ON THE HANG–YANG CONJECTURE FOR GJMS EQUATIONS ON $\mathbb{S}^n$

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ABSTRACT. This work concerns a Liouville type result for positive, smooth solution v to the following higher-order equation

$$\mathbf{P}_n^{2m}(v) = Q_n^{2m}(\varepsilon v + v^{-\alpha})$$

on $\mathbb{S}^n$ with $m \geq 2$, $3 \leq n < 2m$, $0 < \alpha \leq (2m+n)/(2m-n)$, and $\varepsilon > 0$. Here $\mathbf{P}_n^{2m}$ is the GJMS operator of order $2m$ on $\mathbb{S}^n$ and $Q_n^{2m} = \mathbf{P}_n^{2m}(1)$ is constant. We show that if $\varepsilon > 0$ is small and $0 < \alpha \leq (2m+n)/(2m-n)$, then any positive, smooth solution v to the above equation must be constant. The same result remains valid if $\varepsilon = 0$ and $0 < \alpha < (2m+n)/(2m-n)$. In the special case $n = 3$, $m = 2$, and $\alpha = 7$, such Liouville type result was recently conjectured by F. Hang and P. Yang (*Int. Math. Res. Not. IMRN*, 2020). As a by-product, we obtain the sharp (subcritical and critical) Sobolev inequalities

$$\Big(\int_{\mathbb{S}^n} v^{1-\alpha} d\mu_{\mathbb{S}^n}\Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} v\, \mathbf{P}_n^{2m}(v) d\mu_{\mathbb{S}^n} \geq \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)} |\mathbb{S}^n|^{\frac{\alpha+1}{\alpha-1}}$$

for the GJMS operator $\mathbf{P}_n^{2m}$ on $\mathbb{S}^n$ under the conditions $n \geq 3$, $n = 2m-1$, and $\alpha \in (0,1) \cup (1, 2n+1]$. A log-Sobolev type inequality, as the limiting case $\alpha = 1$, is also presented.

1. INTRODUCTION

Let $n \geq 3$ be an odd integer, $2m > n$, and $0 < \alpha \leq (n+2m)/(2m-n)$. In this work, we consider the following equation

$$\mathbf{P}_n^{2m}(v) = Q_n^{2m}(\varepsilon v + v^{-\alpha}) \quad \text{in } \mathbb{S}^n. \tag{1.1$_\varepsilon$}$$

Here $\mathbf{P}_n^{2m}$ is the well-known GJMS operator on $\mathbb{S}^n$ equipped with the standard metric $g_{\mathbb{S}^n}$, which is given as follows

$$\mathbf{P}_n^{2m} := \prod_{i=0}^{m-1} \Big(-\Delta_{g_{\mathbb{S}^n}} - (i+\frac{n}{2})(i-\frac{n}{2}+1)\Big),$$

see [GJMS92], and

$$Q_n^{2m} := \mathbf{P}_n^{2m}(1) = \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)}$$

is a non-zero constant representing the so-called Q-curvature of $(\mathbb{S}^n, g_{\mathbb{S}^n})$, perhaps up to a constant multiple. A special case of the operator $\mathbf{P}_n^{2m}$, which has often been studied over the last two decades, is the well-known Paneitz operator, which is of fourth order. This example of a higher-order conformal operator gains interest because of its role in conformal geometry; see [CGY02, HY16]. On $(\mathbb{S}^3, g_{\mathbb{S}^3})$, the Paneitz operator is given by

$$\mathbf{P}_3^4 = \Delta_{g_{\mathbb{S}^3}}^2 + \frac{1}{2}\Delta_{g_{\mathbb{S}^3}} - \frac{15}{16},$$

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and therefore $Q_3^4 = \Gamma(7/2)/\Gamma(-1/2) = -15/16$. Using the above recursive formula for $\mathbf{P}_n^{2m}$ we can compute higher dimensional cases, for example

$$\mathbf{P}_3^6 = -\Delta_{g_{\mathbb{S}^n}}^3 - \frac{23}{4}\Delta_{g_{\mathbb{S}^n}}^2 - \frac{27}{16}\Delta_{g_{\mathbb{S}^n}} + \frac{315}{64} \quad \text{on } (\mathbb{S}^3, g_{\mathbb{S}^3})$$

with $Q_3^6 = 315/64$ and

$$\mathbf{P}_5^6 = -\Delta_{g_{\mathbb{S}^n}}^3 + \frac{13}{4}\Delta_{g_{\mathbb{S}^n}}^2 + \frac{93}{16}\Delta_{g_{\mathbb{S}^n}} - \frac{945}{64} \quad \text{on } (\mathbb{S}^5, g_{\mathbb{S}^5})$$

with $Q_5^6 = -945/64$. One should pay attention on the sign difference of Q_3^6 and Q_5^6.

Our motivation of working on the equation $(1.1)_\varepsilon$ traces back to a recent conjecture by F. Hang and P. Yang in [HY20] that we are going to describe now. This conjecture concerns the following sharp critical Sobolev inequality on $\mathbb{S}^3$

$$\|\phi^{-1}\|_{L^6(\mathbb{S}^3)}^2 \int_{\mathbb{S}^3} \Big[(\Delta_{g_{\mathbb{S}^3}}\phi)^2 - \frac{1}{2}|\nabla_{g_{\mathbb{S}^3}}\phi|^2 - \frac{15}{16}\phi^2\Big] d\mu_{\mathbb{S}^3} \geq -\frac{15}{16}|\mathbb{S}^3|^{4/3} \tag{1.2}$$

for any $\phi \in H^2(\mathbb{S}^3)$ with $\phi > 0$, which was already proved in [YZ04] by symmetrization argument and in [HY04] by variational argument. Apparently, the inequality (1.2) can be rewritten as follows

$$\|\phi^{-1}\|_{L^6(\mathbb{S}^3)}^2 \int_{\mathbb{S}^3} \phi\, \mathbf{P}_3^4(\phi) d\mu_{\mathbb{S}^3} \geq -\frac{15}{16}|\mathbb{S}^3|^{4/3} \tag{1.3}$$

for any $0 < \phi \in H^2(\mathbb{S}^3)$, because the integral in (1.2) is nothing but $\int_{\mathbb{S}^3} \phi\, \mathbf{P}_3^4(\phi) d\mu_{\mathbb{S}^3}$. In (1.3) and what follows, $|\mathbb{S}^n|$ denotes the surface area of $\mathbb{S}^n$. Besides, by Morrey's theorem, functions in $H^2(\mathbb{S}^3)$ are continuous and therefore the condition $\phi > 0$ is understood in pointwise sense. By direct calculation, one can easily verify that equality in (1.3) occurs if ϕ is any positive constant. This tells us that the Paneitz operator $\mathbf{P}_3^4$ on the standard sphere $\mathbb{S}^3$ is no longer positive; see [XY02] for the assumption on the positivity of the Paneitz operator on closed 3-manifolds.

In an effort to provide a new proof for (1.3) with the sharp constant, the authors in [HY20] propose a new way to prove the above Sobolev inequality by considering the following minimizing problem

$$\inf_{0<\phi\in H^2(\mathbb{S}^3)} \|\phi^{-1}\|_{L^6(\mathbb{S}^3)}^2 \Big[\int_{\mathbb{S}^3} \phi\, \mathbf{P}_3^4(\phi) d\mu_{\mathbb{S}^3} + \varepsilon \int_{\mathbb{S}^3} \phi^2 d\mu_{\mathbb{S}^3}\Big] \tag{1.4}$$

for small $\varepsilon > 0$. Thanks to the small perturbation $\varepsilon\|\phi\|_{L^2(\mathbb{S}^3)}^2$, it is standard and straightforward to verify that the extremal problem (1.4) has a minimizer. Such a minimizer, denoted by v_ε, eventually solves

$$\mathbf{P}_3^4(v_\varepsilon) + \varepsilon v_\varepsilon = -v_\varepsilon^{-7}$$

on $\mathbb{S}^3$, up to a constant. Here is the key observation: if the above equation only admits constant solution for small $\varepsilon > 0$, namely $v_\varepsilon \equiv$ const., then one immediately has

$$\|\phi^{-1}\|_{L^6(\mathbb{S}^3)}^2 \Big[\int_{\mathbb{S}^3} \phi\, \mathbf{P}_3^4(\phi) d\mu_{\mathbb{S}^3} + \varepsilon \int_{\mathbb{S}^3} \phi^2 d\mu_{\mathbb{S}^3}\Big] \geq |\mathbb{S}^3|^{1/3} \Big[\int_{\mathbb{S}^3} \mathbf{P}_3^4(1) d\mu_{\mathbb{S}^3} + \varepsilon|\mathbb{S}^3|\Big]$$

for any $0 < \phi \in H^2(\mathbb{S}^3)$. Having this and as $\mathbf{P}_3^4(1) = Q_3^4 = -15/16$, letting $\varepsilon \searrow 0$ yields (1.3). The novelty of this new approach is that it automatically implies the sharp form of (1.3) with the precise sharp constant.

The above observation leads Hang and Yang to propose the following conjecture.

The Hang–Yang conjecture ([HY20, page 3299]). *Let $\varepsilon > 0$ be a small number. If v is a positive smooth solution to*

$$\mathbf{P}_3^4(v) + \varepsilon v = -v^{-7}$$

on $\mathbb{S}^3$, then v must be a constant function.

In a recent work Zhang [Zha21] provides an affirmative answer to the above conjecture. The idea behind Zhang's proof is first to transfer the differential equation on $\mathbb{S}^3$ to some differential equation on $\mathbf{R}^3$ and then to classify solutions to that equation on $\mathbf{R}^3$. More precisely, let $\pi_N : \mathbb{S}^3 \to \mathbf{R}^3$ be the stereographic projection from the north pole N; see subsection 2.1 below. The pullback $(\pi_N^{-1})^*$ enjoys

$$(\phi^{-1})^*(g_{\mathbb{S}^3}) = \Big(\frac{2}{1+|x|^2}\Big)^2 dx^2$$

and for any smooth solution v on $\mathbb{S}^3$ there holds

$$\mathbf{P}_n^{2m}(v) \circ \pi_N^{-1} = \Big(\frac{2}{1+|x|^2}\Big)^{-7/2} \Delta^2 \Big(\Big(\frac{2}{1+|x|^2}\Big)^{-1/2} v \circ \pi_N^{-1} \Big).$$

(Here and in the sequel, Δ is the usual Laplacian on Euclidean spaces.) Therefore, under the following change of variable

$$u(x) := \Big(\frac{1+|x|^2}{2}\Big)^{1/2} \big(v \circ \pi_N^{-1}\big)(x), \tag{1.5}$$

if v solves $\mathbf{P}_3^4(v) + \varepsilon v = -v^{-7}$ in $\mathbb{S}^3$, then u solves

$$\Delta^2 u(x) = \varepsilon \Big(\frac{2}{1+|x|^2}\Big)^4 u(x) + u^{-7}(x)$$

in $\mathbf{R}^3$. Via a dedicated argument based on the method of moving planes and techniques from potential theory, which are rather involved, it is proved that u is radially symmetric. Finally, with the help of a Kazdan–Warner type identity, the function v must be constant.

Inspired by the work of Zhang described above, we are interested in Hang–Yang's conjecture in higher dimensional cases, namely we want to seek for a suitable Liouville type result for positive, smooth solution to equations involving GJMS operators. This leads us to investigate solutions to $(1.1)_\varepsilon$. Very similar to situation studied by Hang and Yang, our motivation to study the equation $(1.1)_\varepsilon$ comes from the higher-order sharp critical Sobolev inequality; see Theorem 1.2 below. Using the perturbation approach introduced in [HY20], we are forced to establish a Liouville type result for solutions to $(1.1)_\varepsilon$.

Toward a suitable Liouville type result, let us first describe some preliminary results on $(1.1)_\varepsilon$. Our first observation concerns the admissible range for ε. As the perturbation approach is being used, we require the condition $\varepsilon \geq 0$; see the proof of Lemma 5.1. Now, by integrating both sides of $(1.1)_\varepsilon$ over $\mathbb{S}^n$ and as $Q_n^{2m} \neq 0$ we conclude that

$$(1-\varepsilon)\int_{\mathbb{S}^n} v \, d\mu_{\mathbb{S}^n} = \int_{\mathbb{S}^n} v^{-\alpha} d\mu_{\mathbb{S}^n}.$$

This immediately tells us that $\varepsilon < 1$. Thus, the admissible range for ε is $0 \leq \varepsilon < 1$. Having this, let us now state the main result of this paper.

Theorem 1.1. *Let $n \geq 3$ be odd and $m > n/2$. Then there exists $\varepsilon_* \in (0,1)$ such that under one of the following conditions*

(1) *either $\varepsilon \in (0, \varepsilon_*)$ and $0 < \alpha \leq (n+2m)/(2m-n)$*
(2) *or $\varepsilon = 0$ and $0 < \alpha < (n+2m)/(2m-n)$*

any positive, smooth solution to $(1.1)_\varepsilon$ must be constant.

We have the following remarks:

- The above result again confirms the Hang–Yang conjecture for the Paneitz operator on $\mathbb{S}^3$, and generalizes the result of Zhang in the critical setting in higher dimensional cases.

- Theorem 1.1 can be compared with the Liouville type results obtained by Véron and Véron in [VV91, Theorem 6.1] for the Emden equation, see also the work of Gidas and Spruck in [GS81]. Note that the condition $\alpha < (n+2m)/(2m-n)$ is sharp for $\varepsilon = 0$ as the result does not hold if $\alpha = (n+2m)/(2m-n)$. This is because in this limiting case the equation $(1.1)_0$ is conformally invariant; see section 3.
- The threshold ε_* is given in Lemma 4.3.
- Although for any $0 \leq \varepsilon < 1$, equation $(1.1)_\varepsilon$ always admits the trivial solution $v_\varepsilon \equiv (1-\varepsilon)^{-1/(\alpha+1)}$, but it is not clear whether or not the above Liouville type result still holds for $\varepsilon \in [\varepsilon_*, 1)$. This seems to be an interesting open question.

To prove Theorem 1.1, we adopt the strategy used by Zhang. Such strategy can be formulated as the following two main steps: first to transfer $(1.1)_\varepsilon$ in $\mathbb{S}^n$ to the two equations $(1.8)_\varepsilon$ and the corresponding integral equation in $\mathbf{R}^n$, then to study symmetry properties of solutions to these equations for small $\varepsilon > 0$. However, to be able to handle higher-order cases, our approach is significantly different from Zhang. One major reason is that less results is known for the higher-order cases compared to the case $m = 2$. For example, we do not know if the preliminary results of Hang and Yang mentioned in [Zha21, section 2] are available for $m \geq 3$. Because of this difficulty, instead of the differential equation $(1.8)_\varepsilon$, we mainly work on the corresponding integral equation on $\mathbf{R}^n$, and directly prove compactness results and symmetry properties of solutions. As pays off, our analysis is much simpler, and could handle higher-order cases efficiently.

As the operator $\mathbf{P}_n^{2m}$ is conformally covariant, for any smooth function φ on $\mathbb{S}^n$ we have the following identity (π denotes the stereographic projection from $\mathbb{S}^n$ to $\mathbf{R}^n$ with respect to either the north or the south pole)

$$\mathbf{P}_n^{2m}(\varphi) \circ \pi^{-1} = \Big(\frac{2}{1+|x|^2}\Big)^{-\frac{n+2m}{2}} (-\Delta)^m \Big(\Big(\frac{2}{1+|x|^2}\Big)^{\frac{n-2m}{2}} \varphi \circ \pi^{-1} \Big);$$

see e.g. [Han07, Section 2]. Then, similar to (1.5), by setting

$$u(x) := \Big(\frac{2}{1+|x|^2}\Big)^{\frac{n-2m}{2}} \big(v \circ \pi^{-1}\big) \tag{1.6}$$

and

$$F_{\varepsilon,u}(x) := \varepsilon \Big(\frac{2}{1+|x|^2}\Big)^{2m} u(x) + \Big(\frac{2}{1+|x|^2}\Big)^{\frac{n+2m}{2} + \alpha\frac{n-2m}{2}} u(x)^{-\alpha} \tag{1.7}$$

we see that u satisfies

$$(-\Delta)^m u = Q_n^{2m} F_{\varepsilon,u} \quad \text{in } \mathbf{R}^n. \tag*{$(1.8)_\varepsilon$}$$

In view of (1.6), we know that the function u on $\mathbf{R}^n$ has exact growth $|x|^{2m-n}$ at infinity. This additional information allows us to transfer the differential equation $(1.8)_\varepsilon$ into the following integral equation

$$u(x) = \gamma_{2m,n} \int_{\mathbf{R}^n} |x-y|^{2m-n} F_{\varepsilon,u}(y) dy \quad \text{on } \mathbf{R}^n$$

for some constant $\gamma_{2m,n} > 0$; see Theorem 2.2 below. Notice that in general there might be more solutions to $(1.8)_\varepsilon$ than the above integral equation, see e.g. [HW19] and [DN22].

Let us emphasize that transferring to an equivalent integral equation on $\mathbf{R}^n$ also appears in the work of Zhang, but the proof provided in [Zha21] does not seem to work in our case. Similar integral representation in the fractional setting also appears in [FKT22]. In our work, by exploiting some nice structures on $\mathbb{S}^n$ as well as some intriguing properties of the stereographic projection, we offer a completely new argument, which is surprisingly simpler; see section 2.

Having the above integral equation in hand, we use a variant of the method of moving planes in the integral form to show that any positive smooth solution u to the above integral equation with exact growth $|x|^{2m-n}$ at infinity must be radially symmetric. The symmetry of solutions to the integral equation helps us to conclude that the corresponding function v, appeared as in (1.6), must be constant. The strategy we just describe seems to be very simple and straightforward at the first glance, but there are two major difficulties that we want to highlight. First, it is worth emphasizing that the method of moving planes and its variants work well in the case of equations with positive exponents; unfortunately, our equations, both differential and integral forms, have a negative exponent. Second, by analyzing the form of $F_{\varepsilon,u}$ in (1.7), one immediately notices that because of our special choice of perturbation, there are two powers of u, whose exponents have opposite sign. Unless $\varepsilon = 0$, otherwise to run the method of moving planes, one needs to establish certain compactness result for solutions to $(1.1)_\varepsilon$ for suitable small ε, which costs us some energy.

Concerning classification of solutions to $(1.8)_\varepsilon$ with $\varepsilon = 0$ and with the RHS depending only on u, that is equation of the form $(-\Delta)^m u = cu^{-\alpha}$ we refer to [**HW19, Ngo18, Li04**] and the references therein.

Finally, to illustrate our finding on a Liouville type result for solutions to $(1.1)_\varepsilon$, we revisit the sharp critical Sobolev inequality for $\mathbf{P}_n^{2m}$ on $\mathbb{S}^n$ proved in [**Han07**]. In fact, we offer both critical and subcritical inequalities at once.

Theorem 1.2. *Let $n \geq 3$ be an odd integer and $m = (n+1)/2$. Then, for any $\phi \in H^m(\mathbb{S}^n)$ with $\phi > 0$ and any $\alpha \in (0,1) \cup (1, 2n+1]$, we have the following sharp Sobolev inequality*

$$\Big(\int_{\mathbb{S}^n} \phi^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} \phi \, \mathbf{P}_n^{2m}(\phi) d\mu_{\mathbb{S}^n} \geq \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)} |\mathbb{S}^n|^{\frac{\alpha+1}{\alpha-1}}. \tag{1.9}$$

Moreover, the equality occurs if ϕ is any positive constant.

Let us make a few remarks:

- Apparently, by chosing $\alpha = (n+2m)/(2m-n) = 2n+1$, our inequality (1.9) includes the following critical Sobolev inequality

$$\Big(\int_{\mathbb{S}^n} \phi^{-\frac{2n}{2m-n}} d\mu_{\mathbb{S}^n} \Big)^{\frac{2m-n}{n}} \int_{\mathbb{S}^n} \phi \, \mathbf{P}_n^{2m}(\phi) d\mu_{\mathbb{S}^n} \geq \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)} |\mathbb{S}^n|^{\frac{2m}{n}}, \tag{1.10}$$

  which was already proved in [**Han07**], see also [**HY04**] and [**FKT22**].
- Although the condition $n = 2m-1$ is not required in Theorem 1.1, but in our proof of (1.9) we heavily use it as in this case we have the advantage of Q-curvature Q_n^{2m} being negative. In general, the inequality (1.9) is not true for $n < 2m-3$, see e.g. [**FKT22**].
- For $1 < \alpha < (2m+n)/(2m-n)$, one cannot directly derive the subcritical inequality (1.9) from the critical inequality (1.10) by Hölder's inequality in the following way

$$\Big(\int_{\mathbb{S}^n} \phi^{-\frac{2n}{2m-n}} d\mu_{\mathbb{S}^n} \Big)^{\frac{2m-n}{n}} \lesssim \Big(\int_{\mathbb{S}^n} \phi^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}}.$$

  The reason is because of $-2n/(2m-n) < 1-\alpha < 0$. This is one of many analytical differences between problems with positive and negative exponents.

Note that our inequality (1.9) can be rewritten as

$$\Big(\fint_{\mathbb{S}^n} \phi^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \fint_{\mathbb{S}^n} \phi \, \mathbf{P}_n^{2m}(\phi) d\mu_{\mathbb{S}^n} \geq \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)},$$

where $\fint_{\mathbb{S}^n} := |\mathbb{S}^n|^{-1} \int_{\mathbb{S}^n}$ denotes the average. Using this new form one can easily compute the limit as $\alpha \to 1$ to obtain the following corollary.

Corollary 1.3. *Let $n \geq 3$ be an odd integer and $m = (n+1)/2$. Then, for any $\phi \in H^m(\mathbb{S}^n)$ with $\phi > 0$, we have the following sharp Sobolev inequality*

$$\exp\Big(-2 \fint_{\mathbb{S}^n} \log \phi d\mu_{\mathbb{S}^n} \Big) \fint_{\mathbb{S}^n} \phi P_n^{2m}(\phi) d\mu_{\mathbb{S}^n} \geq \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)}. \tag{1.11}$$

Moreover, the equality occurs if ϕ is any positive constant.

Since (1.11) is a direct consequence of (1.9), we omit its proof. Without using averages, (1.11) can be rewritten as follows

$$\exp\Big(-\frac{2}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} \log \phi d\mu_{\mathbb{S}^n} \Big) \int_{\mathbb{S}^n} \phi P_n^{2m}(\phi) d\mu_{\mathbb{S}^n} \geq \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)} |\mathbb{S}^n|.$$

To the best of our knowledge, the above inequality (or the inequality (1.11)) seems to be new.

Our final comment concerns a possible generalization to the fractional setting. Indeed, it seems that part of our argument can be quickly extended to the case of fractional operators of order $2s > n$ instead of GJMS operators of integer order $2m > n$. However, to maintain our work in a reasonable length, we leave this future research.

Before closing this section, let us mention the organization of the paper.

Contents

2. Some auxiliary results

2.1. **Basics of the stereographic projection.** As routine, we denote by π_N and π_S the stereographic projections from the north pole N and from the south pole S of the sphere $\mathbb{S}^n$ respectively. If we denote by (x, x_{n+1}) a general point in $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$, then we have the following expressions for π_N

$$\pi_N(x, x_{n+1}) = \frac{x}{1 - x_{n+1}}, \quad \pi_N^{-1}(x) = \Big(\frac{2x}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1} \Big).$$

Likewise, we also have similar expressions for π_S. But these expressions for π_S can be derived quickly from those for π_N by changing the sign of the last coordinate. In this sense, we arrive at

$$\pi_S(x, x_{n+1}) = \frac{x}{1 + x_{n+1}}, \quad \pi_S^{-1}(x) = \Big(\frac{2x}{|x|^2 + 1}, -\frac{|x|^2 - 1}{|x|^2 + 1} \Big).$$

The following observation plays some role in our analysis.

Lemma 2.1. *There holds*

$$\pi_N^{-1}(x) = \pi_S^{-1}\big(\frac{x}{|x|^2}\big), \quad \pi_S^{-1}(x) = \pi_N^{-1}\big(\frac{x}{|x|^2}\big)$$

in $\mathbf{R}^n \setminus \{0\}$.

Proof. These identities follows from the above expressions for π_N and π_S.

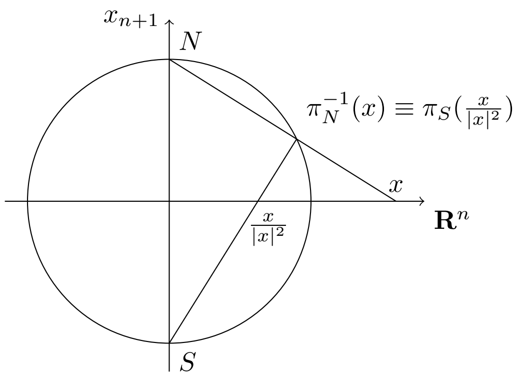

FIGURE 1. Relation between π_N^{-1} and π_S.

We leave the details for interested readers; also see Figure 1 above. □

2.2. **From differential equations to integral equations.** Let v be a positive, smooth solution to (1.1). Recall from $(1.8)_\varepsilon$ that the projected function u, defined by (1.6), solves

$$(-\Delta)^m u = Q_n^{2m} F_{\varepsilon,u} \quad \text{in } \mathbf{R}^n.$$

The main result of this subsection is to show that u actually solves the corresponding integral equation (2.1). To achieve this goal, we need certain preparation including the introduction of a uniform constant that we are going to describe now.

Since n is an odd integer, for some dimensional constant $c_{2m,n} \neq 0$ we have

$$(-\Delta)^m \big(c_{2m,n} |x|^{2m-n} \big) = \delta_0,$$

where δ_0 is the Dirac measure at the origin. For convenience, set

$$\gamma_{2m,n} := c_{2m,n} Q_n^{2m}.$$

For simplicity, throughout the paper, we often denote by C a generic constant whose value could vary from estimate to estimate. We now state our main result in this subsection.

Theorem 2.2. *We have*

$$\gamma_{2m,n} > 0$$

and

$$u(x) = \gamma_{2m,n} \int_{\mathbf{R}^n} |x-y|^{2m-n} F_{\varepsilon,u}(y) dy \tag{2.1}$$

where $F_{\varepsilon,u}$ *is given by* (1.7).

Notice that the integral in (2.1) is well-defined everywhere in $\mathbf{R}^n$. Indeed, as v is positive everywhere on $\mathbb{S}^n$, we have from (1.6) that $u(x) \approx |x|^{2m-n}$ for $|x| \gg 1$, and hence

$$(1+|x|^{2m-n}) F_{\varepsilon,u}(x) \leq \frac{C}{1+|x|^{2n}}. \tag{2.2}$$

In order to prove the above theorem we define the following functions associated with the projections π_N and π_S:

$$u_N(x) := \Big(\frac{1+|x|^2}{2}\Big)^{\frac{2m-n}{2}} (v \circ \pi_N^{-1})(x)$$

and

$$u_S(x) := \Big(\frac{1+|x|^2}{2}\Big)^{\frac{2m-n}{2}} (v \circ \pi_S^{-1})(x)$$

in $\mathbf{R}^n$. In view of the integral equation (2.1), we denote

$$\widetilde{u}_N(x) := \gamma_{2m,n} \int_{\mathbf{R}^n} |x-y|^{2m-n} F_{\varepsilon,u_N}(y) dy$$

and

$$\widetilde{u}_S(x) := \gamma_{2m,n} \int_{\mathbf{R}^n} |x-y|^{2m-n} F_{\varepsilon,u_S}(y) dy$$

in $\mathbf{R}^n$. Our aim is to show that $u_N \equiv \widetilde{u}_N$ and that $\gamma_{2m,n} > 0$. This will be done through several steps. Our first observation is as follows.

Lemma 2.3. *We have*

$$u_S(x) = |x|^{2m-n} u_N\Big(\frac{x}{|x|^2}\Big), \quad u_N(x) = |x|^{2m-n} u_S\Big(\frac{x}{|x|^2}\Big)$$

in $\mathbf{R}^n$.

Proof. This is elementary. Indeed, let us compute u_S. Clearly, with help of Lemma 2.1, we have

$$\begin{aligned} u_S(x) &= \Big(\frac{1+|x|^2}{2}\Big)^{\frac{2m-n}{2}} v\big(\pi_S^{-1}(x)\big) \\ &= \Big(\frac{1+|x|^2}{2}\Big)^{\frac{2m-n}{2}} v\Big(\pi_N^{-1}\Big(\frac{x}{|x|^2}\Big)\Big) \\ &= |x|^{2m-n} \Big(\frac{1+|x/|x|^2|^2}{2}\Big)^{\frac{2m-n}{2}} v\Big(\pi_N^{-1}\Big(\frac{x}{|x|^2}\Big)\Big) \\ &= |x|^{2m-n} u_N\Big(\frac{x}{|x|^2}\Big), \end{aligned}$$

which gives the desired formula for u_S. The identity for u_N can be verified similarly. $\square$

Our next observation is similar to that in Lemma 2.3.

Lemma 2.4. *We have*

$$\widetilde{u}_S(x) = |x|^{2m-n} \widetilde{u}_N\Big(\frac{x}{|x|^2}\Big), \quad \widetilde{u}_N(x) = |x|^{2m-n} \widetilde{u}_S\Big(\frac{x}{|x|^2}\Big)$$

in $\mathbf{R}^n \setminus \{0\}$.

Proof. This is also elementary but rather involved. Indeed, let us verify the first identity. With a change of variable $y = z/|z|^2$ and help of Lemma 2.3 we easily get

$$\begin{aligned}
|x|^{2m-n}\widetilde{u}_N\Big(\frac{x}{|x|^2}\Big) &= \gamma_{2m,n}|x|^{2m-n}\int_{\mathbf{R}^n}\Big|\frac{x}{|x|^2}-y\Big|^{2m-n}F_{\varepsilon,u_N}(y)dy\\
&= \gamma_{2m,n}|x|^{2m-n}\int_{\mathbf{R}^n}\Big|\frac{x}{|x|^2}-\frac{z}{|z|^2}\Big|^{2m-n}F_{\varepsilon,u_N}\Big(\frac{z}{|z|^2}\Big)\frac{dz}{|z|^{2n}}\\
&= \gamma_{2m,n}\int_{\mathbf{R}^n}|x-z|^{2m-n}F_{\varepsilon,u_S}(z)dz\\
&= \widetilde{u}_S(x),
\end{aligned}$$

where in the second last equality we have used the following facts:

$$\Big|\frac{x}{|x|^2}-\frac{z}{|z|^2}\Big| = \frac{|x-z|}{|x||z|},\quad F_{\varepsilon,u_N}\Big(\frac{z}{|z|^2}\Big) = |z|^{2m+n}F_{\varepsilon,u_S}(z).$$

The second identity can be verified similarly. □

Now we are able to examine $u_N - \widetilde{u}_N$ and $u_S - \widetilde{u}_S$.

Lemma 2.5. *The following functions*

$$P_N := u_N - \widetilde{u}_N,\quad P_S := u_S - \widetilde{u}_S$$

are polynomials in $\mathbf{R}^n$ of degree at most $2m-n$.

Proof. Before proving, we see that both P_N and P_S are well-defined everywhere in $\mathbf{R}^n$. Now it follows from (2.2) that the function $\widetilde{u}_N$ satisfies

$$\widetilde{u}_N(x) \le C(1+|x|^{2m-n})\quad \text{for } x\in\mathbf{R}^n.$$

This together with the growth of u_N implies that $|P_N(x)| \le C(1+|x|^{2m-n})$. Since

$$\Delta^m P_N = \Delta^m u_N - \Delta^m \widetilde{u}_N = 0,$$

we conclude that P_N is a polynomial in $\mathbf{R}^n$ of degree at most $2m-n$; see [**Mar09**, Theorem 5]. A similar argument applies to P_S yielding the same conclusion for P_S. □

Finally, we are in a position to prove Theorem 2.2, which simply follows from the next two lemmas.

Lemma 2.6. *There hold $u_N \equiv \widetilde{u}_N$ and $u_S \equiv \widetilde{u}_S$ everywhere.*

Proof. As

$$u_S(x) = |x|^{2m-n}u_N\Big(\frac{x}{|x|^2}\Big),\quad \widetilde{u}_S(x) = |x|^{2m-n}\widetilde{u}_N\Big(\frac{x}{|x|^2}\Big)$$

we obtain

$$P_S(x) = |x|^{2m-n}P_N\Big(\frac{x}{|x|^2}\Big),$$

which is a polynomial (of degree at most $2m-n$). Surely, as n is odd, this is impossible unless $P_N \equiv P_S \equiv 0$, which implies that $u_N \equiv \widetilde{u}_N$ and $u_S \equiv \widetilde{u}_S$. This completes the proof. □

Lemma 2.7. *There hold $\gamma_{2m,n} > 0$.*

Proof. The claim $\gamma_{2m,n} > 0$ follows trivially by seeing the both sides of (2.1) as $v \equiv 1$ is a solution to $(1.1)_0$ and $F_{\varepsilon,u_N} > 0$. More precisely, one has the following identity

$$\Big(\frac{2}{1+|x|^2}\Big)^{\frac{n-2m}{2}} = \gamma_{2m,n} \int_{\mathbf{R}^n} |x-y|^{2m-n} \Big(\frac{2}{1+|y|^2}\Big)^{\frac{n+2m}{2}} dy$$

everywhere in $\mathbf{R}^n$. □

We conclude this subsection by noting that our approach to prove Theorem 2.2 can be used for the case of equations with positive exponent. For example, without using any super polyharmonic property, as in [**CLS22**], our new approach offers a very simple and straightforward proof to convert differential equations on $\mathbb{S}^n$ to the corresponding integral equations on $\mathbf{R}^n$, detail will appear elsewhere.

2.3. **Pohozaev-type identity.** Our last auxiliary result is a Pohozaev-type identity, which shall be used in the proof of a compactness type result; see section 3 below. For simplicity, we let

$$c_\alpha := \alpha \frac{2m-n}{2} - \frac{2m+n}{2} \leq 0. \tag{2.3}$$

Lemma 2.8. *Let $Q \in C^1(\mathbf{R}^n)$ be such that*

$$|Q(x)| \lesssim (1+|x|)^{-n+(\alpha-1)(2m-n)-\delta},$$

for some $\delta > 0$. Let $u \gtrsim (1+|x|)^{2m-n}$ be a regular solution to

$$u(x) = \int_{\mathbf{R}^n} |x-y|^{2m-n} Q(y) u^{-\alpha}(y) dy. \tag{2.4}$$

Then, for $\alpha \neq 1$, there holds

$$\int_{\mathbf{R}^n} (x\cdot\nabla Q) u^{1-\alpha} dx = c_\alpha \int_{\mathbf{R}^n} Q u^{1-\alpha} dx,$$

provided $(x \cdot \nabla Q)u^{1-\alpha} \in L^1(\mathbf{R}^n)$.

Proof. The proof given below is more or less standard. As $x = (1/2)(x+y+x-y)$ and

$$\nabla_x(|x-y|^{2m-n}) = (2m-n)|x-y|^{2m-n-2}(x-y),$$

by differentiating under the integral sign in (2.4), we obtain

$$x\cdot\nabla u(x) = \frac{2m-n}{2} u(x) + \frac{2m-n}{2}\int_{\mathbf{R}^3} \frac{|x|^2-|y|^2}{|x-y|^{n+2-2m}} Q(y)u^{-\alpha}(y)dy.$$

Multiplying the above identity by $Q(x)u^{-\alpha}(x)$, and then integrating the resultant on B_R we arrive at

$$\begin{aligned} \frac{1}{1-\alpha}\int_{B_R} Q(x\cdot\nabla u^{1-\alpha})dx = {} & \frac{2m-n}{2}\int_{B_R} Qu^{1-\alpha} \\ & + \frac{2m-n}{2}\int_{B_R} Q(x)u^{-\alpha}(x)\Big(\int_{\mathbf{R}^n} \frac{|x|^2-|y|^2}{|x-y|^{n+2-2m}} Q(y)u^{-\alpha}(y)dy\Big)dx. \end{aligned}$$

Integration by parts leads to

$$\int_{B_R} Q(x\cdot\nabla u^{1-\alpha})dx = -\int_{B_R} (x\cdot\nabla Q)u^{1-\alpha}dx - n\int_{B_R} Qu^{1-\alpha}dx + R\int_{\partial B_R} Qu^{1-\alpha}dx.$$

Hence,

$$\frac{R}{1-\alpha}\int_{\partial B_R} Qu^{1-\alpha}d\sigma - \frac{2m-n}{2}\int_{B_R}\int_{\mathbf{R}^n}\frac{|x|^2-|y|^2}{|x-y|^{n+2-2m}}Q(y)u^{-\alpha}(y)Q(x)u^{-\alpha}(x)dydx$$
$$= \frac{1}{1-\alpha}\Big[\frac{(2m+n)-\alpha(2m-n)}{2}\int_{B_R} Qu^{1-\alpha}dx + \int_{B_R}(x\cdot\nabla Q)u^{1-\alpha}dx\Big]. \tag{2.5}$$

Thanks to the decay assumption on Q, we easily get

$$\lim_{R\to\infty}\Big(R\int_{\partial B_R} Qu^{1-\alpha}d\sigma\Big) = 0,$$

and clearly

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|x|^2-|y|^2}{|x-y|^{n+2-2m}}Q(y)u^{-\alpha}(y)Q(x)u^{-\alpha}(x)dydx = 0$$

due to the antisymmetry of the integrand. Hence, by sending $R\to\infty$, we conclude that the LHS of (2.5) vanishes, giving the desired identity. This completes the proof. □

3. Compactness results

This section is devoted to a compactness type result for solutions to $(1.1)_\varepsilon$, which is of interest itself; see Theorem 3.1 below. Heuristically, one should study the compactness result for fixed ε and α. However, to derive useful estimates for our analysis, one needs certain compactness result which is independent of ε; see the proof of Lemmas 4.2 and 4.3 below.

Theorem 3.1. *Let $\varepsilon^* \in (0,1)$ and $\alpha \in (0,(2m+n)/(2m-n)]$ be arbitrary but fixed. Assume that $v_k = v_{\varepsilon_k}$ is a sequence of positive regular solutions to $(1.1)_{\varepsilon_k}$ for some $\varepsilon_k \in (0,\varepsilon^*)$. Then there exists $C = C(\varepsilon^*) > 0$ such that*

$$\frac{1}{C} \le v_k \le C \quad \text{in } \mathbb{S}^n$$

for all k. The same conclusion holds true for $\varepsilon_k \in [0,\varepsilon^*)$ if $\alpha \in (0,(2m+n)/(2m-n))$.

It is worth noting that the above compactness fails for solutions to $(1.1)_0$ in the case $\alpha = (n+2m)/(2m-n)$ due to the conformally invariant property of the underlying equation. More specifically, fixing any solution v to

$$\mathbf{P}_n^{2m}(v) = Q_n^{2m} v^{\frac{n+2m}{2m-n}} \quad \text{in } \mathbb{S}^n$$

and let

$$v_\phi = (v\circ\phi)|\det(d\phi)|^{-\frac{1}{2n}},$$

where ϕ is any conformal transformation on $\mathbb{S}^n$. Then, it is well-known that v_ϕ solves the same equation in $\mathbb{S}^n$. Hence, if one choose a sequence of ϕ in such a way that $|\det(d\phi)| \searrow 0$, then the sequence v_ϕ is unbounded in $\mathbb{S}^n$.

In order to prove the above theorem we first need to rule out the possibility that the sequence v_k will eventually touch zero. This in particular implies the lower estimate in the theorem.

Lemma 3.2. *Under the hypothesis of Theorem 3.1 above, we have*

$$\inf_{k\ge 1}\min_{\mathbb{S}^n} v_k > 0.$$

Proof. We assume by contradiction that the lemma is false. Then, up to a subsequence, we assume that

$$\min_{\mathbb{S}^3} v_k \to 0 \quad \text{as } k \to \infty.$$

Without loss of generality we can further assume that the minimum of v_k is attained at the south pole. Let u_k be defined by (1.6) using π_N, and let $F_k := F_{\varepsilon_k, u_k}$ as in (1.7). In view of (1.6) and $2m > n$, the function u_k achieves its minimum at 0. By Theorem 2.2, the function u_k satisfies

$$u_k(x) = \gamma_{2m,n} \int_{\mathbf{R}^n} |x-y|^{2m-n} F_k(y) dy. \tag{3.1}$$

To show that this is also not the case, we use the Pohozaev-type result in Lemma 2.8 and the role played by ε_k and α. Indeed, as $F_k > 0$ we first obtain

$$u_k(0) = \gamma_{2m,n} \int_{\mathbf{R}^n} |y|^{2m-n} F_k(y) dy = o(1)_{k\to\infty}. \tag{3.2}$$

Using this one can show that

$$\lim_{k\to\infty} u_k(x) = \infty \quad \text{for each } x \in \mathbf{R}^n \setminus \{0\}. \tag{3.3}$$

Indeed, by way of contradiction suppose that there is some $x_0 \in \mathbf{R}^n \setminus \{0\}$ such that $u_k(x_0) = O(1)_{k\to\infty}$. As

$$\begin{aligned} \frac{u_k(x_0)}{\gamma_{2m,n}} &= \int_{\mathbf{R}^n} |x_0 - y|^{2m-n} F_k(y) dy \\ &\geq 2^{-2m+n+1} \int_{\mathbf{R}^n} |x_0|^{2m-n} F_k(y) dy - \int_{\mathbf{R}^n} |y|^{2m-n} F_k(y) dy \end{aligned}$$

we obtain

$$\int_{\mathbf{R}^n} F_k(y) dy = O(1)_{k\to\infty},$$

thanks to $u_k(0) = O(1)_{k\to\infty}$. Hence

$$\int_{\mathbf{R}^n} (1 + |y|^{2m-n}) F_k(y) dy = O(1)_{k\to\infty}. \tag{3.4}$$

Consequently, for any $x \in \mathbf{R}^n$, one can estimate

$$\frac{u_k(x)}{\gamma_{2m,n}} = \int_{\mathbf{R}^n} |x-y|^{2m-n} F_k(y) dy \leq 2^{2m-n-1} \int_{\mathbf{R}^n} (|x|^{2m-n} + |y|^{2m-n}) F_k(y) dy,$$

which leads to

$$u_k(x) \leq C(1 + |x|^{2m-n}) \quad \text{in } \mathbf{R}^n$$

for some constant $C > 0$. Having this, one can bound F_k from below near the origin. For example, for any $x \in B_2$, we easily get

$$F_k(x) \geq \Big(\frac{2}{1+|x|^2}\Big)^{-c_\alpha} u_k(x)^{-\alpha} \geq \frac{1}{C^\alpha} \Big(\frac{2}{1+|x|^2}\Big)^{\frac{n+2m}{2}} \geq \frac{1}{C^\alpha} \Big(\frac{2}{5}\Big)^{\frac{n+2m}{2}},$$

thanks to $u_k(x) \leq C(1+|x|^2)^{(2m-n)/2}$ in $\mathbf{R}^n$. However, this violates the fact that $u_k(0) = o(1)_{k\to\infty}$. Indeed,

$$\frac{u_k(0)}{\gamma_{2m,n}} \geq \int_{B_2 \setminus B_1} |y|^{2m-n} F_k(y) dy \geq \frac{1}{C^\alpha} \Big(\frac{2}{5}\Big)^{\frac{n+2m}{2}} \int_{B_2 \setminus B_1} |y|^{2m-n} dy > 0$$

for all k. Thus, no such a point x_0 could exist, hence (3.3) must hold. Notice that the above proof also reveals the fact that

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} F_k(y) dy = \infty, \tag{3.5}$$

otherwise by (3.2) one would again have (3.4) and again this leads to a contradiction. Now we normalize u_k and F_k as follows

$$\widetilde{u}_k := \frac{u_k}{\gamma_{2m,n}\int_{\mathbf{R}^n} F_k dy}, \quad \widetilde{F}_k := \frac{F_k}{\int_{\mathbf{R}^n} F_k dy}.$$

Then

$$\widetilde{u}_k(x) = \int_{\mathbf{R}^n} |x-y|^{2m-n}\widetilde{F}_k(y)dy, \quad \int_{\mathbf{R}^n} \widetilde{F}_k dy = 1.$$

Having (3.5), it is clear that $\widetilde{u}_k(0) \to 0$ and

$$|\nabla \widetilde{u}_k(x)| \le (2m-n)\int_{\mathbf{R}^n} |x-y|^{2m-n-1}\widetilde{F}_k(y)dy \le C(1+|x|^{2m-n-1}) \quad \text{in } \mathbf{R}^n.$$

Notice that because of (3.5) for large k there holds $\widetilde{F}_k(x) \le F_k(x)$ everywhere. This and (3.2) now implies the following

$$\lim_{k\to\infty} \int_{\mathbf{R}^n\setminus B_\delta} \widetilde{F}_k(y)dy \to 0 \quad \text{for any fixed } \delta > 0.$$

Once we have the above limit in hand and seeing $\widetilde{u}$ as a convolution, by standard argument, we get that

$$\widetilde{u}_k \to \widetilde{u} := |x|^{2m-n} \quad \text{in } C^0_{\mathrm{loc}}(\mathbf{R}^n) \tag{3.6}$$

and at the same time

$$\frac{1}{C}|x|^{2m-n} \le \widetilde{u}_k \le C|x|^{2m-n} \quad \text{in } \mathbf{R}^n \setminus B_1 \tag{3.7}$$

for some $C > 0$. Notice that we can write F_k as

$$F_k = \left(\varepsilon_k f^{2m} u_k^{1+\alpha} + f^{-c_\alpha}\right) u_k^{-\alpha} =: Q_k u_k^{-\alpha},$$

where we denote

$$f(x) := \frac{2}{1+|x|^2}.$$

By the Pohozaev-type identity in Lemma 2.8, we get

$$\int_{\mathbf{R}^n} (x\cdot\nabla Q_k)u_k^{1-\alpha}dx = c_\alpha \int_{\mathbf{R}^n} Q_k u_k^{1-\alpha}dx. \tag{3.8}$$

(Here, the multiplicative constant $\gamma_{2m,n} \neq 0$ cancels out from the both sides, thanks to Theorem 2.2.) Let us first compute

$$\nabla\left(\varepsilon_k f^{2m} u_k^{1+\alpha}\right) = 2m\varepsilon_k f^{2m-1}u_k^{1+\alpha}\nabla f + \frac{1+\alpha}{2}\varepsilon_k f^{2m}u_k^{\alpha-1}\nabla u_k^2$$

and

$$\nabla(f^{-c_\alpha}) = -c_\alpha f^{-c_\alpha-1}\nabla f,$$

leading us to

$$x\cdot\nabla Q_k = \Big[\left(2m\varepsilon_k f^{2m-1}u_k^2 - c_\alpha f^{-c_\alpha-1}u_k^{1-\alpha}\right)(x\cdot\nabla f) + \frac{1+\alpha}{2}\varepsilon_k f^{2m}(x\cdot\nabla u_k^2)\Big]u_k^{\alpha-1}.$$

Therefore, from (3.8) we get

$$\begin{aligned} c_\alpha \int_{\mathbf{R}^n} \left[\varepsilon_k f^{2m}u_k^2 + f^{-c_\alpha}u_k^{1-\alpha}\right]dx &= \int_{\mathbf{R}^n} \left[2m\varepsilon_k f^{2m-1}u_k^2 - c_\alpha f^{-c_\alpha-1}u_k^{1-\alpha}\right](x\cdot\nabla f)dx \\ &\quad + \frac{1+\alpha}{2}\varepsilon_k \int_{\mathbf{R}^n} f^{2m}(x\cdot\nabla u_k^2)dx \\ &= \int_{\mathbf{R}^n} m\varepsilon_k(1-\alpha)f^{2m-1}u_k^2(x\cdot\nabla f)dx \\ &\quad + \int_{\mathbf{R}^n} \varepsilon_k \frac{1+\alpha}{2}u_k^2(x\cdot\nabla f^{2m})dx \end{aligned}$$

$$- c_\alpha \int_{\mathbf{R}^n} f^{-c_\alpha - 1} u_k^{1-\alpha} (x \cdot \nabla f) dx$$
$$+ \frac{1+\alpha}{2} \varepsilon_k \int_{\mathbf{R}^n} f^{2m} (x \cdot \nabla u_k^2) dx.$$

By integration by parts, we note that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \big[u_k^2 (x \cdot \nabla f^{2m}) + f^{2m} (x \cdot \nabla u_k^2)\big] dx \\
&= \lim_{R \to \infty} \sum_{i=1}^n \Big[\int_{B_R} \big[- u_k^2 f^{2m} \big] dx + \frac{1}{R} \int_{\partial B_R} x_i^2 f^{2m} u_k^2 d\sigma \Big] \\
&= \lim_{R \to \infty} \Big[-n \int_{B_R} u_k^2 f^{2m} dx + R \int_{\partial B_R} f^{2m} u_k^2 d\sigma \Big] \\
&= -n \int_{\mathbf{R}^n} f^{2m} u_k^2 dx.
\end{aligned}$$

Putting the above estimates together we arrive at

$$\begin{aligned}
\varepsilon_k \int_{\mathbf{R}^n} f^{2m-1} u_k^2 \Big[m(1-\alpha)(x \cdot \nabla f) - \Big(\frac{n(1+\alpha)}{2} + c_\alpha \Big) f \Big] dx& \\
= c_\alpha \int_{\mathbf{R}^n} f^{-c_\alpha - 1} u_k^{1-\alpha} \left(x \cdot \nabla f + f \right) dx.&
\end{aligned} \tag{3.9}$$

Since

$$x \cdot \nabla f + f = f \frac{1 - |x|^2}{1 + |x|^2},$$

and

$$m(1-\alpha) + n \frac{1+\alpha}{2} + c_\alpha = 0,$$

the identity (3.9) can be rewritten as

$$\varepsilon_k m (1-\alpha) \int_{\mathbf{R}^n} f^{2m} u_k^2 \frac{1 - |x|^2}{1 + |x|^2} dx = c_\alpha \int_{\mathbf{R}^n} f^{-c_\alpha} u_k^{1-\alpha} \frac{1 - |x|^2}{1 + |x|^2} dx. \tag{3.10}$$

Our next step is to show that for large k, the two integrals in (3.10) are non-zero with different sign.

Estimate of the LHS of (3.10). Concerning the integral on the LHS of (3.10), a simple calculation shows that

$$\begin{aligned}
&\frac{1}{M_k^2} \int_{\mathbf{R}^n} f^{2m}(x) u_k^2(x) \frac{1 - |x|^2}{1 + |x|^2} dx \\
&= \int_{\mathbf{R}^n} \big(\frac{2}{1 + |x|^2} \big)^{2m} \widetilde{u}_k^2(x) \frac{1 - |x|^2}{1 + |x|^2} dx \\
&= \int_{B_1} \big(\frac{2}{1 + |x|^2} \big)^{2m} \frac{1 - |x|^2}{1 + |x|^2} \Big(\widetilde{u}_k^2(x) - |x|^{4m-2n} \widetilde{u}_k^2 \big(\frac{x}{|x|^2} \big) \Big) dx,
\end{aligned}$$

here we have converted the integral on $\mathbf{R}^n \setminus B_1$ into B_1 using Kelvin's transformation. In $B_1 \setminus \{0\}$, it follows from (3.6) and (3.7) that

$$\widetilde{u}_k^2(x) - |x|^{4m-2n} \widetilde{u}_k^2 \big(\frac{x}{|x|^2} \big) \to |x|^{4m-2n} - 1 \leq 0 \quad \text{as } k \to \infty.$$

Notice that

$$\begin{aligned}
&\lim_{k \to \infty} \int_{B_1} \big(\frac{2}{1 + |x|^2} \big)^{2m} \frac{1 - |x|^2}{1 + |x|^2} \Big(\widetilde{u}_k^2(x) - |x|^{4m-2n} \widetilde{u}_k^2 \big(\frac{x}{|x|^2} \big) \Big) dx \\
&= \int_{B_1} \big(\frac{2}{1 + |x|^2} \big)^{2m} \frac{1 - |x|^2}{1 + |x|^2} \left(|x|^{4m-2n} - 1 \right) dx < 0.
\end{aligned}$$

This and $\varepsilon_k > 0$ give the strictly negative of the integral on the LHS of (3.10) for large k.

Estimate of the RHS of (3.10). Reasoning as in the previous step we should have

$$\begin{aligned}\frac{1}{M_k^{1-\alpha}}&\int_{\mathbf{R}^n} f^{-c_\alpha}(x)u_k^{1-\alpha}(x)\frac{1-|x|^2}{1+|x|^2}dx\\&=\int_{B_1}\big(\frac{2}{1+|x|^2}\big)^{-c_\alpha}\frac{1-|x|^2}{1+|x|^2}\Big(\widetilde{u}_k^{1-\alpha}(x)-|x|^{-2c_\alpha-2n}\widetilde{u}_k^{1-\alpha}\big(\frac{x}{|x|^2}\big)\Big)dx.\end{aligned}$$

In B_1, it follows from (3.6) that

$$\widetilde{u}_k^{1-\alpha}(x)-|x|^{-2c_\alpha-2n}\widetilde{u}_k^{1-\alpha}\big(\frac{x}{|x|^2}\big)\to|x|^{(2m-n)(1-\alpha)}-1\geq 0\quad\text{as } k\to\infty.$$

Now observe that for $\alpha > 1$

$$\int_{B_1}\big(\frac{2}{1+|x|^2}\big)^{-c_\alpha}\frac{1-|x|^2}{1+|x|^2}\big(|x|^{(2m-n)(1-\alpha)}-1\big)dx>0,$$

giving the strictly positive of the integral on the RHS of (3.10) for large k (for certain $\alpha > 1$, the preceding integral could be infinity). Now going back to (3.10), we easily obtain a contradiction for $\alpha > 1$. Indeed, if $\varepsilon_k > 0$ and for large k, then as $\varepsilon_k m(1-\alpha) < 0$, the LHS of (3.10) becomes strictly positive. However, as $c_\alpha \leq 0$, the RHS of (3.10) becomes non-positive. This is a contradiction. If $\varepsilon_k \geq 0$ and for large k, then the LHS of (3.10) becomes non-negative. However, as $c_\alpha < 0$, the RHS of (3.10) becomes strictly negative. This is again a contradiction. And this completes our proof of the compactness for $\alpha > 1$.

Finally we consider the case $0 < \alpha \leq 1$. We set

$$\eta_k(x):=\frac{u_k(r_kx)}{u_k(0)},\quad r_k:=u_k(0)^{\frac{1+\alpha}{2m}}\to 0.$$

Then η_k satisfies $\eta_k \geq \eta_k(0) = 1$, and

$$\eta_k(x)=\gamma_{2m,n}\int_{\mathbf{R}^n}|x-y|^{2m-n}\left(\varepsilon_k r_k^{2m}f^{2m}(r_ky)\eta_k(y)+\frac{f^{-c_\alpha}(r_ky)}{\eta_k^\alpha(y)}\right)dy.\tag{3.11}$$

Then it follows that

$$\int_{\mathbf{R}^n}|y|^{2m-n}\left(\varepsilon_k r_k^{2m}f^{2m}(r_ky)\eta_k(y)+\frac{f^{-c_\alpha}(r_ky)}{\eta_k^\alpha(y)}\right)dy=\frac{\eta_k(0)}{\gamma_{2m,n}}\leq C,\tag{3.12}$$

and together with $\eta_k \geq 1$,

$$\int_{\mathbf{R}^n}\big(1+|y|^{2m-n}\big)\frac{f^{-c_\alpha}(r_ky)}{\eta_k^\alpha(y)}dy\leq C.\tag{3.13}$$

Therefore,

$$\eta_k(x)=\gamma_{2m,n}\varepsilon_k r_k^{2m}\int_{B_1}|x-y|^{2m-n}f^{2m}(r_ky)\eta_k(y)dy+O(1)\quad\text{for } x\in B_1.$$

Integrating the above identity with respect to x in B_1, and using that $f(r_ky) = 2 + o(1)$ on B_1, we obtain

$$\int_{B_1}\eta_k(x)dx=o(1)\int_{B_1}\eta_k(y)dy+O(1),$$

and hence

$$\int_{B_1}\eta_k dx\leq C.$$

Combining the above estimates

$$\int_{\mathbf{R}^n}\big(1+|y|^{2m-n}\big)\left(\varepsilon_k r_k^{2m}f^{2m}(r_ky)\eta_k(y)+\frac{f^{-c_\alpha}(r_ky)}{\eta_k^\alpha(y)}\right)dy\leq C.\tag{3.14}$$

This yields

$$|\nabla \eta_k(x)| \le C(1+|x|^{2m-n-1}), \quad \frac{1}{C}\left(1+|x|^{2m-n}\right) \le \eta_k(x) \le C\left(1+|x|^{2m-n}\right). \tag{3.15}$$

Hence, up to a subsequence,

$$\eta_k \to \eta \quad \text{in } C^0_{loc}(\mathbf{R}^n).$$

From Fatou's lemma, we get that

$$\int_{\mathbf{R}^n} \frac{|y|^{2m-n}}{\eta^\alpha(y)} dy < \infty,$$

thanks to (3.15). Since η satisfies the second estimate in (3.15), we necessarily have that

$$(\alpha-1)(2m-n) > n,$$

a contradiction to $0<\alpha\le 1$. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Since $\varepsilon_k \in [0,\varepsilon^*)$ and $0<\varepsilon^*<1$, integrating (1.1) on $\mathbb{S}^n$ we get that

$$0 \le \int_{\mathbb{S}^n} v_k d\mu_{\mathbb{S}^n} \le \frac{1}{1-\varepsilon^*}\int_{\mathbb{S}^n} v_k^{-\alpha} d\mu_{\mathbb{S}^n} = O(1)_{k\to\infty},$$

thanks to Lemma 3.2. Therefore, we arrive at

$$\mathbf{P}^{2m}_n(v_k) - \varepsilon_k Q^{2m}_n v_k = O(1)_{k\to\infty} \quad \text{in } \mathbb{S}^n$$

with $\|v_k\|_{L^1(\mathbb{S}^n)} = O(1)_{k\to\infty}$. The theorem follows from standard elliptic estimates. □

4. Moving plane arguments and proof of the main result

This section is devoted to the proof of Theorem 1.1. To obtain the symmetry of solutions, our approach is based on the method of moving planes with some new ingredients. The major difficulty is how to handle the negative exponent. As far as we know, although the method of moving planes can be effectively applied to nonlinear equations with positive exponents, see [**CL91, WX99, CLO06, CLS22**] and the references therein, its applications to equations with negative exponents are very rare.

Let us recall some notation and convention often used in the method of moving planes; see Figure 2 below. For $\lambda \in \mathbf{R}$ we set

$$\Sigma_\lambda := \{x\in\mathbf{R}^n : x_1>\lambda\}, \quad T_\lambda := \partial\Sigma_\lambda.$$

Also for any $\lambda\in\mathbf{R}$ we let x^λ be the reflection of $x\in\mathbf{R}^n$ about the plane T_λ, namely

$$x^\lambda := (2\lambda - x_1, x_2, x_3, \dots, x_n).$$

Also for any function f we let f_λ be the reflection of f about the plane T_λ, namely

$$f_\lambda(x) := f(x^\lambda) = f(2\lambda - x_1, x_2, x_3, \dots, x_n).$$

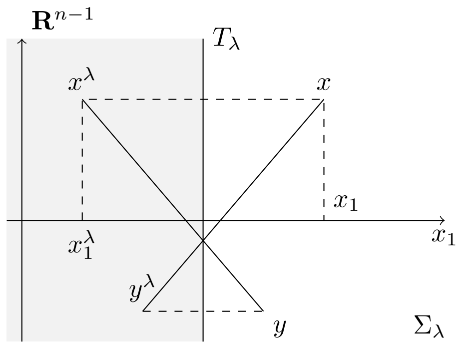

FIGURE 2. Reflection in the method of moving planes

Throughout this section we let $u = u_\varepsilon > 0$ be a (smooth) solution to (2.1) with $F_\varepsilon := F_{\varepsilon,u}$ as in (1.7) for fixed $0 < \varepsilon < \varepsilon^*$. For simplicity, we set

$$w_{\varepsilon,\lambda}(x) := u_\varepsilon(x) - u_\varepsilon(x^\lambda) \quad \text{for all } x \in \mathbf{R}^n.$$

To start moving planes, the following lemma is often required.

Lemma 4.1. *There hold*

$$w_{\varepsilon,\lambda}(x) = \gamma_{2m,n} \int_{\mathbf{R}^n} \big[|x-y|^{2m-n} - |x^\lambda - y|^{2m-n}\big] F_\varepsilon(y)dy \tag{4.1}$$

and

$$w_{\varepsilon,\lambda}(x) = \gamma_{2m,n} \int_{\Sigma_\lambda} \big[|x^\lambda-y|^{2m-n} - |x - y|^{2m-n}\big] [F_\varepsilon(y^\lambda) - F_\varepsilon(y)]dy \tag{4.2}$$

for any $\lambda \in \mathbf{R}$.

Proof. The first identity is obvious from the definition of $w_{\varepsilon,\lambda}$. The second identity follows from variable changes. Indeed, one can write

$$\begin{aligned} u_\varepsilon(x) &= \Big(\int_{\Sigma_\lambda} + \int_{\mathbf{R}^n \setminus \Sigma_\lambda}\Big)|x-y|^{2m-n} F_\varepsilon(y)dy \\ &= \int_{\Sigma_\lambda} |x-y|^{2m-n} F_\varepsilon(y)dy + \int_{\Sigma_\lambda} |x-y^\lambda|^{2m-n} F_\varepsilon(y^\lambda)dy \\ &= \int_{\Sigma_\lambda} |x-y|^{2m-n} F_\varepsilon(y)dy + \int_{\Sigma_\lambda} |x^\lambda-y|^{2m-n} F_\varepsilon(y^\lambda)dy. \end{aligned}$$

Similarly, one has

$$u_\varepsilon(x^\lambda) = \int_{\Sigma_\lambda} |x^\lambda-y|^{2m-n} F_\varepsilon(y)dy + \int_{\Sigma_\lambda} |x-y|^{2m-n} F_\varepsilon(y^\lambda)dy.$$

By putting the above identities together we arrive at the second identity. □

Our next step is to show that the method of moving planes can start from a very large $\lambda_0 > 0$, where λ_0 is independent of ε.

Lemma 4.2. *Let* $\varepsilon^* \in (0,1)$ *be fixed. Then there exists* $\lambda_0 \gg 1$ *such that for every* $\varepsilon \in [0, \varepsilon^*]$ *we have*

$$w_{\varepsilon,\lambda}(x) \geq 0 \quad \textit{in } \Sigma_\lambda$$

for $\lambda \geq \lambda_0$.

Proof. We start the proof by observing the existence of some constant $C > 0$ such that for each $\varepsilon \in [0, \varepsilon^*]$ we have

$$\frac{1}{C}\frac{1}{1+|y|^{2m+n}} \le F_\varepsilon(y) \le C\frac{1}{1+|y|^{2m+n}} \quad \text{in } \mathbf{R}^n; \tag{4.3}$$

see (2.2) for a similar estimate. In the case $\varepsilon > 0$, this simply follows from the uniform bound for v_ε with respect to $\varepsilon \in (0, \varepsilon^*]$ as given by Theorem 3.1. In the case $\varepsilon = 0$, the above estimate is trivial because $u(x) \approx |x|^{2m-n}$ for $|x| \gg 1$. By a simple algebraic computations we have

$$|x-y|^{2m-n} - |x^\lambda - y|^{2m-n} = \frac{|x-y|^2 - |x^\lambda - y|^2}{|x-y|^{2m-n} + |x^\lambda - y|^{2m-n}} \widetilde{P}_\lambda(x,y),$$

where the function $\widetilde{P}_\lambda$ is given as follows

$$\widetilde{P}_\lambda(x,y) := \sum_{k=0}^{2m-n-1} |x-y|^{2(2m-n-1-k)} |x^\lambda - y|^{2k}.$$

(It is clear that $\widetilde{P}_\lambda \equiv 1$ if $2m-n=1$.) Using (4.1) and

$$|x-y|^2 - |x^\lambda - y|^2 = 4(x_1 - \lambda)(\lambda - y_1)$$

we can write

$$|x|^{2+n-2m} \frac{w_{\varepsilon,\lambda}(x)}{x_1 - \lambda} = \int_{\mathbf{R}^n} (\lambda - y_1) P_\lambda(x,y) F_\varepsilon(y) dy =: U_\varepsilon(x),$$

where

$$P_\lambda(x,y) := 4\gamma_{2m,n} \frac{|x|^{2+n-2m}}{|x-y|^{2m-n} + |x^\lambda - y|^{2m-n}} \widetilde{P}_\lambda(x,y). \tag{4.4}$$

For later use, we note that for $x, y \in \Sigma_\lambda$ there holds

$$\begin{aligned} P_\lambda(x,y) &\le C|x|^{2+n-2m} \frac{|x-y|^{2(2m-n-1)} + |x^\lambda - y|^{2(2m-n-1)}}{|x-y|^{2m-n} + |x^\lambda - y|^{2m-n}} \\ &\le C \begin{cases} \dfrac{|x|}{|x-y|} & \text{for } 2m-n=1 \\ 1 + |x|^{2+n-2m}|y|^{2m-n-2} & \text{for } 2m-n \ge 3 \end{cases} \\ &\le C \begin{cases} \dfrac{|x|}{|x-y|} & \text{for } 2m-n=1 \\ |y|^{2m-n-2} & \text{for } 2m-n \ge 3. \end{cases} \end{aligned} \tag{4.5}$$

To conclude the lemma, it suffices to show the existence of $\lambda_0 \gg 1$ such that

$$U_\varepsilon(x) > 0 \quad \text{for any } x \in \Sigma_\lambda \cup T_\lambda$$

for every $\lambda \ge \lambda_0$. With help of (4.3) we can roughly estimate

$$\begin{aligned} U_\varepsilon(x) &= \int_{B_1} (\lambda - y_1) P_\lambda(x,y) F_\varepsilon(y) dy + \int_{\mathbf{R}^n \setminus B_1} (\lambda - y_1) P_\lambda(x,y) F_\varepsilon(y) dy \\ &\ge \frac{1}{C} \int_{B_1} (\lambda - y_1) P_\lambda(x,y) dy + \int_{y_1 > \lambda} (\lambda - y_1) P_\lambda(x,y) F_\varepsilon(y) dy \\ &\ge \frac{1}{C} \int_{B_1} (\lambda - y_1) P_\lambda(x,y) dy - C \int_{y_1 > \lambda} \frac{P_\lambda(x,y)}{1 + |y|^{2m+n-1}} dy \\ &=: I_1(x) - I_2(x). \end{aligned}$$

Here to get the term I_2 we have used the estimates $0 \le y_1 - \lambda \le y_1 \le |y|$ in the region $\{y \in \mathbf{R}^n : y_1 > \lambda\}$ and $|y|/(1+|y|^{2m+n}) \le 2/(1+|y|^{2m+n-1})$ for all y. Next, we estimate I_1 from below and I_2 from above. For I_1, we note that

$$P_\lambda(x,y) \ge \frac{1}{C} \quad \text{for } y \in B_1,\ x \in \Sigma_\lambda,\ \lambda \ge \lambda_0 \gg 1.$$

From this we deduce

$$I_1(x) \ge \frac{\lambda}{C}.$$

We now estimate I_2. For $2m-n \ge 3$ and as $|y|^{2m-n-2}/(1+|y|^{2m+n-1}) \le 2/(1+|y|^{2n+1})$ and $|y| \ge y_1 > \lambda$ we can estimate

$$I_2(x) \le C\int_{y_1>\lambda} \frac{|y|^{2m-n-2}dy}{1+|y|^{2m+n-1}} \le C\int_{y_1>\lambda} \frac{dy}{1+|y|^{2n+1}} \le \frac{C}{\lambda^{n+1}} \le C.$$

For $2m-n=1$, we split $\{y_1>\lambda\}$ as follows

$$\{y_1>\lambda\} \subset A_1 \cup A_2 \cup A_3$$

where

$$A_1 := \{y : \lambda < |y| \le |x|/2\}, \quad A_2 := B_{2|x|} \setminus B_{|x|/2}, \quad A_3 := \mathbf{R}^n \setminus B_{2|x|}.$$

(Although $|x| > \lambda$ as $x \in \Sigma_\lambda$, the set A_1 could be empty if $|x| < 2\lambda$, but it is not important.) Since $|x-y| \ge |x|/2$ on $A_1 \cup A_3$ and again $|y| \ge y_1 > \lambda$, we can estimate

$$\int_{A_1\cup A_3} \frac{|x|}{|x-y|}\frac{dy}{1+|y|^{2m+n-1}} \le \frac{C}{\lambda^{2m-1}}.$$

On the remaining set A_2 as $|x|/2 \le |y| \le 2|x|$ we easily get

$$\int_{A_2} \frac{|x|}{|x-y|}\frac{dy}{1+|y|^{2m+n-1}} \le \frac{C}{|x|^{2m+n-2}}\int_{A_2}\frac{dy}{|x-y|} \le \frac{C}{|x|^{2m-1}} \le \frac{C}{\lambda^{2m-1}} \le C.$$

Putting the above estimate together, we arrive at

$$U_\varepsilon(x) \ge I_1(x) - I_2(x) \ge \frac{\lambda}{C} - C$$

for some constant $C>0$. Thus, the lemma follows by letting λ_0 large enough. $\square$

In Lemma 4.2, we have compared $u_\varepsilon(x)$ and $u_\varepsilon(x^\lambda)$, via $w_{\varepsilon,\lambda}(x)$, in Σ_λ. As there was no restriction on $\varepsilon > 0$, our comparison requires large $\lambda > 0$ to hold. In the next lemma, we compare $F_\varepsilon(x)$ and $F_\varepsilon(x^\lambda)$ in Σ_λ. As there will be no restriction on $\lambda > 0$, our comparison now requires small $\varepsilon > 0$, and this is the place where the constant ε_* appears. Due to the form of F_ε to achieve the goal we need the compactness result established earlier; see section 3.

Lemma 4.3. *There exists $\varepsilon_* \in (0,\varepsilon^*)$ small enough such that for arbitrary $\lambda \in (0,\lambda_0]$ but fixed, the conclusion if*

$$w_{\varepsilon,\lambda} \ge 0 \quad \textit{in } \Sigma_\lambda, \tag{4.6}$$

then

$$F_\varepsilon(x) - F_\varepsilon(x^\lambda) \le 0 \quad \textit{in } \Sigma_\lambda \tag{4.7}$$

holds for each $\varepsilon \in [0,\varepsilon_*)$. In addition, if the inequality (4.6) is strict, then so is the inequality (4.7).

Proof. Let us first be interested in the existence of ε_* and $\varepsilon \in (0,\varepsilon_*)$. As $|x^\lambda| < |x|$ for $\lambda > 0$ and $x \in \Sigma_\lambda$, we obtain

$$F_\varepsilon(x) - F_\varepsilon(x^\lambda) = \varepsilon\big(\frac{2}{1+|x|^2}\big)^{2m}u_\varepsilon(x) - \varepsilon\big(\frac{2}{1+|x^\lambda|^2}\big)^{2m}u_\varepsilon(x^\lambda)$$

$$
\begin{aligned}
&+ \Big(\frac{2}{1+|x|^2}\Big)^{-c_\alpha} \frac{1}{u_\varepsilon^\alpha(x)} - \Big(\frac{2}{1+|x^\lambda|^2}\Big)^{-c_\alpha} \frac{1}{u_\varepsilon^\alpha(x^\lambda)} \\
\le\, & \varepsilon \Big(\frac{2}{1+|x|^2}\Big)^{2m} (u_\varepsilon(x) - u_\varepsilon(x^\lambda)) \\
&+ \Big(\frac{2}{1+|x|^2}\Big)^{-c_\alpha} \Big(\frac{1}{u_\varepsilon^\alpha(x)} - \frac{1}{u_\varepsilon^\alpha(x^\lambda)}\Big),
\end{aligned}
$$

where the constant $c_\alpha \le 0$ is already given in (2.3). Hence, to prove (4.7) in Σ_λ, it suffices to prove that

$$
\frac{u_\varepsilon^\alpha(x) - u_\varepsilon^\alpha(x^\lambda)}{u_\varepsilon(x) - u_\varepsilon(x^\lambda)} \frac{1}{u_\varepsilon^\alpha(x) u_\varepsilon^\alpha(x^\lambda)} \ge \varepsilon \Big(\frac{2}{1+|x|^2}\Big)^{(2m-n)\frac{1+\alpha}{2}} \quad \text{in } \Sigma_\lambda, \tag{4.8}
$$

where we have used that

$$
2m + c_\alpha = (2m-n)\frac{1+\alpha}{2}.
$$

To this end, for some $R \gg 1$ to be specified later, we first split Σ_λ into two parts as follows:

$$
\Sigma_\lambda = \big[\Sigma_\lambda \cap B_R\big] \cup \big[\Sigma_\lambda \setminus B_R\big].
$$

In the region $\Sigma_\lambda \setminus B_R$, there exists some $\varepsilon_1 > 0$ such that (4.8) holds. To see this we need to use uniform bounds with respect to $\varepsilon > 0$, see Theorem 3.1, to obtain

$$
\frac{u_\varepsilon^\alpha(x) - u_\varepsilon^\alpha(x^\lambda)}{u_\varepsilon(x) - u_\varepsilon(x^\lambda)} \ge \varepsilon_1 \Big(\frac{2}{1+|x|^2}\Big)^{(2m-n)\frac{1-\alpha}{2}}
$$

and

$$
\frac{1}{u_\varepsilon^\alpha(x) u_\varepsilon^\alpha(x^\lambda)} \ge \varepsilon_1 \Big(\frac{2}{1+|x|^2}\Big)^{\alpha}
$$

for some small $\varepsilon_1 \in (0,1)$. This is mainly because when R is large enough, we have $|x| \approx |x^\lambda|$ for $|x| > R$ and $\lambda \in (0, \lambda_0]$. In the region $\Sigma_\lambda \cap B_R$, by the smoothness of u_ε, there exists some small $\varepsilon_2 \in (0,1)$ such that

$$
\frac{u_\varepsilon^\alpha(x) - u_\varepsilon^\alpha(x^\lambda)}{u_\varepsilon(x) - u_\varepsilon(x^\lambda)} \frac{1}{u_\varepsilon^\alpha(x) u_\varepsilon^\alpha(x^\lambda)} \ge \varepsilon_2 \Big(\frac{2}{1+|x|^2}\Big)^{(2m-n)\frac{1+\alpha}{2}} \tag{4.9}
$$

for any $x \in B_R$. Hence, combining (4.8) and (4.9) yields the desired estimate (4.7) with

$$
\varepsilon_* = \frac{1}{2} \min\{\varepsilon_1, \varepsilon_2\}.
$$

Now we consider the remaining case $\varepsilon = 0$. However, this case is trivial because

$$
F_0(x) - F_0(x^\lambda) = \Big(\frac{2}{1+|x|^2}\Big)^{-c_\alpha} \Big(\frac{1}{u_0^\alpha(x)} - \frac{1}{u_0^\alpha(x^\lambda)}\Big) \le 0
$$

whenever $w_{0,\lambda}(x) = u_0(x) - u_0(x^\lambda) \ge 0$. Finally, from the above calculation, it is clear that if the inequality (4.6) is strict, then the inequality (4.7) is also strict. Hence, the lemma is proved. □

Thanks to Lemma 4.2, for each $\varepsilon > 0$ we can set

$$
\overline{\lambda}_\varepsilon := \inf \big\{ \lambda > 0 : w_{\varepsilon,\mu} \ge 0 \;\text{ in }\; \Sigma_\mu \;\text{ for every }\; \mu \ge \lambda \big\}.
$$

Then, still by Lemma 4.2, we necessarily have

$$
0 \le \overline{\lambda}_\varepsilon \le \lambda_0.
$$

By decreasing λ down to zero we eventually show that $\overline{\lambda}_\varepsilon = 0$. This can be done through two steps. First we show that if $\overline{\lambda}_\varepsilon > 0$, then we must have $w_{\varepsilon,\overline{\lambda}_\varepsilon} \equiv 0$ in $\Sigma_{\overline{\lambda}_\varepsilon}$; see Lemma 4.5. Finally, we show that $\overline{\lambda}_\varepsilon = 0$; see Lemma 4.6.

Our next lemma is of importance to achieve the first step as it allows us to move λ to the left.

Lemma 4.4. *Let $\varepsilon \in [0, \varepsilon_*)$ and $\bar\lambda \in (0, \lambda_0]$ be such that*

$$0 \not\equiv w_{\varepsilon,\bar\lambda} \geq 0 \quad \text{in } \Sigma_{\bar\lambda}.$$

Then, there exist $R \gg 1$ and $\delta > 0$ small, both may depend on $w_{\varepsilon,\bar\lambda}$, such that for every $\lambda \in (\bar\lambda - \delta, \bar\lambda)$ we have

$$w_{\varepsilon,\lambda} > 0 \quad \text{in } \Sigma_\lambda \setminus B_R.$$

Proof. Using the representation (4.2) and as in the first part of the proof of Lemma 4.2, we have

$$w_{\varepsilon,\lambda}(x)\frac{|x|^{2+n-2m}}{x_1 - \lambda} = \int_{\Sigma_\lambda} (y_1 - \lambda) P_\lambda(x, y)[F_\varepsilon(y^\lambda) - F_\varepsilon(y)]dy, \tag{4.10}$$

where P_λ is given by (4.4). In view of (4.10), it suffices to show that its RHS is positive in $\Sigma_\lambda \setminus B_R$ for suitable $R > 0$. For convenience, we recall the following formula for P_λ

$$P_\lambda(x, y) = 4\gamma_{2m,n} \frac{|x|^{2+n-2m}}{|x-y|^{2m-n} + |x^\lambda - y|^{2m-n}} \sum_{k=0}^{2m-n-1} |x-y|^{2(2m-n-1-k)} |x^\lambda - y|^{2k}.$$

Hence, there exists some $\theta > 0$ such that for every $R_1 > 0$ fixed

$$P_\lambda(x, y) \rightrightarrows \theta \quad \text{uniformly in } y \in B_{R_1} \tag{4.11}$$

as $|x| \to \infty$. This is because $|x| \approx |x - y| \approx |x^\lambda - y|$ for large $|x|$. From (4.7) we know that

$$0 \not\equiv F_\varepsilon(y^{\bar\lambda}) - F_\varepsilon(y) \geq 0 \quad \text{for } y \in \Sigma_{\bar\lambda},$$

which implies

$$\int_{\Sigma_{\bar\lambda}} (y_1 - \bar\lambda)[F_\varepsilon(y^{\bar\lambda}) - F_\varepsilon(y)]dy \geq 2c_0 > 0,$$

for some small constant $c_0 > 0$. Thus, by the dominated convergence theorem, we can find some $\delta > 0$ such that

$$\int_{\Sigma_\lambda} (y_1 - \lambda)[F_\varepsilon(y^\lambda) - F_\varepsilon(y)]dy \geq c_0 > 0, \tag{4.12}$$

for every $|\lambda - \bar\lambda| < \delta$. To obtain the positivity of the right hand side of (4.10), we split the integral $\int_{\Sigma_\lambda}$ into two parts as follows

$$\int_{\Sigma_\lambda} = \int_{\Sigma_\lambda \setminus B_{R_2}} + \int_{\Sigma_\lambda \cap B_{R_2}}$$

for some $R_2 > 0$ to be determined later and estimate these integrals term by term; see the two estimates (4.14) and (4.15) below. Our aim is to show that the integral $\int_{\Sigma_\lambda \setminus B_{R_2}}$ is negligible.

We assume for a moment that such a constant R_2 exists. We now estimate the integral $\int_{\Sigma_\lambda \setminus B_{R_2}}$. First we initially choose new $R_0 \gg 1$ in such a way that $|y_1 - \lambda| < 2|y|$ for all $|y| \geq R_0$. Then we find some $R_1 \gg R_0$ such that in $\Sigma_\lambda \setminus B_{R_1}$ we have

$$\int_{\Sigma_\lambda \setminus B_{R_1}} \frac{dy}{1 + |y|^{2m+n-1}} \leq \frac{\theta c_0}{16C} \tag{4.13}$$

and

$$F_\varepsilon(y) + F_\varepsilon(y^\lambda) \leq \frac{C}{1 + |y|^{2m+n}}$$

for some $C > 0$ because $|y| \approx |y^\lambda|$. By the estimate (4.5) for P_λ, we now claim that there are some $R_3 \gg 1$ and $R_2 \gg R_1$ such that

$$\int_{\Sigma_\lambda \setminus B_{R_2}} (y_1 - \lambda) P_\lambda(x,y)[F_\varepsilon(y^\lambda) + F_\varepsilon(y)]dy \leq \frac{\theta c_0}{4} \tag{4.14}$$

for $|x| \geq R_3$. To see this, for clarity, we consider the two cases $2m-n=1$ and $2m-n \geq 3$ separately.

Case 1. Suppose $2m - n = 1$. In this case our estimate for P_λ becomes $P_\lambda(x,y) \leq C|x|/|x-y|$. Consequently, there holds

$$\int_{\Sigma_\lambda \setminus B_{R_2}} (y_1 - \lambda) P_\lambda(x,y)[F_\varepsilon(y^\lambda) + F_\varepsilon(y)]dy \leq C \int_{\Sigma_\lambda \setminus B_{R_2}} \frac{|x|}{|x-y|} \frac{|y|}{1+|y|^{2m+n}} dy.$$

For $|x| \geq R_3 \gg 2R_2$ to be determined later, we now split $\int_{\Sigma_\lambda \setminus B_{R_2}}$ as follows

$$\int_{\Sigma_\lambda \setminus B_{R_2}} = \int_{[\Sigma_\lambda \setminus B_{R_2}] \cap [B_{|x|/2} \cup (\mathbf{R}^n \setminus B_{2|x|})]} + \int_{[\Sigma_\lambda \setminus B_{R_2}] \setminus [B_{|x|/2} \cup (\mathbf{R}^n \setminus B_{2|x|})]}.$$

Thanks to (4.13), we get

$$C \int_{[\Sigma_\lambda \setminus B_{R_2}] \cap [B_{|x|/2} \cup (\mathbf{R}^n \setminus B_{2|x|})]} \frac{|x|}{|x-y|} \frac{|y|}{1+|y|^{2m+n}} dy < \frac{\theta c_0}{8}.$$

For the remaining integral on $[\Sigma_\lambda \setminus B_{R_2}] \setminus [B_{|x|/2} \cup (\mathbf{R}^n \setminus B_{2|x|})]$ which is a subset of $B_{2|x|} \setminus B_{|x|/2}$ because $|x| \geq 2R_2$, we estimate as follows

$$C \int_{[\Sigma_\lambda \setminus B_{R_2}] \setminus [B_{|x|/2} \cup (\mathbf{R}^n \setminus B_{2|x|})]} \leq \frac{C|x|^2}{1+|x|^{2m+n}} \int_{B_{2|x|} \setminus B_{|x|/2}} \frac{dy}{|x-y|}.$$

Since the last integral is of order $|x|^n$ and $m \geq 2$ we can find some $R_3 \gg 1$ such that

$$\frac{C|x|^2}{1+|x|^{2m+n}} \int_{B_{2|x|} \setminus B_{|x|/2}} \frac{dy}{|x-y|} \leq \frac{\theta c_0}{8}$$

for all $x \in \Sigma_\lambda \cap B_{R_3}$. Combining the two estimates above gives (4.14). This completes the first case.

Case 2. Suppose $2m - n \geq 3$. This case is easy to handle. Recall that our estimate for P_λ becomes $P_\lambda(x,y) \leq C|y|^{2m-n-2}$. Consequently, there holds

$$\int_{\Sigma_\lambda \setminus B_{R_2}} (y_1 - \lambda) P_\lambda(x,y)[F_\varepsilon(y^\lambda) + F_\varepsilon(y)]dy \leq C \int_{\Sigma_\lambda \setminus B_{R_2}} \frac{|y|^{2m-n-1}}{1+|y|^{2m+n}} dy.$$

Seeing (4.13) or as in the proof of Lemma 4.2, we easily obtain the desired estimate.

Hence, up to this point, we have already shown that there are some $R_2 \gg 1$ and $R_3 \gg 1$ such that the estimate (4.14) holds for $|x| \geq R_3$. Now we estimate the integral $\int_{\Sigma_\lambda \cap B_{R_2}}$. Keep using the constant R_2. By the uniform convergence in (4.11), we can choose $R_4 \gg R_2$ such that

$$P_\lambda(x,y) \geq \frac{1}{2}\theta \quad \text{for } |x| \geq R_4 \text{ and } |y| \leq R_2.$$

This and (4.12) imply that

$$\int_{\Sigma_\lambda \cap B_{R_2}} (y_1 - \lambda) P_\lambda(x,y)[F_\varepsilon(y^\lambda) - F_\varepsilon(y)]dy \geq \frac{\theta c_0}{2} \tag{4.15}$$

for $|x| \geq R_4$. We conclude the lemma by combing the two estimates (4.14) and (4.15) and choosing $R = \max\{R_3, R_4\}$. □

We are now in a position to complete the first step, namely, to show that $\overline{\lambda}_\varepsilon = 0$. To this purpose, we must rule out the case $\overline{\lambda}_\varepsilon > 0$ and this is the content of the next two lemmas. First, we characterize the function $w_{\varepsilon,\overline{\lambda}_\varepsilon}$ in case $\overline{\lambda}_\varepsilon > 0$.

Lemma 4.5. *If $\overline{\lambda}_\varepsilon > 0$ for some $\varepsilon \in [0, \varepsilon_*)$, then $w_{\varepsilon,\overline{\lambda}_\varepsilon} \equiv 0$ in $\Sigma_{\overline{\lambda}_\varepsilon}$.*

Proof. Let $\overline{\lambda}_\varepsilon > 0$ for some $\varepsilon \in [0, \varepsilon_*)$ and assume by contradiction that $w_{\varepsilon,\overline{\lambda}_\varepsilon} \not\equiv 0$ in $\Sigma_{\overline{\lambda}_\varepsilon}$. This and the definition of $\overline{\lambda}_\varepsilon$ imply that

$$0 \not\equiv w_{\varepsilon,\overline{\lambda}_\varepsilon} \geq 0 \quad \text{in } \Sigma_{\overline{\lambda}_\varepsilon}.$$

By Lemma 4.4, there exist $R \gg 1$ and $\delta > 0$ small enough such that

$$w_{\varepsilon,\lambda} > 0 \quad \text{in } \Sigma_\lambda \setminus B_R \quad \text{for every } \lambda \in (\overline{\lambda}_\varepsilon - \delta, \overline{\lambda}_\varepsilon).$$

Take a sequence $(\mu_k)_k$ convergent to $\overline{\lambda}_\varepsilon$ such that $\mu_k \in (\overline{\lambda}_\varepsilon - \delta, \overline{\lambda}_\varepsilon)$. Still by the definition of $\overline{\lambda}_\varepsilon$ and as $\mu_k < \overline{\lambda}_\varepsilon$ we know that w_{ε,μ_k} is negative somewhere in Σ_{μ_k}. Since outside B_R, the function w_{ε,μ_k} is strictly positive, for each k there is some $x_k \in \Sigma_{\mu_k} \cap \overline{B_R}$ such that

$$w_{\varepsilon,\mu_k}(x_k) = \min_{\Sigma_{\mu_k}} w_{\varepsilon,\mu_k} < 0.$$

In particular, there holds

$$\frac{w_{\varepsilon,\mu_k}(x_k)}{(x_k)_1 - \mu_k} < 0.$$

Obviously, the sequence (x_k) is bounded as $x_k \in \overline{B_R}$. Also note that $\Sigma_{\overline{\lambda}_\varepsilon} \subset \Sigma_{\mu_k}$ and $\Sigma_{\mu_k} \searrow \Sigma_{\overline{\lambda}_\varepsilon}$ as $k \nearrow +\infty$. Therefore, up to a subsequence, we have

$$\Sigma_{\overline{\lambda}_\varepsilon} \cup T_{\overline{\lambda}_\varepsilon} \ni x_\infty := \lim_{k \to \infty} x_k.$$

In particular, by passing to the limit as $k \to \infty$, there holds $w_{\varepsilon,\overline{\lambda}_\varepsilon}(x_\infty) \leq 0$. This and (4.10) implies that

$$0 \geq w_{\varepsilon,\overline{\lambda}_\varepsilon}(x_\infty) \frac{|x_\infty|^{2+n-2m}}{(x_\infty)_1 - \overline{\lambda}_\varepsilon} = \int_{\Sigma_{\overline{\lambda}_\varepsilon}} (y_1 - \overline{\lambda}_\varepsilon) P_{\overline{\lambda}_\varepsilon}(x_\infty, y)[F_\varepsilon(y^{\overline{\lambda}_\varepsilon}) - F_\varepsilon(y)]dy \geq 0,$$

thanks to $|x_\infty| > 0$ and $F_\varepsilon(y^{\overline{\lambda}_\varepsilon}) \geq F_\varepsilon(y)$ in $\Sigma_{\overline{\lambda}_\varepsilon}$ by Lemma 4.3. Thus, we must have

$$F_\varepsilon(y^{\overline{\lambda}_\varepsilon}) - F_\varepsilon(y) = 0 \quad \text{for any } y \in \Sigma_{\overline{\lambda}_\varepsilon},$$

which, by (4.10), now yields $w_{\varepsilon,\overline{\lambda}_\varepsilon} \equiv 0$ in $\Sigma_{\overline{\lambda}_\varepsilon}$. However, this is a contradiction. The proof is complete. □

From the characterization of $w_{\varepsilon,\overline{\lambda}_\varepsilon}$ in the case $\overline{\lambda}_\varepsilon > 0$, we are able to show that in fact the case $\overline{\lambda}_\varepsilon > 0$ cannot happen.

Lemma 4.6. *Let $\varepsilon \in [0, \varepsilon_*)$. There holds $\overline{\lambda}_\varepsilon = 0$. In particular, the function u_ε is symmetric with respect to the hyperplane $\{x \in \mathbf{R}^n : x_1 = 0\}$.*

Proof. By way of contradiction, assume that $\overline{\lambda}_\varepsilon > 0$. In view of Lemma 4.5, we must have

$$0 = w_{\varepsilon,\overline{\lambda}_\varepsilon}(x) = u_\varepsilon(x) - u_\varepsilon(x^{\overline{\lambda}_\varepsilon})$$

in $\Sigma_{\overline{\lambda}_\varepsilon}$. This and (4.2) tell us that

$$\int_{\Sigma_{\overline{\lambda}_\varepsilon}} \big[|x^{\overline{\lambda}_\varepsilon} - y|^{2m-n} - |x - y|^{2m-n}\big][F_\varepsilon(y^{\overline{\lambda}_\varepsilon}) - F_\varepsilon(y)]dy = 0$$

for any $x \in \Sigma_{\overline{\lambda}_\varepsilon}$, thanks to $\gamma_{2m,n} \neq 0$, see Theorem 2.2. But this cannot happen because $|x-y| \leq |x^{\overline{\lambda}_\varepsilon} - y|$ for any $x, y \in \Sigma_{\overline{\lambda}_\varepsilon}$ and

$$\begin{aligned} F_\varepsilon(x) - F_\varepsilon(x^{\overline{\lambda}_\varepsilon}) = & \varepsilon \Big[\Big(\frac{2}{1+|x|^2}\Big)^{2m} - \Big(\frac{2}{1+|x^{\overline{\lambda}_\varepsilon}|^2}\Big)^{2m} \Big] u_\varepsilon(x) \\ & + \Big[\Big(\frac{2}{1+|x|^2}\Big)^{-c_\alpha} - \Big(\frac{2}{1+|x^{\overline{\lambda}_\varepsilon}|^2}\Big)^{-c_\alpha} \Big] \frac{1}{u_\varepsilon^\alpha(x)} \\ < & \, 0, \end{aligned}$$

everywhere in $\Sigma_{\overline{\lambda}_\varepsilon}$, thanks to the estimates $u_\varepsilon > 0$, $-c_\alpha \geq 0$, and $|x| \leq |x^{\overline{\lambda}_\varepsilon}|$ in $\Sigma_{\overline{\lambda}_\varepsilon}$. (Here we also use the fact that if $\varepsilon = 0$, then $\alpha < (n+2m)/(2m-n)$ in order to guarantee $-c_\alpha > 0$.) Thus, we must have $\overline{\lambda}_\varepsilon = 0$. In particular, we have from the definition of $\overline{\lambda}_\varepsilon$ the following

$$u_\varepsilon(x_1, x_2, ..., x_n) \geq u_\varepsilon(-x_1, x_2, ..., x_n).$$

We now apply the method of moving planes in the opposite direction, namely $\lambda < 0$, to get

$$u_\varepsilon(x_1, x_2, ..., x_n) \leq u_\varepsilon(-x_1, x_2, ..., x_n).$$

Hence

$$u_\varepsilon(x_1, x_2, ..., x_n) = u_\varepsilon(-x_1, x_2, ..., x_n).$$

This establishes the symmetry of u_ε with respect to the hyperplane $\{x \in \mathbf{R}^n : x_1 = 0\}$. The proof is now complete. □

As a consequence of Lemma 4.6 above, we obtain a Liouville type result for positive, smooth solution to $(1.1)_\varepsilon$ for small $\varepsilon > 0$, hence proving Theorem 1.1.

Lemma 4.7. *Any positive, smooth solution v_ε to $(1.1)_\varepsilon$ for small ε must be constant.*

Proof. Let $\varepsilon \in [0, \varepsilon_*)$ be arbitrary. From Lemma 4.6 we know that the corresponding solution u_ε is symmetric with respect to the hyperplane $\{x \in \mathbf{R}^n : x_1 = 0\}$. This together with the relation

$$u_\varepsilon(x) = \Big(\frac{1+|x|^2}{2}\Big)^{\frac{2m-n}{2}} (v_\varepsilon \circ \pi_N^{-1})(x)$$

tells us that v_ε depends only on the last coordinate x_{n+1}. However, as the x_{n+1}-axis is freely chosen, we conclude that v_ε must be constant. This completes the proof. □

Before closing this section, we have a remark. To obtain the symmetry of solutions to $(1.1)_\varepsilon$ for small ε, our approach is based on the method of moving planes in the integral form. A natural question is weather or not one can use the method of moving spheres; see [**LZ95**, **Li04**]. Due to the presence of the weight $2/(1+|x|^2)$ in (1.7), it is natural to ask whether or not the method of moving spheres can still be used. Toward a possible answer to this question, we refer the reader to the work [**JLX08**].

5. Application to the sharp Sobolev inequality

This section is devoted to a proof of Theorem 1.2 which concerns a sharp (critical or subcritical) Sobolev inequality. Let $\varepsilon \in (0,1)$ and inspired by (1.4) consider the following variational problem

$$\mathcal{S}_\varepsilon = \inf_{0<\phi \in H^m(\mathbb{S}^n)} \Big(\int_{\mathbb{S}^n} \phi^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} \big[\phi \, \mathbf{P}_n^{2m}(\phi) - \varepsilon Q_n^{2m} \phi^2 \big] d\mu_{\mathbb{S}^n} \tag{5.1}$$

with $m = (n+1)/2$ and $\alpha \in (0,1) \cup (1, 2n+1]$. Now as

$$\mathbf{P}_n^{2m}(1) - \varepsilon Q_n^{2m} = (1-\varepsilon) Q_n^{2m} \neq 0$$

by testing (5.1) with constant functions we conclude from (5.1) that

$$\mathcal{S}_\varepsilon \le (1-\varepsilon) Q_n^{2m} |\mathbb{S}^n|^{\frac{\alpha+1}{\alpha-1}} < 0,$$

however, $\mathcal{S}_\varepsilon$ could be $-\infty$. Next we show that $\mathcal{S}_\varepsilon$ is finite and is achieved by some smooth positive function.

Lemma 5.1. *Assume that $m = (n+1)/2$ and $\alpha \in (0,1) \cup (1, 2n+1]$. Then, the constant $\mathcal{S}_\varepsilon$ in (5.1) is finite and there exists some $v_\varepsilon \in C^\infty(\mathbb{S}^n)$ such that $v_\varepsilon > 0$ and*

$$\Big(\int_{\mathbb{S}^n} v_\varepsilon^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} \big[v_\varepsilon \mathbf{P}_n^{2m}(v_\varepsilon) - \varepsilon Q_n^{2m} v_\varepsilon^2 \big] d\mu_{\mathbb{S}^n} = \mathcal{S}_\varepsilon.$$

In particular, v_ε solves

$$\mathbf{P}_n^{2m}(v_\varepsilon) - \varepsilon Q_n^{2m} v_\varepsilon = S_\varepsilon v_\varepsilon^{-\alpha}$$

in $\mathbb{S}^n$ with

$$S_\varepsilon = \frac{\mathcal{S}_\varepsilon}{\|v_\varepsilon^{-1}\|_{L^{\alpha-1}(\mathbb{S}^n)}^{\alpha+1}}.$$

Proof. Let $(v_k)_k$ be a positive, smooth minimizing sequence in $H^{2m}(\mathbb{S}^n)$, that is

$$\Big(\int_{\mathbb{S}^n} v_k^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} \big[v_k \mathbf{P}_n^{2m}(v_k) - \varepsilon Q_n^{2m} v_k^2 \big] d\mu_{\mathbb{S}^n} \searrow \mathcal{S}_\varepsilon$$

as $k \to \infty$. By the scaling invariant we can assume $\max_{\mathbb{S}^n} v_k = 1$ which then yields

$$\|v_k\|_{L^2(\mathbb{S}^n)}^2 \le |\mathbb{S}^n|.$$

By seeing $\mathbf{P}_n^{2m}$ as a polynomial of $-\Delta_{g_{\mathbb{S}^n}}$, whose coefficient of the leading term is positive, it is easy to get that

$$\int_{\mathbb{S}^n} v_k \mathbf{P}_n^{2m}(v_k) d\mu_{\mathbb{S}^n} \ge c_1 \|v_k\|_{H^m(\mathbb{S}^n)}^2 - c_2 \|v_k\|_{L^2(\mathbb{S}^n)}^2 \ge c_1 \|v_k\|_{H^m(\mathbb{S}^n)}^2 - c_2 |\mathbb{S}^n|$$

for some $c_1 > 0$ and $c_2 > 0$. Note that $\mathcal{S}_\varepsilon < 0$ and $Q_n^{2m} < 0$ would imply

$$\int_{\mathbb{S}^n} v_k \mathbf{P}_n^{2m} v_k d\mu_{\mathbb{S}^n} < 0.$$

Therefore, the previous estimate leads to

$$c_1 \|v_k\|_{H^m(\mathbb{S}^n)}^2 \le c_2 |\mathbb{S}^n|,$$

giving the boundedness of the sequence (v_k) in $H^m(\mathbb{S}^n)$. Hence, after passing to a subsequence if necessary, there exists some $v_\varepsilon \in H^m(\mathbb{S}^n)$ such that

$$v_k \to v_\varepsilon \ge 0 \text{ uniformly in } C(\mathbb{S}^n)$$

by Morrey's inequality and the Arzelà–Ascoli lemma, and

$$v_k \rightharpoonup v_\varepsilon \text{ weakly in } H^m(\mathbb{S}^n).$$

In particular, there holds $\max_{\mathbb{S}^n} v_\varepsilon = 1$. As $v_\varepsilon \ge 0$, there are two possibilities. First, let us assume that v_ε vanishes somewhere on $\mathbb{S}^n$. By assuming this we shall obtain a contradiction, therefore we must have $v_\varepsilon > 0$. Indeed, as $n = 2m-1$, we can make use of [**Han07**, Corollary 3.1] to conclude that

$$\int_{\mathbb{S}^n} v_\varepsilon \mathbf{P}_n^{2m}(v_\varepsilon) d\mu_{\mathbb{S}^n} \ge 0.$$

This together with $\varepsilon Q_n^{2m} < 0$ and $\int_{\mathbb{S}^n} v_\varepsilon^2 d\mu_{\mathbb{S}^n} > 0$ help us to get

$$0 < \int_{\mathbb{S}^n} \big[v_\varepsilon \mathbf{P}_n^{2m}(v_\varepsilon) - \varepsilon Q_n^{2m} v_\varepsilon^2 \big] d\mu_{\mathbb{S}^n} \le \liminf_{k \nearrow +\infty} \int_{\mathbb{S}^n} \big[v_k \mathbf{P}_n^{2m}(v_k) - \varepsilon Q_n^{2m} v_k^2 \big] d\mu_{\mathbb{S}^n}.$$

This is a contradiction to $\mathcal{S}_\varepsilon < 0$. Thus, $v_\varepsilon > 0$ everywhere. Then, this allows us to gain

$$v_k^{-1} \to v_\varepsilon^{-1} \quad \text{uniformly in } C(\mathbb{S}^n)$$

and consequently

$$\int_{\mathbb{S}^n} v_k^{1-\alpha} d\mu_{\mathbb{S}^n} \to \int_{\mathbb{S}^n} v_\varepsilon^{1-\alpha} d\mu_{\mathbb{S}^n}.$$

Putting these facts together, we obtain

$$\begin{aligned}
\mathcal{S}_\varepsilon &\leq \Big(\int_{\mathbb{S}^n} v_\varepsilon^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} \big[v_\varepsilon \, \mathbf{P}_n^{2m}(v_\varepsilon) - \varepsilon Q_n^{2m} v_\varepsilon^2 \big] d\mu_{\mathbb{S}^n} \\
&\leq \liminf_{k \nearrow +\infty} \Big[\Big(\int_{\mathbb{S}^n} v_k^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} \big[v_k \, \mathbf{P}_n^{2m}(v_k) - \varepsilon Q_n^{2m} v_k^2 \big] d\mu_{\mathbb{S}^n} \Big] \\
&= \mathcal{S}_\varepsilon.
\end{aligned} \tag{5.2}$$

Hence, on one hand implies that $\mathcal{S}_\varepsilon$ must be finite, on the other hand, yields that v_ε is a minimizer for (5.1). Rest of the proof follows immediately. $\square$

Having Lemma 5.1 in hand, we are able to prove Theorem 1.2 as we shall do now. By seeing our Liouville type result in Theorem 1.1, this is the place we need the smallness of ε.

Proof of Theorem 1.2. Let $\varepsilon > 0$ and $\alpha \in (0,1) \cup (1, 2n+1]$. By Lemma 5.1, there is some positive, smooth function v_ε satisfying

$$\int_{\mathbb{S}^n} v_\varepsilon^{1-\alpha} d\mu_{\mathbb{S}^n} = 1$$

and

$$\int_{\mathbb{S}^n} \big[v_\varepsilon \, \mathbf{P}_n^{2m}(v_\varepsilon) - \varepsilon Q_n^{2m} v_\varepsilon^2 \big] d\mu_{\mathbb{S}^n} = \mathcal{S}_\varepsilon.$$

Then, up to a constant multiple, v_ε solves $(1.1)_\varepsilon$ in $\mathbb{S}^n$. Therefore, for small $\varepsilon > 0$, it follows from Theorem 1.1 that v_ε is constant. Keep in mind that $\alpha \neq 1$. Hence, on one hand, as $Q_n^{2m} = \mathbf{P}_n^{2m}(1)$, we can compute to get

$$\mathcal{S}_\varepsilon = (1-\varepsilon) Q_n^{2m} |\mathbb{S}^n|^{\frac{\alpha+1}{\alpha-1}},$$

on the other hand, by the definition of $\mathcal{S}_\varepsilon$ we get

$$\Big(\int_{\mathbb{S}^n} \phi^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} \big[\phi \, \mathbf{P}_n^{2m}(\phi) - \varepsilon Q_n^{2m} \phi^2 \big] d\mu_{\mathbb{S}^n} \geq (1-\varepsilon) Q_n^{2m} |\mathbb{S}^n|^{\frac{\alpha+1}{\alpha-1}}$$

for any $\phi \in H^m(\mathbb{S}^n)$ with $\phi > 0$. Now letting $\varepsilon \searrow 0$ we obtain

$$\Big(\int_{\mathbb{S}^n} \phi^{1-\alpha} d\mu_{\mathbb{S}^n} \Big)^{\frac{2}{\alpha-1}} \int_{\mathbb{S}^n} \phi \, \mathbf{P}_n^{2m}(\phi) d\mu_{\mathbb{S}^n} \geq Q_n^{2m} |\mathbb{S}^n|^{\frac{\alpha+1}{\alpha-1}}.$$

Recall that $Q_n^{2m} = \mathbf{P}_n^{2m}(1) = \Gamma(n/2+m)/\Gamma(n/2-m)$. This completes the proof of Theorem 1.2. $\square$

DATA AVAILABILITY

Data sharing is not applicable to this article as no dataset was generated or analysed during the current study.

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