Fract. Calc. Appl. Anal. manuscript No.
(will be inserted by the editor)

Time-fractional integro-differential equations in power growth function spaces

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Received: ...... / Revised: ...... / Accepted: ......

Abstract In this paper we study the global solvability of Riemann-Liouville and Caputo time-fractional integro-differential equations in a space of functions with square average power growth. We prove the uniqueness of corresponding inverse problems for one point observation.

Keywords Functions with square average power growth (primary) · Laplace transform · Caputo fractional derivative · Riemann-Liouville fractional derivative · Time-fractional integro-differential equation · Inverse fractional equation

Mathematics Subject Classification (2010) 44A10, 26A33 (primary) · 45J05

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with a sufficiently smooth boundary $\partial\Omega$. We consider a time-fractional differential equation

$$\partial_t^\alpha u(x,t) = -kAu(x,t) + F(x,t), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.1}$$

where ∂_t^α is either the Caputo or the Riemann-Liouville fractional derivative of order $\alpha > 0$, with respect to the time variable t [13], [14], and $-A$ is a uniform elliptic differential operator.

Equations such as (1.1) describe anomalous diffusion processes and wave propagation in viscoelastic materials [3], [5], [6], [15], [16]. They have attracted increasing interest in the physical, chemical and engineering literature [2], [11], [18], [22]. It has been shown by Adams and Gelhar [2] that field data show anomalous diffusion in a highly heterogeneous aquifer, which cannot be modelled by a classical advection-diffusion equation which corresponds to $\alpha = 1$. In [18] Nigmatullin introduced a fractional diffusion equation to describe diffusions in media with fractal geometry. In [11] Ginoa, Cerbelli and Roman have derived a fractional equation for relaxation in complex viscoelastic materials by employing a formal analogy with a diffusion process in the corresponding disordered structure. Roman and Alemany [22] studied continuous-time random walks on fractals and showed that the average probability density of random walks on fractals obeys a diffusion equation with a fractional time derivative asymptotically.

The existence, uniqueness, and inverse problems for space-time fractional differential equations have been considered by many authors, see for example [1], [10], [14], [24], [26], [27]. The book [14] contains a very intensive treatment of equation (1.1) of order $0 < \alpha < 1$ in a classical space on the domain $\Omega \times [0,T]$. At the same time, in the paper [25], V.K. Tuan considered the time-fractional partial integro-differential equation

$$\partial_t^\alpha u(x,t) = k\Delta u(x,t) - \int_0^t g(t-\tau)u(x,\tau)d\tau, \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.2}$$

satisfying the Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+,$$

and a suitable initial condition that depends on whether ∂_t^α is the Riemann-Liouville derivative or the Caputo derivative of order $0 < \alpha < 1$, and showed the existence of the solution for large time, which means its global existence in $\Omega \times \mathbb{R}_+$ in the so called Wiener space of functions with bounded square averages. Equation (1.1) is a well known model for heat distribution of a visco-elastic material with memory and has important applications in material science [3], [5], [6], [23].

In spite of the importance, to the authors' best knowledge, there are not any works published concerning the unique existence of the global solution to equation (1.2) for $1 < \alpha < 2$. This motivates us to consider the time-fractional integro-differential equations of order $1 < \alpha < 2$ in this work.

The first purpose of this article is to prove the global existence of solutions in direct problems in the space of functions with square average power growth for the case of $1 < \alpha < 2$, global existence results presented here are new and do not rely on semi-group techniques [9]. Secondly we solve the inverse problem of reconstructing the fractional order α, the parameter k, and the memory function g from a single observation of the solution at one point inside or on the boundary of the domain Ω.

2 Preliminaries

In this section, we provide some auxiliary notations and facts that will be used in the sequel.

We first recall a class of functions which plays very important role in study fractional integro-differential equations and time-fractional integro-differential equations [19], [25], [28], [30] on the whole $\mathbb{R}_+$.

Definition 1 [28] By $BSA_p(\mathbb{R}_+)$, a linear space of functions with square average power growth of order $p \geq 0$, we denote the set of all locally integrable functions f on $\mathbb{R}_+$ such that

$$\sup_{T>0} \frac{1}{(T+1)^p} \int_0^T |f(t)|^2 \, dt < \infty,$$

and

$$BSA_\infty(\mathbb{R}_+) = \bigcup_{p>0} BSA_p(\mathbb{R}_+).$$

We say $f \in BSA_p^m(\mathbb{R}_+)$ if $f, f', ..., f^{(m)} \in BSA_p(\mathbb{R}_+)$.

It was shown [28] that $f \in BSA_p(\mathbb{R}_+)$ if, and only if, $F(s)$, its Laplace transform, defined by

$$F(s) = (\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t) \, dt,$$

is holomorphic in the right-half plane $\operatorname{Re} s > 0$, and

$$\sup_{x>0} \left(\frac{x}{x+1}\right)^p \int_{-\infty}^\infty |F(x+iy)|^2 \, dy < \infty.$$

Denote by $\mathcal{L}^{-1}$ the inverse Laplace transform [29]

$$f(t) = (\mathcal{L}^{-1}F)(t) := \frac{1}{2\pi i} \int_{\operatorname{Re} s = d} F(s) \, e^{st} \, ds.$$

The following theorem is from [19, Theorem 3] for the case $1 < \alpha < 2$ and [28, Theorem 3.1 and Corollary 3.2] for the case $0 < \alpha \leq 1$

Theorem 1 *Let $k > 0, 0 < \alpha < 2$, and $g \in L^1(\mathbb{R}_+)$, such that $\|g\|_1 < \frac{|\tan(\frac{\alpha\pi}{2})|}{1+|\tan(\frac{\alpha\pi}{2})|}k$, if $\alpha > 1$, and $\|g\|_1 < k$, if $\alpha \le 1$, then $\frac{1}{s^\alpha+k+G(s)}$ is holomorphic in the domain $\operatorname{Re} s > 0$, and there exists $M > 0$ such that*

$$\left|\frac{1}{s^\alpha + k + G(s)}\right| < \frac{M}{|s|^\alpha},$$

and the inverse Laplace transform

$$f := \mathcal{L}^{-1}\left(\frac{s^\beta}{s^\alpha + k + G(s)}\right)$$

is from $BSA_{2(\alpha-\beta)-1}(\mathbb{R}_+)$, where $\beta < \alpha - \frac{1}{2}$.

The importance of the space $BSA_p(\mathbb{R}_+)$ can be seen when we consider the following Caputo and Riemann-Liouville fractional integro-differential initial value problems

$$\begin{aligned} {}^{\mathcal{C}}\partial_t^\alpha f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \\ f^{(i)}(0+) = f_i, \quad i = 0, 1, ..., m-1, \end{aligned} \tag{2.1}$$

$$\begin{aligned} D_{0+}^\alpha f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \\ I_{0+}^{m-i-\alpha} f(0+) = f_i, \quad i = 0, 1, ..., m-1, \end{aligned} \tag{2.2}$$

$$\alpha \in (m-1, m], \quad m \in \mathbb{N}, \quad m \ge 1,$$

where $k \in \mathbb{R}_+,\ g, h \in L^1(\mathbb{R}_+)$ are given, and f is an unknown function. Here and throughout this paper D_{0+}^α is the Riemann-Liouville fractional derivative [13], [27] and ${}^{\mathcal{C}}\partial_t^\alpha$ is the Caputo fractional derivative [13], [27].

From [19, Theorems 4 and 5], we can see that if $\frac{3}{2} < \alpha < 2$, $g, h \in L^1(\mathbb{R}_+)$, and $\|g\|_1 < \gamma(\alpha)\,k$ with $\gamma(\alpha) = \frac{|\tan(\frac{\alpha\pi}{2})|}{1+|\tan(\frac{\alpha\pi}{2})|}$, then the Caputo fractional integro-differential equation (2.1) has a unique solution f from $BSA_{2\alpha+3}(\mathbb{R}_+)$, and the Riemann-Liouville fractional integro-differential equation (2.2) has a unique solution f from $BSA_{2\alpha-1}(\mathbb{R}_+)$.

Next, we recall the two-parametric Mittag-Leffler function [12] that plays a crucial role in the study of time-fractional integro-differential equations

$$E_{\alpha,\beta}(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta > 0.$$

The following lemmas collect important properties of the Mittag-Leffler functions, that will be used in the sequel.

Lemma 1 *[20, Theorem 1.6] Let $x \geq 0$, $0 < \alpha < 2$ and $\beta > 0$. Then there exists a positive constant M such that*

$$|E_{\alpha,\beta}(-x)| \leq \frac{M}{1+x}.$$

In particular, $|E_{\alpha,\beta}(-x)| \leq M_1$.

One can prove that $M_1 = \frac{1}{\Gamma(\beta)}$ if $\beta > 0$.
Here and throughout the paper we use M, M_1, and M_2 to denote universal positive constants that can be distinct in different contexts.

Lemma 2 *Let $x \geq 0$ and $1 < \alpha < 2$. Then*

$$x|E_{\alpha,2}(-x^\alpha)| \leq -2\cos\frac{\pi}{\alpha}.$$

Proof Recall the integral representation [12, page 111, Exercise 4.11.16]

$$x^{\beta-1}E_{\alpha,\beta}(-x^\alpha) = \frac{1}{\pi}\int_0^\infty e^{-rx}\frac{r^{2\alpha-\beta}\sin(\beta\pi) - r^{\alpha-\beta}\sin[(\alpha-\beta)\pi]}{1+2r^\alpha\cos(\alpha\pi)+r^{2\alpha}}dr.$$

Taking $\beta = 2$, we get

$$xE_{\alpha,2}(-x^\alpha) = \frac{1}{\pi}\int_0^\infty e^{-rx}\,\frac{-r^{\alpha-2}\sin(\alpha\pi)}{1+2r^\alpha\cos(\alpha\pi)+r^{2\alpha}}dr, \quad 1<\alpha<2,$$

then

$$x|E_{\alpha,2}(-x^\alpha)| \leq \frac{-\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{r^{\alpha-2}}{1+2r^\alpha\cos(\alpha\pi)+r^{2\alpha}}dr.$$

Using formula [21, page 311, formula 2.2.9.24]

$$\int_0^\infty \frac{x^{\rho-1}}{x^2+2xy\cos\gamma+y^2}\,dx = \frac{\pi\sin[(1-\rho)\gamma]}{y^{2-\rho}\sin\gamma\sin\rho\pi}, \quad y>0,\ 0<\gamma<\pi,\ 0<\rho<2,$$

we obtain

$$\int_0^\infty \frac{r^{\alpha-2}}{1+2r^\alpha\cos(\alpha\pi)+r^{2\alpha}}dr = \int_0^\infty \frac{t^{-\frac{1}{\alpha}}}{t^2+2t\,\cos(\pi\alpha-2\pi)+1}dt$$
$$= \frac{\pi}{\sin\frac{2\pi}{\alpha}}\sin(\alpha\pi)\sin\frac{\pi}{\alpha}.$$

Thus

$$x|E_{\alpha,2}(-x^\alpha)| \leq \frac{-\sin(\alpha\pi)}{\pi}\,\frac{\pi\sin\frac{2\pi}{\alpha}}{\sin(\alpha\pi)\sin\frac{\pi}{\alpha}} = -2\cos\frac{\pi}{\alpha}, \quad 1<\alpha<2.$$

□

3 Time-fractional Integro-differential Equations

In this section we shall study the following time-fractional integro-differential equation of the form

$$\begin{aligned} &\partial_t^\alpha u(x,t) = k\Delta u(x,t) - \int_0^t g(t-\eta)u(x,\eta)d\eta, \\ &(x,t) := (x_1,\dots,x_d,t) \in \Omega \times \mathbb{R}_+, \quad 1 < \alpha < 2, \end{aligned} \tag{3.1}$$

with the Dirichlet boundary condition

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{3.2}$$

where $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain with smooth boundary $\partial\Omega \in C^{[\frac{d}{2}]+1}$, and $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ is the Laplacian in $\mathbb{R}^d$.

Concerning initial conditions, they depend on the definition of fractional derivative ∂_t^α. If ∂_t^α is the Riemann-Liouville fractional derivative D_{0+}^α of order $\alpha \in (1,2)$, the initial conditions are

$$I_{0+}^{2-\alpha}u(x,0) = f(x), \quad D_{0+}^{\alpha-1}u(x,0) = h(x), \qquad x \in \Omega, \tag{3.3}$$

and the initial-boundary value problem (3.1)-(3.2)-(3.3) is called Problem (I).

If ∂_t^α is the Caputo fractional derivative ${}^C\partial_t^\alpha$ of order $\alpha \in (1,2)$, the initial conditions become

$$u(x,0) = f(x), \quad \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = h(x), \qquad x \in \Omega, \tag{3.4}$$

and the initial-boundary value problem (3.1)-(3.2)-(3.4) is called Problem (II).

We will show the observability of the solution for large time, which means its global existence. The fractional nature of the problem will require us to use the space $BSA_p(\mathbb{R}_+)$ of functions with square averages power growth.

To study Problems (I):(3.1)-(3.2)-(3.3) and (II):(3.1)-(3.2)-(3.4) we shall use the eigenfunctions expansion method. For the Laplacian under Dirichlet boundary condition, denote its eigenvalues, indexed in the ascending order and counting their multiplicity, by λ_j and associated eigenfunctions by φ_j, i.e.

$$\begin{cases} \Delta\varphi_j(x) = -\lambda_j\varphi_j(x), & \text{in } \Omega, \\ \varphi_j(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

It is known [17] that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots$, with $\lim_{j\to\infty} \lambda_j = \infty$, and the set $\{\varphi_j\}_{j\geq 1}$, normalized by $||\varphi_j||_{L^2(\Omega)} = 1$, is an orthonormal basis for $L^2(\Omega)$. Moreover, $\varphi_j \in C^\infty(\Omega)$, and the smoothness condition on the boundary guarantees that $\varphi_j \in C(\overline{\Omega})$ (see [17, Theorem 7, Section 2, Chapter IV]).

We derive first a result about the global existence of classical solutions of Problem (I):(3.1)-(3.2)-(3.3). Because initial conditions depend on the form of the fractional derivative on the left hand side of the time-fractional integro-differential equation (3.1), the corresponding representations of solutions differ. We demonstrate the derivation in the case of the Rieman-Liouville time-fractional derivative. The case of the Caputo time-fractional derivative can be treated in a similar way, and therefore will be mentioned without much details.

First we look for a particular solution of (3.1) in the form

$$u(x,t) = [c_j(t) + d_j(t)]\varphi_j(x), \tag{3.6}$$

satisfying $I_{0+}^{2-\alpha}u(x,0) = D_{0+}^{\alpha-1}u(x,0) = \varphi_j(x)$. Plugging (3.6) in the equation (3.1), thanks to (3.5), we obtain that c_j and d_j satisfy the fractional integro-differential equations, respectively,

$$\begin{aligned} &D_{0+}^{\alpha}c_j(t) = -k\lambda_j c_j(t) - \int_0^t g(t-s)c_j(s)\,ds, \\ &I_{0+}^{2-\alpha}c_j(0) = 1, \quad D_{0+}^{\alpha-1}c_j(0) = 0, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &D_{0+}^{\alpha}d_j(t) = -k\lambda_j d_j(t) - \int_0^t g(t-s)d_j(s)\,ds, \\ &I_{0+}^{2-\alpha}d_j(0) = 0, \quad D_{0+}^{\alpha-1}d_j(0) = 1. \end{aligned} \tag{3.8}$$

To solve (3.7) and (3.8), one shall use the Laplace transform. Formally we would have

$$s^{\alpha}C_j(s) - s = s^{\alpha}C_j(s) - s\,I_{0+}^{2-\alpha}c_j(0) - D_{0+}^{\alpha-1}c_j(0) = -k\lambda_j C_j(s) - G(s)C_j(s),$$

and

$$s^{\alpha}D_j(s) - 1 = s^{\alpha}D_j(s) - s\,I_{0+}^{2-\alpha}d_j(0) - D_{0+}^{\alpha-1}d_j(0) = -k\lambda_j D_j(s) - G(s)D_j(s),$$

therefore,

$$C_j(s) = \frac{s}{s^{\alpha} + \lambda_j k + G(s)}, \quad c_j(t) = \left(\mathcal{L}^{-1}C_j\right)(s), \tag{3.9}$$

$$D_j(s) = \frac{1}{s^{\alpha} + \lambda_j k + G(s)}, \quad d_j(t) = \left(\mathcal{L}^{-1}D_j\right)(s). \tag{3.10}$$

By $f \in BSA_p^{\alpha}(\mathbb{R}_+)$ for $1 < \alpha < 2$, we mean both f, $D_{0+}^{\alpha}f \in BSA_p(\mathbb{R}_+)$. We have

Theorem 2 *Let* $\frac{3}{2} < \alpha < 2$, $\|g\|_1 < \gamma(\alpha)k\lambda_1$ *with*

$$\gamma(\alpha) = \frac{|\tan(\frac{\alpha\pi}{2})|}{1 + |\tan(\frac{\alpha\pi}{2})|}. \tag{3.11}$$

Then $c_j(t)$ *and* $d_j(t)$*, defined by* (3.9) *and* (3.10)*, belong to* $BSA_{2\alpha-3}^{\alpha}(\mathbb{R}_+)$ *and* $BSA_{2\alpha-1}^{\alpha}(\mathbb{R}_+)$*, respectively.*

Proof Since $\|g\|_1 < \gamma(\alpha)\lambda_1 k < \gamma(\alpha)\lambda_j k$, for $j = 2, 3, ...$, Theorem 1 and formulas (3.9), (3.10) show that $c_j(t) \in BSA_{2\alpha-3}(\mathbb{R}_+)$ and $d_j(t) \in BSA_{2\alpha-1}(\mathbb{R}_+)$ for $j = 1, 2, \cdots$.

We have

$$(\mathcal{L}D^{\alpha}_{0+}c_j)(s) = s^{\alpha}\, C_j(s) - s\, I^{2-\alpha}_{0+}c_j(0) - D^{\alpha-1}_{0+}c_j(0) = -\frac{s(\lambda_j k + G(s))}{s^{\alpha} + \lambda_j k + G(s)},$$

and

$$(\mathcal{L}D^{\alpha}_{0+}d_j)(s) = s^{\alpha}\, D_j(s) - s\, I^{2-\alpha}_{0+}d_j(0) - D^{\alpha-1}_{0+}d_j(0) = -\frac{\lambda_j k + G(s)}{s^{\alpha} + \lambda_j k + G(s)}.$$

Since $|G(s)| \leq \|g\|_1 < \gamma(\alpha)\lambda_1 k \leq \gamma(\alpha)\lambda_j k$, by Theorem 1, we get $D^{\alpha}_{0+}c_j(t) \in BSA_{2\alpha-3}(\mathbb{R}_+)$ and $D^{\alpha}_{0+}d_j(t) \in BSA_{2\alpha-1}(\mathbb{R}_+)$.
Thus $c_j(t) \in BSA^{\alpha}_{2\alpha-3}(\mathbb{R}_+)$ and $d_j(t) \in BSA^{\alpha}_{2\alpha-1}(\mathbb{R}_+)$. □

If we take $f(x) = h(x) = \varphi_j(x)$, then $u(x,t)$, defined by (3.6), (3.9) and (3.10), satisfies (3.1). Hence, if we take $f(x) = \sum_{j=1}^m a_j\varphi_j(x)$, $h(x) = \sum_{j=1}^m b_j\varphi_j(x)$ then $u(x,t)$ defined by

$$u(x,t) = \sum_{j=1}^{m}[a_j c_j(t) + b_j d_j(t)]\varphi_j(x), \tag{3.12}$$

formulas (3.9) and (3.10), satisfies (3.1).

As $c_j \in BSA^{\alpha}_{2\alpha-3}(\mathbb{R}_+) \subset BSA^{\alpha}_{2\alpha-1}(\mathbb{R}_+)$ and $\varphi_j \in C^{\infty}(\Omega) \cap C(\overline{\Omega})$ we arrive at

Theorem 3 *Let* $\frac{3}{2} < \alpha < 2$, $\|g\|_1 < \gamma(\alpha)k\lambda_1$ *with* $\gamma(\alpha)$ *being given by* (3.11), *and* $f(x) = \sum_{j=1}^m a_j\varphi_j(x)$, $h(x) = \sum_{j=1}^m b_j\varphi_j(x)$. *Then* (3.12) *is the classical solution to Problem (I):*(3.1)-(3.2)-(3.3), *exists for all* $t > 0$, *and belongs to* $C^{\infty}(\Omega) \cap C(\overline{\Omega}) \times BSA^{\alpha}_{2\alpha-1}(\mathbb{R}_+)$.

Remark 1 This function theory approach gives a new method to prove global existence of solutions of evolution equations that is not based on semi-group theory which usually provides local existence of the solution. Also we need only $g \in L^1(\mathbb{R}_+)$ and no sign condition, $g > 0$, is needed.

For Problem (II), we again look for a particular solution of (3.1) in the form (3.12), but under the initial conditions $u(x,0) = \frac{\partial u(x,t)}{\partial t}|_{t=0} = \varphi_j(x)$.
To find the exact expressions of $c_j(t)$ and $d_j(t)$, use (3.1), (3.5), to deduce

$${}^{C}\partial^{\alpha}_t c_j(t) = -k\lambda_j c_j(t) - \int_0^t g(t-s)c_j(s)\,ds, \quad c_j(0) = 1, \quad c'_j(0) = 0, \tag{3.13}$$

and

$${}^{C}\partial^{\alpha}_t d_j(t) = -k\lambda_j d_j(t) - \int_0^t g(t-s)d_j(s)\,ds, \quad d_j(0) = 0, \quad d'_j(0) = 1. \tag{3.14}$$

The solutions of (3.13) and (3.14) can be written in the form

$$c_j(t) = \left(\mathcal{L}^{-1}C_j\right)(s), \quad C_j(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda_j k + G(s)} \tag{3.15}$$

and

$$d_j(t) = \left(\mathcal{L}^{-1}D_j\right)(s), \quad D_j(s) = \frac{s^{\alpha-2}}{s^\alpha + \lambda_j k + G(s)}, \tag{3.16}$$

respectively.

If $\|g\|_1 < \gamma(\alpha)k\lambda_1$, $\frac{3}{2} < \alpha < 2$, then Theorem 1 shows that $c_j(t) \in BSA_1(\mathbb{R}_+) \subset BSA_{2\alpha-1}(\mathbb{R}_+)$ and $d_j(t) \in BSA_3(\mathbb{R}_+)$. Moreover, we have

$$\begin{aligned}(\mathcal{L}c_j')(s) &= s\,C_j(s) - c_j(0) = -\frac{\lambda_j k + G(s)}{s^\alpha + \lambda_j k + G(s)},\\ (\mathcal{L}d_j')(s) &= s\,D_j(s) - d_j(0) = \frac{s^{\alpha-1}}{s^\alpha + \lambda_j k + G(s)},\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}(\mathcal{L}c''{}_j)(s) &= s^2\,C_j(s) - s\,c_j(0) - c_j'(0) = -\frac{s(\lambda_j k + G(s))}{s^\alpha + \lambda_j k + G(s)},\\ (\mathcal{L}d''{}_j)(s) &= s^2\,D_j(s) - s\,d_j(0) - d_j'(0) = -\frac{\lambda_j k + G(s)}{s^\alpha + \lambda_j k + G(s)}.\end{aligned} \tag{3.18}$$

From Theorem 1, we derive $c_j'(t)$ and $c_j''(t)$ belong to $BSA_{2\alpha-1}(\mathbb{R}_+)$ and $BSA_{2\alpha-3}(\mathbb{R}_+) \subset BSA_{2\alpha-1}(\mathbb{R}_+)$, respectively.

Thus $c_j(t) \in BSA^2_{2\alpha-1}(\mathbb{R}_+) \subset BSA^2_3(\mathbb{R}_+)$.

Similarly, $d_j'(t)$ and $d_j''(t)$ belong to $BSA_1(\mathbb{R}_+) \subset BSA_3(\mathbb{R}_+)$ and $BSA_{2\alpha-1}(\mathbb{R}_+) \subset BSA_3(\mathbb{R}_+)$, respectively. Thus $d_j(t) \in BSA^2_3(\mathbb{R}_+)$.

Now if we take $f(x) = h(x) = \varphi_j(x)$, then $u(x,t)$ defined by (3.12), (3.15), and (3.16), satisfies (3.1). Thus, we have proved

Theorem 4 *Assume that the assumptions in Theorem 3 hold, then* (3.12)*, together with* (3.15)*and* (3.16)*, is the classical solution to Problem (II):* (3.1)-(3.2)-(3.4)*, exists for all* $t > 0$*, and belongs to* $C^\infty(\Omega) \cap C(\overline{\Omega}) \times BSA^2_3(\mathbb{R}_+)$.

Now we go to the general case. Let $H^m_0(\Omega)$ be the Sobolev space of functions with compact supports in Ω with generalized derivatives up to order $m \geq 0$. We obtain the following theorem for Problem (I)

Theorem 5 *Let* $g \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+)$*,* $f \in H^m_0(\Omega)$*,* $h \in H^{m-1}_0(\Omega)$*,* $\|g\|_1 < \gamma(\alpha)k\lambda_1$ *with* $\gamma(\alpha)$ *being defined by* (3.11)*,* $\frac{3}{2} < \alpha < 2$*, and* c_j *and* d_j *be defined by* (3.9) *and* (3.10)*, respectively.*

a) If $\partial\Omega \in C^m$ *then the following series converges in* $H^{m-1}(\Omega)$ *norm for each* $t > 0$

$$u(x,t) := \sum_{j=1}^{\infty}[f_j c_j(t) + h_j d_j(t)]\varphi_j(x), \tag{3.19}$$

where f_j, h_j *are the* j^{th} *Fourier coefficients of* $f, h \in L^2(\Omega)$ *in the basis* $\{\varphi_j\}_{j\geq 1}$, *namely* $\begin{Bmatrix} f_j \\ h_j \end{Bmatrix} = \int_\Omega \begin{Bmatrix} f(x) \\ h(x) \end{Bmatrix} \varphi_j(x)\, dx$. *If, moreover,* $m \geq \left[\frac{d}{2}\right]+2$, *then* $u(\cdot, t) \in C^{m-\left[\frac{d}{2}\right]-2}(\overline{\Omega})$.

b) If $m > \frac{d}{2}+1$, *then the series* (3.19) *converges absolutely on* $Q := \Omega \times \mathbb{R}_+$.

c) If $m > \frac{3d+1}{2}$, *then the series* (3.19) *converges uniformly on any compact subset of* Q. *Moreover, if* $\partial\Omega \in C^{\left[\frac{d}{2}\right]+1}$, *then* $u \in C(\overline{\Omega}) \times BSA_{2\alpha-1}(\mathbb{R}_+)$.

The absolute convergence of (3.19) should be understood in the following unconventional way. With the presence of multiple eigenvalues, let us regroup all eigenvalues into a strictly increasing sequence $\mu_1 < \mu_2 < \cdots$ such that the sets $\{\lambda_1, \lambda_2, \cdots, \lambda_j, \cdots\}$ and $\{\mu_1, \mu_2, \cdots, \mu_l, \cdots\}$ coincide. Then the absolute convergence of (3.19) means the convergence of the series

$$\sum_{l=1}^{\infty} \left| \sum_{\lambda_j = \mu_l} [f_j c_j(t) + h_j d_j(t)]\varphi_j(x) \right|.$$

Proof Consider the equation

$$D^\alpha_{0+} y(t) = -\lambda y(t) + f(t), \quad D^{\alpha-1}_{0+} y(0) = y_0,\ I^{2-\alpha}_{0+} y(0) = y_1,\ \alpha \in (1,2), \tag{3.20}$$

its solution has the form [13]

$$\begin{aligned} y(t) =& y_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + y_1 t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha) \\ &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-\lambda(t-\tau)^\alpha] f(\tau)\, d\tau. \end{aligned} \tag{3.21}$$

Applying (3.20), (3.21) to (3.7) and (3.8) with $f(t) = -(g * c_j)(t)$, $f(t) = -(g * d_j)(t)$, respectively, and λ being replaced by $k\lambda_j$, we obtain

$$\begin{aligned} c_j(t) &= t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_j t^\alpha) \\ &\quad - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-k\lambda_j (t-\tau)^\alpha] \int_0^\tau g(\tau-\eta) c_j(\eta)\, d\eta\, d\tau \\ &= t^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_j t^\alpha) - \int_0^t \beta(t,\eta) c_j(\eta)\, d\eta, \end{aligned}$$

and

$$\begin{aligned} d_j(t) &= t^{\alpha-2} E_{\alpha,\alpha-1}(-k\lambda_j t^\alpha) \\ &\quad - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-k\lambda_j (t-\tau)^\alpha] \int_0^\tau g(\tau-\eta) d_j(\eta)\, d\eta\, d\tau \\ &= t^{\alpha-2} E_{\alpha,\alpha-1}(-k\lambda_j t^\alpha) - \int_0^t \beta(t,\eta) d_j(\eta)\, d\eta, \end{aligned}$$

where

$$\beta(t,\eta) = \int_\eta^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[-k\lambda_j (t-\tau)^\alpha] g(\tau-\eta)\, d\tau. \tag{3.22}$$

From Lemma 1 we have

$$t^{\alpha-1}|E_{\alpha,\alpha}(-k\lambda_j t^{\alpha})| \le M_1 t^{\alpha-1}, \quad t^{\alpha-2}|E_{\alpha,\alpha-1}(-k\lambda_j t^{\alpha})| \le M\frac{t^{\alpha-2}}{1+k\lambda_j t^{\alpha}},$$

and

$$|\beta(t,\eta)| \le M_1\|g\|_\infty \int_\eta^t (t-\tau)^{\alpha-1}d\tau = \frac{(t-\eta)^\alpha}{\alpha}M_1\|g\|_\infty. \tag{3.23}$$

Consequently,

$$|c_j(t)| \le M_1 t^{\alpha-1} + \frac{M_1\|g\|_\infty}{\alpha}\int_0^t (t-\eta)^\alpha |c_j(\eta)|d\eta, \quad t>0, \tag{3.24}$$

and

$$|d_j(t)| \le M\frac{t^{\alpha-2}}{1+k\lambda_j t^\alpha} + \frac{M_1\|g\|_\infty}{\alpha}\int_0^t (t-\eta)^\alpha |d_j(\eta)|d\eta, \quad t>0. \tag{3.25}$$

Applying the Gronwall inequality for fractional integral [31, Corollary 2] and using monotonicity of $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, to (3.24), we obtain

$$|c_j(t)| \le M_1 t^{\alpha-1}E_\alpha\left(M_1\|g\|_\infty t^{\alpha+1}\Gamma(\alpha)\right) := N(t), \quad t>0. \tag{3.26}$$

Now applying the generalized Gronwall inequality [14, Lemma A.2] to (3.25) yields

$$|d_j(t)| \le M\frac{t^{\alpha-2}}{1+k\lambda_j t^\alpha} + M_1 e^{M_2 t}\int_0^t (t-\eta)^\alpha \frac{\eta^{\alpha-2}}{1+k\lambda_j\eta^\alpha}d\eta.$$

Note that $\max\limits_{x>0}\frac{x^p}{1+x} = p^p(1-p)^{1-p}$, where $0<p<1$, we have

$$\begin{aligned}
\lambda_j^p|d_j(t)| &\le Mk^{-p}t^{\alpha-2-\alpha p}\frac{k^p\lambda_j^p t^{\alpha p}}{1+k\lambda_j t^\alpha} \\
&+ M_1k^{-p}e^{M_2 t}\int_0^t (t-\eta)^\alpha \eta^{\alpha-2-\alpha p}\frac{k^p\lambda_j^p\eta^{\alpha p}}{1+k\lambda_j\eta^\alpha}d\eta \\
&\le Mk^{-p}t^{\alpha-2-\alpha p}p^p(1-p)^{1-p} \\
&+ \frac{M_1p^p(1-p)^{1-p}}{k^p}e^{M_2t}\int_0^t (t-\eta)^\alpha\eta^{\alpha-2-\alpha p}d\eta \\
&= \frac{Mt^{\alpha-2-\alpha p}p^p}{k^p(1-p)^{p-1}} + \frac{M_1p^p\Gamma(\alpha+1)\Gamma(\alpha-\alpha p-1)}{k^p(1-p)^{p-1}\Gamma(2\alpha-\alpha p)}t^{2\alpha-\alpha p-1}e^{M_2t} \\
&= N^*(t), \quad 0<p<1-\frac{1}{\alpha}.
\end{aligned}$$

Therefore

$$|d_j(t)| \le \lambda_j^{-p}N^*(t) \le \lambda_1^{-p}N^*(t), \quad 0<p<1-\frac{1}{\alpha}. \tag{3.27}$$

Thus, $\{c_j(t)\}_{j\geq 1}$ and $\{d_j(t)\}_{j\geq 1}$ are uniformly bounded on any compact subset of $\mathbb{R}_+$.

a) Since $f \in H_0^m(\Omega)$, $h \in H_0^{m-1}(\Omega)$ and $\partial\Omega \in C^m$, then by [17, Theorem, Chapter IV],

$$\sum_{j=1}^{\infty} f_j^2 \lambda_j^m \leq M_1 \|f\|^2_{H^m_\Omega} < \infty, \quad \sum_{j=1}^{\infty} h_j^2 \lambda_j^{m-1} \leq M_2 \|h\|^2_{H^{m-1}_\Omega} < \infty,$$

and the series $\sum_{j=1}^{\infty} f_j \varphi_j(x)$ and $\sum_{j=1}^{\infty} h_j \varphi_j(x)$ converge to $f(x)$ and $h(x)$ in $H^m(\Omega)$ and $H^{m-1}(\Omega)$ norms, respectively. Together with the uniform boundedness of c_j and d_j on $[\epsilon, T]$, $0 < \epsilon < T < \infty$, it yields

$$\begin{aligned}
&\sum_{j=1}^{\infty} (f_j c_j(t) + h_j d_j(t))^2 \lambda_j^{m-1} \\
&\leq 2\left(\sum_{j=1}^{\infty} f_j^2 c_j^2(t) \lambda_j^{m-1} + \sum_{j=1}^{\infty} h_j^2 d_j^2(t) \lambda_j^{m-1}\right) \\
&\leq 2\left(N^2(t) \sum_{j=1}^{\infty} f_j^2 \lambda_j^{m-1} + \lambda_1^{-2p} (N^*(t))^2 \sum_{j=1}^{\infty} h_j^2 \lambda_j^{m-1}\right) \\
&\leq 2\left(N^2(t) \lambda_1^{-1} \sum_{j=1}^{\infty} f_j^2 \lambda_j^{m} + \lambda_1^{-2p} (N^*(t))^2 \sum_{j=1}^{\infty} h_j^2 \lambda_j^{m-1}\right) < \infty, \\
&0 < \epsilon \leq t \leq T < \infty.
\end{aligned}$$

In other words, the series (3.19) converges in $H^{m-1}(\Omega)$ norm, and $u(\cdot, t) \in H^{m-1}(\Omega)$ for any $t > 0$. On the other hand, when $m \geq \left[\frac{d}{2}\right] + 2$, we have [17] $H^{m-1}(\Omega) \subset C^{m-\left[\frac{d}{2}\right]-2}(\overline{\Omega})$, therefore, $u(\cdot, t) \in C^{m-\left[\frac{d}{2}\right]-2}(\overline{\Omega})$.

b) If $m > \frac{d}{2} + 1$, $f \in H_0^m(\Omega)$ and $h \in H_0^{m-1}(\Omega)$, then the series $\sum_{j=1}^{\infty} f_j \varphi_j(x)$ and $\sum_{j=1}^{\infty} h_j \varphi_j(x)$ converge absolutely to f and h on any compact subset of Ω, respectively [4], see also [25, Lemma 4.1 a]. Noticing that $c_j(t) = c_{j'}(t)$ and $d_j(t) = d_{j'}(t)$ if $j = j'$ we arrive at

$$\begin{aligned}
&\sum_{l=1}^{\infty} \left| \sum_{\lambda_j = \mu_l} [f_j c_j(t) + h_j d_j(t)] \varphi_j(x) \right| \\
&\leq \sum_{l=1}^{\infty} \left| \sum_{\lambda_j = \mu_l} f_j c_j(t) \varphi_j(x) \right| + \sum_{l=1}^{\infty} \left| \sum_{\lambda_j = \mu_l} h_j d_j(t) \varphi_j(x) \right| \\
&\leq N(t) \sum_{l=1}^{\infty} \left| \sum_{\lambda_j = \mu_l} f_j \varphi_j(x) \right| + \lambda_1^{-p} N^*(t) \sum_{l=1}^{\infty} \left| \sum_{\lambda_j = \mu_l} h_j \varphi_j(x) \right| < \infty,
\end{aligned}$$

i.e., the absolute convergence of (3.19).

c) Recall the Weyl's law for the asymptotics of the eigenvalues λ_j [7,8]

$$\lambda_j \simeq \delta j^{\frac{2}{d}}, \qquad j \to \infty, \quad \text{where} \quad \delta = \left[\frac{(2\sqrt{\pi})^{-d}}{\Gamma\left(\frac{d}{2}+1\right)} \int_\Omega \omega^{\frac{d}{2}}(x)\, dx \right]^{-\frac{2}{d}}. \tag{3.28}$$

For the eigenfunctions $\varphi_j(x)$ the following asymptotics formula holds uniformly on any compact subset K of Ω [4]

$$\sum_{|\sqrt{\lambda_j}-\lambda|\le 1} \varphi_j^2(x) = O\left(\lambda^{d-1}\right), \quad \lambda \to \infty.$$

In particular,

$$\varphi_j(x) = O\left(\lambda_j^{\frac{d-1}{4}}\right) = O\left(j^{\frac{d-1}{2d}}\right), \quad j \to \infty. \tag{3.29}$$

Now, if $f \in H_0^m(\Omega)$, then its Fourier coefficient f_j has the asymptotics [4]

$$f_j = O\left(\lambda_j^{-\frac{m}{2}}\right) = O\left(j^{-\frac{m}{d}}\right), \quad j \to \infty. \tag{3.30}$$

Similarly, if $h \in H_0^{m-1}(\Omega)$ for $m \geq 1$, then its Fourier coefficient h_j has the asymptotics

$$h_j = O\left(\lambda_j^{-\frac{m-1}{2}}\right) = O\left(j^{-\frac{m-1}{d}}\right), \quad j \to \infty. \tag{3.31}$$

From (3.29), (3.30), and (3.31) we have

$$[f_j c_j(t) + h_j d_j(t)]\varphi_j(x) = O\left(j^{\frac{d-1-2m}{2d}}\right) + O\left(j^{\frac{d+1-2m}{2d}}\right), \tag{3.32}$$

uniformly on $K \times [\epsilon, T]$, where K is any compact subset of Ω. Since $m > \frac{3d+1}{2}$, then $\frac{d-1-2m}{2d} < \frac{d+1-2m}{2d} < -1$, and therefore, the series (3.19) converges uniformly on $K \times [\epsilon, T]$.

From Theorem 1, (3.9), and (3.10) we have

$$|C_j(s)| \le \frac{M_1}{|s|^{\alpha-1}}, \quad |D_j(s)| \le \frac{M_2}{|s|^{\alpha}}, \quad \operatorname{Re} s > 0,$$

where M_1 and M_2 are independent of j. Hence, Hölder's inequality and the Parseval formula for the Laplace transform in $L^2(\mathbb{R}_+)$, see [29], give

$$\begin{aligned} &\left[\int_0^\infty e^{-xt}|c_j(t)|dt\right]^2 \le \int_0^\infty e^{-xt}dt \int_0^\infty e^{-xt}|c_j(t)|^2 dt \\ &= \frac{1}{2\pi x}\int_{-\infty}^{+\infty} \left|C_j\left(\frac{x}{2}+iy\right)\right|^2 dy \le \frac{M_1^2}{2\pi x}\int_{-\infty}^{+\infty} \frac{1}{\left|\frac{x}{2}+iy\right|^{2(\alpha-1)}} dy \\ &= \frac{M_1^2 2^{2\alpha-4}\Gamma(\alpha-\frac{3}{2})}{x^{2\alpha-2}\sqrt{\pi}\Gamma(\alpha-1)}, \quad x > 0, \end{aligned}$$

and

$$\begin{aligned}&\left[\int_0^\infty e^{-xt}|d_j(t)|dt\right]^2 \le \int_0^\infty e^{-xt}dt \int_0^\infty e^{-xt}|d_j(t)|^2 dt\\&= \frac{1}{2\pi x}\int_{-\infty}^{+\infty}\left|D_j\left(\frac{x}{2}+iy\right)\right|^2 dy \le \frac{M_2^2}{2\pi x}\int_{-\infty}^{+\infty}\frac{1}{\left|\frac{x}{2}+iy\right|^{2\alpha}}dy\\&= \frac{M_2^2 2^{2\alpha-2}\Gamma(\alpha-\frac{1}{2})}{x^{2\alpha}\sqrt{\pi}\Gamma(\alpha)}, \quad x>0.\end{aligned}$$

Consequently,

$$\begin{aligned}&\sum_{j=1}^\infty \int_0^\infty e^{-xt}|(f_j c_j(t)+h_j d_j(t))\varphi_j(x)|dt\\&\le \sum_{j=1}^\infty |f_j\varphi_j(x)|\int_0^\infty e^{-xt}|c_j(t)|dt + \sum_{j=1}^\infty |h_j\varphi_j(x)|\int_0^\infty e^{-xt}|d_j(t)|dt\\&\le \frac{1}{x^{\alpha-1}}\sum_{j=1}^\infty O\left(j^{\frac{d-1-2m}{2d}}\right) + \frac{1}{x^\alpha}\sum_{j=1}^\infty O\left(j^{\frac{d+1-2m}{2d}}\right) < \infty, \quad x>0.\end{aligned}$$

Thus, we can apply Lebesgue's dominated convergence theorem to obtain

$$(\mathcal{L}u(x,\cdot))(s) = \sum_{j=1}^\infty \left[f_j\varphi_j(x)(\mathcal{L}c_j)(s) + h_j\varphi_j(x)(\mathcal{L}d_j)(s)\right], \quad \operatorname{Re} s>0.$$

In other words,

$$\begin{aligned}U(x,s) &= \sum_{j=1}^\infty [f_j C_j(s) + h_j D_j(s)]\varphi_j(x)\\&= \sum_{j=1}^\infty O\left(j^{\frac{d-1-2m}{2d}}\right) O\left(\frac{1}{s^{\alpha-1}}\right) + \sum_{j=1}^\infty O\left(j^{\frac{d+1-2m}{2d}}\right) O\left(\frac{1}{s^\alpha}\right) \qquad (3.33)\\&= O\left(\frac{1}{s^{\alpha-1}}\right) + O\left(\frac{1}{s^\alpha}\right).\end{aligned}$$

By Theorem 1 we have $u(x,\cdot) \in BSA_{2\alpha-1}(\mathbb{R}_+)$.

Now, $m > \frac{3d+1}{2} \ge \left[\frac{d}{2}\right]+2$, therefore, combining with Part a) we arrive at $u \in C(\overline{\Omega}) \times BSA_{2\alpha-1}(\mathbb{R}_+)$.

Theorem 5 is proved. □

Similarly, we have the following theorem for Problem (II):(3.1)-(3.2)-(3.4)

Theorem 6 *Let $g \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+)$, $f \in H_0^m(\Omega)$, $h \in H_0^{m-1}(\Omega)$, $\|g\|_1 < \gamma(\alpha)k\lambda_1$ with $\frac{3}{2} < \alpha < 2$, and c_j and d_j be defined by (3.15) and (3.16), respectively.*

a) If $\partial\Omega \in C^m$ then the series

$$u(x,t) := \sum_{j=1}^{\infty} \left(f_j c_j(t) + h_j d_j(t)\right) \varphi_j(x) \tag{3.34}$$

converges in $H^m(\Omega)$ norm for each $t > 0$. If, moreover, $m \geq \left[\frac{d}{2}\right] + 1$, then $u(\cdot, t) \in C^{m-\left[\frac{d}{2}\right]-1}(\overline{\Omega})$.

b) If $m > \frac{d}{2} + 1$, then the series (3.34) *converges absolutely on $Q := \Omega \times \mathbb{R}_+$.*

c) If $m > \frac{3d+1}{2}$, then the series (3.34) *converges uniformly on any compact subset of Q. Moreover, if $\partial\Omega \in C^{\left[\frac{d}{2}\right]+1}$, then $u \in C(\overline{\Omega}) \times BSA_3(\mathbb{R}_+)$.*

Proof Consider the equation

$$^{C}\partial_t^{\alpha} y(t) = -\lambda y(t) + f(t), \quad y(0) = y_0, \quad y'(0) = y_1, \quad \alpha \in (1,2), \tag{3.35}$$

its solution has the explicit form [13]

$$y(t) = y_0 E_\alpha(-\lambda t^\alpha) + y_1 t E_{\alpha,2}(-\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda (t-\tau)^\alpha\right) f(\tau)\, d\tau. \tag{3.36}$$

Applying (3.35) and (3.36) to (3.13) and (3.14) with $f(t) = -(g * c_j)(t)$, $f(t) = -(g * d_j)(t)$, respectively, and λ being replaced by $k\lambda_j$, we obtain

$$\begin{aligned} c_j(t) &= E_\alpha(-k\lambda_j t^\alpha) \\ &\quad - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k\lambda_j (t-\tau)^\alpha) \int_0^\tau g(\tau-\eta) c_j(\eta)\, d\eta\, d\tau \\ &= E_\alpha(-k\lambda_j t^\alpha) - \int_0^t \beta(t,\eta) c_j(\eta)\, d\eta, \end{aligned}$$

and

$$d_j(t) = t E_{\alpha,2}(-k\lambda_j t^\alpha) - \int_0^t \beta(t,\eta) d_j(\eta)\, d\eta,$$

where $\beta(t,\eta)$ is defined (3.22). From Lemma 1 and Lemma 2 we have

$$|E_\alpha(-k\lambda_j t^\alpha)| \leq M_1, \quad t|E_{\alpha,2}(-k\lambda_j t^\alpha)| \leq \left(-2\cos\frac{\pi}{\alpha}\right)(k\lambda_j)^{-\frac{1}{\alpha}}.$$

Together with (3.23), we derive

$$|c_j(t)| \leq M_1 + \frac{M_1 \|g\|_\infty}{\alpha} \int_0^t (t-\eta)^\alpha |c_j(\eta)| d\eta,$$

and

$$|d_j(t)| \leq \left(-2\cos\frac{\pi}{\alpha}\right)(k\lambda_j)^{-\frac{1}{\alpha}} + \frac{M_1 \|g\|_\infty}{\alpha} \int_0^t (t-\eta)^\alpha |d_j(\eta)| d\eta.$$

Applying the Gronwall inequality for fractional integral [31, Corollary 2], we obtain

$$|c_j(t)| \le M_1 E_{\alpha+1}\left(M_1\|g\|_\infty \Gamma(\alpha) t^{\alpha+1}\right) \le M_1 E_{\alpha+1}\left(M_1\|g\|_\infty \Gamma(\alpha) T^{\alpha+1}\right) =: N_T,\ t\in[0,T] \tag{3.37}$$

and

$$\begin{aligned}|d_j(t)| &\le \left(-2\cos\frac{\pi}{\alpha}\right)(k\lambda_j)^{-\frac{1}{\alpha}} E_{\alpha+1}\left(M_1\|g\|_\infty \Gamma(\alpha) t^{\alpha+1}\right)\\ &\le \left(-2\cos\frac{\pi}{\alpha}\right) k^{-\frac{1}{\alpha}} E_{\alpha+1}\left(M_1\|g\|_\infty \Gamma(\alpha) T^{\alpha+1}\right)\lambda_j^{-\frac{1}{\alpha}}\\ &=: N_T^*\lambda_j^{-\frac{1}{\alpha}} \le N_T^*\lambda_1^{-\frac{1}{\alpha}},\ t\in[0,T].\end{aligned} \tag{3.38}$$

Thus, $\{c_j(t)\}_{j\ge1}$ and $\{d_j(t)\}_{j\ge1}$ are uniformly bounded on any interval $[0,T]$, $T<\infty$.

a) Similar to the proof of Theorem 5 Part a), we have

$$\begin{aligned}\sum_{j=1}^{\infty}(f_jc_j(t)+h_jd_j(t))^2\lambda_j^m &\le 2\left(\sum_{j=1}^{\infty} f_j^2c_j^2(t)\lambda_j^m + \sum_{j=1}^{\infty} h_j^2d_j^2(t)\lambda_j^m\right)\\ &\le 2\left(N_T^2\sum_{j=1}^{\infty} f_j^2\lambda_j^m + (N_T^*)^2\sum_{j=1}^{\infty} h_j^2\lambda_j^{m-1}\lambda_j^{1-\frac{2}{\alpha}}\right)\\ &\le 2\left(N_T^2\sum_{j=1}^{\infty} f_j^2\lambda_j^m + (N_T^*)^2\lambda_1^{1-\frac{2}{\alpha}}\sum_{j=1}^{\infty} h_j^2\lambda_j^{m-1}\right)<\infty.\end{aligned}$$

In other words, the series (3.34) converges in $H^m(\Omega)$ norm, and $u(\cdot,t)\in H^m(\Omega)$ for any $t\ge0$. On the other hand, when $m\ge\left[\frac{d}{2}\right]+1$, we have [17] $H^m(\Omega)\subset C^{m-\left[\frac{d}{2}\right]-1}(\overline{\Omega})$, therefore, $u(\cdot,t)\in C^{m-\left[\frac{d}{2}\right]-1}(\overline{\Omega})$.

b) Combining [25, Lemma 4.1 a)], and noticing that $c_j(t)=c_{j'}(t)$ and $d_j(t)=d_{j'}(t)$ if $\lambda_j=\lambda_{j'}$, we arrive at

$$\begin{aligned}&\sum_{l=1}^{\infty}\left|\sum_{\lambda_j=\mu_l}(f_jc_j(t)+h_jd_j(t))\varphi_j(x)\right|\\ &\le \sum_{l=1}^{\infty}\left|\sum_{\lambda_j=\mu_l} f_jc_j(t)\varphi_j(x)\right| + \sum_{l=1}^{\infty}\left|\sum_{\lambda_j=\mu_l} h_jd_j(t)\varphi_j(x)\right|\\ &\le N_T\sum_{l=1}^{\infty}\left|\sum_{\lambda_j=\mu_l} f_j\varphi_j(x)\right| + N_T^*\lambda_1^{-\frac{1}{\alpha}}\sum_{l=1}^{\infty}\left|\sum_{\lambda_j=\mu_l} h_j\varphi_j(x)\right|<\infty,\end{aligned}$$

i.e., the absolute convergence of (3.34).

c) Similarly to the proof of Theorem 5 Part c), the series (3.34) converges uniformly on $K \times [0,T]$, where K is any compact subset of Ω.

From Theorem 1, (3.15) and (3.16) we have

$$|C_j(s)| \le \frac{M_1}{|s|}, \quad |D_j(s)| \le \frac{M_2}{|s|^2}, \quad \operatorname{Re} s > 0,$$

where M_1 and M_2 are independent of j. Hence, Hölder's inequality and the Parseval formula for the Laplace transform in $L^2(\mathbb{R}_+)$ give

$$\left[\int_0^\infty e^{-xt}|c_j(t)|dt\right]^2 \le \int_0^\infty e^{-xt}dt \int_0^\infty e^{-xt}|c_j(t)|^2 dt$$
$$= \frac{1}{2\pi x}\int_{-\infty}^{+\infty} \left|C_j\left(\frac{x}{2}+iy\right)\right|^2 dy \le \frac{M_1^2}{2\pi x}\int_{-\infty}^{+\infty} \frac{1}{\left|\frac{x}{2}+iy\right|^2}dy = \frac{M_1^2}{x^2}, \quad x>0,$$

and

$$\left[\int_0^\infty e^{-xt}|d_j(t)|dt\right]^2 \le \int_0^\infty e^{-xt}dt \int_0^\infty e^{-xt}|d_j(t)|^2 dt$$
$$= \frac{1}{2\pi x}\int_{-\infty}^{+\infty} \left|D_j\left(\frac{x}{2}+iy\right)\right|^2 dy \le \frac{M_2^2}{2\pi x}\int_{-\infty}^{+\infty} \frac{1}{\left|\frac{x}{2}+iy\right|^4}dy = \frac{2M_2^2}{x^4}, \quad x>0.$$

Consequently, if $m > \frac{3d+1}{2}$, then

$$\sum_{j=1}^\infty \int_0^\infty e^{-xt}|(f_j c_j(t) + h_j d_j(t))\varphi_j(x)|dt$$
$$\le \sum_{j=1}^\infty |f_j\varphi_j(x)| \int_0^\infty e^{-xt}|c_j(t)|dt + \sum_{j=1}^\infty |h_j\varphi_j(x)|\int_0^\infty e^{-xt}|d_j(t)|dt$$
$$\le \frac{M_1}{x}\sum_{j=1}^\infty O\left(j^{\frac{d-1-2m}{2d}}\right) + \frac{\sqrt{2}M_2}{x^2}\sum_{j=1}^\infty O\left(j^{\frac{d+1-2m}{2d}}\right) < \infty, \quad x>0.$$

Thus, we can apply Lebesgue's dominated convergence theorem to obtain

$$(\mathcal{L}u(x,\cdot))(s) = \sum_{j=1}^\infty \left[f_j\varphi_j(x)(\mathcal{L}c_j)(s) + h_j\varphi_j(x)(\mathcal{L}d_j)(s)\right], \quad \operatorname{Re} s > 0.$$

In other words,

$$\begin{aligned} U(x,s) &= \sum_{j=1}^\infty [f_j C_j(s) + h_j D_j(s)]\varphi_j(x) \\ &= \sum_{j=1}^\infty O\left(j^{\frac{d-1-2m}{2d}}\right) O\left(\frac{1}{s}\right) + \sum_{j=1}^\infty O\left(j^{\frac{d+1-2m}{2d}}\right) O\left(\frac{1}{s^2}\right) \qquad (3.39) \\ &= O\left(\frac{1}{s}\right) + O\left(\frac{1}{s^2}\right). \end{aligned}$$

By Theorem 1 we have $u(x,\cdot) \in BSA_3(\mathbb{R}_+)$.

Now, $m > \frac{3d+1}{2} \geq \left[\frac{d}{2}\right] + 1$, therefore, combining with Part a) we arrive at $u \in C(\overline{\Omega}) \times BSA_3(\mathbb{R}_+)$.

Theorem 6 is proved. □

Remark 2 The series (3.19) converges in $H^{m-1}(\Omega)$ norm, but the series (3.34) converges in a stronger $H^m(\Omega)$ norm.

Now we are ready to prove the main theorems of this section about the global existence of solutions of Problem (I):(3.1)-(3.2)-(3.3) and Problem (II):(3.1)-(3.2)-(3.4).

For Problem (I) we have

Theorem 7 *Let* $g \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+)$, $f \in H_0^m(\Omega)$, $h \in H_0^{m-1}(\Omega)$, $\partial\Omega \in C^m$ *with* $m > \frac{3d+5}{2}$, $\frac{3}{2} < \alpha < 2$, *and* $\|g\|_1 < \gamma(\alpha)k\lambda_1$. *Then* $u(x,t)$, *defined by* (3.19), *is the unique classical solution of Problem (I):*(3.1)-(3.2)-(3.3) *in* $C^2(\overline{\Omega}) \times BSA^{\alpha}_{2\alpha-1}(\mathbb{R}_+)$.

Proof Since $m - \left[\frac{d}{2}\right] - 2 > \frac{3d+5}{2} - \left[\frac{d}{2}\right] - 2 \geq 2$, by Theorem 5 a) we have $u(\cdot,t) \in C^2(\overline{\Omega})$. Moreover, from (3.28), (3.32) and $\frac{d+3-2m}{2d} < \frac{d+5-2m}{2d} < -1$, we obtain

$$\sum_{j=1}^{\infty} |(f_j c_j(t) + h_j d_j(t))\Delta\varphi_j(x)| = \sum_{j=1}^{\infty} |\lambda_j [f_j c_j(t) + h_j d_j(t)]\varphi_j(x)|$$

$$= \sum_{j=1}^{\infty} O\left(j^{\frac{d+3-2m}{2d}}\right) + \sum_{j=1}^{\infty} O\left(j^{\frac{d+5-2m}{2d}}\right) < \infty,$$

uniformly on any compact subset of $K \times [\epsilon, T]$. Hence,

$$\Delta u(x,t) = \sum_{j=1}^{\infty} (f_j c_j(t) + h_j d_j(t))\Delta\varphi_j(x) = -\sum_{j=1}^{\infty} \lambda_j (c_j(t) + d_j(t))\varphi_j(x). \tag{3.40}$$

From (3.20) and (3.26), (3.27), we get

$$|D^{\alpha}_{0+} c_j(t)| \leq k\lambda_j N(t) + N(t)\int_0^t |g(t-s)|ds$$
$$\leq N(t)(k\lambda_j + \|g\|_1) = O(\lambda_j) = O(j^{\frac{2}{d}}), \quad t \in [\epsilon, T]$$

and

$$|D^{\alpha}_{0+} d_j(t)| \leq k\lambda_j \lambda_1^{-p} N^*(t) + \lambda_1^{-p} N^*(t)\int_0^t |g(t-s)|ds$$
$$\leq \lambda_1^{-p} N^*(t)(k\lambda_j + \|g\|_1) = O(\lambda_j) = O(j^{\frac{2}{d}}), \quad t \in [\epsilon, T].$$

Consequently,

$$\sum_{j=1}^{\infty} |D^{\alpha}_{0+}[f_j c_j(t) + h_j d_j(t)]\varphi_j(x)| = \sum_{j=1}^{\infty} O\left(j^{\frac{d+3-2m}{2d}}\right) + \sum_{j=1}^{\infty} O\left(j^{\frac{d+5-2m}{2d}}\right) < \infty,$$

uniformly on $[\epsilon, T]$ for any $0 < \epsilon < T < \infty$, and it yields

$$D_{0+}^{\alpha} u(x,t) = \sum_{j=1}^{\infty} [f_j D_{0+}^{\alpha} c_j(t) + h_j D_{0+}^{\alpha} d_j(t)] \varphi_j(x). \tag{3.41}$$

It is obvious that

$$\begin{aligned} &\int_0^t g(t-\tau) u(x,\tau) d\tau \\ &= \sum_{j=1}^{\infty} \left[f_j \int_0^t g(t-\tau) c_j(\tau) d\tau + h_j \int_0^t g(t-\tau) d_j(\tau) d\tau \right] \varphi_j(x). \end{aligned} \tag{3.42}$$

Combining (3.40), (3.41), and (3.42) we arrive that

$$\begin{aligned} &D_{0+}^{\alpha} u(x,t) - k\Delta u(x,t) + \int_0^t g(t-\tau) u(x,\tau) d\tau \\ &= \sum_{j=1}^{\infty} f_j \varphi_j(x) \left[D_{0+}^{\alpha} c_j(t) + k\lambda_j c_j(t) + \int_0^t g(t-\tau) c_j(\tau) d\tau \right] \\ &+ \sum_{j=1}^{\infty} h_j \varphi_j(x) \left[D_{0+}^{\alpha} d_j(t) + k\lambda_j d_j(t) + \int_0^t g(t-\tau) d_j(\tau) d\tau \right] = 0. \end{aligned}$$

Since $\varphi_j(x) = 0$ on $\partial\Omega$, then

$$u(x,t) = \sum_{j=1}^{\infty} [f_j c_j(t) + h_j d_j(t)] \varphi_j(x) = 0, \quad x \in \partial\Omega.$$

Because $I_{0+}^{2-\alpha} c_j(0) = 1$, $D_{0+}^{\alpha-1} c_j(0) = 0$ and $I_{0+}^{2-\alpha} d_j(0) = 0$, $D_{0+}^{\alpha-1} d_j(0) = 1$, by the fact that the series $\sum_{j=1}^{\infty} f_j \varphi_j(x)$, $\sum_{j=1}^{\infty} h_j \varphi_j(x)$ converge absolutely to f and h on any compact subset of Ω, respectively see [25, Lemma 4.1 a)],

$$I_{0+}^{2-\alpha} u(x,0) = \sum_{j=1}^{\infty} f_j I_{0+}^{2-\alpha} c_j(0) \varphi_j(x) = \sum_{j=1}^{\infty} f_j \varphi_j(x) = f(x), \quad x \in \Omega, \tag{3.43}$$

and

$$D_{0+}^{\alpha-1} u(x,0) = \sum_{j=1}^{\infty} h_j D_{0+}^{\alpha-1} d_j(0) \varphi_j(x) = \sum_{j=1}^{\infty} h_j \varphi_j(x) = h(x), \quad x \in \Omega. \tag{3.44}$$

Thus, $u(x,t)$ defined by (3.19), is a classical solution of Problem (I).

Taking into account (3.33) and (3.43), (3.44) obtain

$$\begin{aligned}
&(\mathcal{L}D_{0+}^{\alpha}u(x,t))(s)\\
&= s^{\alpha}U(x,s) - s\, I_{0+}^{2-\alpha}u(x,0) - D_{0+}^{\alpha-1}u(x,0)\\
&= s^{\alpha}\sum_{j=1}^{\infty}[f_jC_j(s)+h_jD_j(s)]\varphi_j(x) - s\sum_{j=1}^{\infty}f_j\varphi_j(x) - \sum_{j=1}^{\infty}h_j\varphi_j(x)\\
&= \sum_{j=1}^{\infty}f_j\varphi_j(x)(s^{\alpha}C_j(s)-s) + \sum_{j=1}^{\infty}h_j\varphi_j(x)(s^{\alpha}D_j(s)-1)\\
&= \sum_{j=1}^{\infty}f_j\varphi_j(x)\frac{-s(\lambda_jk+G(s))}{s^{\alpha}+\lambda_jk+G(s)} + \sum_{j=1}^{\infty}h_j\varphi_j(x)\frac{-(\lambda_jk+G(s))}{s^{\alpha}+\lambda_jk+G(s)}.
\end{aligned}$$

Using Theorem 1 and Weyl's law (3.28) we get

$$\left|\frac{s(\lambda_jk+G(s))}{s^{\alpha}+\lambda_jk+G(s)}\right| \le \frac{M_1\lambda_j}{|s|^{\alpha-1}} \le \frac{M_1 j^{\frac{2}{d}}}{|s|^{\alpha-1}},\ \left|\frac{\lambda_jk+G(s)}{s^{\alpha}+\lambda_jk+G(s)}\right| \le \frac{M_2\lambda_j}{|s|^{\alpha}} \le \frac{M_2 j^{\frac{2}{d}}}{|s|^{\alpha}}.$$

Together with (3.29), (3.30), it yields

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left|f_j\varphi_j(x)\frac{s(\lambda_jk+G(s))}{s^{\alpha}+\lambda_jk+G(s)}\right| + \sum_{j=1}^{\infty}\left|h_j\varphi_j(x)\frac{\lambda_jk+G(s)}{s^{\alpha}+\lambda_jk+G(s)}\right|\\
&\le \frac{M_1}{|s|^{\alpha-1}}\sum_{j=1}^{\infty}j^{\frac{d+3-2m}{2d}} + \frac{M_2}{|s|^{\alpha}}\sum_{j=1}^{\infty}j^{\frac{d+1-2m}{2d}} \le \frac{M_1}{|s|^{\alpha-1}} + \frac{M_2}{|s|^{\alpha}}
\end{aligned}$$

because $\frac{d+1-2m}{2d} < \frac{d+3-2m}{2d} < \frac{d+5-2m}{2d} < -1$. By Theorem 1, we get $D_{0+}^{\alpha}u(x,t) \in BSA_{2\alpha-1}(\mathbb{R}_+)$. Together with $u(x,t) \in BSA_{2\alpha-1}(\mathbb{R}_+)$ by Theorem 5 c) it yields $u(x,t) \in BSA_{2\alpha-1}^{\alpha}(\mathbb{R}_+)$. Thus, $u \in C^2(\overline{\Omega}) \times BSA_{2\alpha-1}^{\alpha}(\mathbb{R}_+)$.

Let $u, \tilde{u} \in C^2(\overline{\Omega}) \times BSA_{2\alpha-1}^{\alpha}(\mathbb{R}_+)$ be two solutions of Problem (I). Then $w = u - \tilde{u} \in C^2(\overline{\Omega}) \times BSA_{2\alpha-1}^{\alpha}(\mathbb{R}_+)$ is a solution of

$$\begin{cases}
D_{0+}^{\alpha}w(x,t) = k\Delta w(x,t) - \displaystyle\int_0^t g(t-\eta)w(x,\eta)d\eta, \quad (x,t) \in \Omega\times\mathbb{R}^+,\\
w(x,t) = 0, \qquad (x,t) \in \partial\Omega\times\mathbb{R}^+,\\
I_{0+}^{2-\alpha}w(x,0) = D_{0+}^{\alpha-1}w(x,0) = 0, \qquad x \in \Omega.
\end{cases} \tag{3.45}$$

Taking the Laplace transform of (3.45) we get

$$\begin{cases}
\Delta W(x,s) = \frac{G(s)+s^{\alpha}}{k}W(x,s), \quad x \in \Omega,\\
W(x,s) = 0, \quad x \in \partial\Omega.
\end{cases}, W(x,s) \in C^2(\overline{\Omega}),\ \mathrm{Re}\, s > 0. \tag{3.46}$$

If $s \in \left(\|g\|_1^{\frac{1}{\alpha}}, \infty\right)$, then $-\frac{G(s)+s^{\alpha}}{k} < 0$ cannot be an eigenvalue of the Dirichlet Laplacian (3.5), therefore the Schrödinger equation with Dirichlet's boundary condition (3.46) has only trivial solution $W(x,s) = 0$, $x \in \Omega$ [17], for such s.

But for a fixed parameter $x \in \Omega$, $W(x,s)$, as a function of s, is analytic in $\operatorname{Re} s > 0$. Hence, $w(x,t) = 0$, and we obtain the uniqueness of u. The theorem is proved. □

Similarly, for Problem (II) we obtain

Theorem 8 *Assume that the assumptions in Theorem 7 hold, then $u(x,t)$, defined by* (3.34)*, is the unique classical solution of Problem (II):* (3.1)-(3.2)-(3.4) *in* $C^3(\overline{\Omega}) \times BSA^2_3(\mathbb{R}_+)$.

Proof Since $m - \left[\frac{d}{2}\right] - 1 > \frac{3d+5}{2} - \left[\frac{d}{2}\right] - 1 \geq 3$, by Theorem 6 a) we have $u(\cdot,t) \in C^3(\overline{\Omega})$. Moreover, from $\frac{d+3-2m}{2d} < \frac{d+5-2m}{2d} < -1$,

$$\sum_{j=1}^{\infty} |(f_j c_j(t) + h_j d_j(t))\Delta\varphi_j(x)| = \sum_{j=1}^{\infty} |\lambda_j [f_j c_j(t) + h_j d_j(t)]\varphi_j(x)|$$
$$= \sum_{j=1}^{\infty} O\left(j^{\frac{d+3-2m}{2d}}\right) + \sum_{j=1}^{\infty} O\left(j^{\frac{d+5-2m}{2d}}\right) < \infty,$$

uniformly on any compact subset of Q. Hence,

$$\Delta u(x,t) = \sum_{j=1}^{\infty} (f_j c_j(t) + h_j d_j(t))\Delta\varphi_j(x) = -\sum_{j=1}^{\infty} \lambda_j (c_j(t) + d_j(t))\varphi_j(x). \tag{3.47}$$

From (3.35), (3.37) and (3.38) we get

$$|{}^{\mathcal{C}}\partial_t^\alpha c_j(t)| \leq k\lambda_j N_T + N_T \int_0^T |g(t-s)|ds \leq N_T(k\lambda_j + \|g\|_1)$$
$$= O(\lambda_j) = O(j^{\frac{2}{d}}), \quad t \in [0,T]$$

and

$$|{}^{\mathcal{C}}\partial_t^\alpha d_j(t)| \leq k\lambda_j\, \lambda_1^{-\frac{1}{\alpha}} N_T^* + \lambda_1^{-\frac{1}{\alpha}} N_T^* \int_0^T |g(t-s)|ds \leq \lambda_1^{-\frac{1}{\alpha}} N_T^*(k\lambda_j + \|g\|_1)$$
$$= O(\lambda_j) = O(j^{\frac{2}{d}}), \quad t \in [0,T].$$

Consequently,

$$\sum_{j=1}^{\infty} |{}^{\mathcal{C}}\partial_t^\alpha [f_j c_j(t) + h_j d_j(t)]\varphi_j(x)| = \sum_{j=1}^{\infty} O\left(j^{\frac{d+3-2m}{2d}}\right) + \sum_{j=1}^{\infty} O\left(j^{\frac{d+5-2m}{2d}}\right) < \infty,$$

uniformly on $[0,T]$ for any $T > 0$, and it yields

$${}^{\mathcal{C}}\partial_t^\alpha u(x,t) = \sum_{j=1}^{\infty} [f_j\, {}^{\mathcal{C}}\partial_t^\alpha c_j(t) + h_j\, {}^{\mathcal{C}}\partial_t^\alpha d_j(t)]\varphi_j(x). \tag{3.48}$$

It is obvious that

$$\begin{aligned}&\int_0^t g(t-\tau)u(x,\tau)d\tau\\&=\sum_{j=1}^{\infty}\left[f_j\int_0^t g(t-\tau)c_j(\tau)d\tau+h_j\int_0^t g(t-\tau)d_j(\tau)d\tau\right]\varphi_j(x).\end{aligned}\tag{3.49}$$

Combining (3.47), (3.48), and (3.49) we arrive that

$$\begin{aligned}&{}^{C}\partial_t^{\alpha}u(x,t)-k\Delta u(x,t)+\int_0^t g(t-\tau)u(x,\tau)d\tau\\&=\sum_{j=1}^{\infty}f_j\varphi_j(x)\left[{}^{C}\partial_t^{\alpha}c_j(t)+k\lambda_j c_j(t)+\int_0^t g(t-\tau)c_j(\tau)d\tau\right]\\&+\sum_{j=1}^{\infty}h_j\varphi_j(x)\left[{}^{C}\partial_t^{\alpha}d_j(t)+k\lambda_j d_j(t)+\int_0^t g(t-\tau)d_j(\tau)d\tau\right]=0.\end{aligned}$$

Since $\varphi_j(x)=0$ on $\partial\Omega$, then

$$u(x,t)=\sum_{j=1}^{\infty}[f_j c_j(t)+h_j d_j(t)]\varphi_j(x)=0,\quad x\in\partial\Omega.$$

Because $c_j(0)=1$, $c_j'(0)=0$ and $d_j(0)=0$, $d_j'(0)=1$, by the fact that the series $\sum_{j=1}^{\infty}f_j\varphi_j(x)$, $\sum_{j=1}^{\infty}h_j\varphi_j(x)$ converge absolutely to f and h on any compact subset of Ω, respectively see [25, Lemma 4.1 a)]

$$u(x,0)=\sum_{j=1}^{\infty}f_j c_j(0)\varphi_j(x)=\sum_{j=1}^{\infty}f_j\varphi_j(x)=f(x),\quad x\in\Omega,$$

$$u_t(x,0)=\sum_{j=1}^{\infty}h_j d_j'(0)\varphi_j(x)=\sum_{j=1}^{\infty}h_j\varphi_j(x)=h(x),\quad x\in\Omega.$$

Thus, $u(x,t)$ defined by (3.34), is a classical solution of Problem (II).

Taking into account (3.39), (3.17) and (3.18) we obtain

$$\begin{aligned}(\mathcal{L}u_t(x,t))(s)&=sU(x,s)-u(x,0)\\&=s\sum_{j=1}^{\infty}[f_jC_j(s)+h_jD_j(s)]\varphi_j(x)-\sum_{j=1}^{\infty}f_j\varphi_j(x)\\&=\sum_{j=1}^{\infty}f_j\varphi_j(x)(sC_j(s)-1)+\sum_{j=1}^{\infty}h_j\varphi_j(x)sD_j(s)\\&=\sum_{j=1}^{\infty}f_j\varphi_j(x)\frac{-(\lambda_jk+G(s))}{s^{\alpha}+\lambda_jk+G(s)}+\sum_{j=1}^{\infty}h_j\varphi_j(x)\frac{s^{\alpha-1}}{s^{\alpha}+\lambda_jk+G(s)},\end{aligned}$$

and

$$\begin{aligned}(\mathcal{L}u_{tt}(x,t))(s) &= s^2\,U(x,s) - s\,u(x,0) - u_t(x,0)\\ &= s^2\sum_{j=1}^{\infty}[f_jC_j(s)+h_jD_j(s)]\varphi_j(x) - s\sum_{j=1}^{\infty}f_j\varphi_j(x) - \sum_{j=1}^{\infty}h_j\varphi_j(x)\\ &= \sum_{j=1}^{\infty}f_j\varphi_j(x)(s^2C_j(s)-s) + \sum_{j=1}^{\infty}h_j\varphi_j(x)(s^2D_j(s)-1)\\ &= \sum_{j=1}^{\infty}f_j\varphi_j(x)\frac{-s(\lambda_jk+G(s))}{s^{\alpha}+\lambda_jk+G(s)} + \sum_{j=1}^{\infty}h_j\varphi_j(x)\frac{-(\lambda_jk+G(s))}{s^{\alpha}+\lambda_jk+G(s)}.\end{aligned}$$

Using Theorem 1 and (3.28) we get

$$\left|\frac{\lambda_jk+G(s)}{s^{\alpha}+\lambda_jk+G(s)}\right| \le \frac{M_1\lambda_j}{|s|^{\alpha}} \le \frac{M_1 j^{\frac{2}{d}}}{|s|^{\alpha}}, \quad \left|\frac{s^{\alpha-1}}{s^{\alpha}+\lambda_jk+G(s)}\right| \le \frac{M_2}{|s|}.$$

Together with (3.29), (3.30), it yields

$$\begin{aligned}&\sum_{j=1}^{\infty}\left|f_j\varphi_j(x)\frac{\lambda_jk+G(s)}{s^{\alpha}+\lambda_jk+G(s)}\right| + \sum_{j=1}^{\infty}\left|h_j\varphi_j(x)\frac{s^{\alpha-1}}{s^{\alpha}+\lambda_jk+G(s)}\right|\\ &\le \frac{M_1}{|s|^{\alpha}}\sum_{j=1}^{\infty}j^{\frac{d+3-2m}{2d}} + \frac{M_2}{|s|}\sum_{j=1}^{\infty}j^{\frac{d+1-2m}{2d}} \le \frac{M_1}{|s|^{\alpha}} + \frac{M_2}{|s|},\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{\infty}\left|f_j\varphi_j(x)\frac{s(\lambda_jk+G(s))}{s^{\alpha}+\lambda_jk+G(s)}\right| + \sum_{j=1}^{\infty}\left|h_j\varphi_j(x)\frac{\lambda_jk+G(s)}{s^{\alpha}+\lambda_jk+G(s)}\right|\\ &\le \frac{M_1}{|s|^{\alpha-1}}\sum_{j=1}^{\infty}j^{\frac{d+3-2m}{2d}} + \frac{M_2}{|s|^{\alpha}}\sum_{j=1}^{\infty}j^{\frac{d+5-2m}{2d}} \le \frac{M_1}{|s|^{\alpha-1}} + \frac{M_2}{|s|^{\alpha}},\end{aligned}$$

because $\frac{d+1-2m}{2d} < \frac{d+3-2m}{2d} < \frac{d+5-2m}{2d} < -1$. By Theorem 1, we obtain $u_t(x,t), u_{tt}(x,t) \in BSA_{2\alpha-1}(\mathbb{R}_+) \subset BSA_3(\mathbb{R}_+)$. Together with $u(x,t) \in BSA_3(\mathbb{R}_+)$ by Theorem 6 c) it yields $u(x,t) \in BSA_3^2(\mathbb{R}_+)$. Thus, $u \in C^3(\overline{\Omega}) \times BSA_3^2(\mathbb{R}_+)$.

Let $u, \tilde{u} \in C^3(\overline{\Omega}) \times BSA_3^2(\mathbb{R}_+)$ be two solutions of Problem (II). Then $w = u - \tilde{u} \in C^2(\overline{\Omega}) \times BSA_3^2(\mathbb{R}_+)$ is a solution of

$$\begin{cases} {}^{C}\partial_t^{\alpha}w(x,t) = k\Delta w(x,t) - \displaystyle\int_0^t g(t-\eta)w(x,\eta)d\eta, & (x,t) \in \Omega \times \mathbb{R}^+,\\ w(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, & \\ w(x,0) = 0, \quad \frac{\partial w(x,t)}{\partial t}\Big|_{t=0} = 0, \quad x \in \Omega. & \end{cases} \tag{3.50}$$

Taking the Laplace transform of (3.50) we get the Dirichlet Schrödinger problem (3.46), and the uniqueness of u follows. □

4 Inverse Problems

In the previous section we proved the global existence of classical solutions for Problems (I) and (II). We now focus on the inverse problems of Problem (I) and (II), that is to find initial functions $f(x), h(x)$, so that we can reconstruct the constants $\{\alpha, k\}$ and the memory function g uniquely from a single observation of the solution of Problem (I) and Problem (II), respectively, at one point $\{u(b,t)\}_{t>0}$, where $b \in \Omega$ is an arbitrary point.

We start with the inverse problem of Problem (I). Choose the initial conditions $f(x) = h(x) = \varphi_1(x)$. Then the observation $u(b,t)$ is given by

$$u(b,t) = (c_1(t) + d_1(t))\varphi_1(b)$$

with

$$I_{0+}^{2-\alpha} c_1(0) = 1,\ D_{0+}^{\alpha-1} c_1(0) = 0,\ I_{0+}^{2-\alpha} d_1(0) = 0,\ D_{0+}^{\alpha-1} d_1(0) = 1, \quad \text{where } b \in \Omega.$$

Recall that $\varphi_1(b) \neq 0$, as the principal eigenfunction of the Laplacian never vanishes inside Ω [17], and so the observation is not trivial.

Taking the Laplace transform of the observation $u(b,t)$ with respect to t, and recalling (3.9) and (3.10), we have

$$U(b,s) = \frac{s+1}{s^\alpha + \lambda_1 k + G(s)} \varphi_1(b).$$

Consequently,

$$\frac{(s+1)\varphi_1(b)}{U(b,s)} = s^\alpha + \lambda_1 k + G(s) \sim s^\alpha, \quad s \to \infty,$$

and therefore, α can be recovered from the observation as follows

$$\alpha = \lim_{s\to\infty} \frac{\ln\left(\dfrac{(s+1)\varphi_1(b)}{U(b,s)}\right)}{\ln s}. \tag{4.1}$$

Once α is recovered, k can be obtained as

$$k = \lim_{s\to\infty} \frac{1}{\lambda_1}\left[\frac{(s+1)\varphi_1(b)}{U(b,s)} - s^\alpha\right], \tag{4.2}$$

and $G(s)$ as

$$G(s) = \frac{(s+1)\varphi_1(b)}{U(b,s)} - s^\alpha - k\lambda_1, \quad \operatorname{Re} s > 0. \tag{4.3}$$

The memory kernel $g(t)$ can be recovered by taking the Laplace inverse transform of $G(s)$. Thus we have proved

Theorem 9 *Let $f(x) = h(x) = \varphi_1(x)$, then one observation of u at a single point $b \in \Omega$ determines uniquely the fractional order α by* (4.1)*, the parameter k by* (4.2)*, and the memory function g by taking the Laplace inverse of $G(s)$ from* (4.3).

Similarly, from the observation $u(b,t)$ of Problem (II) one can find

$$\alpha = -\lim_{s\to\infty} \frac{\ln\left(\dfrac{(s+1)\varphi_1(b)}{s^2\,U(b,s)} - 1\right)}{\ln s},$$

$$k = \lim_{s\to\infty} \frac{s^\alpha}{\lambda_1}\left[\frac{(s+1)\varphi_1(b)}{s^2\,U(b,s)} - 1\right],$$

and

$$G(s) = s^\alpha\left[\frac{(s+1)\varphi_1(b)}{s^2\,U(b,s)} - 1\right] - k\lambda_1, \quad \operatorname{Re} s > 0, \quad g(t) = (\mathcal{L}^{-1}G)(t).$$

Assume now that the observation point b is on the boundary $\partial\Omega$. Since $u(b,t) = 0$ when $b \in \partial\Omega$, so instead of $u(b,t)$ we should observe $\frac{\partial u(b,t)}{\partial\nu}$, the outer normal derivative of u at the boundary point b.

For the inverse problem of Problem (I) with the special initial conditions $I_{0+}^{2-\alpha}u(x,0) = D_{0+}^{\alpha-1}u(x,0) = \varphi_1(x)$ the solution $u(x,t) = (c_1(t)+d_1(t))\varphi_1(x) \in C^1(\overline{\Omega})$ for each $t > 0$ when $\partial\Omega \in C^{[\frac{d}{2}]+2}$ [17]. Since $\frac{\partial\varphi_1(b)}{\partial\nu} \neq 0$ [17], the observation $\frac{\partial u(b,t)}{\partial\nu}$ is meaningful.

Taking the Laplace transform of the observation $u(b,t)$ with respect to t, and recalling (3.9) and (3.10), we have

$$\frac{\partial U(b,s)}{\partial\nu} = \frac{s+1}{s^\alpha + \lambda_1 k + G(s)}\frac{\partial\varphi_1(b)}{\partial\nu}.$$

Therefore,

$$\alpha = \lim_{s\to\infty} \frac{\ln\left(\dfrac{(s+1)\frac{\partial\varphi_1(b)}{\partial\nu}}{\frac{\partial U(b,s)}{\partial\nu}}\right)}{\ln s}, \tag{4.4}$$

$$k = \lim_{s\to\infty} \frac{1}{\lambda_1}\left[\frac{(s+1)\frac{\partial\varphi_1(b)}{\partial\nu}}{\frac{\partial U(b,s)}{\partial\nu}} - s^\alpha\right], \tag{4.5}$$

and

$$G(s) = \frac{(s+1)\frac{\partial\varphi_1(b)}{\partial\nu}}{\frac{\partial U(b,s)}{\partial\nu}} - s^\alpha - k\lambda_1, \quad \operatorname{Re} s > 0. \tag{4.6}$$

Then we obtain

Theorem 10 *Let $\partial\Omega \in C^{[\frac{d}{2}]+2}$. Taking $f(x) = h(x) = \varphi_1(x)$ then using one observation $\frac{\partial u(b,t)}{\partial\nu}$ at a single point $b \in \partial\Omega$ we can reconstruct uniquely the fractional order α by (4.4), the parameter k by (4.5), and the memory function g by taking the Laplace inverse of $G(s)$ from (4.6).*

In the case of the inverse problem for Problem (II), if $\partial\Omega \in C^{\left[\frac{d}{2}\right]+2}$ and $b \in \partial\Omega$, then from the observation $\frac{\partial u(b,t)}{\partial\nu}$ of Problem (II) we can find

$$\alpha = -\lim_{s\to\infty} \frac{\ln\left(\dfrac{(s+1)\frac{\partial\varphi_1(b)}{\partial\nu}}{s^2\,\frac{\partial U(b,s)}{\partial\nu}} - 1\right)}{\ln s},$$

$$k = \lim_{s\to\infty} \frac{s^\alpha}{\lambda_1}\left[\frac{(s+1)\frac{\partial\varphi_1(b)}{\partial\nu}}{s^2\,\frac{\partial U(b,s)}{\partial\nu}} - 1\right],$$

and

$$G(s) = s^\alpha\left[\frac{(s+1)\frac{\partial\varphi_1(b)}{\partial\nu}}{s^2\,\frac{\partial U(b,s)}{\partial\nu}} - 1\right] - k\lambda_1, \quad \operatorname{Re} s > 0, \quad g(t) = (\mathcal{L}^{-1}G)(t).$$

Acknowledgements The first three authors would like to thank the Vietnam Institute for Advanced Study in Mathematics VIASM for support during their visit to VIASM in Summer 2022. The work of the last three authors was supported in part by the research project 13Mat1276-02 under Kingdom of Saudi Arabia National Science, Technology and Innovation Plan.

Conflict of interest

The authors declare that they have no conflict of interest.

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