are different from those in [22,23].

Lemma 3.5 ([13]) *Assume that Condition 2 holds. Let $\{\tau_n\}$ be the sequence generated by (5). Then*

$$\lim_{n\to\infty} \tau_n = \tau \text{ with } \tau \in \left[\min\left\{\tau_1, \frac{\mu}{L}\right\}, \tau_1 + \alpha\right],$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Moreover

$$2\langle Fu_n - Fq_n, x_{n+1} - q_n\rangle \leq \frac{\mu}{\tau_{n+1}}(\|u_n - q_n\|^2 + \|x_{n+1} - q_n\|^2). \tag{6}$$

Theorem 3.1 *Assume that Conditions 1–3 hold. If the mapping $F : H \to H$ satisfies the $*$-property then the sequence $\{x_n\}$, generated by Algorithm 3.1, converges strongly to an element $z \in Sol(C,F)$, where $z = P_{Sol(C,F)}(0)$.*

Proof To improve readability, we split the proof of our main theorem into some parts.
Claim 1.

$$\|x_{n+1} - z\|^2 \leq \|u_n - z\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1} - q_n\|^2.$$

Since $z \in C \subset T_n$ and P_{T_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 =& \|P_{T_n}(u_n - \tau_n Fq_n) - P_{T_n} z\|^2 \leq \langle x_{n+1} - z, u_n - \tau_n Fq_n - z\rangle \\
=& \frac{1}{2}\|x_{n+1} - z\|^2 + \frac{1}{2}\|u_n - \tau_n Fq_n - z\|^2 - \frac{1}{2}\|x_{n+1} - u_n + \tau_n Fq_n\|^2 \\
=& \frac{1}{2}\|x_{n+1} - z\|^2 + \frac{1}{2}\|u_n - z\|^2 + \frac{1}{2}\tau_n^2\|Fq_n\|^2 - \langle u_n - z, \tau_n Fq_n\rangle \\
&- \frac{1}{2}\|x_{n+1} - u_n\|^2 - \frac{1}{2}\tau_n^2\|Fq_n\|^2 - \langle x_{n+1} - u_n, \tau_n Fq_n\rangle \\
=& \frac{1}{2}\|x_{n+1} - z\|^2 + \frac{1}{2}\|u_n - z\|^2 - \frac{1}{2}\|x_{n+1} - u_n\|^2 - \langle x_{n+1} - z, \tau_n Fq_n\rangle.
\end{aligned}$$

This implies that

$$\|x_{n+1} - z\|^2 \leq \|u_n - z\|^2 - \|x_{n+1} - u_n\|^2 - 2\langle x_{n+1} - z, \tau_n Fq_n\rangle. \tag{7}$$

Since z is the solution of VI, we have $\langle Fz, x - z\rangle \geq 0$ for all $x \in C$. By the pseudomontonicity of F on C we have $\langle Fx, x - z\rangle \geq 0$ for all $x \in C$. Taking $x := q_n \in C$ we get

$$\langle Fq_n, z - q_n\rangle \leq 0.$$

Thus,

$$\langle Fq_n, z - x_{n+1}\rangle = \langle Fq_n, z - q_n\rangle + \langle Fq_n, q_n - x_{n+1}\rangle \leq \langle Fq_n, q_n - x_{n+1}\rangle. \tag{8}$$

From (7) and (8) we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 \leq& \|u_n - z\|^2 - \|x_{n+1} - u_n\|^2 + 2\tau_n\langle Fq_n, q_n - x_{n+1}\rangle \\
=& \|u_n - z\|^2 - \|x_{n+1} - q_n\|^2 - \|q_n - u_n\|^2 - 2\langle x_{n+1} - q_n, q_n - u_n\rangle \\
&+ 2\tau_n\langle Fq_n, q_n - x_{n+1}\rangle \\
=& \|u_n - z\|^2 - \|x_{n+1} - q_n\|^2 - \|q_n - u_n\|^2 + 2\langle u_n - \tau_n Fq_n - q_n, x_{n+1} - q_n\rangle.
\end{aligned} \tag{9}$$

Since $q_n = P_{T_n}(u_n - \tau_n F u_n)$ and $x_{n+1} \in T_n$ we have

$$\begin{aligned}
2\langle u_n - \tau_n F q_n - q_n, x_{n+1} - q_n\rangle \\
&= 2\langle u_n - \tau_n F u_n - q_n, x_{n+1} - q_n\rangle + 2\tau_n\langle F u_n - F q_n, x_{n+1} - q_n\rangle \\
&\le 2\tau_n\langle F u_n - F q_n, x_{n+1} - q_n\rangle. \qquad (10)
\end{aligned}$$

It follows from (6) that

$$2\tau_n\langle F u_n - F q_n, x_{n+1} - q_n\rangle \le \mu\frac{\tau_n}{\tau_{n+1}}\|u_n - q_n\|^2 + \mu\frac{\tau_n}{\tau_{n+1}}\|q_n - x_{n+1}\|^2. \qquad (11)$$

Combining (10) and (11), we obtain

$$2\langle u_n - \tau_n F q_n - q_n, x_{n+1} - q_n\rangle \le \mu\frac{\tau_n}{\tau_{n+1}}\|u_n - q_n\|^2 + \mu\frac{\tau_n}{\tau_{n+1}}\|q_n - x_{n+1}\|^2. \qquad (12)$$

Substituting (12) into (9) we obtain

$$\|x_{n+1} - z\|^2 \le \|u_n - z\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1} - q_n\|^2.$$

Claim 2. The sequence $\{x_n\}$ is bounded. Indeed, we have

$$\begin{aligned}
\|u_n - z\| &= \|(1-\gamma_n)(x_n + \theta_n(x_n - x_{n-1})) - z\| \\
&= \|(1-\gamma_n)(x_n - z) + (1-\gamma_n)\theta_n(x_n - x_{n-1}) - \gamma_n z\| \\
&\le (1-\gamma_n)\|x_n - z\| + (1-\gamma_n)\theta_n\|x_n - x_{n-1}\| + \gamma_n\|z\| \\
&= (1-\gamma_n)\|x_n - z\| + \gamma_n[(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| + \|z\|]. \qquad (13)
\end{aligned}$$

On the other hand, since (4) we have

$$\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| \le \frac{\varepsilon_n}{\gamma_n} \to 0,$$

which implies that $\lim_{n\to\infty}\left[(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| + \|z\|\right] = \|z\|$, hence there exists $M > 0$ such that

$$(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| + \|z\| \le M. \qquad (14)$$

Combining (13) and (14) we obtain

$$\|u_n - z\| \le (1-\gamma_n)\|x_n - z\| + \gamma_n M.$$

Moreover, we have $\lim_{n\to\infty}(1 - \mu\frac{\tau_n}{\tau_{n+1}}) = 1 - \mu > \frac{1-\mu}{2}$, hence there exists $n_0 \in \mathbb{N}$ such that $1 - \mu\frac{\tau_n}{\tau_{n+1}} > 0 \ \forall n \ge n_0$. By Claim 1 we obtain

$$\|x_{n+1} - z\| \le \|u_n - z\| \ \ \forall n \ge n_0. \qquad (15)$$

Thus

$$\begin{aligned}
\|x_{n+1} - z\| \le &(1-\gamma_n)\|x_n - z\| + \gamma_n M \\
&\le \max\{\|x_n - z\|, M\} \le \ldots \le \max\{\|x_{n_0} - z\|, M\}.
\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded.

Claim 3.

$$(1-\mu\frac{\tau_n}{\tau_{n+1}})\|q_n-u_n\|^2+(1-\mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1}-q_n\|^2 \\ \leq \|x_n-z\|^2-\|x_{n+1}-z\|^2+\gamma_n M_1.$$

Indeed, we have $\|u_n-z\|\leq(1-\gamma_n)\|x_n-z\|+\gamma_n M$, this implies that

$$\begin{aligned}\|u_n-z\|^2 &\leq (1-\gamma_n)^2\|x_n-z\|^2+2\gamma_n(1-\gamma_n)M\|x_n-z\|+\gamma_n^2M^2\\ &\leq \|x_n-z\|^2+\gamma_n[2(1-\gamma_n)M\|x_n-z\|+\gamma_n M^2]\\ &\leq \|x_n-z\|^2+\gamma_n M_1, \end{aligned} \tag{16}$$

where $M_1:=\max\{2(1-\gamma_n)M\|x_n-z\|+\gamma_n M^2 : \ n\in\mathbb{N}\}$. Substituting (16) into Claim 1 we get

$$\|x_{n+1}-z\|^2\leq\|x_n-z\|^2+\gamma_nM_1-(1-\mu\frac{\tau_n}{\tau_{n+1}})\|q_n-u_n\|^2-(1-\mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1}-q_n\|^2.$$

Or equivalently

$$(1-\mu\frac{\tau_n}{\tau_{n+1}})\|q_n-u_n\|^2+(1-\mu\frac{\tau_n}{\tau_{n+1}})\|x_{n+1}-q_n\|^2 \\ \leq \|x_n-z\|^2-\|x_{n+1}-z\|^2+\gamma_n M_1.$$

Claim 4.

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq &(1-\gamma_n)\|x_n-z\|^2+\gamma_n\Big[2(1-\gamma_n)\|x_n-z\|\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|\\ &+\theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|+2\|z\|\|u_n-x_{n+1}\|+2\langle -z,x_{n+1}-z\rangle\Big],\end{aligned}$$

$\forall n\geq n_0$. Indeed, using Lemma 2.2 ii) and (15) we get

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq &\|u_n-z\|^2 \ \ \forall n\geq n_0\\ =&\|(1-\gamma_n)(x_n-z)+(1-\gamma_n)\theta_n(x_n-x_{n-1})-\gamma_n z\|^2 \ \ \forall n\geq n_0\\ \leq&\|(1-\gamma_n)(x_n-z)+(1-\gamma_n)\theta_n(x_n-x_{n-1})\|^2+2\gamma_n\langle -z,u_n-z\rangle \ \ \forall n\geq n_0\\ \leq&(1-\gamma_n)^2\|x_n-z\|^2+2(1-\gamma_n)\theta_n\|x_n-z\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2\\ &+2\gamma_n\langle -z,u_n-x_{n+1}\rangle+2\gamma_n\langle -z,x_{n+1}-z\rangle \ \ \forall n\geq n_0\\ \leq&(1-\gamma_n)\|x_n-z\|^2+\gamma_n\Big[2(1-\gamma_n)\|x_n-z\|\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|\\ &+\theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|+2\|z\|\|u_n-x_{n+1}\|+2\langle -z,x_{n+1}-z\rangle\Big] \ \ \forall n\geq n_0.\end{aligned}$$

Claim 5. $\{\|x_n-z\|^2\}$ converges to zero.
Indeed, by Lemma 2.4 it suffices to show that $\limsup_{k\to\infty}\langle -z,x_{n_k+1}-z\rangle\leq 0$ and $\limsup_{k\to\infty}\|u_{n_k}-x_{n_k+1}\|\leq 0$ for every subsequence $\{\|x_{n_k}-z\|\}$ of $\{\|x_n-z\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)\geq 0.$$

For this purpose, suppose that $\{\|x_{n_k}-z\|\}$ is a subsequence of $\{\|x_n-z\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)\geq 0$. Then

$$\liminf_{k\to\infty}(\|x_{n_k+1}-z\|^2-\|x_{n_k}-z\|^2)=\liminf_{k\to\infty}[(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)(\|x_{n_k+1}-z\|+\|x_{n_k}-z\|)]\geq 0.$$

By Claim 3 we obtain

$$\begin{aligned}
\limsup_{k\to\infty}&\left[(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}})\|u_{n_k}-q_{n_k}\|^2+(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}})\|x_{n_k+1}-q_{n_k}\|^2\right]\\
&\leq\limsup_{k\to\infty}\left[\|x_{n_k}-z\|^2-\|x_{n_k+1}-z\|^2+\gamma_{n_k}M_1\right]\\
&\leq\limsup_{k\to\infty}\left[\|x_{n_k}-z\|^2-\|x_{n_k+1}-z\|^2\right]+\limsup_{k\to\infty}\gamma_{n_k}M_1\\
&=-\liminf_{k\to\infty}\left[\|x_{n_k+1}-z\|^2-\|x_{n_k}-z\|^2\right]\\
&\leq 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|q_{n_k}-u_{n_k}\|=0 \text{ and } \lim_{k\to\infty}\|x_{n_k+1}-q_{n_k}\|=0.$$

Thus

$$\lim_{k\to\infty}\|x_{n_k+1}-u_{n_k}\|=0. \tag{17}$$

Now, we show that

$$\|x_{n_k+1}-x_{n_k}\|\to 0 \;\text{ as }\; k\to\infty. \tag{18}$$

Indeed, using $\lim_{n\to\infty}\gamma_n=0$ we have

$$\begin{aligned}
\|x_{n_k}-u_{n_k}\|&=\|(1-\gamma_{n_k})(x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1}))-x_{n_k}\|\\
&=\|\theta_{n_k}(x_{n_k}-x_{n_k-1})-\gamma_{n_k}(x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1}))\|\\
&\leq\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|+\gamma_{n_k}\|x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1})\|\\
&=\gamma_{n_k}\frac{\theta_{n_k}}{\gamma_{n_k}}\|x_{n_k}-x_{n_k-1}\|+\gamma_{n_k}\|x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1})\|\to 0.
\end{aligned} \tag{19}$$

From (17) and (19), we get

$$\|x_{n_k+1}-x_{n_k}\|\leq\|x_{n_k+1}-u_{n_k}\|+\|x_{n_k}-u_{n_k}\|\to 0.$$

Since the sequence $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z^*\in H$, such that

$$\limsup_{k\to\infty}\langle -z,x_{n_k}-z\rangle=\lim_{j\to\infty}\langle -z,x_{n_{k_j}}-z\rangle=\langle -z,z^*-z\rangle. \tag{20}$$

Using (19), we get

$$u_{n_k}\rightharpoonup z^* \text{ as } k\to\infty,$$

Using (17), we obtain

$$x_{n_k}\rightharpoonup z^* \text{ as } k\to\infty.$$

Now, we show that $z^* \in Sol(C,F)$. Indeed, since $q_{n_k} = P_C(u_{n_k} - \tau_{n_k} F u_{n_k})$, we have

$$\langle u_{n_k} - \tau_{n_k} F u_{n_k} - q_{n_k}, x - q_{n_k} \rangle \leq 0 \ \ \forall x \in C,$$

or equivalently

$$\frac{1}{\tau_{n_k}} \langle u_{n_k} - q_{n_k}, x - q_{n_k} \rangle \leq \langle F u_{n_k}, x - q_{n_k} \rangle \ \ \forall x \in C.$$

Consequently

$$\frac{1}{\tau_{n_k}} \langle u_{n_k} - q_{n_k}, x - q_{n_k} \rangle + \langle F u_{n_k}, q_{n_k} - u_{n_k} \rangle \leq \langle F u_{n_k}, x - u_{n_k} \rangle \ \ \forall x \in C. \tag{21}$$

Being weakly convergent, $\{u_{n_k}\}$ is bounded. Then, by the Lipschitz continuity of F, $\{F u_{n_k}\}$ is bounded. As $\|u_{n_k} - q_{n_k}\| \to 0$, $\{q_{n_k}\}$ is also bounded and $\tau_{n_k} \geq \min\{\tau_1, \frac{\mu}{L}\}$. Passing (21) to limit as $k \to \infty$, we get

$$\liminf_{k\to\infty} \langle F u_{n_k}, x - u_{n_k} \rangle \geq 0 \ \ \forall x \in C. \tag{22}$$

Moreover, we have

$$\langle F q_{n_k}, x - q_{n_k} \rangle = \langle F q_{n_k} - F u_{n_k}, x - u_{n_k} \rangle + \langle F u_{n_k}, x - u_{n_k} \rangle + \langle F q_{n_k}, u_{n_k} - q_{n_k} \rangle. \tag{23}$$

Since $\lim_{k\to\infty} \|u_{n_k} - q_{n_k}\| = 0$ and F is L-Lipschitz continuous on H, we get

$$\lim_{k\to\infty} \|F u_{n_k} - F q_{n_k}\| = 0$$

which, together with (22) and (23) implies that

$$\liminf_{k\to\infty} \langle F q_{n_k}, x - q_{n_k} \rangle \geq 0.$$

Next, we choose a sequence $\{\varepsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle F q_{n_j}, x - q_{n_j} \rangle + \varepsilon_k \geq 0 \ \ \forall j \geq N_k. \tag{24}$$

Since $\{\varepsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{q_{N_k}\} \subset C$ we can suppose $F q_{N_k} \neq 0$ (otherwise, q_{N_k} is a solution) and, setting

$$v_{N_k} = \frac{F q_{N_k}}{\|F q_{N_k}\|^2},$$

we have $\langle F q_{N_k}, v_{N_k} \rangle = 1$ for each k. Now, we can deduce from (24) that for each k

$$\langle F q_{N_k}, x + \varepsilon_k v_{N_k} - q_{N_k} \rangle \geq 0.$$

From F is pseudomonotone on H, we get

$$\langle F(x + \varepsilon_k v_{N_k}), x + \varepsilon_k v_{N_k} - q_{N_k} \rangle \geq 0.$$

This implies that

$$\langle F x, x - q_{N_k} \rangle \geq \langle F x - F(x + \varepsilon_k v_{N_k}), x + \varepsilon_k v_{N_k} - q_{N_k} \rangle - \varepsilon_k \langle F x, v_{N_k} \rangle. \tag{25}$$

Now, we show that $\lim_{k\to\infty}\varepsilon_k v_{N_k}=0$. Indeed, since $u_{n_k}\rightharpoonup z^*$ and $\lim_{k\to\infty}\|u_{n_k}-q_{n_k}\|=0$, we obtain $q_{N_k}\rightharpoonup z^*$ as $k\to\infty$. By $\{q_n\}\subset C$, we obtain $z^*\in C$. Since F has $*$-property, we have

$$0<\|Fz^*\|\le\liminf_{k\to\infty}\|Fq_{n_k}\|.$$

Since $\{q_{N_k}\}\subset\{q_{n_k}\}$ and $\varepsilon_k\to 0$ as $k\to\infty$, we obtain

$$0\le\limsup_{k\to\infty}\|\varepsilon_k v_{N_k}\|=\limsup_{k\to\infty}\left(\frac{\varepsilon_k}{\|Fq_{n_k}\|}\right)\le\frac{\limsup_{k\to\infty}\varepsilon_k}{\liminf_{k\to\infty}\|Fq_{n_k}\|}=0,$$

which implies that $\lim_{k\to\infty}\varepsilon_k v_{N_k}=0$.

Now, letting $k\to\infty$, then the right hand side of (25) tends to zero by F is uniformly continuous, $\{u_{N_k}\},\{v_{N_k}\}$ are bounded and $\lim_{k\to\infty}\varepsilon_k v_{N_k}=0$. Thus, we get

$$\liminf_{k\to\infty}\langle Fx,x-q_{N_k}\rangle\ge 0.$$

Hence, for all $x\in C$ we have

$$\langle Fx,x-z^*\rangle=\lim_{k\to\infty}\langle Fx,x-q_{N_k}\rangle=\liminf_{k\to\infty}\langle Fx,x-q_{N_k}\rangle\ge 0.$$

By Lemma 2.1, we get

$$z^*\in\mathrm{Sol(C,F)}.$$

Since (20) and the definition of $z=P_{Sol(C,F)}(0)$, we have

$$\limsup_{k\to\infty}\langle -z,x_{n_k}-z\rangle=\langle -z,z^*-z\rangle\le 0. \tag{26}$$

Combining (18) and (26), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle -z,x_{n_k+1}-z\rangle&\le\limsup_{k\to\infty}\langle -z,x_{n_k}-z\rangle\\&=\langle -z,z^*-z\rangle\\&\le 0.\end{aligned}\tag{27}$$

Hence, by (27), $\lim_{n\to\infty}\dfrac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|=0$, $\lim_{k\to\infty}\|x_{n_k+1}-u_{n_k}\|=0$, Claim 5 and Lemma 2.4, we have $\lim_{n\to\infty}\|x_n-z\|=0$, which was to be proved.

Remark 3.2 It should be noted that if the operator F is monotone, the $*$ property is redundant, see [7,21].

4 Convergence rate

In this section we establish a convergence rate for the so-called relaxed inertial subgradient extragradient method. Actually, we consider the following modification of Algorithm 3.1:

Algorithm 4.2 *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers which satisfies $\sum_{n=1}^{\infty}\alpha_n < +\infty$. Given $\theta \in [0,1), \gamma \in (0,\frac{1}{2}),\ \mu \in (0,1)$ and $\tau_1 > 0$, Let $x_0, x_1 \in H$ be arbitrary. Let*

$$\begin{aligned}
&u_n = x_n + \theta(x_n - x_{n-1}),\\
&q_n = P_C(u_n - \tau_n F u_n),\\
&z_n = P_{T_n}(u_n - \tau_n F q_n),\\
&\text{where } T_n := \{x \in H | \langle u_n - \tau_n F u_n - q_n, x - q_n\rangle \le 0\},\\
&x_{n+1} = (1-\gamma)x_n + \gamma z_n.\\
&\text{Update}\\
&\tau_{n+1} := \begin{cases} \min\{\mu \dfrac{\|u_n - q_n\|^2 + \|z_n - q_n\|^2}{2\langle F u_n - F q_n, z_n - q_n\rangle}, \tau_n + \alpha_n\} \text{ if } \langle F u_n - F q_n, z_n - q_n\rangle > 0,\\ \tau_n + \alpha_n \text{ otherwise.}\end{cases}
\end{aligned}$$

Throughout this section, the operator F is assumed to be L-Lipschitz continuous on H and δ-strongly pseudo-monotone on C. We now prove that the iterative sequence generated by Algorithm 4.2 converges strongly to the unique solution of problem (VI) with an R-linear rate.

Theorem 4.2 *Assume that $F : H \to H$ is L-Lipschitz continuous on H and δ-strongly pseudo-monotone on C. Let $\theta \in \left[0, \dfrac{\delta}{L+\delta}\right)$, $\mu \in \left(\dfrac{\theta}{1+\theta}\dfrac{L}{\delta}, \dfrac{1-\theta}{1+\theta}\right)$ and $\tau_1 > \dfrac{\mu}{L}$. Then the sequence $\{x_n\}$ generated by Algorithm 4.2 converges in norm with an R-linear convergence rate to the unique element z in* Sol(C,F).

Proof Since $\langle Fz, q_n - z\rangle \ge 0$, the δ-strong pseudo-monotonicity of F on C yields the inequality

$$\langle F q_n, q_n - z\rangle \ge \delta\|q_n - z\|^2.$$

This implies that

$$\langle F q_n, z - z_n\rangle = \langle F q_n, z - q_n\rangle + \langle F q_n, q_n - z_n\rangle \le -\delta\|q_n - z\|^2 + \langle F q_n, q_n - z_n\rangle. \tag{28}$$

Now, using (28) and a similar argument as in Claim 1 of Theorem 3.1, we get

$$\begin{aligned}
\|z_n - z\|^2 \le& \|u_n - z\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|z_n - q_n\|^2 - 2\delta\tau_n\|q_n - z\|^2\\
\le& \|u_n - z\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\|q_n - u_n\|^2 - 2\delta\tau_n\|q_n - z\|^2.
\end{aligned}$$

Since $\theta < \dfrac{\delta}{L+\delta}$, it follows that

$$\frac{\theta}{1+\theta}\frac{L}{\delta} < \frac{1-\theta}{1+\theta}$$

therefore there always exists

$$\mu \in \left(\frac{\theta}{1+\theta}\frac{L}{\delta}, \frac{1-\theta}{1+\theta}\right).$$

From $\mu < \frac{1-\theta}{1+\theta}$, one finds $\frac{1-\mu}{2} > \frac{\theta}{1+\theta}$ and $\mu > \frac{\theta}{1+\theta}\frac{L}{\delta}$ implies that $\delta\frac{\mu}{L} > \frac{\theta}{1+\theta}$. Fix $\varepsilon \in \left(\frac{\theta}{1+\theta}, \min\left\{\frac{1-\mu}{2}, \delta\frac{\mu}{L}\right\}\right)$. We have

$$\lim_{n\to\infty}(1-\mu\frac{\tau_n}{\tau_{n+1}}) = 1-\mu > 2\varepsilon$$

and

$$\lim_{n\to\infty}\delta\tau_n = \delta\tau \geq \delta\min\left\{\tau_1, \frac{\mu}{L}\right\} = \delta\frac{\mu}{L} > \varepsilon.$$

Therefore, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we get

$$\begin{aligned}\|z_n - z\|^2 \leq& \|u_n - z\|^2 - 2\varepsilon\|q_n - u_n\|^2 - 2\varepsilon\|q_n - z\|^2\\ \leq& \|u_n - z\|^2 - \varepsilon\|u_n - z\|^2\\ =& (1-\varepsilon)\|u_n - z\|^2.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|(1-\gamma)x_n + \gamma z_n - z\|^2\\ &= \|(1-\gamma)(x_n - z) + \gamma(z_n - z)\|^2\\ &= (1-\gamma)\|x_n - z\|^2 + \gamma\|z_n - z\|^2 - (1-\gamma)\gamma\|x_n - z_n\|^2\\ &= (1-\gamma)\|x_n - z\|^2 + \gamma\|z_n - z\|^2 - \frac{1-\gamma}{\gamma}\|x_{n+1} - x_n\|^2\\ &\leq (1-\gamma)\|x_n - z\|^2 + \gamma(1-\varepsilon)\|u_n - z\|^2 - \frac{1-\gamma}{\gamma}\|x_{n+1} - x_n\|^2 \ \forall n \geq N.\end{aligned}$$

We also have

$$\begin{aligned}\|u_n - z\|^2 &= \|(1+\theta)(x_n - z) - \theta(x_{n-1} - z)\|^2\\ &= (1+\theta)\|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 + \theta(1+\theta)\|x_n - x_{n-1}\|^2.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|x_{n+1} - z\|^2 \leq& (1-\gamma)\|x_n - z\|^2 + \gamma(1-\varepsilon)[(1+\theta)\|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 + \theta(1+\theta)\|x_n - x_{n-1}\|^2]\\ &- \frac{1-\gamma}{\gamma}\|x_{n+1} - x_n\|^2 \ \forall n \geq N\\ &\leq (1-\gamma(1-(1-\varepsilon)(1+\theta)))\|x_n - z\|^2 - \gamma(1-\varepsilon)\theta\|x_{n-1} - z\|^2\\ &\quad + \gamma(1-\varepsilon)\theta(1+\theta)\|x_n - x_{n-1}\|^2 - \frac{1-\gamma}{\gamma}\|x_{n+1} - x_n\|^2 \ \forall n \geq N\\ &\leq (1-\gamma(1-(1-\varepsilon)(1+\theta)))\|x_n - z\|^2 + \gamma(1-\varepsilon)\theta(1+\theta)\|x_n - x_{n-1}\|^2\\ &- \frac{1-\gamma}{\gamma}\|x_{n+1} - x_n\|^2 \ \forall n \geq N.\end{aligned}$$

Since $\gamma \in (0, \frac{1}{2})$, it implies $\frac{1-\gamma}{\gamma} > 1$. Hence, we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 + \|x_{n+1} - x_n\|^2 \leq& \|x_{n+1} - z\|^2 + \frac{1-\gamma}{\gamma}\|x_{n+1} - x_n\|^2\\ \leq& (1-\gamma(1-(1-\varepsilon)(1+\theta)))\|x_n - z\|^2\\ &+ \gamma(1-\varepsilon)\theta(1+\theta)\|x_n - x_{n-1}\|^2 \ \forall n \geq N.\end{aligned}$$

This follows that

$$\|x_{n+1}-z\|^2+\|x_{n+1}-x_n\|^2 \le (1-\gamma(1-(1-\varepsilon)(1+\theta)))\Big[\|x_n-z\|^2 + \frac{1}{(1-\gamma(1-(1-\varepsilon)(1+\theta)))}\gamma(1-\varepsilon)\theta(1+\theta)\|x_n-x_{n-1}\|^2\Big]\ \forall n \ge N.$$

We now show that

$$1-\gamma(1-(1-\varepsilon)(1+\theta)) \in (0,1)$$

and

$$\frac{1}{1-\gamma(1-(1-\varepsilon)(1+\theta))}\gamma(1-\varepsilon)\theta(1+\theta) \in (0,1).$$

Indeed, since $\varepsilon \in (\frac{\theta}{1+\theta}, \min\left\{\frac{1-\mu}{2}, \delta\frac{\mu}{L}\right\})$, this implies that $\varepsilon > \frac{\theta}{1+\theta}$, or, $1-\varepsilon < \frac{1}{1+\theta}$ that is $(1-\varepsilon)(1+\theta) < 1$, hence $1-\gamma(1-(1-\varepsilon)(1+\theta)) \in (0,1)$. It is easy to see that

$$\frac{1}{1-\gamma(1-(1-\varepsilon)(1+\theta))}\gamma(1-\varepsilon)\theta(1+\theta) \in (0,1).$$

Therefore, we deduce

$$\|x_{n+1}-z\|^2+\|x_{n+1}-x_n\|^2 \le (1-\gamma(1-(1-\varepsilon)(1+\theta)))[\|x_n-z\|^2+\|x_n-x_{n-1}\|^2]\ \forall n \ge N.$$

Letting $a_n := \|x_n-z\|^2+\|x_n-x_{n-1}\|^2$ and $\xi := (1-\gamma(1-(1-\varepsilon)(1+\theta)))$, we get

$$\|x_{n+1}-z\|^2 \le a_{n+1} \le \xi a_n \le \xi^{n-N+1}a_N = \frac{\xi}{\xi^N}a^N\xi^n.$$

Thus, the sequence $\{x_n\}$ converges R-linearly to z, as desired.

Remark 4.3 It should be emphasized that we obtain the linear convergence rate of Algorithm 4.2 instead of the strong convergence as in [18].

Remark 4.4 In Theorem 4.2, the $*$- property of F is not assumed.

5 Numerical Illustrations

In this section, we present some numerical experiments in solving variational inequality problems. In the first example, we compare the proposed algorithm with two well-known algorithms including Algorithm 2 of Yang, J. et al. in [22] and the modified Halpern subgradient extragradient method (HSEGM) of R. Kraikaew and S. Saejung in [12, Section 4] of Kraikaew et al.. In the second example, we compare the proposed algorithm with Algorithm 1 of Yang, J. et al. in [22], Algorithm 3.1 of Yang, J. et al. in [23], and the subgradient extragradient algorithm (SEGM) of Censor et al. in [2] and illustrate the convergence of the proposed algorithm. All the numerical experiments are performed on a HP laptop with Intel(R) Core(TM)i5-6200U CPU 2.3GHz with 4 GB RAM. The programs are written in Matlab2015a. We use the sequence $D_n = ||x_n - x^*||^2,\ n = 0,1,2,\ldots$ to check the convergence of $\{x_n\}$, where x^* is the solution of the problem. The convergence of $\{D_n\}$ to 0 implies that $\{x_n\}$ converges to the solution of the problem.

Remark 5.5 We usually choose $\alpha_n = \theta_0$ for Algorithm 3.1 of Yang, J. et al. in [23] because α_n and θ_n have similar roles in their algorithm as well as in our proposed algorithm. Similarly, we take $\lambda = \tau_0$ for the subgradient extragradient algorithm of Censor et al. in [2].

In numerical experiments, we choose $\mu = 0.5$ for the proposed algorithm, Algorithms 1, 2 of Yang, J. et al. in [22], Algorithm 3.1 of Yang, J. et al. in [23], and the other parameters as follows:

Proposed algorithm: $\gamma_n = \dfrac{1}{n+1}, \theta_0 = 0.5, \theta_n = \begin{cases} \min\{\theta_0, \frac{\gamma_n^2}{\|x_n - x_{n-1}\|}\}, \text{if } x_n \neq x_{n-1} \\ \theta_0, \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$,

Algorithm 3.1: $\alpha_n = \theta_0 = 0.5$

Example 1 Suppose that $H = L^2([0,1])$ with the inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt, \forall x, y \in H \tag{29}$$

and the included norm

$$\|x\| := (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}, \forall x \in H \tag{30}$$

Let $C := \{x \in H : \|x\| \le 1\}$ be the unit ball and define an operator $F : C \to H$ by

$$Fx(t) = \max\{0, x(t)\}.$$

It is easy to see that F is 1-Lipschitz continuous and monotone on C and so F is pseudomonotone.

All the integrals in equations (29) - (30) and others are computed by the trapezoidal formula with the stepsize $t = 0.001$ using function trapz of matlab. The starting points are $x_0 = \frac{(t^2 - \exp(-t))}{525}$ or $x_0 = \frac{(sin(-3*t)+cos(-10*t))}{600}$ and $x_1 = 0.5x_0$ for the proposed algorithm, $x_0 = x_1$ for other algorithms. We take $\lambda_0 = \lambda = 0.5, \alpha_n = \frac{1}{n+1}$ for Algorithm 2 of Yang, J. et al. in [22], Halpern subgradient extragradient method (HSEGM) of Kraikaew et al. in [12], respectively, and $\tau_0 = 0.5$ for the proposed algorithm. We use stopping rule $D_n < 10^{-10}$ or iterations ≥ 2000 for all algorithms. The numerical results are described in Table 5.1 and Figs. 1 - 2.

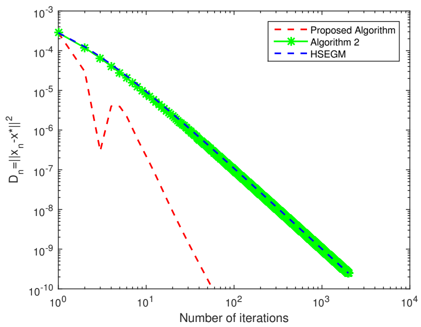

Fig. 1: Comparison of all algorithms with $x_0 = \frac{(t^2 - \exp(-t))}{525}$.

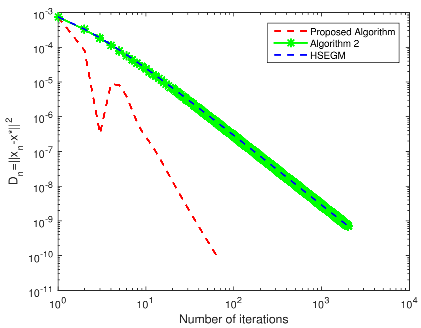

Fig. 2: Comparison of all algorithms with $x_0 = \frac{(sin(-3*t)+cos(-10*t))}{600}$.

Table 5.1: Numerical results of all algorithms with different x_0

Methods	$x_0 = \frac{(t^2-\exp(-t))}{525}$			$x_0 = \frac{(sin(-3*t)+cos(-10*t))}{600}$		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Proposed Alg	0.04	57	9.9976e-11	0.05	64	9.7417e-11
Algorithm 2	1.1	2000	2.4867e-10	1.1	2000	7.0810e-10
HSEGM	0.7	2000	2.4867e-10	0.7	2000	7.0810e-10

Table 5.1 and Figures 1-2 give the errors of the proposed algorithm, Algorithm 2 of Yang, J. et al. in [22], and Kraikaew and Saejung's algorithm [12] as well as their execution times. They show that the proposed algorithm is less time consuming and more accurate than those of Yang, J. et al. in [22], Kraikaew and Saejung's algorithm in [12] .

Example 2 Assume that $F : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $F(x) = Mx+q$ with $M = NN^T +S+D$, N is an $m\times m$ matrix, S is an $m\times m$ skew-symmetric matrix, D is an $m\times m$ diagonal matrix, whose diagonal entries are positive (so M is positive definite), q is a vector in $\mathbb{R}^m$, and

$$C := \{x \in \mathbb{R}^m : x_i \geq -1, i = 1,\cdots,m\}.$$

It is clear that F is monotone and Lipschitz continuous with the Lipschitz constant $L = \|M\|$. For $q = 0$, the unique solution of the corresponding variational inequality is $\{0\}$.

For experiment, all entries of B, S and D are generated randomly from a normal distribution with mean zero and unit variance. The process is started with the initial $x_0 = (1,...,1)^T \in \mathbb{R}^m$ and $x_1 = 0.9x_0$. To terminate algorithms, we use the condition $D_n \leq \varepsilon$ with $\varepsilon = 10^{-6}$ or the number of iterations ≥ 2000 for all algorithms.

Case 1: We take $\lambda = \frac{0.7}{\|M\|}$ for the subgradient extragradient algorithm of Censor et al. in [2] and $\tau_0 = \frac{0.7}{\|M\|}$ for Algorithm 1 of Yang, J. et al. in [22], Algorithm 3.1 of Yang, J. et al. in [23] and the proposed algorithm. The numerical results are described in Table 5.2 and Figs. 3 - 4.

Table 5.2: Numerical results obtained by other algorithms

Methods	m=50			m=100		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Proposed Alg	0.11	31	9.9773e-07	0.75	43	9.4795e-07
Algorithm 1	5.8	2000	0.0029	34	2000	0.0384
Algorithm 3.1	5.9	2000	4.9922e-05	35	2000	0.0043
SEGM	5.6	2000	0.0012	30	2000	0.0272

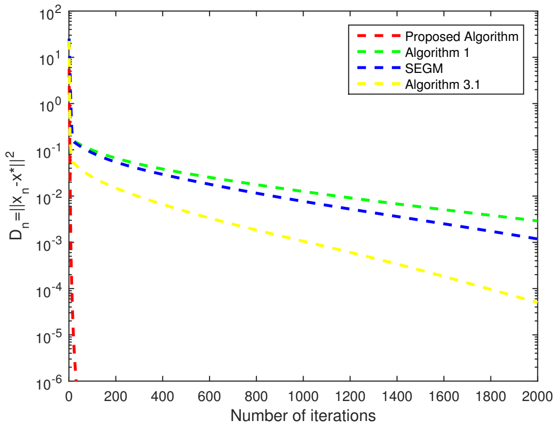

Fig. 3: Comparison of all algorithms with $m = 50$

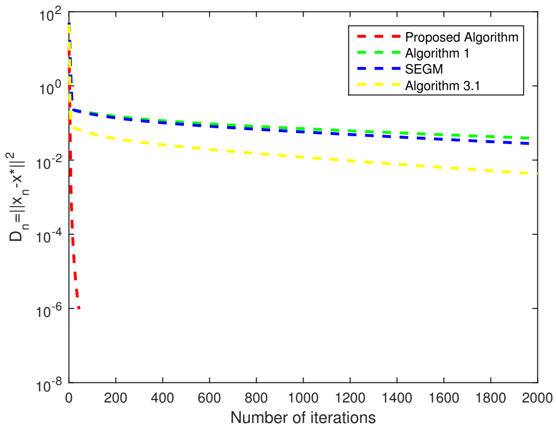

Fig. 4: Comparison of all algorithms with $m = 100$.

Case 2: We take $\lambda = \frac{0.9}{\|M\|}$ for the subgradient extragradient algorithm of Censor et al. in [2] and $\tau_0 = \frac{0.9}{\|M\|}$ for Algorithm 1 of Yang, J. et al. in [22], Algorithm 3.1 of Yang, J. et al. in [23] and the proposed algorithm. The numerical results are described in Table 5.3 and Figs. 5 - 6.

Table 5.3: Numerical results obtained by other algorithms

Methods	m=50			m=100		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Proposed Alg	0.12	34	9.3168e-07	0.76	45	9.8653e-07
Algorithm 1	5.62	2000	0.0034	34	2000	0.0409
Algorithm 3.1	5.7	2000	8.2121e-05	35	2000	0.0054
SEGM	5.59	2000	3.6547e-04	30	2000	0.0184

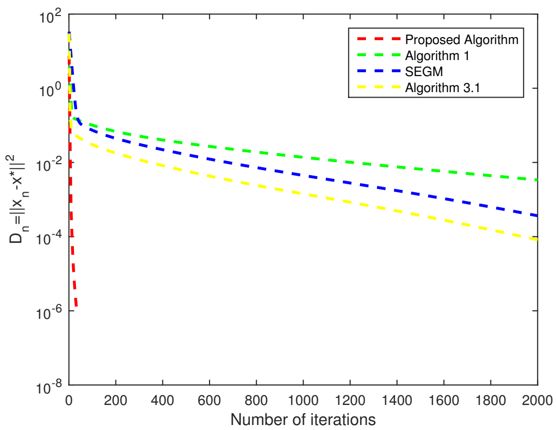

Fig. 5: Comparison of all algorithms with $m = 50$.

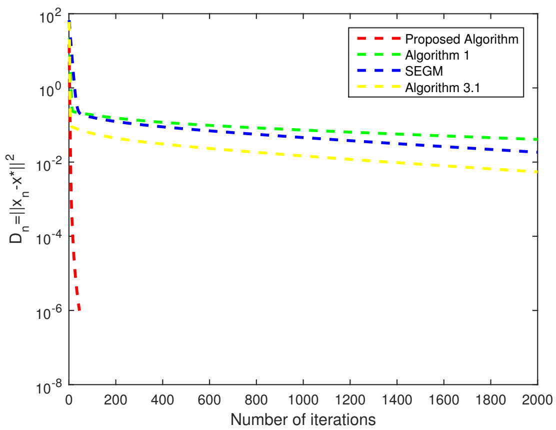

Fig. 6: Comparison of all algorithms with $m = 100$.

Table 5.2 - 5.3 and Figures 3 - 6 give the errors of the proposed algorithm, algorithm of Censor et al. in [2], Algorithm 1 of Yang, J. et al. in [22], Algorithm 3.1 of Yang, J. et al. in [23] as well as their execution times. They show that the proposed algorithm is less time consuming and more accurate than those of Yang, J. et al. in [22], [23], Censor et al. in [2].

In Fig. 7 we illustrate the convergence rate of the proposed algorithm for different choices of the θ with $\lambda = \frac{0.7}{\|M\|}, \mu = 0.5, m = 50$ and $\gamma_n = \frac{1}{n+1}$.

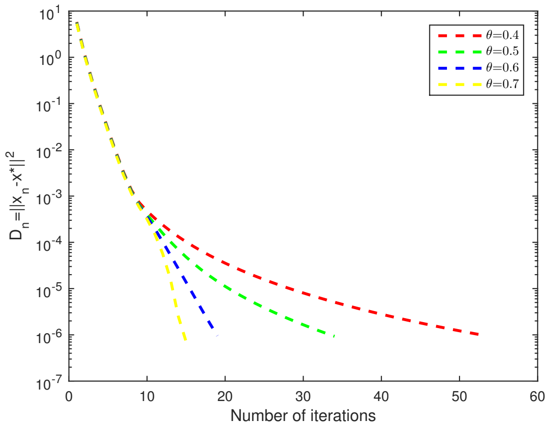

Fig. 7: Convergence rate of the proposed algorithms for different choice of the θ.

In Fig. 8 we illustrate the convergence rate of the proposed algorithm for different choices of the γ_n with $\lambda = \frac{0.7}{\|M\|}, \mu = 0.5, m = 50$ and $\theta = 0.5$.

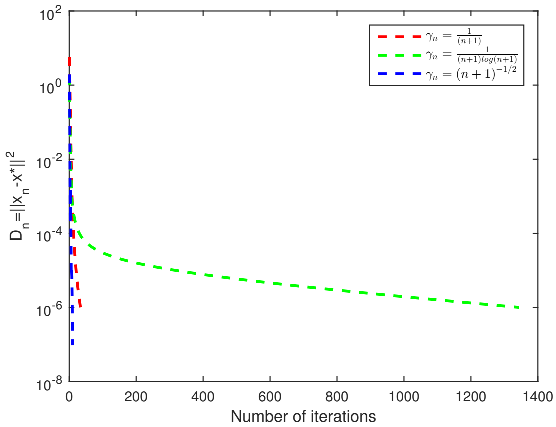

Fig. 8: Convergence rate of the proposed algorithm for different choice of the γ_n.

Figures 7 - 8 show that the rate of convergence of the proposed algorithm in general depends strictly on the convergent rate of sequence of γ_n and θ.

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