Some recent results on regularity theory for linear parabolic equations with singular-degenerate coefficients

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Topics and Outline

Topics: Investigate regularity estimates in Sobolev spaces of solutions with degenerate or singular coefficients.

Outline:

- Classical results for equations with uniformly elliptic and bounded coefficients:
  - Harnack's inequality, Hölder's estimates.
  - Estimates in Sobolev spaces.
- Equations with degenerate or singular coefficients:
  - Definitions, some motivations, examples.
  - Known results: Harnack's inequality, Hölder's estimates.
- L^p-theory for equations with degenerate or singular coefficients:
  - Fabes-Kenig-Serapioni singular-degenerate type equations.
  - Equations of Caffarelli-Silvestre extensional type equations.
  - Equations with degenerate coefficients (degenerate viscous Hamilton- Jacobi equation).

Classical results: Harnack's inequality, Hölder's estimates

- Elliptic equations in divergence form

$$-D_i(a_{ij}(x)D_ju) = D_iF_i(x) + f(x) \quad \text{in} \quad \Omega \subset \mathbb{R}^d$$

or in non-divergence form

$$-a_{ij}(x)D_{ij}u(x) = f(x) \quad \text{in} \quad \Omega \subset \mathbb{R}^d$$

where Enstein's summation convention is used and

$$D_i = \frac{\partial}{\partial x_i}.$$

- Uniformly elliptic and boundedness condition: $\exists \nu \in (0,1)$ such that

$$\nu|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \quad \text{and} \quad |a_{ij}(x)| \leq \nu^{-1},$$

for all $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d$ and all x.

Harnack's inequality, Hölder's estimates (cont.)

- Results: We have Harnack's inequality for non-negative solutions and Hölder's regularity estimates for solutions: there is $\alpha = \alpha(d, \nu) \in (0, 1)$

  $$\|u\|_{C^\alpha(B_{1/2})} \leq N(d, \nu)\|u\|_{L^2(B_1)}, \quad \text{(assuming } F_i = f = 0\text{)}$$

  Divergence case: De Giorgi - Nash - Moser, 1960s.

  Non-divergence case: Krylov-Safanov, 1980s.

- Similar results also hold for parabolic equations.

Classical results: Estimates in Sobolev spaces

With additional regularity conditions on a_{ij}: a_{ij} are uniformly continuous or VMO, i.e.

$$\fint_{B_r(x_0)} |a_{ij}(x) - (a_{ij})_{B_r(x_0)}| dx \to 0 \quad \text{as} \quad r \to 0^+ \text{ uniformly in } x_0$$

where

$$(a_{ij})_{B_r(x_0)} = \fint_{B_r(x_0)} a_{ij}(x) dx = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} a_{ij}(x) dx$$

Then, for $p \in (1, \infty)$

$$\|Du\|_{L^p} \leq N\Big[\|F\|_{L^p} + \|f\|_{L^p}\Big] \quad \text{for equations in divergence form}$$

and

$$\|D^2 u\|_{L^p} \leq N \|f\|_{L^p} \quad \text{for equations in non-divergence form.}$$

where $N = N(d, \nu, p) > 0$.

Estimates in Sobolev spaces (cont.)

- Classical proofs: Based on solution representation/formula and applications of Calderón-Zygmund theorem and Coifman-Rochberg-Weiss commutator theorem:
  - Chiarenza-Frasca-Longo (1991, 1993): W^2_p-estimate for non-divergence form elliptic equations with uniformly elliptic, bounded VMO leading coefficients.
  - Di Fazio (1996): W^1_p-estimate for divergence form elliptic equations with uniformly elliptic, bounded VMO leading coefficients.
  - Bramanti and Cerutti (1993): non-divergence form parabolic equations with uniformly elliptic, bounded VMO leading coefficients.

  The method can't be extended to quasilinear/nonlinear equations.
- Modern proofs: Krylov (2000s): used mean oscillation; and Cafferalli-Peral (2000s) used level set. Both methods can be used for nonlinear, fully nonlinear equations.

Degenerate or singular coefficients

Recall: uniformly elliptic and boundedness condition: $\exists \nu \in (0,1)$ such that

$$\nu|\xi|^2 \leq a_{ij}\xi_i\xi_j \quad \text{and} \quad |a_{ij}| \leq \nu^{-1}, \quad \forall \xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d.$$

Definition

- The coefficient (a_{ij}) is degenerate if the condition: $\exists \nu \in (0,1)$ such that

$$\nu|\xi|^2 \leq a_{ij}\xi_i\xi_j \quad \xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d$$

  does not hold.
- The coefficient (a_{ij}) is singular if it is not bounded.

Degenerate or singular coefficients (cont.)

Degenerate or singular coefficients appear naturally in many areas/problems

- Porous media, fast diffusion, composite materials, ...
- Mathematical finance.
- Mathematical biology (see, for instance, Epstein and Mazzeo's book)
- Probability: degenerate diffusion processes
- Geometric PDEs
- Purely mathematics:
  + Changes of variables: Extension operators associate with fractional Laplacian or fractional heat operator (Caffarelli -Silvestre, Comm. PDE, 2007).

$$-D_i(x_d^{\alpha} D_i u) = f \qquad x = (x', x_d) \in \mathbb{R}^d_+ := \mathbb{R}^{d-1} \times (0, \infty)$$

  + Regularization (approximation): Singular/degenerate viscous Hamilton-Jacobi equations

$$-\epsilon x_d^{\alpha} \Delta u + H(x, Du) + u = f, \quad x = (x', x_d) \in \mathbb{R}^d_+.$$

Two simple examples

- For $\alpha \in (0,1)$, consider the equation

$$(x^\alpha u_x)_x = 0, \quad x > 0.$$

The equation is degenerate at only one point $x = 0$. Note that $u(x) = x^{1-\alpha}$ is a solution which satisfies the boundary condition $u(0) = 0$. However,

$$u_x(x) \notin L^p(0,1) \text{ for large } p$$

i.e. $W^{1,p}$-estimate does not hold.

- Consider $u_{xx} = 0$ in 2D. This equation is degenerate in the whole y-variable direction. Note that $u = u(y)$ is a solution which we can not expect to have any regularity.

Conclusion: Has to impose certain conditions on the equations (i.e. coefficients) to derive/find suitable L^p-estimates.

Known results for equations with degenerate or singular coefficients: Harnack's inequality and Hölder estimates

Harnack's inequality and Hölder estimates for equations in divergence form $D_i(a_{ij}(x)D_j u) = 0 \quad \Omega \subset \mathbb{R}^d$.

- Trudinger (1971, ARMA): Denote $\lambda(x)$ the smallest eigenvalue of $(a_{ij}(x))$ and $\mu(x) = \sup |a_{ij}(x)|$. If

$$\lambda^{-1} \in L^p \quad \text{and} \quad \mu \in L^q \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} < \frac{2}{d}$$

  then Harnack's inequality and Hölder's estimates hold.
- Bella and Schaffner (2021, CPAM) improved Trudinger's results with

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$$

  which is optimal due to a counterexample of Franchi, Serapioni, and Serra Cassano (1998).
- Related work: Kruzkov (1963), Murthy and Stampacchia (1968)

Known results: Harnack's inequality and Hölder estimates (cont.)

Harnack's inequality and Hölder estimates for equations in divergence form

$$D_i(a_{ij}(x)D_j u) = 0 \quad \Omega \subset \mathbb{R}^d.$$

- Fabes, Kenig, and Serapioni (1982): If

$$\nu\omega(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \nu^{-1}\omega(x)|\xi|^2$$

  where $\omega \in A_2$ Muckenhoupt weight. Then Harnack's inequality and Hölder's estimates hold.
- Note that for $p \in (1, \infty)$, a non-negative locally integrable function $\omega : \mathbb{R}^d \to (0, \infty)$ is said to be in A_p if

$$\sup_{\rho>0, x_0 \in \mathbb{R}^d} \left(\fint_{B_\rho(x_0)} \omega(x) dx \right) \left(\fint_{B_\rho(x_0)} \omega(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty.$$

- Example: $\omega(x) = |x|^\alpha \in A_p$ if and only if $\alpha \in (-d, p(d-1))$.

Other known related results and remarks

- Harnack's inequality, Hölder/ Schauder estimates: proved for various nonlinear, fully nonlinear elliptic and parabolic equations with specific structures: Keldys (1957); Fichera (1965); Oleinik-Radkevic (1973); Chiarenza-Serapioni (1987); Chen-DiBenedetto (1988); Gutierrez-Wheeden (1991); F.-H. Lin (1989); F.-H. Lin-L. Wang (1998); H. Koch (1999); Daskalopoulos-Hamilton (2000); Daskalopoulos-Hamilton-Lee (2001); Feehan-Pop (2014); Le-Savin (2017); Sire-Terracini-Vita (2021); Jin-Xiong (2019, 2022),...

Remark

- *There is no version of Krylov-Safanov's result for equations in non-divergence. Mainly, no known APB (Alexandrov-Bakelmann-Pucci) estimate in this case.*
- *In contrast to the theory on Hölder/ Schauder estimates, theory and estimates in Sobolev spaces for equations with singular/degenerate coefficients are not known much.*

Singular-degenerate coefficients of Fabes-Kenig-Serapioni type: Cao-Mengesha-P.'s work

- Consider the equation

$$\begin{cases} D_i(a_{ij}(x)D_j u) = D_i F_i & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{0.1}$$

- Assume as in Fabes-Kenig-Serapioni's work: there exist $\nu \in (0, 1)$ and $\omega \in A_2$ such that

$$\nu\omega(x)|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \nu^{-1}\omega(x)|\xi|^2$$

- Goal: Study the theory of the equation in Sobolev space.

  What is the Sobolev counterpart of the Fabes - Kenig - Serapioni's result?

  What are the conditions on (a_{ij})?

Cao-Mengesha-P.'s work for Fabes-Kenig-Serapioni type

Theorem (Cao-Mengesha-P. - IUMJ (2018))

Let $p \in (1, \infty)$ and $\nu, r_0 \in (0, 1)$. Then, there is $\delta = \delta(p, \nu) > 0$ sufficiently small such that if $\Omega \subset \mathbb{R}^d$ is open, bounded and sufficiently smooth and (r_0, δ)-weighted VMO condition for (a_{ij}) holds, for $F = (F_1, \ldots, F_d) \in L^p(\Omega)^d$, there exists unique weak solution $u \in W_0^{1,p}(\Omega, \omega)$ to the equation (0.1) and

$$\|Du\|_{L^p(\Omega,\omega)} \leq N\|F/\omega\|_{L^p(\Omega,\omega)}$$

where $N = N(p, r_0, \nu, \Omega) > 0$ and

$$\|Du\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |Du(x)|^p \omega(x) dx\right)^{1/p}.$$

(r_0, δ)-weighted VMO condition

Definition

The coefficient (a_{ij}) is said to satisfy the (r_0, δ)-weighted VMO condition with some $r_0, \delta \in (0, 1)$ if

$$\frac{1}{\omega(B_r(x_0))} \int_{B_r(x_0)} \frac{|a_{ij}(x) - (a_{ij})_{B_r(x_0)}|^2}{\omega(x)} dx \leq \delta,$$

$\forall\ x_0 \in \overline{\Omega},\ \forall\ r \in (0, r_0), \forall\ i, j = 1, \ldots, d$, where

$$(a_{ij})_{B_r(x_0)} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} a_{ij}(x) dx.$$

Notes

- The condition is invariant under the scaling (of the PDE).
- The relation between (a_{ij}) and ω is encoded implicitly.

Remarks on Cao-Menghesha-P.'s result

- The (r_0, δ)-weighted VMO condition is optimal (a counter example was provided in Cao-Menghesha-P.'s work).
- Consider $\omega(x) = |x|^\alpha$ which is in A_2 and only if $\alpha \in (-1, 1)$. However, it satisfies the (r_0, δ)-weighted VMO condition with sufficiently small δ if and only if $|\alpha|$ is sufficiently small.
- Recent paper by A. Kh. Balci, L. Diening, R. Giova, and A. P. di Napoli (SIMA - 2022) for equations with matrix weights, and p-Laplace equations.

Equations of Caffarelli-Silvestre extensional type

- Caffarelli-Silvestre (CPDE - 2007): Studied the nonlocal equation:
$$(-\Delta)^s u = f \quad \text{in} \quad \mathbb{R}^{d-1},$$
where $s \in (0,1)$. The operator can be defined using Fourier transform of some fractional integrations.
- The solution u is a "trace" of the solution U of
$$-\text{div}(x_d^\alpha \nabla U) = g \quad \text{in} \quad \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0,\infty).$$
where $\alpha = 1 - 2s$, $x = (x', x_d) \in \mathbb{R}^{d-1} \times (0,\infty)$.
- Similar situations also hold for the equation $\mathcal{L}^s u = f$ where
$$\mathcal{L} = -\sum_{i,j=1}^d D_i(a_{ij}(x) D_j u) \quad \text{or} \quad \mathcal{L} = u_t - \sum_{i,j=1}^d D_i(a_{ij}(z) D_j u)$$

Refs: A. Banerjee and N. Garofalo (Adv. Math., 2018), P. R. Stinga and J. L. Torrea (SIMA, 2017).

Problem setting

- Denote $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, \infty)$ the upper half space and

$$z = (t, x), \quad x = (x', x_d) \in \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, \infty), \quad t \in (-\infty, T).$$

- Equations of extensional type in divergence form

$$x_d^\alpha\big(u_t + \lambda u\big) - D_i[x_d^\alpha(a_{ij}(z)D_j u + F_i)] = x_d^\alpha f(z)$$

and in (non-divergence form

$$u_t + \lambda u - a_{ij}(z)D_{ij}u - \frac{\alpha}{x_d} a_{dj}(z)D_j u = f(z)$$

- Boundary condition on $\partial\mathbb{R}^d_+ = \{x_d = 0\}$:

$$\begin{array}{lll} \text{Conormal}: & \lim_{x_d \to 0^+} x_d^\alpha(a_{dj}(z)D_j u(z) + F_d) = 0 & \text{when } \alpha > -1 \\ \text{Dirichlet}: & u(t, x', 0) = 0 & \text{when } \alpha < 1. \end{array}$$

- The coefficient (a_{ij}) is bounded, and uniformly elliptic.

Problem setting and questions (cont.)

- Question: L^p-theory for the PDE
  - Do we have the L^p theory (unweighted) for the PDEs? No, in general. Recall the example $(x^\alpha u_x) = 0$.
  - We need to consider L^p theory with weights. What are the suitable weights?
  - What are the right (or minimal) conditions on a_{ij}?
- Issue: The coefficients $x_d^\alpha a_{ij}(z)$ may not bounded, not uniformly elliptic, and not sufficiently smooth (even not locally integrable as $\alpha \leq -1$), given $a_{ij}(z)$ are uniformly elliptic and bounded.
- Recall Cao-Mengehsa-P.'s result requires α to be sufficiently small, which is not valid in our setting.
  Recalling that for fractional s- Laplacian, $\alpha = 1 - 2s \in (-1, 1)$ for $s \in (0, 1)$.

Energy estimates: what are the right choices of weights and right conditions on coefficients?

- Consider a solution $u \in C_0^\infty(\overline{\mathbb{R}^{d+1}}_+)$ of
$$x_d^\alpha(u_t + \lambda u) - D_i[x_d^\alpha a_{ij}(z) D_j u + F_i] = x_d^\alpha f(z)$$
with either the Dirichlet or the conormal B.C. on $\{x_d = 0\}$.
- Testing the equation with u, and using the integration by parts, and Cauchy-Schwartz inequality, we obtain
$$\int_{\mathbb{R}^{d+1}_+} |Du(z)|^2 x_d^\alpha dxdt + \lambda \int_{\mathbb{R}^{d+1}_+} |u(z)|^2 x_d^\alpha dxdt$$
$$\leq N \int_{\mathbb{R}^{d+1}_+} \Big[|F|^2 + \lambda^{-1}|f(z)|^2\Big] x_d^\alpha dxdt.$$
- This suggests that x_d^α is a natural weight. *Can we use the weight x_d^α to derive the L^p-theory for $p \in (1, \infty)$? What conditions are needed for the coefficients a_{ij}?*
- Note that x_d^α is only locally integrable when $\alpha > -1$.

L^p-theory in divergence case: Conormal B.C.

Theorem (Dong-P. (CVPDE 2021, IUMJ (accepted)- 2022))

Let $\alpha \in (-1,\infty)$, $p \in (1,\infty)$, and $d\mu = x_d^\alpha dz$. Then if $\lambda \geq \lambda_0$, and $f, F \in L^p(\Omega_T, \mu)$ and a_{ij} are partially VMO in (t, x') with respect to $d\mu$, there exists a unique weak solution u of

$$\begin{cases} x_d^\alpha(u_t + \lambda u) - D_i[x_d^\alpha a_{ij}(D_j u + F_i)] = x_d^\alpha f(z) \quad \text{in} \quad \Omega_T \\ \lim\limits_{x_d \to 0^+} x_d^\alpha [a_{dj}(x_d) D_j u(z) + F_d] = 0. \end{cases}$$

Moreover, there is $N = N(d, p, \nu, \alpha) > 0$ such that

$$\begin{aligned} &\|Du\|_{L^p(\Omega_T,\mu)} + \lambda^{1/2}\|u\|_{L^p(\Omega_T,\mu)} \\ &\leq N\Big[\|F\|_{L^p(\Omega_T,\mu)} + \lambda^{-1/2}\|f\|_{L^p(\Omega_T,\mu)}\Big]. \end{aligned}$$

Recall

$$\|f\|_{L^p(\Omega_T,\mu)} = \left(\int_{\Omega_T} |f(t,x)|^p x_d^\alpha dxdt\right)^{1/p}.$$

Partially weighted VMO

- We say that the coefficients a_{ij} satisfy the partially VMO in (t, x') with respect to the weight $d\mu$ if for any $z_0 = (z_0', x_{d0}) \in \mathbb{R}^d \times \overline{\mathbb{R}_+}$ and for $r \in (0, r_0)$

$$\fint_{Q_r^+(z_0)} |a_{ij}(z) - [a_{ij}]_{Q_r'(z_0')}(x_d)| d\mu(z) \leq \delta$$

  for sufficiently small $\delta > 0$, where $Q_r^+(z_0)$ denotes the parabolic cylinder in $\mathbb{R}^d \times \mathbb{R}_+$, $Q_r'(z_0')$ denotes the parabolic cylinder in $\mathbb{R}^d$, and

$$[a_{ij}]_{Q_r'(z_0')}(x_d) = \fint_{Q_r'(z_0')} a_{ij}(t, x', x_d) dt dx'$$

- When $\mu \equiv 1$, the partially VMO condition was introduced by Kim-Krylov (2007) to study equations with uniformly elliptic and bounded coefficients.

L^p-theory in divergence case: Dirichlet B.C.

- It turned out the that the Dirichlet B.C. is surprisingly complicated.

- Recall the example $(x^\alpha u_x)_x = 0$ with solution $u(x) = x^{1-\alpha}$ which satisfies the B.C. $u(0) = 0$ when $\alpha < 0$. However,

$$\int_0^1 |u_x|^p x^\alpha dx \sim \int_0^1 x^{\alpha(1-p)} dx$$

- Therefore, with the weight x^α, we need the condition $\alpha(1 - p) > -1$ or $p < 1 + \frac{1}{\alpha}$.

- In contrast to the Conormal/Neumann B.C., for the Dirichlet one, the weight x^α does not seem to be desirable. *What are the right weights and right functional spaces to look for solutions to the PDE?*

L^p-theory in divergence case: Dirichlet B.C. (cont.)

Theorem (Dong-P. (2022 - TAMS))

Let $\alpha \in (-\infty, 1)$, $p \in (1, \infty)$, $d\mu_1 = x_d^{-\alpha} dxdt$ and a_{ij} are partially VMO in (t, x') with respect to $d\mu_1$. Assume $f, F \in L^p(\Omega_T, \mu_1)$ and $\lambda \geq \lambda_0$, then there exists a unique weak solution u of

$$x_d^{\alpha}(u_t + \lambda u) - D_i[x_d^{\alpha} a_{ij}(D_j u + F_i)] = x_d^{\alpha} f(z) \quad \text{in} \quad \Omega_T$$

with $u = 0$ on $\{x_d = 0\}$. Moreover, there is $N = N(d, p, \nu, \alpha)$

$$\begin{aligned} &\|x_d^{\alpha} Du\|_{L^p(\Omega_T,\mu_1)} + \lambda^{1/2}\|x_d^{\alpha} u\|_{L^p(\Omega_T,\mu_1)} \\ &\leq N\Big[\|x_d^{\alpha} F\|_{L^p(\Omega_T,\mu_1)} + \lambda^{-1/2}\|x_d^{\alpha} f\|_{L^p(\Omega_T,\mu_1)}\Big] \end{aligned}$$

Note that: When $p = 2$, we recover the L_2 estimate (energy estimate) because

$$x_d^{2\alpha}\mu_1 = x_d^{2\alpha} x_d^{-\alpha} = x_d^{\alpha}.$$

Ingredients for the proofs: main ideas

- To prove the estimates, we use the mean oscillation method introduced by Krylov (2002): Prove pointwise estimates of the sharp function of Du, then apply the Fefferman-Stein theorem and Hardy-Littlewood maximal function theorem:
  - Fefferman-Stein theorem (for sharp function)

$$\|f\|_{L^p(\mathbb{R}^d)} \leq N(p,d)\|f^{\#}\|_{L^p(\mathbb{R}^d)}, \quad \forall\, f \in L^p(\mathbb{R}^d)$$

    where

$$f^{\#}(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - \bar{f}_{B_r(x)}|dy$$

    with $\bar{f}_{B_r(x)}$ the average of f in the ball $B_r(x)$.
  - Hardy-Littlewood maximal function theorem

$$\|\mathcal{M}(f)\|_{L^p(\mathbb{R}^d)} \leq N(d,p)\|f\|_{L^p(\mathbb{R}^d)}, \quad \forall\, f \in L^p(\mathbb{R}^d)$$

    where

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|dy.$$

- To show the existence of solutions, we also used a level set argument due to Caffarelli and Peral (1998).

Ingredients for the proofs: A regularity estimate for α-harmonic functions

Proposition (Dong-P.)

For a solution u of

$$x_d^\alpha u_t - D_i[x_d^\alpha \overline{a}_{ij}(x_d) D_j u] = 0 \quad in \quad Q_2^+ \quad with \quad u(t,x',0) = 0.$$

Then, for a.e. $(z) \in Q_1^+$

$$\sup_{(z)\in Q_1^+} x_d^{\alpha-1}|u(z)| \leq N \left(\int_{Q_2^+} |x_d^\alpha u(t,x)|^2 x_d^{-\alpha} dxdt \right)^{1/2},$$

$$\sup_{(z)\in Q_1^+} x_d^{\alpha}|Du(z)| \leq N \left(\int_{Q_2^+} |x_d^\alpha u(t,x)|^2 x_d^{-\alpha} dxdt \right)^{1/2}.$$

Proof: Energy estimates, Sobolev embedding, and an iteration technique.

Weighted Sobolev embedding (parabolic case)

Lemma (Dong-P)

Let $\alpha \in (-\infty, 1)$ and $q, q^* \in (1, \infty)$ satisfy

$$\begin{cases} \dfrac{1}{q} \leq \dfrac{1}{d+2+\alpha_-} + \dfrac{1}{q^*} & \text{if} \quad d \geq 2, \\ \dfrac{1}{q} \leq \dfrac{1}{4+\alpha_-} + \dfrac{1}{q^*} & \text{if} \quad d = 1. \end{cases} \tag{0.2}$$

Then for any $u \in \mathcal{H}^1_q(Q_2^+, x_d^{\alpha q} d\mu_1)$, we have

$$\|x_d^\alpha u\|_{L^{q^*}(Q_2^+, d\mu_1)} \leq N \|u\|_{\mathcal{H}^1_q(Q_2^+, x_d^{\alpha q} d\mu_1)},$$

where $N = N(d, \alpha, q, q^*) > 0$ is a constant and $\alpha_- = \max\{-\alpha, 0\}$. The result still holds when $q^* = \infty$ and the inequalities in (0.2) are strict.

To prove the result, we used weighted Hardy inequality, Hardy-Littlewood Sobolev inequality.

Ingredients in the proofs: Weighted Sobolev embedding (elliptic case)

Remark (Sobolev imbedding)

Let $\alpha \in (-\infty, 1)$ *and* $q, q^* \in (1, \infty)$ *satisfy*

$$\frac{1}{q} \leq \frac{1}{d + \alpha_-} + \frac{1}{q^*}.$$

Then for any $u \in \mathcal{W}^1_q(B_2^+, x_d^{\alpha q} d\mu_1)$, *we have*

$$\|x_d^{\alpha} u\|_{L^{q^*}(B_2^+, d\mu_1)} \leq N \|u\|_{\mathcal{W}^1_q(B_2^+, x_d^{\alpha q} d\mu_1)},$$

where $N = N(d, \alpha, q, q^*) > 0$ *and* $\alpha_- = \max\{-\alpha, 0\}$. *The result still holds when* $q^* = \infty$ *and the inequality in* (0.2) *is strict.*

Refs: P. Hajlasz. *Sobolev spaces on an arbitrary metric space*, Potential Anal., 5:403-415, 1996.

L^p estimates in non-divergence case?

- Similar results as before can be derived for equations in non-divergence form (both for Dirichlet BC and and Conormal BC): Dong-P. (RMI -2021), Dong-P. (2021, arXiv:2103.08033).
- To demonstrate the ideas, let us just consider

$$u_t + \lambda u - a_{ij} D_{ij}(z) u(z) - \frac{\alpha}{x_d} a_{dj}(z) D_j u = f \qquad (0.3)$$

with the Dirichlet BC: $u = 0$ on $(-\infty, T) \times \partial \mathbb{R}^d_+$ and $\alpha \in (-\infty, 1)$

- Assume a_{dj}/a_{dd} are VMO in all variables and a_{ij}/a_{dd} are partially VMO in (t, x')-variables with respect to $d\mu_2$ for $j = 1, \ldots, d-1$ and $i = 1, \ldots, d$, where

$$d\mu_2 = x_d^{-\gamma_0} \quad \text{with some} \quad \gamma_0 \in (-1, 1-\alpha).$$

- Simplest example:

$$\begin{cases} u_t + u - \Delta u - \frac{\alpha}{x_d} D_d u & = & f & \text{in} & (-\infty, T) \times \mathbb{R}^d_+ \\ u & = & 0 & \text{on} & (-\infty, T) \times \partial \mathbb{R}^d_+. \end{cases}$$

which is the case of the equation (0.3) when $(a_{ij}) = I$.

L^p theory in non-divergence form (Dirichlet BC)

Theorem (Dong-P. (2021, arXiv:2103.08033))

Let $\alpha \in (-\infty, 1)$, $p \in (1, \infty)$, $d\mu_2 = x_d^{\gamma_0} dxdt$ with $\gamma_0 \in (-1, 1-\alpha)$. Then, for $\lambda \geq \lambda_0$ and $f \in L^p(\Omega_T, d\mu_2)$, there exists a unique strong solution $u = u(z)$ to the equation (0.3)*. Moreover, which satisfies*

$$
\begin{aligned}
&\|x_d^\alpha u_t\| + \lambda^{1/2}\|x_d^\alpha Du\| + \|x_d^\alpha DD_{x'}u\| + \|x_d^{\alpha-1}D_{x'}u\| \\
&+ \|D(x_d^\alpha D_d u)\| + \lambda\|x_d^\alpha u\| + \lambda^{1/2}\|x_d^{\alpha-1}u\| \\
&\leq N\|x_d^\alpha f\|
\end{aligned}
$$

where $\|\cdot\| = \|\cdot\|_{L^p(d\mu_2)}$, and $N = N(\nu, p, d, \alpha, \gamma_0) > 0$. where $N = N(d, \alpha, p, \gamma) > 0$.

A few remarks on the result

- In fact, we were able to obtain a much more general result in weighted estimates $L^p(\omega d\mu_2)$ with general weight $\omega \in A_p(\mu_2)$. As a special case, we obtained the estimate

$$
\begin{aligned}
&\|u_t\|_{L^p(x^\gamma)} + \lambda^{1/2}\|Du\|_{L^p(x^\gamma)} + \|DD_{x'}u\|_{L^p(x^\gamma)} + \|x_d^{-1}D_{x'}u\|_{L^p(x^\gamma)} \\
&+ \|x_d^{-\alpha}D(x_d^{\alpha}D_d u)\|_{L^p(x^\gamma)} + \lambda\|u\|_{L^p(x^\gamma)} + \lambda^{1/2}\|x_d^{-1}u\|_{L^p(x^\gamma)} \\
&\leq N\|f\|_{L^p(x^\gamma)}.
\end{aligned}
$$

- A similar result was obtained in Metafune-Negro-Spina (2021) by using an functional analytic approach, and the assumed that the coefficients in the equation are constant.
- When $\alpha = 0$, similar estimates were established by Krylov (1994), Kozlov-Nazarov (2009), and Dong-Kim (2015), by using different methods.
- For Conormal/Neumann BC: We proved similar results but with the underlying measure $d\mu = x_d^{\alpha}dxdt$.

Remarks on our approach

- Our starting point is the weighted L^p estimates for equations of divergence form that we discussed before together with an Agmon's idea on partition of unity, and Hardy's inequality.

- To estimate $x_d^{-1}u, x_d^{-1}D_{x'}u$, we proved Hölder estimates for $x_d^{\alpha-1}u$ for solutions of homogeneous equations.

Equations with degenerate diffusion

- Motivation: degenerate viscous Hamilton- Jacobi equation

$$u_t(z) + \lambda u(z) + H(z, Du) - x_d^\alpha \Delta u = f(z),$$

for $z = (t, x', x_d) \in (-\infty, T) \times \mathbb{R}^{d-1} \times (0, \infty)$ with $u(t, x', 0) = 0$ (Dirichlet BC).

- Wellposedness of viscosity solutions: S. N. Armstrong, H. V. Tran (2015), M. G. Crandall, H. Ishii, P.-L. Lions (1992). Such solutions are in general Lipschitz in z
  Finer regularity of solutions is not very well understood.

Difficulties

- The coefficient x_d^α becomes singular if $\alpha < 0$ or degenerate if $\alpha > 0$ on the boundary $\{x_d = 0\}$.
- The degeneracy/singularity does not appear in a balance way as previously considered equations.
- The scaling is not clear.
- In general, no Harnack inequality (Chiarenza-Serapioni, 1985)

The following equation was studied extensively

$$u_t - x_d \Delta u - \beta D_d u = f$$

for $\beta > 0$ and no boundary condition on $\{x_d = 0\}$.

- Daskalopoulos-Hamilton (1998), Feehan-Pop (2013, 2014). See also Daskalopoulos-Lee (2003) and Lieberman (2016): Schauder estimates in weighted Hölder spaces
- H. Koch (1999): weighted $W^{2,p}$-estimates.

Similarly, the Keldys type equation

$$x_d D_d^2 u + \beta D_d u + \Delta_{x'} u = f$$

was also studied by J. Hong-G. Huang (2012). The weighted Schauder and $W^{2,p}$-estimates were obtained.

Problem setting

- As an initial Step, we study the equation

$$u_t(z) + \lambda u(z) - x_d^\alpha D_i(a_{ij}(z)D_j u + F_i) = f \quad \text{in} \quad \Omega_T$$

with uniformly elliptic and bounded matrix (a_{ij}) and with the boundary condition

$$u(t, x', 0) = 0, \quad t \in (-\infty, T), \quad x' \in \mathbb{R}^{d-1}$$

where $\alpha \in (0, 2)$. In the simplest case, the PDE is written as

$$u_t - x_d^\alpha \Delta u = f.$$

- Observe that the PDE can be written as

$$x_d^{-\alpha}[u_t(z) + \lambda u(z)] - D_i(a_{ij}(z)D_j u + F_i) = x_d^{-\alpha} f, \quad \text{in} \quad \Omega_T.$$

- Even $f \sim 1$ near $\{x_d = 0\}$ and compactly supported, the force term $x_d^{-\alpha} f = x_d^{-\alpha}$ is not locally integrable when $\alpha \in (1, 2)$.

L^p-theory for equations with degenerate coeffcieints

Theorem (Dong-Tran-P. (2021, arXiv:2107.08033))

Let $\alpha \in (0,2), p \in (1,\infty)$, $\omega \in A_p$, and a_{ij} are partial VMO in (t,x') (with Lebesgue measure). Then for $\lambda \geq \lambda_0$, $|F| \in L^p(\omega)$, and $f = f_1 + f_2$ such that

$$g = x_d^{1-\alpha}|f_1| + \lambda^{-1/2}x_d^{-\alpha/2}|f_2| \in L^p(\omega).$$

There exists unique weak solution u and

$$\|Du\|_{L^p(\omega)} + \lambda^{1/2}\|x_d^{-\alpha/2}u\|_{L^p(\omega)} \leq N\Big[\|F\|_{L^p(\omega)} + \|g\|_{L^p(\omega)}\Big].$$

In particular, the theorem holds for $\omega = x_d^{\gamma}$ for $\gamma \in (-1, p-1)$.

Notes: The forcing terms have different scalings near $\{x_d = 0\}$ and near $\{x_d = \infty\}$. Weights only on u, not Du.

A few remarks in the proof

- The equation has the natural scaling

$$(t, x) \mapsto (s^{2-\alpha} t, sx), \quad s > 0.$$

  Anisotropic parabolic cylinders: for $z_0 = (t_0, x_0)$ with $x_0 = (x_0', x_{d0})$ and $\rho > 0$

$$Q_\rho(z_0) = (t_0 - \rho^{2-\alpha}, t_0) \times B_{r(\rho, x_{d0})}(x_0)$$

  where $r(\rho, x_{d0}) = \max\{\rho, x_{d0}\}^{\alpha/2} \rho^{1-\alpha/2}$.

- We constructed a special quasi-metric

$$\rho((t,x),(s,y)) = |t-s|^{1/(2-\alpha)} + \min\{|x-y|, |x-y|^{2/(2-\alpha} \min\{x_d, y_d\}^{-\alpha/(2-\alpha)}\}$$

  and the corresponding filtration of partitions.

- Then, we used the mean oscillation argument that we discussed but with respect to this filtration of partitions.

Thank you for your attention

All papers are available in my homepage

https://web.math.utk.edu/~phan/