Non-abelian Hodge theory and higher Teichmüller spaces

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Surface group representations

- S smooth closed orientable surface of genus $g \geq 2$
- $\pi_1(S)$ fundamental group of S
- Decomposition of S as a $4g$-**gone** with $2g$ **identifications** leads to the presentation

$$\pi_1(S) = \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1 \rangle$$

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- G semisimple (more generally, reductive) Lie group (real or complex)
- **A representation of** $\pi_1(S)$ **in** G is a homomorphism $\rho \colon \pi_1(S) \to G$

Surface group representations

- The set $\mathrm{Hom}(\pi_1(S), G)$ can be identified with

$$\{(A_1, B_1, \cdots, A_g, B_g) \in G^{2g} \quad : \quad [A_1, B_1] \cdots [A_g, B_g] = 1\}$$

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- Interested in the set of **equivalence classes of reps.**: The G-**character variety** of $\pi_1(S)$

$$\mathcal{R}(S, G) := \mathrm{Hom}(\pi_1(S), G)/G$$

  where

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}, \quad g \in G$$

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- $G = \mathrm{U}(1) = \{z \in \mathbb{C}^* \ : \ |z| = 1\}$ **(compact abelian)**
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- This is related to **complex algebraic geometry**
  For this, we need a **complex structure** on S: For every $x \in S$

$$J : T_xS \to T_xS \quad \text{so that} \quad J^2 = -\mathrm{Id}$$

On a surface this is equivalent to having a **conformal structure** (J is a rotation by $\pi/2$)

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Abel–Jacobi Theorem (19th century)

There is a diffeomorphism

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- Let L be the trivial C^∞ line bundle on X

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- $d_A = d + A, \;\; A \in \Omega^1(X)$ (1-form on X)
  Take
  $$\bar\partial_A := \bar\partial + \alpha \;\; \text{with } \alpha = A^{0,1} \in \Omega^{0,1}(X)$$
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- To prove the converse, start with $\bar\partial_L = \bar\partial + \alpha$ with $\alpha \in \Omega^{0,1}(X)$. To recover A:

  $$\text{Take } \; A = \alpha + \alpha^{*_h} \;\text{ where } h = e^u \text{ is a metric on } L$$

  $$d_A \text{ flat} \iff dA = 0 \iff \Delta u = 0 \;\; \textbf{Laplace equation}$$

$G = \mathbb{C}^*$ (abelian complex group)

- The character variety in this case is

$$\mathcal{R}(S, \mathbb{C}^*) = (\mathbb{C}^*)^{2g} \quad \textbf{complex torus}$$

Abel–Jacobi–Hodge Theorem (1930s)

Let J be a complex structure on S and let $X = (S, J)$ be the corresponding Riemann surface. Then there is a diffeomorphism

$$\mathcal{R}(S, \mathbb{C}^*) \cong T^* \operatorname{Jac}(X)$$

- $T^* \operatorname{Jac}(X)$ is the holomorphic cotangent bundle of $\operatorname{Jac}(X)$, whose fibre at $L \in \operatorname{Jac}(X)$ is isomorphic to $H^0(X, K)$, where K is the **canonical line bundle** of X.

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- The proof is along the same lines as the $\mathrm{U}(1)$ case, using Hodge theory on the existence of **harmonic** forms
- $\mathcal{R}(S,\mathbb{C}^*)$ has a complex structure coming from $\mathbb{C}^*$, while $T^*\operatorname{Jac}(X)$ has a complex structure coming from X, and this **are not** isomorphic: **hyperkähler structure**

$G = \mathrm{SU}(2)$ (non abelian compact group)

- Need rank 2 holomorphic vector bundles $E \to X$: $E|_U = U \times \mathbb{C}^2$, $U \subset X$ open and holomorphic transition functions $U \cap V \to \mathrm{SL}(2,\mathbb{C})$
- $\Lambda^2 E \cong \mathcal{O}_X$ trivial holomorphic line bundle $\deg E = 0$ (degree = first Chern class)

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- $\Lambda^2 E \cong \mathcal{O}_X$ trivial holomorphic line bundle $\deg E = 0$ (degree = first Chern class)
- Not every such E arises from a representation $\rho : \pi_1(S) \to \mathrm{SU}(2)$. We need **stability** (in the sense of **Mumford**) for $E \to X$:
  - E is **stable** if $\deg L < 0$ for every line subbundle $L \subset E$
  - E is **polystable** if it is stable or $E = L \oplus L^{-1}$ with $\deg L = 0$.
- Let $M(X, \mathrm{SL}(2, \mathbb{C}))$ be the **moduli space** of isomorphism classes of polystable $\mathrm{SL}(2, \mathbb{C})$-vector bundles on X. This is a **projective complex algebraic variety**.

Narasimhan–Seshadri Theorem (1965)

There is a homeomorphism

$$\mathcal{R}(S, \mathrm{SU}(2)) \cong M(X, \mathrm{SL}(2,\mathbb{C}))$$

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- Donaldson (1983) gave another proof using gauge theory bulding upon work of Atiyah–Bott
- The theorem was generalized to any connected compact Lie group by Ramanathan (1975) who proved

  $$\mathcal{R}(S, G) \cong M(X, G^{\mathbb{C}}),$$

  where $G^{\mathbb{C}}$ is the complexification of G, and $M(X, G^{\mathbb{C}})$ is the moduli space of polystable **principal $G^{\mathbb{C}}$-bundles** over X.

$G = \mathrm{SL}(2, \mathbb{C})$ (complex non abelian group)

- Need **Higgs bundles** (Hitchin 1987): Pairs (E, φ)
  E is a holomorphic $\mathrm{SL}(2, \mathbb{C})$-vector bundle over X
  $\varphi : E \to E \otimes K$, with $\mathrm{Tr}(\varphi) = 0$ (**Higgs field**)

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- ρ is **reductive** if it is **irreducible** or a direct sum of irreducible representations (in this case, of two 1-dimensional representations)

- **Correct definition** of the G-character variety:

$$\mathcal{R}(S,G) := \mathrm{Hom}(\pi_1(S),G)^{\text{reductive}}/G = \mathrm{Hom}(\pi_1(S),G)/\!\!/G$$

Non-abelian Hodge correspondence

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Theorem (Hitchin 1987, Donaldson 1987)

There is a homeomorphism

$$\mathcal{R}(S,\mathrm{SL}(2,\mathbb{C})) \cong \mathcal{M}(X,\mathrm{SL}(2,\mathbb{C})).$$

- The proof is a combination of two existence theorems for two systems of **non-linear PDEs**

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Theorem (Simpson 1988, Corlette 1988)

Let G be a semisimple complex Lie group, and let $\mathcal{M}(X,G)$ be the moduli space of G-Higgs bundles. Then there is a homeomorphism

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$G = \mathrm{SL}(2,\mathbb{R})$ (non-abelian, non-compact, non-complex, real Lie group)

- $\rho : \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ has a topological invariant (**Euler number**) $d(\rho) \in \pi_1(\mathrm{SL}(2,\mathbb{R})) \cong \pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$

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Theorem (Goldman, 1988; Hitchin, 1987)

- $\mathcal{R}_d$ is connected if $|d| < g - 1$
- $\mathcal{R}_d$ has 2^{2g} connected components if $|d| = g - 1$

$G = \mathrm{SL}(2, \mathbb{R})$

- **Goldman 1980**: Any of the 2^{2g} connected components of $\mathcal{R}_d$ for $|d| = g - 1$, consist of **Fuchsian representations** (discrete and faithful) and, can be identified with $\mathcal{T}(S)$, the **Teichmüller space**, parametrizing complex structures.

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- **Hitchin section**: Let $L = K^{1/2}$ (there are 2^{2g} choices). Take $E = L \oplus L^{-1}$. Let $q \in H^0(X, K^2)$. Taking

$\varphi = \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}$ gives the 2^{2g} components identified to $\mathcal{T}(S)$

Non-abelian Hodge correspondence for real forms

- Let G be a semisimple complex Lie group. A real subgroup $G_{\mathbb{R}} \subset G$ is a **real form** of G if $G_{\mathbb{R}} = G^{\sigma}$, the fixed point subgroup of an **antiholomorphic involution** of G.
  Example: $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ are real forms of $\mathrm{SL}(2,\mathbb{C})$

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  Example: $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ are real forms of $\mathrm{SL}(2,\mathbb{C})$
- Let $G_{\mathbb{R}} \subset G$ be a real form. One can define $G_{\mathbb{R}}$-Higgs bundles over X. These are pairs (E,φ):
  - E is an principal H-bundle over X, where H is the complexification of a maximal compact subgroup $H_{\mathbb{R}} \subset G_{\mathbb{R}}$
  - φ is a section of $E(\mathfrak{m}) \otimes K$, where $E(\mathfrak{m})$ is the vector bundle associated to the isotropy representation of H $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m})$

Non-abelian Hodge correspondence for a real form $G_{\mathbb{R}}$

Let $\mathcal{M}(X,G_{\mathbb{R}})$ be the moduli space of $G_{\mathbb{R}}$-Higgs bundles. Then there is a homeomorphism

$$\mathcal{R}(S,G_{\mathbb{R}}) \cong \mathcal{M}(X,G_{\mathbb{R}}).$$

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- **topological invariant** of $\rho : \pi_1(S) \to G_{\mathbb{R}}$ given by $c(\rho) \in \pi_1(G^{\mathbb{R}}) \cong \pi_1(H_{\mathbb{R}})$. Define the subvariety

$$\mathcal{R}_c(S, G^{\mathbb{R}}) := \{\rho \in \mathcal{R}(S, G^{\mathbb{R}}) \ : \ c(\rho) = c\}$$

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- **Topological invariant** of (E, φ) is given by $c(E, \varphi) \in \pi_1(H) \cong \pi_1(H_{\mathbb{R}})$. Define the subvariety

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- **Topological invariant** of (E, φ) is given by $c(E, \varphi) \in \pi_1(H) \cong \pi_1(H_{\mathbb{R}})$. Define the subvariety

$$\mathcal{M}_c(X, G^{\mathbb{R}}) := \{(E, \varphi) \in \mathcal{M}(X, G^{\mathbb{R}}) \ : \ c(E, \varphi) = c\}$$

- Of course, the non-abelian Hodge correspondence restricts to give a homeomorphism

$$\mathcal{R}_c(S, G^{\mathbb{R}}) \cong \mathcal{M}_c(X, G^{\mathbb{R}})$$

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- We approach the problem from the point of view of Higgs bundles

$G_{\mathbb{R}}$ split real form of G

- Every semisimple complex Lie group has a split real form
- Split real forms for the classical groups: $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}(p,p)$, $\mathrm{SO}(p,p+1)$, $\mathrm{Sp}(2n,\mathbb{R})$

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- **Hitchin map**: $h : \mathcal{M}(G) \to B(G) = \oplus H^0(X, K^{m_i})$
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- Labourie (2006) introduced the theory of **Anosov representations**, and using this proved that the Hitchin component consists entirely of discrete and faithful representations. So, it is a **higher Teichmüller space**

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- The symmetric space $G_{\mathbb{R}}/H_{\mathbb{R}}$ is a Kähler manifold
  Example: $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ **real hyperbolic plane**

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- The torsion free part of $\pi_1(G_{\mathbb{R}})$ is isomorphic to $\mathbb{Z}$ and hence the natural invariant associated to a representation and Higgs bundle in this case is an integer d : the **Toledo invariant**
- **Milnor–Wood inequality**:

$$|d| \leq \mathrm{rank}(G_{\mathbb{R}}/H_{\mathbb{R}})(g-1)$$

  Proved for the classical groups by Domic–Toledo, Turaev, Bradlow-G-Gothen and in general by Burger–Iozzi–Wienhard and Biquard–G–Rubio.

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- Consider **maximal Toledo invariant**:

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- Tube type: $\mathrm{SU}(p,p)$, $\mathrm{SO}(2,n)$, $\mathrm{SO}^*(4m)$, $\mathrm{Sp}(2n,\mathbb{R})$, real form of E_7

Theorem (Cayley correspondence)

Let $G_{\mathbb{R}}$ be of tube type and $\mathcal{M}_{K^2}(G'_{\mathbb{R}})$ be the moduli space of K^2-twisted $G'_{\mathbb{R}}$-Higgs bundles. There is an isomorphism of complex algebraic varieties

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- This correspondence exhibits **hidden topological invariants**
- Burger–Iozzi–Labourie–Wienhard proved that if $G_{\mathbb{R}}$ is of tube type $\mathcal{R}_{\max}(G_{\mathbb{R}})$ consists entirely of discrete and faithful representations: **higher Teichmüller spaces**

Higher Teichmüller spaces for other groups?

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- Indeed, in Aparicio–Bradlow–Collier–G–Gothen–Oliveira, Invent. Math. (2019) a **generalized Cayley correspondence** was given for $\mathcal{M}(\mathrm{SO}(p,q))$ exhibiting the existence of special components for $G_{\mathbb{R}} = \mathrm{SO}(p,q)$.

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- More recently (2021), a **full classification** of real simple groups $G_{\mathbb{R}}$ for which there are higher Teichmüller spaces has been given. Combination of two developments:
  1. The notion of **positivity** for certain real groups introduced by Guichard–Wienhard (2016), generalizing Lusztig's total positivity for split real forms. And the study of **positive representations** (which, in particular are discrete and faithful) by Guichard–Labourie–Wienhard (2021).

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  2. Notion of **magical $\mathfrak{sl}_2$-triples** and the **general Cayley correspondence** given by Bradlow–Collier–G–Gothen–Oliveira (2021).

General Cayley correspondence

The list of groups $G_{\mathbb{R}}$ admitting a positive structure and magical $\mathfrak{sl}_2$-triples are the following:

1. Split real forms
2. Hermitian real forms of tube type
3. $\mathrm{SO}(p,q)$
4. **Quaternionic real forms** of E_6, E_7, E_8 and F_4

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General Cayley correspondence (BCGGO, 2021)

Let $G_{\mathbb{R}}$ be as in the list. The magical $\mathfrak{sl}_2$-triple defines integers m_i and a group $G'_{\mathbb{R}}$. Then there is a subvariety $\mathcal{C}(G_{\mathbb{R}}) \subset \mathcal{M}(G_{\mathbb{R}})$ defined as the image of a map

$$\mathcal{M}_{K^{m_c}}(G'_{\mathbb{R}}) \times \bigoplus_{i=1,\, i\neq c} H^0(K^{m_i}) \hookrightarrow \mathcal{M}(G_{\mathbb{R}})$$

which is an isomorphism onto its image, open and closed in $\mathcal{M}(G^{\mathbb{R}})$. Hence $\mathcal{C}(G_{\mathbb{R}})$ is a union of connected components.

- The connected components of $\mathcal{C}(G_{\mathbb{R}})$ are called **Cayley components**

Cayley components = Higher Teichmüller components

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Theorem (Guichard–Labourie–Wienhard, 2021)

- Positive representations are **Anosov** (hence discrete and faithful)
- The set of positive representations is **open** in $\mathcal{R}(G_{\mathbb{R}})$ (this was known)
- The set of positive representations is **closed** in $\mathcal{R}^{\text{irreducible}}(G_{\mathbb{R}}) \subset \mathcal{R}(G_{\mathbb{R}})$

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- In [BCGGO,2021] we prove that $\mathcal{C}(G_{\mathbb{R}})$ contains positive representations and that $\mathcal{C}(G_{\mathbb{R}}) \subset \mathcal{R}^{\text{irreducible}}(G_{\mathbb{R}})$. By the openness ans closedness conditions of positive reps, $\mathcal{C}(G_{\mathbb{R}})$ consists entirely of positive representations. Hence
- **Cayley comp. = Higher Teichmüller comp.**