DATA ASSIMILATION FOR THE THREE-DIMENSIONAL LERAY-α MODEL USING LOCAL OBSERVABLES ON ANY TWO COMPONENTS OF THE VELOCITY FIELD

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Abstract. We study continuous data assimilation for the three-dimensional Leray-α model using local observables and using only coarse mesh observations of any two components of the three-dimensional velocity field, and without any information of the rest component. We prove that, with the spatial resolution N and the analyticity radius σ are sufficiently large (in the periodic boundary conditions case), the complement of the full domain Ω_0 and the sub-domain Ω is small enough (in the no-slip boundary conditions case), and with the relaxation (nudging) parameter μ is sufficiently large, and the spatial mesh resolution h is sufficiently small, we can approximately recover the unknown reference solution corresponding to the measurements by the approximating solution.

1. Introduction and statement main results

Data assimilation is a methodology to estimate weather or ocean variables combining (synchronizing) information from observational data with a numerical dynamical (forecast) model. In recent years, data assimilation problems for many important equations in fluid mechanics have been extensively studied by Edriss Titi and his coauthors, see e.g. [2, 8, 22, 23, 24, 27, 29, 30, 32]. We also refer the interested reader to [1, 4, 5, 6, 10, 11, 31] for some results of other authors.

In verry recent years, Biswas et al., [12] studied the data assimilation for the two-dimensional Navier-Stokes equations using local observables was studied recently in [12]. To overcome the difficulty due to the local observations, the authors used spectral inequality and Gevrey property of solutions in the periodic case, and the assumption of the complement of the full domain Ω_0 and the sub-domain Ω is small enough in the no-slip boundary conditions case.

The Leray-α model was introduced in [17]. In the last years, the mathematical questions related to the Leray-α model, including existence, regularity, convergence and long-time behavior of solutions as well as controllability property, has attracted the attention of many mathematicians, see e.g., [3, 7, 14, 16, 28, 33, 35] and references therein.

In this paper, we will investigate the data assimilation for the 3D Leray-α model using local observables on any two components of the three-dimensional velocity field, and without any information of the rest component. In the case of periodic boundary conditions, besides Gevrey assumption of solution, the key ingredient is a spectral inequality due to Egidi and Veselić [19, 20] which bounds the L^2 norm over the full domain in terms of that over a sub-domain, enabling us to use the local data obtained from the sub-domain for global assimilation of the system. In the case of no-slip boundary conditions, we require the sub-domains occupies almost the full domain. In what follows, we will explain the problem to be investigated.

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Suppose that the evolution of u is governed by the three-dimensional Leray-α model [17, 25], subject to periodic boundary conditions on $\Omega_0 = [0, L]^3$ or no-slip boundary conditions ($u = \Delta u = 0$ on $\partial\Omega_0$) if Ω_0 is a C^2 bounded domain in $\mathbb{R}^3$

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (u \cdot \nabla)v + \nabla p = f, \\ \nabla \cdot u = \nabla \cdot v = 0, \\ v = u - \alpha^2 \Delta u, \end{cases} \tag{1.1}$$

on the interval $[0, \infty)$, where the initial data $v(0) = v_0$ is unknown. In the Leray-α model (1.1), u represents the velocity of the fluid, called the *filtered velocity* and $\alpha > 0$ is a scale parameter with dimension of length, p is the pressure, and f is a body force which is assumed, for simplicity, to be time-independent.

We will study a data assimilation algorithm for three-dimensional Leray-α model on the domain Ω_0 using local observables on a sub-domain on Ω. Observations are limited to an open set Ω compactly contained in Ω_0. Here, the reference solution is given by a solution v of (1.1) for which the initial data is missing. We have the following type 1 local interpolant operator (see [12] for more details) which satisfies the following property:

$$\|I_{h,\Omega}(\varphi) - \varphi\|^2_{L^2(\Omega)} \leq c_0 h^2 \|\varphi\|^2_{H^1(\Omega_0)}. \tag{1.2}$$

From the bounded property of the interpolant operator, the local interpolant operator $I_{h,\Omega} : L^2(\Omega_0) \to L^2(\Omega)$ satisfies the following estimate for some positive constant c_1

$$\|I_{h,\Omega}(\varphi)\|^2_{L^2(\Omega)} \leq c_1 \|\varphi\|^2_{L^2(\Omega_0)}. \tag{1.3}$$

For any positive integral N, we denote $I_{h,N,\Omega} = P_N I_{h,\Omega}$ where P_N is the projection onto the first N eigenvectors of the Stokes operator (see in Section 2)

Periodic boundary conditions case. We now follow the approach in [12] to introduce the following data assimilation algorithm for finding an approximate solution z_N of the unknown reference solution v: Given information about a reference solution v by using the interpolant operator $I_{h,N,\Omega}$, we look for a function $z_N = (z_N^1, z_N^2, z_N^3)$ satisfies the same boundary conditions for $v = (v_1, v_2, v_3)$, and the following system

$$\begin{cases} \dfrac{\partial z_N^1}{\partial t} - \nu\Delta z_N^1 + P_N(w_N \cdot \nabla)z_N^1 + P_N \partial_{x_1} q = P_N f_1 - \mu I_{h,N,\Omega}(z_N^1 - v_1), \\ \dfrac{\partial z_N^2}{\partial t} - \nu\Delta z_N^2 + P_N(w_N \cdot \nabla)z_N^2 + P_N \partial_{x_2} q = P_N f_2 - \mu I_{h,N,\Omega}(z_N^2 - v_2), \\ \dfrac{\partial z_N^3}{\partial t} - \nu\Delta z_N^3 + P_N(w_N \cdot \nabla)z_N^3 + P_N \partial_{x_3} q = P_N f_3, \\ \nabla \cdot w_N = \nabla \cdot z_N = 0, \\ z_N = w_N - \alpha^2 \Delta w_N, \\ z_N(0) = 0. \end{cases} \tag{1.4}$$

Here the samples used to drive z_N are confined to the sub-domain $\Omega \subset \Omega_0$ and z_N lives in $\text{span}(\phi_1, \ldots, \phi_N)$. ν and f are the same kinematic viscosity parameter and forcing term from (1.1), q is a modified pressure, and $\mu > 0$ is a relaxation (nudging) parameter. The purpose of μ is to force the coarse spatial scales of z_N toward those of the reference solution v.

We will show that the data assimilation equation (1.4) has a unique solution z_N, and within any given tolerance ε, under suitable conditions (which are provided by Ω, ν, the Grashof number G and ε) of N, μ (sufficiently large) and h (sufficiently small), this approximate solution will capture the long time properties of the reference solution v of the three-dimensional Leray-α equations. Here, we have to require the solution v and consequently the forcing f to be uniformly in the L^2

based Gevrey class, i.e., $v \in L^\infty((0,\infty); D(e^{\sigma A^{1/2}}))$, with σ sufficiently large as determined by Ω, ν, G and ε. We will recall the definition of $D(e^{\sigma A^{1/2}})$ in Section 2.

The approximate convergence for local observations is given in the following theorem.

Theorem 1.1. *Let Ω be an open set in $[0,L]^3$. Let v be the weak solution to* (1.1) *for some $v_0 \in H$ and $f \in H$. Assume additionally that*

$$\mathcal{C} = \limsup_{t\to\infty} \|v(t)\|_{D(e^{\sigma A^{1/2}})} = \limsup_{t\to\infty} |e^{\sigma A^{1/2}} v| < \infty. \tag{1.5}$$

Let $\varepsilon > 0$ be given. If

$$\sigma > \frac{C_\Omega L}{2\pi}, \tag{1.6}$$

$$N \geq \max\left\{ \left(\frac{L}{2\pi\sigma}\right)^3 \ln^3\left(\frac{cM_1^2 C_\Omega \mathcal{C}^2}{\alpha^3\nu^3\lambda_1^2 L^3}\right), \left(\frac{-\alpha^2 M_0^2 + \sqrt{\alpha^{-4}M_0^4 + 2^{-1/4}c^{-1}\alpha^{-3/2}M_0 M_1}}{2^{3/4}\alpha^{-3/2}M_0 M_1}\right)^4 - 1 \right\}, \tag{1.7}$$

$$\mu = \frac{cM_1^2 C_\Omega e^{C_\Omega \sqrt[3]{N}}}{\alpha^3 \nu^2 \lambda_1}, \tag{1.8}$$

$$h \leq \left(\frac{\nu}{2c_0\mu}\right)^{1/2} \min\left\{\lambda_1^{1/2}, 2^{-1/2}\right\}, \tag{1.9}$$

then there exists a unique global solution z_N to (1.4) *such that*

$$|z_N(t) - v(t)| < \varepsilon$$

for sufficiently large t.

Here, C_Ω presents a positive constant in (2.10) *which is independent of N, the constants M_0, M_1 are defined the upper bounds for u (see* (2.6)-(2.7)*).*

An exact convergence result follows as a corollary of Theorem 1.1. In particular, we can construct a vector field z that converges to v as $t \to \infty$ in an appropriate average by increasing the sample size in Theorem 1.1

Corollary 1.2. *Under the assumptions of Theorem 1.1, there exists a vector field $z \in L^\infty(0,\infty;H) \cap L^2(0,T;V)$ for all $T > 0$ so that z is a limit (in an appropriate sense) of a sequence of vector fields satisfying* (1.4) *and for every measurable set U we have*

$$\lim_{t\to\infty} \int_U (z(x,t) - v(x,t))dx = 0,$$

at an exponential rate.

Remark 1.1. As in [12] for the case of two dimensional Navier-Stokes equations, we do not know the precise dynamics of z since we have not obtained a governing system for z via the limiting process. This comes from the term $\mu_\varepsilon I_{h_\varepsilon, N_\varepsilon, \Omega}(v)$ can go to ∞ as $\mu_\varepsilon \to \infty$.

No-slip boundary conditions case. Now we consider the data assimilation problem for the three-dimensional Leray-α model in the case of no-slip boundary conditions. The interpolant operators are localized but we require the sub-domains occupies almost the full domain. This requirement enables us to use a helpful inequality (see (2.10) below) as same as in the smooth boundary case in [9, Lemma 1]. The variant of the local data assimilation equation is given by

$$\begin{cases} \dfrac{\partial z_1}{\partial t} - \nu\Delta z_1 + (w\cdot\nabla)z_1 + \partial_{x_1} q = f_1 - \mu I_{h,\Omega}(z_1 - v_1), \\ \dfrac{\partial z_2}{\partial t} - \nu\Delta z_2 + (w\cdot\nabla)z_2 + \partial_{x_2} q = f_2 - \mu I_{h,\Omega}(z_2 - v_2), \\ \dfrac{\partial z_3}{\partial t} - \nu\Delta z_3 + (w\cdot\nabla)z_3 + \partial_{x_3} q = f_3, \\ \nabla\cdot w = \nabla\cdot z = 0, \\ z = w - \alpha^2\Delta w, \\ z(0) = 0. \end{cases} \tag{1.10}$$

We have the following result about the exact convergence for large sub-domains.

Theorem 1.3. *Let Ω be a sub-domain of Ω_0. Let v be the solution to* (1.1) *for some $v_0 \in H$ and $f \in H$.*

If

$$\mu \geq c\nu, \tag{1.11}$$

$$h \leq \left(\frac{\nu}{2c_0\mu}\right)^{1/2} \min\left\{\lambda_1^{1/2}, 1\right\} \tag{1.12}$$

and

$$d_H(\partial\Omega, \partial\Omega_0) \sim \frac{\alpha^{3/2}\nu\lambda_1^{1/2}}{M_1} \tag{1.13}$$

then there exists a unique solution z of (1.10) *satisfying*

$$|z(t) - v(t)| \to 0, \quad \textit{as} \quad t\to\infty \tag{1.14}$$

at an exponential rate, where d_H denotes the Hausdorff distance.

The rest of the paper is organized as follows. In Section 2, for the convenience of the reader, we recall the functional setting and some results on the three-dimensional Leray-α model which will be used in the proof of main results. Section 3 is devoted to proving Theorem 1.1, Corollary 1.2, and Theorem 1.3. In the Appendix, we give the proof of Theorem 2.1 for the case of no-slip boundary conditions.

2. Preliminaries

We begin by defining a suitable domain Ω_0 and space $\mathcal{V}$ of smooth functions which satisfy each type of boundary conditions.

- In the periodic boundary condition case: $\Omega_0 = [0, L]^3$, we denote by $\mathcal{V}$ the set of all vector valued trigonometric polynomials defined in Ω_0, which are divergence-free and have average zero.
- In the homogeneous Dirichlet boundary condition case: Let Ω_0 be an open, bounded and connected domain with C^2 boundary. We denote by $\mathcal{V}$ the set of all C^∞ vector fields from Ω_0 to $\mathbb{R}^3$ that are divergence free and compactly supported.

Then we denote by H and V the closures of $\mathcal{V}$ in the $L^2(\Omega_0)^3$ and $H^1(\Omega_0)^3$, respectively. Then H and V are Hilbert spaces with inner products given by

$$(u, v) = \sum_{i=1}^{3}\int_{\Omega_0} u_i v_i dx \text{ and } ((u, v)) = \sum_{i,j=1}^{3}\int_{\Omega_0} \partial_j u_i \partial_j v_i dx,$$

respectively, and the associated norms

$$|u| = (u, u)^{1/2} \text{ and } \|u\| = ((u, u))^{1/2}.$$

With the Leray projector $\mathcal{P}$, we denote the Stokes operator $A = -\mathcal{P}\Delta$ with domain $D(A) = H^2(\Omega_0)^3 \cap V$. In the case of periodic boundary conditions, $A = -\Delta|_{D(A)}$. The Stokes operator A is a positive self-adjoint operator with compact inverse. Hence there exists a complete orthonormal set of eigenfunctions $\{\phi_j\}_{j=1}^{\infty} \subset H$, such that $A\phi_j = \lambda_j \phi_j$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty.$$

Moreover, we have $\lambda_j \sim \lambda_1^{-1} j^{2/3}$.

We have the following versions of the Poincaré inequalities:

$$\|u\|_{V'}^2 \leq \lambda_1^{-1}|u|^2, \ \forall u \in H, \tag{2.1}$$

$$|u|^2 \leq \lambda_1^{-1}\|u\|^2, \ \forall u \in V. \tag{2.2}$$

For all $v = u - \alpha^2\Delta u, v \in H$, we have

$$\begin{aligned}|v|^2 &= (u - \alpha^2\Delta u, u - \alpha^2\Delta u)\\ &= (u,u) - 2\alpha^2(u,\Delta u) + \alpha^4(\Delta u, \Delta u)\\ &= |u|^2 + 2\alpha^2\|u\|^2 + \alpha^4|\Delta u|^2.\end{aligned}$$

Thus,

$$|u| \leq |v|, \|u\| \leq 2^{-1/2}\alpha^{-1}|v|, |\Delta u| \leq \alpha^{-2}|v|. \tag{2.3}$$

For $u, v, w \in V$ we have that

$$\langle (u\cdot\nabla)v, w\rangle_{V',V} = -\langle (u\cdot\nabla)w, v\rangle_{V',V},$$

and consequently

$$\langle (u\cdot\nabla)v, v\rangle_{V',V} = 0. \tag{2.4}$$

Furthermore,

$$|\langle (u\cdot\nabla)v, w\rangle_{V',V}| \leq c\|u\|^{1/2}\|u\|_{H^2}^{1/2}|v|\|w\|, \ \forall u \in D(A), v \in H, w \in V. \tag{2.5}$$

For external force $f \in H$, we define the Grashof number in three dimension as follows

$$G = \frac{|f|}{\nu^2\lambda_1^{3/4}}.$$

We need the following result for the solutions to the Leray-α model (1.1) whose proof is given in the Appendix.

Theorem 2.1. *Let $f \in H$ and $v_0 \in H$. Then the system* (1.1) *with the initial data $v(0) = v_0$ subject to both periodic boundary conditions and no-slip boundary conditions, has a unique weak solution v that satisfies*

$$v \in C([0,\infty); H) \cap L^2_{loc}(0,\infty; V), \frac{dv}{dt} \in L^2_{loc}(0,\infty; V').$$

Furthermore, the associated semigroup $S(t) : H \to H$ has a global attractor $\mathcal{A}$ in H. Additionally, for any $v \in \mathcal{A}$, we have

$$|v| \leq M_0 := \frac{\sqrt{2}\nu G}{\lambda_1^{1/4}}, \tag{2.6}$$

$$\|v\| \leq M_1 := e^{\frac{|f|^2}{2\nu}}\left(1 + \frac{2}{\nu\lambda_1} + \frac{2\tilde{c}}{\nu\alpha^3\lambda_1}\right)^{1/2}\nu\lambda_1^{1/2}G, \tag{2.7}$$

for some dimensionless constant $\tilde{c} > 0$.

Now we recall about L^2 Gevrey classes. Let $\varphi \in H$ then

$$\varphi(x) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \hat{\varphi}_k e^{2\pi i \frac{k}{L}\cdot x},$$

where

$$\hat{\varphi}_k = \int_{\Omega_0} \varphi(y) e^{-2\pi i \frac{k}{L}\cdot y} dy.$$

On the periodic box $\Omega_0 = [0, L]^3$, with $\sigma > 0$, the Gevrey space $D(e^{\sigma A^s})$ is the set of elements of H satisfying

$$\|\varphi\|^2_{D(e^{\sigma A^s})} := L^3 \sum_{k\in\mathbb{Z}^3} e^{2\sigma|2\pi\frac{k}{L}|^{2s}} |\hat{\varphi}_k|^2 < \infty.$$

Note that for Gevrey class forcing, a solution v to (1.1) becomes and remains Gevrey regular for positive times (see [36, Theorem 1.3]. More precisely, for $v_0 \in V$ and $f \in D(e^{\sigma A^{1/2}})$ then $v \in D(A^{1/2}e^{\sigma A^{1/2}})$ and then (1.5) is satisfied. However, here we only assume $v \in D(e^{\sigma A/2})$. Then from condition (1.5), we have

$$\limsup_{t\to\infty} |\hat{v}_k(t)|^2 \le \frac{\mathcal{C}^2}{L^3} e^{-4\pi\sigma\frac{|k|}{L}}. \tag{2.8}$$

Now, we recall some spectral inequalities applying to thick sets (see [19, 20]). A set S is thick in $\mathbb{R}^3$ if there exists $\gamma \in (0, 1]$ and $a = (a_1, a_2, a_3)$ where $a_i > 0$ so that for every $x \in \mathbb{R}^3$,

$$|(S + x) \cap ([0, a_1] \times [0, a_2] \times [0, a_3])| \ge \gamma a_1 a_2 a_3.$$

Any open set in $[0, L]^3$ which is periodically extended to $\mathbb{R}^3$ is thick. We recall the spectral theorem on the torus.

Theorem 2.2. [20] *Let $\varphi \in L^2(\Omega_0)$ where Ω_0 denotes the torus $[0, L_1] \times [0, L_2] \times [0, L_3]$. Assume supp $\hat{\varphi} \subset J$ where J is a rectangle in $\mathbb{R}^3$ with sides parallel to coordinate axes and of length b_1, b_2 and b_3. Set $b = (b_1, b_2, b_3)$. Let $S \subset \mathbb{R}^3$ be a (γ, a)-thick set with $a = (a_1, a_2, a_3)$ so that $0 < a_i < 2\pi L_i$ for $i = 1, 2, 3$. Then*

$$\|\varphi\|_{L^2(\Omega_0)} \le C\gamma^{-ca\cdot b-\frac{19}{2}} \|\varphi\|_{L^2(S\cap\Omega_0)},$$

where c is a numerical constant and $a \cdot b$ stands for the Euclidean inner product in $\mathbb{R}^3$.

We note that here, for any $\varphi \in \text{span}(\phi_1, \dots, \phi_N)$, there exists $K \sim \sqrt[3]{N}$ such that $\hat{\varphi}$ is supported in $[-K, K]^3$. Then similar to [12], from Theorem 2.2 we have the following inequality: If $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \text{span}(\phi_1, \dots, \phi_N)$, then

$$\|\varphi_i\|^2_{L^2(\Omega_0)} \le C_\Omega e^{C_\Omega \sqrt[3]{N}} \|\varphi_i\|^2_{L^2(\Omega)}, \quad \text{for } i = 1, 2, 3. \tag{2.9}$$

where C_Ω presents a positive constant which is independent of N.

We define $\lambda_1(\Omega_0\setminus\Omega)$ is the first eigenvalue of the Laplace operator on the domain $\Omega_0 \setminus \overline{\Omega}$ with no-slip boundary conditions, i.e.,

$$\lambda_1(\Omega_0 \setminus \Omega) := \inf \left\{ \int_{\Omega_0\setminus\Omega} |\nabla\varphi|^2 dx \mid \forall\varphi \in H^1_0(\Omega_0 \setminus \Omega) \text{ with } \int_{\Omega_0\setminus\Omega} |\varphi|^2 dx = 1 \right\}.$$

Then we have the following lemma whom proof is same as in [9, Lemma 1] (one can see in [34]).

Lemma 2.3. *Let Ω and Ω_0 be bounded domains with smooth boundary so that $\Omega \subset \Omega_0$. For any $\varepsilon > 0$, there exists $\ell_0 = \ell_0(\varepsilon) > 0$ so that for $\ell > \ell_0$, the following inequality holds*

$$\int_{\Omega_0} \left(|\nabla\varphi|^2 + \ell\chi_\Omega|\varphi|^2\right) dx \ge (\lambda_1(\Omega_0 \setminus \Omega) - \varepsilon) \int_{\Omega_0} |\varphi|^2 dx, \tag{2.10}$$

for $\varphi \in H_0^1(\Omega_0)$.

We note that here

$$\lambda_1(\Omega_0 \setminus \Omega) \geq C \left(\sup_{x\in\Omega_0\setminus\Omega} \operatorname{dist}(x, \partial\Omega_0) \right)^{-2}. \tag{2.11}$$

3. Proof of main theorems

3.1. **Proof of Theorem 1.1.** We will prove this theorem in two steps.

Step 1. Global existence of z_N. We first prove that z_N exists globally. To do this, we only need prove that z_N is uniformly bounded in H for h is sufficiently small.

Multiplying $(1.4)_1$, $(1.4)_2$ and $(1.4)_3$ by z_N^1, z_N^2 and z_N^3 respectively, then integrating over Ω_0 we obtain

$$\frac{1}{2}\frac{d}{dt}|z_N|^2 + \nu\|z_N\|^2 = (f, z_N) + \mu\sum_{i=1}^{2}\left(I_{h,N,\Omega}(v_i), z_N^i\right) - \mu\sum_{i=1}^{2}\left(I_{h,N,\Omega}(z_N^i), z_N^i\right). \tag{3.1}$$

Using the Cauchy inequality and the Poincaré inequality (2.1), we obtain

$$|(f, z_N)| \leq \frac{1}{\nu\lambda_1}|f|^2 + \frac{\nu}{4}\|z_N\|^2. \tag{3.2}$$

Using the Cauchy inequality and the Poincaré inequality (2.1), the fact that v_N is projected onto the first N modes and (1.3), we obtain

$$\begin{aligned}
\mu\sum_{i=1}^{2}\left(I_{h,N,\Omega}(v_i), z_N^i\right) &\leq \frac{\mu^2}{\nu\lambda_1}\sum_{i=1}^{2}\|I_{h,N,\Omega}(v_i)\|^2_{L^2(\Omega)} + \frac{\nu}{4}\|z_N\|^2 \\
&\leq \frac{\mu^2}{\nu\lambda_1}\sum_{i=1}^{2}\|I_{h,\Omega}(v_i)\|^2_{L^2(\Omega)} + \frac{\nu}{4}\|z_N\|^2 \\
&\leq \frac{c_1\mu^2}{\nu\lambda_1}\left(\|v_1\|^2_{L^2(\Omega_0)} + \|v_2\|^2_{L^2(\Omega_0)}\right) + \frac{\nu}{4}\|z_N\|^2 \\
&\leq \frac{c_1\mu^2}{\nu\lambda_1}|v|^2 + \frac{\nu}{4}\|z_N\|^2.
\end{aligned} \tag{3.3}$$

Using the Cauchy inequality, (1.2) and the Poincaré inequality (2.2), we obtain

$$\begin{aligned}
-\mu\sum_{i=1}^{2}\left(I_{h,N,\Omega}(z_N^i), z_N^i\right) &= -\mu\sum_{i=1}^{2}(I_{h,\Omega}(z_N^i), z_N^i) \\
&= -\mu\sum_{i=1}^{2}\left(I_{h,\Omega}(z_N^i) - \chi_\Omega z_N^i, z_N^i\right) - \mu\sum_{i=1}^{2}\|z_N^i\|^2_{L^2(\Omega)} \\
&\leq \mu\sum_{i=1}^{2}\|I_{h,\Omega}(z_N^i) - z_N^i\|_{L^2(\Omega)}\|z_N^i\|_{L^2(\Omega)} - \mu\sum_{i=1}^{2}\|z_N^i\|^2_{L^2(\Omega)} \\
&\leq \frac{\mu c_0 h^2}{2\lambda_1}\sum_{i=1}^{2}\|z_N^i\|^2_{H^1(\Omega_0)} - \frac{\mu}{2}\sum_{i=1}^{2}\|z_N^i\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.4}$$

If h is sufficiently small so that $\frac{\mu c_0 h^2}{2\lambda_1} \leq \frac{\nu}{4}$, we deduce from (3.4) that

$$-\mu\sum_{i=1}^{2}\left(I_{h,N,\Omega}(z_N^i), z_N^i\right) \leq \frac{\nu}{4}\|z_N\|^2 - \frac{\mu}{2}\sum_{i=1}^{2}\|z_N^i\|^2_{L^2(\Omega)}. \tag{3.5}$$

Substituting estimates (3.2), (3.3) and (3.5) into (3.1), we deduce that

$$\frac{1}{2}\frac{d}{dt}|z_N|^2 + \frac{\nu}{4}\|z_N\|^2 \le \frac{1}{\nu\lambda_1}|f|^2 + \frac{c_1\mu^2}{\nu\lambda_1}|v|^2 - \frac{\mu}{2}\sum_{i=1}^{2}\|z_N^i\|^2_{L^2(\Omega)}.$$

Using the Poincaré inequality (2.2) and dropping the last term, we have

$$\frac{d}{dt}|z_N|^2 + \frac{\nu\lambda_1}{2}|z_N|^2 \le \frac{2}{\nu\lambda_1}|f|^2 + \frac{2c_1\mu^2}{\nu\lambda_1}|v|^2. \tag{3.6}$$

Because the right hand side is uniformly bounded in t, using the Gronwall inequality for (3.6), we have a uniform in time and independent of N bound on $|z_N|$.

Step 2. Approximate convergence. Let ε be given and we set $\bar{\varepsilon} = \frac{\varepsilon\nu\lambda_1}{8}$. Let v and z_N be as in the statement of Theorem 1.1. Note that for any $N \in \mathbb{N}$, $P_N v$ satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}P_N v_1 - \nu\Delta(P_N v_1) + P_N(P_N u\cdot\nabla)v_1 + P_N\partial_{x_1}p = P_N f_1 - P_N(Q_N u\cdot\nabla)v_1, \\ \dfrac{\partial}{\partial t}P_N v_2 - \nu\Delta(P_N v_2) + P_N(P_N u\cdot\nabla)v_2 + P_N\partial_{x_2}p = P_N f_2 - P_N(Q_N u\cdot\nabla)v_2, \\ \dfrac{\partial}{\partial t}P_N v_3 - \nu\Delta(P_N v_3) + P_N(P_N u\cdot\nabla)v_3 + P_N\partial_{x_3}p = P_N f_3 - P_N(Q_N u\cdot\nabla)v_3, \end{cases} \tag{3.7}$$

where $Q_N = I - P_N$. Let $\delta = z_N - P_N v, \eta = w_N - P_N u$, we have $\delta = \eta - \alpha^2\Delta\eta$. Subtracting (1.4) from (3.7) to obtain

$$\begin{cases} \dfrac{\partial\delta_1}{\partial t} - \nu\Delta\delta_1 + P_N(w_N\cdot\nabla)z_N^1 - P_N(P_N u\cdot\nabla)v_1 + P_N\partial_{x_1}q \\ \qquad\qquad = -\mu I_{h,N,\Omega}(\delta_1) + \mu I_{h,N,\Omega}(Q_N v_1) + P_N(Q_N u\cdot\nabla)v_1, \\ \dfrac{\partial\delta_2}{\partial t} - \nu\Delta\delta_2 + P_N(w_N\cdot\nabla)z_N^2 - P_N(P_N u\cdot\nabla)v_2 + \partial_{x_2}q \\ \qquad\qquad = -\mu I_{h,N,\Omega}(\delta_2) + \mu I_{h,N,\Omega}(Q_N v_2) + P_N(Q_N u\cdot\nabla)v_2, \\ \dfrac{\partial\delta_3}{\partial t} - \nu\Delta\delta_3 + P_N(w_N\cdot\nabla)z_N^3 - P_N(P_N u\cdot\nabla)v_2 + \partial_{x_3}q \\ \qquad\qquad = -\mu I_{h,N,\Omega}(\delta_3) + \mu I_{h,N,\Omega}(Q_N v_3) + P_N(Q_N u\cdot\nabla)v_3. \end{cases} \tag{3.8}$$

Multiplying the first, the second and the third equation in (3.8) by δ_1, δ_2 and δ_3 respectively, then integrating over Ω_0, summing up and using (2.4), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\delta|^2 + \nu\|\delta\|^2 + ((\eta\cdot\nabla)P_N v,\delta) - ((Q_N u\cdot\nabla)v,\delta) - ((P_N u\cdot\nabla)Q_N v,\delta) \\ &= -\mu\sum_{i=1}^{2}(I_{h,N,\Omega}(\delta_i),\delta_i) + \mu\sum_{i=1}^{2}(I_{h,N,\Omega}(Q_N v_i),\delta_i), \end{aligned} \tag{3.9}$$

where we have used the fact that

$$P_N(w_N\cdot\nabla)z_N - P_N(P_N u\cdot\nabla)v = P_N(P_N u\cdot\nabla)\delta + P_N(\eta\cdot\nabla)\delta + P_N(\eta\cdot\nabla)P_N v.$$

Using the Cauchy inequality and (1.2), we have

$$\begin{aligned} -\mu\sum_{i=1}^{2}(I_{h,N,\Omega}(\delta_i),\delta_i) &= \mu\sum_{i=1}^{2}(\chi_\Omega\delta_i - I_{h,\Omega}(\delta_i),\delta_i) - \mu\sum_{i=1}^{2}\|\delta_i\|^2_{L^2(\Omega)} \\ &\le \mu\sum_{i=1}^{2}\|I_{h,\Omega}(\delta_i) - \delta_i\|_{L^2(\Omega)}\,\|\delta_i\|_{L^2(\Omega)} - \mu\sum_{i=1}^{2}\|\delta_i\|^2_{L^2(\Omega)} \\ &\le \mu c_0 h^2\sum_{i=1}^{2}\|\delta_i\|^2 - \frac{3\mu}{4}\sum_{i=1}^{2}\|\delta_i\|^2_{L^2(\Omega)} \end{aligned}$$

$$\leq \mu c_0 h^2\|\delta\|^2 - \frac{3\mu}{4}\sum_{i=1}^{2}\|\delta_i\|^2_{L^2(\Omega)}. \tag{3.10}$$

We now have

$$((\eta\cdot\nabla)P_N v,\delta) = \sum_{i=1}^{3} I_{1i} + \sum_{i=1}^{3} I_{2i} + \sum_{i=1}^{3} I_{3i}, \tag{3.11}$$

where

$$I_{1i} = (\eta_i\partial_{x_i}P_N v_1,\delta_1),\ I_{2i} = (\eta_i\partial_{x_i}P_N v_2,\delta_2),\ I_{3i} = (\eta_i\partial_{x_i}P_N v_3,\delta_3),\ i=1,2,3.$$

By the Hölder inequality, the Agmon inequality, inequalities (2.3) and the Cauchy inequality we have

$$\begin{aligned}\sum_{i=1}^{3} I_{1i} &\leq \sum_{i=1}^{3}\|\eta_i\|_{L^\infty(\Omega_0)}\|\partial_{x_i}P_N v_1\|_{L^2(\Omega_0)}\|\delta_1\|_{L^2(\Omega_0)}\\
&\leq c\sum_{i=1}^{3}\|\eta_i\|^{1/2}_{H^1(\Omega_0)}\|\eta_i\|^{1/2}_{H^2(\Omega_0)}\|\partial_{x_i}P_N v_1\|_{L^2(\Omega_0)}\|\delta_1\|_{L^2(\Omega_0)}\\
&\leq c\alpha^{-3/2}\sum_{i=1}^{3}\|\delta_i\|_{L^2(\Omega_0)}\|\partial_{x_i}P_N v_1\|_{L^2(\Omega_0)}\|\delta_1\|_{L^2(\Omega_0)}\\
&\leq c\alpha^{-3/2}\lambda_1^{-1/2}\sum_{i=1}^{3}\|\nabla\delta_i\|_{L^2(\Omega_0)}\|\partial_{x_i}P_N v_1\|_{L^2(\Omega_0)}\|\delta_1\|_{L^2(\Omega_0)}\\
&\leq \frac{\nu}{8}\sum_{i=1}^{3}\|\nabla\delta_i\|^2_{L^2(\Omega_0)} + c\alpha^{-3}\nu^{-1}\lambda_1^{-1}\sum_{i=1}^{3}\|\partial_{x_i}P_N v_1\|^2_{L^2(\Omega_0)}\|\delta_1\|^2_{L^2(\Omega_0)}.\end{aligned}$$

Hence,

$$\sum_{i=1}^{3} I_{1i} \leq \frac{\nu}{16}\|\delta\|^2 + c\alpha^{-3}\nu^{-1}\lambda_1^{-1}\|\nabla v_1\|^2_{L^2(\Omega_0)}\|\delta_1\|^2_{L^2(\Omega_0)}. \tag{3.12}$$

By the same as above we have

$$\sum_{i=1}^{3} I_{2i} \leq \frac{\nu}{16}\|\delta\|^2 + c\alpha^{-3}\nu^{-1}\lambda_1^{-1}\|\nabla v_2\|^2_{L^2(\Omega_0)}\|\delta_2\|^2_{L^2(\Omega_0)}, \tag{3.13}$$

and

$$\begin{aligned}\sum_{i=1}^{2} I_{3i} &\leq \frac{\nu}{16}\|\delta_3\|^2 + c\alpha^{-3}\nu^{-1}\lambda_1^{-1}\sum_{i=1}^{2}\|\partial_{x_i}v_3\|^2_{L^2(\Omega_0)}\|\delta_i\|^2_{L^2(\Omega_0)}\\
&\leq \frac{\nu}{16}\|\delta_3\|^2 + c\alpha^{-3}\nu^{-1}\lambda_1^{-1}\|\nabla v_3\|^2_{L^2(\Omega_0)}\left(\|\delta_1\|^2_{L^2(\Omega_0)} + \|\delta_2\|^2_{L^2(\Omega_0)}\right).\end{aligned} \tag{3.14}$$

For the last term, integrating by parts and using $\nabla\cdot\eta = \nabla\cdot\delta = 0$, we have

$$\begin{aligned}I_{33} &= (\eta_3\partial_{x_3}P_N v_3,\delta_3) = -(P_N v_3,\partial_{x_3}(\eta_3\delta_3))\\
&= -(P_N v_3,\delta_3\partial_{x_3}\eta_3) - (P_N v_3,\eta_3\partial_{x_3}\delta_3)\\
&= (P_N v_3,\delta_3(\partial_{x_1}\eta_1 + \partial_{x_2}\eta_2)) + (P_N v_3,\eta_3(\partial_{x_1}\delta_1 + \partial_{x_2}\delta_2)).\end{aligned}$$

Integrating by parts once again and using the Hölder inequality, the Agmon inequality, inequalities (2.3) and the Cauchy inequality we deduce that

$$(P_N v_3,\delta_3(\partial_{x_1}\eta_1 + \partial_{x_2}\eta_2)) = -\sum_{i=1}^{2}(\delta_3\partial_{x_i}P_N v_3,\eta_i) - \sum_{i=1}^{2}(P_N v_3\partial_{x_i}\delta_3,\eta_i)$$

$$\leq \sum_{i=1}^{2}\left[\|\delta_3\|_{L^2(\Omega_0)}\|\partial_{x_i}P_N v_3\|_{L^2(\Omega_0)} + \|P_N v_3\|_{L^2(\Omega_0)}\|\partial_{x_i}\delta_3\|_{L^2(\Omega)}\right]\|\eta_i\|_{L^\infty(\Omega_0)}$$

$$\le c\sum_{i=1}^{2}\left[\|\delta_3\|_{L^2(\Omega_0)}\|\partial_{x_i}v_3\|_{L^2(\Omega_0)}+\|v_3\|_{L^2(\Omega_0)}\|\partial_{x_i}\delta_3\|_{L^2(\Omega)}\right]\|\eta_i\|_{H^1(\Omega_0)}^{1/2}\|\eta_i\|_{H^2(\Omega_0)}$$

$$\le c\alpha^{-\frac{3}{2}}\lambda_1^{-\frac{1}{2}}\sum_{i=1}^{2}\left[\|\nabla\delta_3\|_{L^2(\Omega_0)}\|\partial_{x_i}v_3\|_{L^2(\Omega_0)}+\|\nabla v_3\|_{L^2(\Omega_0)}\|\partial_{x_i}\delta_3\|_{L^2(\Omega_0)}\right]\|\delta_i\|_{L^2(\Omega_0)}.$$

Hence, by the Cauchy inequality one obtains that

$$(P_Nv_3,\delta_3(\partial_{x_1}\eta_1+\partial_{x_2}\eta_2))$$
$$\le\frac{\nu}{32}\|\nabla\delta_3\|_{L^2(\Omega_0)}^2+c\alpha^{-3}\lambda_1^{-1}\nu^{-1}\|\nabla v_3\|_{L^2(\Omega_0)}^2\left(\|\delta_1\|_{L^2(\Omega_0)}^2+\|\delta_2\|_{L^2(\Omega_0)}^2\right). \quad (3.15)$$

By similarly, we have

$$(P_Nv_3,\eta_3(\partial_{x_1}\delta_1+\partial_{x_2}\delta_2))$$
$$\le\frac{\nu}{32}\|\nabla\delta_3\|_{L^2(\Omega_0)}^2+c\alpha^{-3}\lambda_1^{-1}\nu^{-1}\|\nabla v_3\|_{L^2(\Omega_0)}^2\left(\|\delta_1\|_{L^2(\Omega_0)}^2+\|\delta_2\|_{L^2(\Omega_0)}^2\right). \quad (3.16)$$

From (3.15) and (3.16) then we have

$$I_{33}\le\frac{\nu}{16}\|\nabla\delta_3\|_{L^2(\Omega_0)}^2+c\alpha^{-3}\lambda_1^{-1}\nu^{-1}\|\nabla v_3\|_{L^2(\Omega_0)}^2\left(\|\delta_1\|_{L^2(\Omega_0)}^2+\|\delta_2\|_{L^2(\Omega_0)}^2\right). \quad (3.17)$$

Substituting estimates (3.12), (3.13), (3.14) and (3.17) into (3.11) we infer that

$$((\eta\cdot\nabla)P_Nv,\delta)\le\frac{\nu}{4}\|\delta\|^2+c\alpha^{-3}\lambda_1^{-1}\nu^{-1}\|v\|^2\left(\|\delta_1\|_{L^2(\Omega_0)}^2+\|\delta_2\|_{L^2(\Omega_0)}^2\right). \quad (3.18)$$

Using (2.9) and (2.7), we obtain from (3.18) that

$$((\eta\cdot\nabla)P_Nv,\delta)\le\frac{\nu}{4}\|\delta\|^2+\frac{cM_1^2C_\Omega e^{C_\Omega\sqrt[3]{N}}}{\alpha^3\nu\lambda_1}\left(\|\delta_1\|_{L^2(\Omega)}^2+\|\delta_2\|_{L^2(\Omega)}^2\right). \quad (3.19)$$

Combining (2.5), (2.3), (2.6), (2.7) and the Cauchy inequality, we obtain

$$((Q_Nu\cdot\nabla)v,\delta)+((P_Nu\cdot\nabla)Q_Nv,\delta)$$
$$\le c\left(\|Q_Nu\|^{1/2}|AQ_Nu|^{1/2}|v|+\|P_Nu\|^{1/2}|AP_Nu|^{1/2}|Q_Nv|\right)\|\delta\|$$
$$\le c\left(\lambda_{N+1}^{-1/4}|Au||v|+\lambda_{N+1}^{-1/2}\|u\|^{1/2}|Au|^{1/2}\|v\|\right)\|\delta\|$$
$$\le c\left(\lambda_{N+1}^{-1/4}\alpha^{-2}|v|^2+\lambda_{N+1}^{-1/2}2^{-1/4}\alpha^{-3/2}|v|\|v\|\right)\|\delta\|$$
$$\le c\left(\lambda_{N+1}^{-1/4}\alpha^{-2}M_0^2+\lambda_{N+1}^{-1/2}2^{-1/4}\alpha^{-3/2}M_0M_1\right)\|\delta\|$$
$$\le c^2\left(\lambda_{N+1}^{-1/4}\alpha^{-2}M_0^2+\lambda_{N+1}^{-1/2}2^{-1/4}\alpha^{-3/2}M_0M_1\right)^2+\frac{\nu}{4}\|\delta\|^2. \quad (3.20)$$

The last term is bounded uniformly in time for sufficiently large times by our assumption on Gevrey bounds for v (see (1.5) and then (2.8)) as following estimate

$$\mu\sum_{i=1}^{2}(I_{h,N,\Omega}(Q_Nv_i),\delta_i)\le\mu\sum_{i=1}^{2}\|I_{h,\Omega}(Q_Nv_i)\|_{L^2(\Omega)}\|\delta_i\|_{L^2(\Omega)}$$
$$\le\mu c_1^{1/2}\sum_{i=1}^{2}\|Q_Nv_i\|_{L^2(\Omega_0)}\|\delta_i\|_{L^2(\Omega)}$$
$$\le c_1\mu|Q_Nv|^2+\frac{\mu}{4}(\|\delta_1\|_{L^2(\Omega)}^2+\|\delta_2\|_{L^2(\Omega)}^2)$$
$$\le c_1\mu\sum_{\sqrt[3]{N}\le|k|}|\hat{v}_k|^2+\frac{\mu}{4}(\|\delta_1\|_{L^2(\Omega)}^2+\|\delta_2\|_{L^2(\Omega)}^2)$$

$$\leq c_1\mu \sum_{\sqrt[3]{N}\leq|k|} \frac{\mathcal{C}^2}{L^3} e^{-4\pi\sigma\frac{|k|}{L}} + \frac{\mu}{4}(\|\delta_1\|^2_{L^2(\Omega)} + \|\delta_2\|^2_{L^2(\Omega)}). \tag{3.21}$$

Now, substituting (3.10), (3.19), (3.20) and (3.21) into (3.9) to deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\delta|^2 + \frac{\nu}{2}\|\delta\|^2 \leq \mu c_0 h^2\|\delta\|^2 &+ \left(\frac{cM_1^2 C_\Omega e^{C_\Omega\sqrt[3]{N}}}{\alpha^3\nu^2\lambda_1} - \frac{\mu}{2}\right)(\|\delta_1\|^2_{L^2(\Omega)} + \|\delta_2\|^2_{L^2(\Omega)})\\ &+ c^2\left(\lambda_{N+1}^{-1/4}\alpha^{-2}M_0^2 + \lambda_{N+1}^{-1/2}2^{-1/4}\alpha^{-3/2}M_0M_1\right)^2\\ &+ c_1\mu\sum_{\sqrt[3]{N}\leq|k|}\frac{\mathcal{C}^2}{L^3}e^{-4\pi\sigma\frac{|k|}{L}}.\end{aligned} \tag{3.22}$$

Chose N large enough satisfying condition (1.7), we have

$$c^2\left(\lambda_{N+1}^{-1/4}\alpha^{-2}M_0^2 + \lambda_{N+1}^{-1/2}2^{-1/4}\alpha^{-3/2}M_0M_1\right)^2 \leq \frac{\bar{\varepsilon}}{2}.$$

From condition (1.8) on μ and (1.9) on h, then from (3.22) we have

$$\frac{1}{2}\frac{d}{dt}|\delta|^2 + \frac{\nu}{4}\|\delta\|^2 \leq c_1\mu\sum_{\sqrt[3]{N}\leq|k|}\frac{\mathcal{C}^2}{L^3}e^{-4\pi\sigma\frac{|k|}{L}} + \frac{\bar{\varepsilon}}{2}.$$

From condition (1.8) on μ then

$$\begin{aligned}c_1\mu\sum_{\sqrt[3]{N}\leq|k|}\frac{\mathcal{C}^2}{L^3}e^{-4\pi\sigma\frac{|k|}{L}} &= \frac{cM_1^2C_\Omega}{\alpha^3\nu^2\lambda_1}\sum_{\sqrt[3]{N}\leq|k|}\frac{\mathcal{C}^2}{L^3}e^{C_\Omega\sqrt[3]{N}-4\pi\sigma\frac{|k|}{L}}\\ &= \frac{cM_1^2C_\Omega\mathcal{C}^2}{\alpha^3\nu^2\lambda_1L^3}\sum_{\sqrt[3]{N}\leq|k|}e^{C_\Omega\sqrt[3]{N}-2\pi\sigma\frac{|k|}{L}}e^{-2\pi\sigma\frac{|k|}{L}}\\ &\leq \frac{cM_1^2C_\Omega\mathcal{C}^2}{\alpha^3\nu^2\lambda_1L^3e^{2\pi\sigma\frac{\sqrt[3]{N}}{L}}}\sum_{\sqrt[3]{N}\leq|k|}e^{C_\Omega\sqrt[3]{N}-2\pi\sigma\frac{|k|}{L}}.\end{aligned}$$

Here we have used the fact that

$$\sum_{\sqrt[3]{N}\leq|k|}e^{C_\Omega\sqrt[3]{N}-2\pi\sigma\frac{|k|}{L}}e^{-2\pi\sigma\frac{|k|}{L}} \leq \frac{1}{e^{2\pi\sigma\frac{\sqrt[3]{N}}{L}}}\sum_{\sqrt[3]{N}\leq|k|}e^{C_\Omega\sqrt[3]{N}-2\pi\sigma\frac{|k|}{L}}.$$

Taking σ satisfying (1.6) we have

$$\sum_{\sqrt[3]{N}\leq|k|}e^{C_\Omega\sqrt[3]{N}-2\pi\sigma\frac{|k|}{L}} \leq 1.$$

Hence

$$\frac{1}{2}\frac{d}{dt}|\delta|^2 + \frac{\nu}{4}\|\delta\|^2 \leq \bar{\varepsilon} \tag{3.23}$$

provided N is large enough satisfying (1.7), i.e.,

$$N \geq \left(\frac{L}{2\pi\sigma}\right)^3\ln^3\left(\frac{cM_1^2C_\Omega\mathcal{C}^2}{\alpha^3\nu^2\lambda_1L^3\bar{\varepsilon}}\right).$$

Using (2.2), we obtain

$$\frac{d}{dt}|\delta|^2 + \frac{\nu}{2}\lambda_1|\delta|^2 \leq 2\bar{\varepsilon}.$$

Hence, by using Gronwall inequality and the fact that $\bar{\varepsilon} = \frac{\varepsilon\nu\lambda_1}{8}$, we deduce that

$$|\delta(t)|^2 \leq |v_0|^2e^{-\nu\lambda_1t/2} + \frac{4\bar{\varepsilon}}{\nu\lambda_1}(1 - 2^{-\nu\lambda_1t/2}) \leq |v_0|^2e^{-\nu\lambda_1t/2} + \frac{\varepsilon}{2}. \tag{3.24}$$

Note that

$$|v(t) - z_N(t)|^2 \le |\delta(t)|^2 + |Q_N v(t)|^2 \le |v_0|^2 e^{-\nu\lambda_1 t/2} + \frac{3\varepsilon}{4}, \tag{3.25}$$

provided N is taken large enough so that

$$|Q_N v|^2 \le \frac{\varepsilon}{4}.$$

Thus,

$$|v(t) - z_N(t)| < \varepsilon,$$

for t sufficiently large.

Remark 3.1. In Theorem 1.1 we work with the periodic boundary conditions since to proving this result we have to use the Gevrey property of solution and a spectral inequality in the case of periodic boundary conditions. We note that here we also have the spectral inequality for the no-slip boundary conditions (see [15, Theorem 3.1]). However, in order to overcome the difficulty concerning with $Q_N v$ (see (3.21)), we need to assume that $\hat{v}_N = (v, \phi_N)$ is decay sufficiently fast as $N \to \infty$. More precisely, we need $\hat{v}_N \sim e^{-C\sqrt[3]{N}}$ for some positive constant C. But we do not know when this holds for the case of no-slip boundary conditions. This is true for the case of periodic boundary conditions.

Remark 3.2. The results in Theorem 1.1 also hold for the local interpolant operators of type 2 satisfying (see [12])

$$\|\mathcal{I}_{h,\Omega}(\varphi) - \varphi\|_{L^2(\Omega)}^2 \le c_0^2 \left(h^2 \|\varphi\|_{H^1(\Omega_0)}^2 + h^4 \|\varphi\|_{H^2(\Omega_0)}^2 \right), \ \forall \varphi \in H^2(\Omega_0). \tag{3.26}$$

Indeed, we have the same proof, with a note that by using (3.26) then (3.4) is replaced by the following estimate

$$\begin{aligned}
&- \mu \sum_{i=1}^{2} \left(\mathcal{I}_{h,N,\Omega}(z_N^i), z_N^i \right) \\
&= -\mu \sum_{i=1}^{2} \left(\mathcal{I}_{h,\Omega}(z_N^i) - \chi_\Omega z_N^i, z_N^i \right) - \mu \sum_{i=1}^{2} \|z_N^i\|_{L^2(\Omega)}^2 \\
&\le \mu \sum_{i=1}^{2} \|\mathcal{I}_{h,\Omega}(z_N^i) - z_N^i\|_{L^2(\Omega)} \|z_N^i\|_{L^2(\Omega)} - \mu \sum_{i=1}^{2} \|z_N^i\|_{L^2(\Omega)}^2 \\
&\le \frac{\mu}{2} \sum_{i=1}^{2} c_0^2 \left(h^2 \|z_N^i\|_{H^1(\Omega_0)}^2 + h^4 \|z_N^i\|_{H^2(\Omega_0)}^2 \right) - \frac{\mu}{2} \sum_{i=1}^{2} \|z_N^i\|_{L^2(\Omega)}^2 \\
&\le \frac{\mu c_0^2 h^2 (1 + h^2 \lambda_{N+1})}{2} \|z_N\|^2 - \frac{\mu}{2} \sum_{i=1}^{2} \|z_N^i\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.27}$$

And (3.10) is similarly as (3.27) replaced by

$$\begin{aligned}
-\mu \sum_{i=1}^{2} (\mathcal{I}_{h,N,\Omega}(\delta_i), \delta_i) &= \mu \sum_{i=1}^{2} (\chi_\Omega \delta_i - \mathcal{I}_{h,\Omega}(\delta_i), \delta_i) - \mu \sum_{i=1}^{2} \|\delta_i\|_{L^2(\Omega)}^2 \\
&\le \mu c_0^2 h (1 + h^2 \lambda_{N+1}) \|\delta\|^2 - \frac{3\mu}{4} \sum_{i=1}^{2} \|\delta_i\|_{L^2(\Omega)}^2.
\end{aligned}$$

3.2. **Proof of Corollary 1.2.** From (3.24) in the proof of Theorem 1.1, we have

$$|z_N(t)| \le |z_N(t) - P_N v| + |P_N v| < |v_0| e^{-\frac{\nu\lambda_1 t}{4}} + \varepsilon + |v|.$$

This implies that z_N is bounded in $L^\infty(0,T;H)$ uniformly in μ and N. Moreover, from (3.23) one can see that z_N is bounded in $L^2(0,T;V)$ uniformly in μ and N. Thus, for any $0 < \varepsilon \ll 1$ we can construct a solution z_{N_ε} for parameters $N_\varepsilon, \mu_\varepsilon$ and h_ε to (1.4) with the usual energy class bounds holding independent of N_ε and μ_ε, provided we have knowledge of v at all points in Ω. As $\varepsilon \to 0$, we have $N_\varepsilon, \mu_\varepsilon \to \infty$ while $h_\varepsilon \to 0$. By the Banach-Alaoglu theorem, there exists z so that $z_{N_\varepsilon} \to z$ in the weak-star topology on $L^\infty(0,T;H)$ for every $T > 0$ as well as the weak topology on $L^2(0,T;V)$.

Let U be a fixed measurable set and let $\Delta > 0$ be a given time scale. For any t, we have

$$\int_t^{t+\Delta} \int_U (v - z) dx ds = \int_t^{t+\Delta} \int_U (v - z_{N_\varepsilon}) dx ds + \int_t^{t+\Delta} \int_U (z_{N_\varepsilon} - z) dx ds.$$

Using (3.25) in the proof of Theorem 1.1, we have

$$\begin{aligned} \int_t^{t+\Delta} \int_U (v - z_{N_\varepsilon}) dx ds &\le |U|^{1/2} \left(\sup_{s\in[t,t+\Delta]} |v - z_{N_\varepsilon}|^2(s) \right)^{1/2} \\ &\le |U|^{1/2} \left(\sup_{s\in[t,t+\Delta]} |v_0|^2 e^{-\frac{\nu\lambda_1 s}{2}} + \frac{3}{4}\varepsilon \right)^{1/2}. \end{aligned}$$

Additionally, because of $*$-weak convergence in $L^\infty L^2$, we have

$$\left| \int_t^{t+\Delta} \int_U (z_{N_\varepsilon} - z)(x,s) dx ds \right| \to 0.$$

Thus, we can choose ε so that $\varepsilon < e^{-t}$ and the above quantity is smaller than e^{-t}. So that, we obtain the advertised exponentially decaying bound.

Now we prove the second item in Corollary 1.2. By the Lebesgue differentiation theorem, for almost t we have

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_U (v - z)(x,s) dx ds = \int_U (v - z)(x,t) dx,$$

where the time-scale Δt is depended on t. Denote S the set of times for which this holds. Then $|S^c| = 0$ where $|.|$ denotes Lebesgue measure on the line. Fix $t > 0$. Then for Δt sufficiently small we have

$$\begin{aligned} &\int_U (v - z)(x,t) dx \le \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_U (v - z)(x,s) dx ds + e^{-t} \\ &\le \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_U (v - z_{N_\varepsilon})(x,s) dx ds + \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_U (z_{N_\varepsilon} - z)(x,s) dx ds + e^{-t}. \end{aligned}$$

The first two terms can be made exponentially small that have been already explained. Because this holds for all $t \in S$, we obtain

$$\chi_S(t) \int_U (v - z)(x,t) dx \to 0,$$

at an exponential rate. Redefining v to equal u on S^c completes the proof.

3.3. **Proof of Theorem 1.3.** We prove in two steps as follows.
Step 1. Existence and uniqueness of z. The existence of z is based on the Galerkin approximation method. We give some a priori estimates for z.

Multiplying $(1.10)_1$, $(1.10)_2$ and $(1.10)_3$ by z_1, z_2 and z_3 respectively, then integrating over Ω_0 we obtain

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \nu\|z\|^2 = (f,z) + \mu\sum_{i=1}^{2}(I_{h,\Omega}(v_i), z_i) - \mu\sum_{i=1}^{2}(I_{h,\Omega}(z_i), z_i)\,. \tag{3.28}$$

Using the Cauchy inequality and the Poincaré inequality (2.1), we obtain

$$|(f,z)| \le \frac{1}{\nu\lambda_1}|f|^2 + \frac{\nu}{4}\|z\|^2. \tag{3.29}$$

Using the Cauchy inequality and the Poincaré inequality (2.1) and (3.26), we obtain

$$\begin{aligned}
\mu\sum_{i=1}^{2}(I_{h,\Omega}(v_i), z_i) &\le \frac{\mu^2}{\nu\lambda_1}\sum_{i=1}^{2}\|I_{h,\Omega}(v_i)\|^2_{L^2(\Omega)} + \frac{\nu}{4}\|z\|^2 \\
&\le \frac{c_1\mu^2}{\nu\lambda_1}\left(\|v_1\|^2_{L^2(\Omega_0)} + \|v_2\|^2_{L^2(\Omega_0)}\right) + \frac{\nu}{4}\|z\|^2 \\
&\le \frac{c_1\mu^2}{\nu\lambda_1}|v|^2 + \frac{\nu}{4}\|z\|^2.
\end{aligned} \tag{3.30}$$

Using the Cauchy inequality, (1.2) and the Poincaré inequality (2.2), we obtain

$$\begin{aligned}
-\mu\sum_{i=1}^{2}(I_{h,\Omega}(z_i), z_i) &= -\mu\sum_{i=1}^{2}(I_{h,\Omega}(z_i), z_i) \\
&= -\mu\sum_{i=1}^{2}(I_{h,\Omega}(z_i) - \chi_\Omega z_i, z_i) - \mu\sum_{i=1}^{2}\|z_i\|^2_{L^2(\Omega)} \\
&\le \mu\sum_{i=1}^{2}\|I_{h,\Omega}(z_i) - z_i\|_{L^2(\Omega)}\|z_i\|_{L^2(\Omega)} - \mu\sum_{i=1}^{2}\|z_i\|^2_{L^2(\Omega)} \\
&\le \frac{\mu c_0 h^2}{2\lambda_1}\sum_{i=1}^{2}\|z_i\|^2_{H^1(\Omega_0)} - \frac{\mu}{2}\sum_{i=1}^{2}\|z_i\|^2_{L^2(\Omega)}.
\end{aligned}$$

Since h is sufficiently small satisfying (1.12), then $\frac{\mu c_0 h^2}{2\lambda_1} \le \frac{\nu}{4}$, we have

$$-\mu\sum_{i=1}^{2}(I_{h,\Omega}(z_i), z_i) \le \frac{\nu}{4}\|z\|^2 - \frac{\mu}{2}\sum_{i=1}^{2}\|z_i\|^2_{L^2(\Omega)}. \tag{3.31}$$

Substituting estimates (3.29), (3.30) and (3.31) into (3.28), we deduce that

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \frac{\nu}{4}\|z\|^2 \le \frac{1}{\nu\lambda_1}|f|^2 + \frac{c_1\mu^2}{\nu\lambda_1}|v|^2 - \frac{\mu}{2}\sum_{i=1}^{2}\|z_i\|^2_{L^2(\Omega)}. \tag{3.32}$$

Using the Poincaré inequality (2.2) and dropping the last term, we have

$$\frac{d}{dt}|z|^2 + \frac{\nu\lambda_1}{2}|z|^2 \le \frac{2}{\nu\lambda_1}|f|^2 + \frac{2c_1\mu^2}{\nu\lambda_1}|v|^2. \tag{3.33}$$

From (3.33), using the Gronwall inequality, we conclude that $z \in L^\infty(0,\infty;H)$. Moreover, from (3.32) we obtain that $z \in L^2_{loc}(0,\infty;V)$. From these bounds, one gets the global existence of z. The proof of uniqueness is standard.

Step 2. Prove the convergence (1.14)**.** Consider $\delta = z - v,\ \eta = w - u$, we have $\delta = \eta - \alpha^2 \Delta \eta$. Then δ satisfies

$$\frac{1}{2}\frac{d}{dt}|\delta|^2 + \nu\|\delta\|^2 + ((w\cdot\nabla)z - (u\cdot\nabla)v, \delta) = -\mu\sum_{i=1}^{2}(I_{h,\Omega}(\delta_i), \delta_i). \tag{3.34}$$

Since $(w\cdot\nabla)z - (u\cdot\nabla)v = (w\cdot\nabla)\delta + (\eta\cdot\nabla)v$ and by the property (2.4), we deduce from (3.34) that

$$\frac{1}{2}\frac{d}{dt}|\delta|^2 + \nu\|\delta\|^2 = -\left((\eta\cdot\nabla)v, \delta\right) - \mu\sum_{i=1}^{2}(I_{h,\Omega}(\delta_i), \delta_i). \tag{3.35}$$

By the Cauchy inequality and using (2.3) we deduce that

$$\begin{aligned}
-\mu\sum_{i=1}^{2}(I_{h,\Omega}(\delta_i), \delta_i) &= \mu\sum_{i=1}^{2}(\chi_\Omega\delta_i - I_{h,\Omega}(\delta_i), \delta_i) - \mu\|\delta_i\|^2_{L^2(\Omega)} \\
&\le \frac{\mu}{2}\sum_{i=1}^{2}\|\delta_i - I_{h,\Omega}(\delta_i)\|^2_{L^2(\Omega)} - \frac{\mu}{2}\sum_{i=1}^{2}\|\delta_i\|^2_{L^2(\Omega)} \\
&\le \frac{c_0\mu h^2}{2}\sum_{i=1}^{2}\|\delta_i\|^2 - \frac{\mu}{2}\sum_{i=1}^{2}\|\delta_i\|^2_{L^2(\Omega)} \\
&\le \frac{c_0\mu h^2}{2}\|\delta\|^2 - \frac{\mu}{2}\sum_{i=1}^{2}\|\delta_i\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.36}$$

For the nonlinear term $((\eta\cdot\nabla)v, \delta)$, we process as same as when estimates for $((\eta\cdot\nabla)P_N v, \delta)$ in the proof of Theorem 1.1 (see (3.18)), we infer that

$$((\eta\cdot\nabla)v, \delta) \le \frac{\nu}{4}\|\delta\|^2 + c\alpha^{-3}\lambda_1^{-1}\nu^{-1}\|v\|^2\left(\|\delta_1\|^2_{L^2(\Omega_0)} + \|\delta_2\|^2_{L^2(\Omega_0)}\right). \tag{3.37}$$

Combining (3.36) and (3.37), using (2.3) and note that $\frac{c_0\mu h^2}{2} \le \frac{\nu}{4}$ we obtain from (3.35) that

$$\frac{d}{dt}|\delta|^2 + \nu\|\delta\|^2 \le \frac{cM_1^2}{\alpha^3\nu\lambda_1}\left(\|\delta_1\|^2_{L^2(\Omega_0)} + \|\delta_2\|^2_{L^2(\Omega_0)}\right) - \mu\left(\|\delta_1\|^2_{L^2(\Omega)} + \|\delta_2\|^2_{L^2(\Omega)}\right). \tag{3.38}$$

By using the inequality (2.10) then (3.38) becomes

$$\begin{aligned}
\frac{d}{dt}|\delta|^2 + \nu\|\delta\|^2 \le &\frac{cM_1^2}{\alpha^3\nu\lambda_1}\left(\frac{\ell}{\lambda_1(\Omega_0\setminus\Omega)}\left(\|\delta_1\|^2_{L^2(\Omega)} + \|\delta_2\|^2_{L^2(\Omega)}\right) + \frac{1}{\lambda_1(\Omega_0\setminus\Omega)}\|\delta\|^2\right) \\
&- \mu\left(\|\delta_1\|^2_{L^2(\Omega)} + \|\delta_2\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

So we require $\Omega_0\setminus\Omega$ to be thin enough such that

$$\frac{cM_1^2\ell}{\alpha^3\nu\lambda_1\lambda_1(\Omega_0\setminus\Omega)} \le \frac{\nu}{2}, \tag{3.39}$$

then

$$\frac{d}{dt}|\delta|^2 + \frac{\nu}{2}\|\delta\|^2 \le -\left(\mu - \frac{cM_1^2\ell}{\alpha^3\nu\lambda_1\lambda_1(\Omega_0\setminus\Omega)}\right)\left(\|\delta_1\|^2_{L^2(\Omega)} + \|\delta_2\|^2_{L^2(\Omega)}\right).$$

From (2.11) we see that $d_H(\partial\Omega, \partial\Omega_0) \sim \frac{1}{\sqrt{\lambda_1(\Omega_0\setminus\Omega)}}$. Therefore, if $\Omega_0\setminus\Omega$ is thin enough satisfying (1.13), then from (3.39) and using condition (1.11) we have

$$\mu - \frac{cM_1^2\ell}{\alpha^3\nu\lambda_1\lambda_1(\Omega_0\setminus\Omega)} \ge 0.$$

Thus

$$\frac{d}{dt}|\delta|^2 + \frac{\nu\lambda_1}{2}|\delta|^2 \leq 0.$$

By the Gronwall inequality we obtain the convergence (1.14) at an exponential rate.

Remark 3.3. This theorem is only proved when the local interpolant operator is of type 1, i.e., the operator $I_{h,\Omega}$ satisfies (1.2). The data assimilation for the three-dimensional Leray-α model (1.10) using observables of type 2 is much more complicate even for the global observables cases (see [25]). Here, due to the local observables we have to use estimate (2.10) to treat the local terms. Thus, the data assimilation for the Leray-α model using local in of type 2 for both fully or reduced observables are interesting questions to study.

4. Appendix

Proof of Theorem 2.1. The proof in the case of periodic boundary conditions is given in [17, 21, 25]. For the case of no-slip boundary conditions, the proof is similarly to the case of Navier-Stokes-α equations (see [13, 18]). To convenience for readers, we sketch the proof for the case of no-slip boundary conditions.

The existence and uniqueness of solutions is proved by the Garlenkin approximate method. We only give some uniform bounded estimates for the approximate solution of the following system

$$\begin{cases} \dfrac{\partial v_n}{\partial t} - \nu\Delta v_n + (u_n\cdot\nabla)v_n + \nabla p = f, \\ v_n = u_n - \alpha^2\Delta u_n, \\ \nabla\cdot u_n = \nabla\cdot v_n = 0, \\ u_n = v_n = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

Multiplying the first equation in (1.10) by v_n and integrating in Ω_0, using boundary conditions $(4.1)_3$ and free divergence $(4.1)_2$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|v_n|^2 + \nu\|v_n\|^2 + ((u_n\cdot\nabla)v_n, v_n) = (f, v_n). \tag{4.2}$$

From (2.2) and the Cauchy inequality, we deduce from (4.2) that

$$\frac{1}{2}\frac{d}{dt}|v_n|^2 + \nu\|v_n\|^2 \leq \frac{|f|^2}{2\nu\lambda_1} + \frac{\nu\lambda_1}{2}|v_n|^2.$$

By the Poincaré inequality (2.2) one obtain that

$$\frac{d}{dt}|v_n|^2 + \nu\|v_n\|^2 \leq \frac{|f|^2}{\nu\lambda_1}. \tag{4.3}$$

From this inequality, by using the Poincaré inequality (2.2) once again to deduce

$$|v_n(t)|^2 \leq |v_0|^2 e^{-\nu\lambda_1 t} + \frac{|f|^2}{\nu^2\lambda_1^2}(1 - e^{-\nu\lambda_1 t}).$$

This inequality implies estimate (2.6).

Multiplying the first equation in (4.1) by Δv_n, integrating in Ω_0, using the boundary conditions $(4.1)_3$ and free divergence $(4.1)_2$ we deduce that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_n\|^2 + \nu|\Delta v_n|^2 &= -((u_n\cdot\nabla)v_n, \Delta v_n) + (f, \Delta v_n) \\ &\leq \|u_n\|_{L^\infty(\Omega_0)^3}|\nabla v_n||\Delta v_n| + \frac{|f|^2}{\nu} + \frac{\nu}{4}|\Delta v_n|^2 \\ \text{(by Cauchy's inequality)} &\leq \frac{c}{\nu}\|u_n\|^2_{L^\infty(\Omega_0)^3}\|v_n\|^2 + \frac{\nu}{2}|\Delta v_n|^2 + \frac{|f|^2}{\nu} \end{aligned}$$

$$\text{(by Agmon's inequality)} \leq \frac{c}{\nu}\|u_n\||\Delta u_n|\|v_n\|^2 + \frac{\nu}{2}|\Delta v_n|^2 + \frac{|f|^2}{\nu}$$

$$\text{(by (2.3))} \leq \frac{c}{\nu\alpha^3}|v_n|^2\|v_n\|^2 + \frac{\nu}{2}|\Delta v_n|^2 + \frac{|f|^2}{\nu}.$$

Thus,

$$\frac{d}{dt}\|v_n\|^2 + \nu|\Delta v_n|^2 \leq \frac{c}{\nu\alpha^3}|v_n|^2\|v_n\|^2 + \frac{|f|^2}{\nu}. \tag{4.4}$$

From (4.3) then

$$\int_t^{t+1}\|v_n(s)\|^2 ds \leq \frac{|f|^2}{\nu^2\lambda_1} + \frac{1}{\nu}|v_n(t)|^2. \tag{4.5}$$

Noting that $|v_n(t)|^2 \leq \dfrac{2|f|^2}{\nu^2\lambda_1^2}$ for $t \geq t_0 := t_0(|v_0|^2) > 0$, then from (4.4) and (4.5), we use the uniform Gronwall inequality to deduce that

$$\|v_n(t)\|^2 \leq \left(\frac{1}{\nu^2\lambda_1} + \frac{2}{\nu^3\lambda_1^2} + \frac{2c}{\nu^3\alpha^3\lambda_1^2}\right)|f|^2 e^{\frac{|f|^2}{\nu}}, \ \forall t \geq t_0 + 1.$$

This inequality implies (2.7). □

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