

COMPUTATION OF THE ŁOJASIEWICZ EXPONENTS OF REAL BIVARIATE ANALYTIC FUNCTIONS

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ABSTRACT. The main goal of this paper is to present some explicit formulas for computing the Łojasiewicz exponent in the Łojasiewicz inequality comparing the rate of growth of two real bivariate analytic function germs.

1. INTRODUCTION

The Łojasiewicz inequalities and their variants play an important role in many branches of mathematics. For example, Łojasiewicz inequalities are very useful in the study of continuous regular functions, see [13, 18] for pioneering works and [19] for a survey. Also, Łojasiewicz inequalities, together with Nullstellensatz, are crucial tools for the study of the ring of (bounded) continuous semi-algebraic functions on a semi-algebraic set, see [10, 11, 12].

Let $f, g: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be nonzero real analytic function germs. Assume that $0 \in \{f = 0\} \subset \{g = 0\}$. By the classical Łojasiewicz inequality on comparing the rate of growth, there exist positive constants C, r and α such that

$$|f(x)| \geq C|g(x)|^\alpha \quad \text{for } |x| \leq r. \quad (1) \text{ ?Łojasiewicz?}$$

The infimum of such α is called the *Łojasiewicz exponent of g w.r.t. f* and denoted by $\mathcal{L}_g(f)$.

Note that several versions of the Łojasiewicz inequality have been studied for a special case where g is the distance function to the zero set of f , see [4, 5, 6, 7, 8, 9, 16, 17]. Furthermore, the computation or estimation of Łojasiewicz exponents in this case has been considered in these works.

In [3], the authors provided a global version of the Łojasiewicz inequality on comparing the rate of growth of two polynomial functions in the case the mapping defined by these functions is (Newton) non-degenerate at infinity. However, no computation or estimation of

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Lojasiewicz exponents has been given. In this work, we will address partially to this problem by giving some explicit formulas for computing the Łojasiewicz exponent $\mathcal{L}_g(f)$ when f and g are real bivariate analytic function germs.

The rest of the paper is organized as follows. In Section 2, we recall the notions of Newton polygon relative to an arc and sliding due to Kuo and Parusiński which are crucial in the proof of our formulas for the Łojasiewicz exponent, which are our main results (Theorem 3.1 and Theorem 3.2), whose statements, together with the proofs, will be given in Section 3.

2. THE NEWTON POLYGON RELATIVE TO AN ARC

⟨Section2⟩

The technique of Newton polygons plays an important role in this paper. It is well-known that Newton transformations which arise in a natural way when applying the Newton algorithm provide a useful tool for calculating invariants of singularities. For a complete treatment we refer to [1, 2, 23, 24]. In this section we recall the notion of Newton polygon relative to an arc due to Kuo and Parusiński [20] (see also, [14] and [15]).

Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{C}$ and let $f: (\mathbb{K}^2, 0) \rightarrow (\mathbb{K}, 0)$ denote a nonzero analytic function germ with Taylor expansion:

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots ,$$

where f_k is a homogeneous polynomial of degree k , and $f_m \neq 0$. For the remainder of the paper, we will assume that f is *regular in x of order m* in the sense that $f_m(1, 0) \neq 0$. (This can be achieved by a linear transformation $x' = x, y' = y + cx$, where c is a generic number). Let ϕ be an analytic arc in \mathbb{K}^2 , which is not tangent to the x -axis. Then it can be parametrized by

$$x = c_1 t^{n_1} + c_2 t^{n_2} + \cdots \in \mathbb{K}\{t\} \text{ and } y = t^N$$

and therefore can be identified with a *Puiseux series* (denoted also by ϕ for simplicity of notation)

$$x = \phi(y) = c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots \in \mathbb{K}\{y^{1/N}\}$$

with $N \leq n_1 < n_2 < \cdots$ being positive integers. The changes of variables $X := x - \phi(y)$ and $Y := y$ yield

$$F(X, Y) := f(X + \phi(Y), Y) := \sum c_{ij} X^i Y^{j/N}.$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a *Newton dot*. The set of Newton dots is called the *Newton diagram*. They generate a convex hull, whose boundary is called the *Newton polygon of f relative to ϕ* , to be denoted by $\mathbb{P}(f, \phi)$. Note that this is the Newton polygon of F in the usual sense. If ϕ is a *Newton–Puiseux root* of f (i.e., $f(\phi(y), y) = 0$), then there are no Newton dots on $X = 0$, and vice versa. Assume that ϕ is not a Newton–Puiseux root of f , then the exponents of the series $f(\phi(y), y) = F(0, Y)$ correspond to the Newton

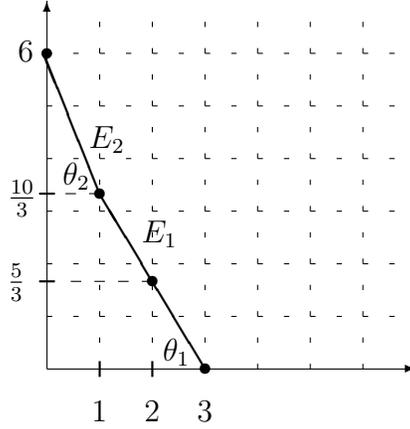


FIGURE 1.

⟨Figure1⟩

dots on the line $X = 0$. In particular, $\text{ord}f(\phi(y), y) = h_0$, where $(0, h_0)$ is the lowest Newton dot on $X = 0$. The *Newton edges* E_1, \dots, E_s and their associated *Newton angles* $\theta_1, \dots, \theta_s$ are defined in an obvious way as illustrated in the following example.

⟨Example1⟩

Example 2.1. Take $f(x, y) := x^3 - y^5 + y^6$ and $\phi(y) := y^{5/3}$. We have

$$F(X, Y) := f(X + \phi(Y), Y) = X^3 + 3X^2Y^{5/3} + 3XY^{10/3} + Y^6.$$

By definition, the Newton polygon of f relative to ϕ has two compact edges E_1, E_2 with $\tan \theta_1 = 5/3$, $\tan \theta_2 = 8/3$ (see Figure 1).

Take any edge E_s . The *associated polynomial* $\mathcal{E}_s(z)$ is defined to be $\mathcal{E}_s(z) := \mathcal{E}_s(z, 1)$, where

$$\mathcal{E}_s(X, Y) := \sum_{(i, j/N) \in E_s} c_{ij} X^i Y^{j/N}.$$

The *highest Newton edge*, denoted by E_H , is defined as follows: If ϕ is a Newton–Puiseux root of f , then E_H is the non-compact edge of the polygon $\mathbb{P}(f, \phi)$ parallel to the y -axis. If ϕ is not a Newton–Puiseux root of f , then E_H is the compact edge of the polygon $\mathbb{P}(f, \phi)$ with a vertex being the lowest Newton dot on $X = 0$. For instance, in Example 2.1, E_2 is the highest Newton edge.

Next, let us recall the notion of *sliding* (see [20]). Suppose that ϕ is not a Newton–Puiseux root of f . Consider the Newton polygon $\mathbb{P}(f, \phi)$. Take any nonzero root c of $\mathcal{E}_H(z) = 0$, the polynomial equation associated to the highest Newton edge E_H . We call

$$\phi_1: x = \phi(y) + cy^{\tan \theta_H}$$

a *sliding* of ϕ along f , where θ_H is the angle associated to E_H . A recursive sliding

$$\phi \rightarrow \phi_1 \rightarrow \phi_2 \rightarrow \dots$$

produces a limit, denoted by ϕ_∞ , which is a Newton–Puiseux root of f . The series ϕ_∞ will be called a *final result of sliding ϕ along f* . Note that ϕ_∞ has the form

$$\phi_\infty: x = \phi(y) + cy^{\tan \theta_H} + \text{higher order terms},$$

due to the following technical lemma.

(Lemma22)

Lemma 2.1. *Let ϕ be a Puiseux series, which is not a Newton–Puiseux root of f . Let θ_H and \mathcal{E}_H be respectively the Newton angle and polynomial associated to the highest Newton edge E_H . Consider a series of the following form*

$$\psi: x = \phi(y) + cy^\rho + \text{higher order terms},$$

where $c \in \mathbb{K}$ and $\rho \in \mathbb{Q}, \rho > 0$. Then the following statements hold:

- (i) *If either c or ρ is generic (i.e., $\arctan \rho$ is not a Newton angle of $\mathbb{P}(f, \phi)$; or $\arctan \rho$ is a Newton angle of $\mathbb{P}(f, \phi)$ but c is not a root of the polynomial associated to the Newton edge with Newton angle $\arctan \rho$), then*

$$\text{ord}f(\psi(y), y) = \min\{a\rho + b \mid (a, b) \in \mathbb{P}(f, \phi)\}.$$

Furthermore,

$$\text{ord}f(\psi(y), y) \leq \text{ord}f(\phi(y), y).$$

In particular, if either $\tan \theta_H < \rho$ or $\tan \theta_H = \rho$ and $\mathcal{E}_H(c) \neq 0$ then $\mathbb{P}(f, \psi) = \mathbb{P}(f, \phi)$, and therefore

$$\text{ord}f(\psi(y), y) = \text{ord}f(\phi(y), y).$$

- (ii) *If $\tan \theta_H = \rho$ and $\mathcal{E}_H(c) = 0$ then*

$$\text{ord}f(\psi(y), y) > \text{ord}f(\phi(y), y).$$

Proof. cf. [1, 2, 23]. For a detailed proof, we refer to [14]. In fact, the special case where $\psi(y) = \phi(y) + cy^{\tan \theta_H}$ was proved in [14, Lemma 2.1]. Then the lemma is deduced by applying the special case (possibly infinitely) many times. \square

(def21)

Definition 2.1. For each Puiseux series $\phi(y) = \sum_i a_i y^{\alpha_i}$ and for each positive real number ρ , the ρ -approximation of $\phi(y)$ is defined to be the series $\sum_{\alpha_i < \rho} a_i y^{\alpha_i} + cy^\rho$, where c is a generic real number. We associate to any Puiseux series ϕ its *real approximation* $\phi^{\mathbb{R}}(y)$ defined to be the ρ -approximation of ϕ , where ρ is the smallest exponent occurring in ϕ with non-real coefficient. It is clear that if φ is *real*, i.e., all coefficients of φ are real, then the real approximation of φ is itself. Now, for $f \in \mathbb{K}\{x, y\}$ which is regular in x , let $\mathcal{V}_{\mathbb{R}}(f)$ be the set of all real approximations of non-real Newton–Puiseux roots of f .

For any two distinct series ϕ_1, ϕ_2 , their *approximation*, denoted by $\phi_{1,2}$, is defined to be the ρ -approximation of ϕ_1 where $\rho := \text{ord}(\phi_1 - \phi_2)$. Let $\mathcal{V}(f)$ be the set of all approximations

of ϕ_1 and ϕ_2 with $\phi_1 \neq \phi_2$ being Newton–Puiseux roots of f . Note that $\mathcal{V}_{\mathbb{R}}(f) \subset \mathcal{V}(f)$ if $f \in \mathbb{R}\{x, y\}$.

The following useful assertion is a direct consequence of Lemma 2.1:

⟨Lemma23⟩

Lemma 2.2. *Assume that $\mathbb{K} = \mathbb{R}$. Let ϕ be a Puiseux series and let E_1, \dots, E_s be the Newton edges of $\mathbb{P}(f, \phi)$. Let θ_i and \mathcal{E}_i be the corresponding Newton angle and polynomial associated to E_i . Then by a permutation of indexes, we have*

$$\pi/2 \geq \theta_1 > \theta_2 > \dots > \theta_s$$

and the following statements hold:

(i) *If \mathcal{E}_i has two distinct roots, there exists $\psi \in \mathcal{V}(f)$ being of the form*

$$\psi(y) = \phi(y) + cy^{\tan \theta_i} + \text{higher order terms},$$

where c is a generic number.

(ii) *If $\theta \neq \theta_1$ is a Newton angle, then there exists $\psi \in \mathcal{V}(f)$ being of the form*

$$\psi(y) = \phi(y) + cy^{\tan \theta} + \text{higher order terms},$$

where c is a generic number.

(iii) *If \mathcal{E}_i has a non-real root, then there exists $\psi \in \mathcal{V}_{\mathbb{R}}(f)$ being of the form*

$$\psi(y) = \phi(y) + cy^{\tan \theta_i} + \text{higher order terms},$$

where c is a generic number.

3. FORMULAS FOR ŁOJASIEWICZ EXPONENTS

⟨Section3⟩

For the remainder of this section, let $f, g: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ be nonzero real analytic function germs, which are regular in x . Assume that $0 \in \{f = 0\} \subset \{g = 0\}$. By the classical Łojasiewicz inequality, there exists positive constants C, r and α such that

$$|f(x, y)| \geq C|g(x, y)|^\alpha \quad \text{for } |x| \leq r.$$

Recall that the Łojasiewicz exponent of g w.r.t. f is denoted by $\mathcal{L}_g(f)$. Take any analytic arc $\phi \in \mathbb{R}^2$ at the origin parametrized by $(x(t), y(t))$. If $g \circ \phi \not\equiv 0$, then we can define the following positive rational number

$$\ell(\phi) := \frac{\text{ord} f(\phi(t))}{\text{ord} g(\phi(t))}.$$

By the Curve Selection Lemma (see [22, Lemma 3.1]), it is not hard to show that

$$\mathcal{L}_g(f) = \sup_{\phi} \ell(\phi), \tag{2} \boxed{\text{Eqn1}}$$

where the supremum is taken over all analytic arcs passing through the origin, which are not contained in the zero locus of g . Furthermore, we have the following.

$\langle \text{Notlie} \rangle$

Lemma 3.1. *The supremum in (2) can be taken over all real analytic arcs passing through the origin not contained in the zero locus of yg .*

Proof. By the assumption $0 \in \{f = 0\} \subset \{g = 0\}$, we have

$$m := \text{ord } f > 0 \quad \text{and} \quad n := \text{ord } g > 0.$$

Let f_m and g_n be the homogeneous parts of degree m of f and of degree n of g respectively. Pick a constant $c \neq 0$ such that $f_m(1, c) \neq 0$ and $g_n(1, c) \neq 0$. Then

$$\text{ord } f(x, 0) = m = \text{ord } (x^m f(1, c)) = \text{ord } f(x, cx).$$

Similarly $\text{ord } g(x, 0) = \text{ord } g(x, cx)$. Hence $\ell((x, 0)) = \ell((x, cx))$. The lemma follows. \square

3.1. First formula for the Lojasiewicz exponents. Let β_j , $j = 1, \dots, k$ be the common real Newton–Puiseux roots of f and g of multiplicities m_j and n_j respectively. Let $\mathcal{V}(fg)$ be the set given by Definition 2.1. For any $\phi \in \mathcal{V}(fg)$, we will write $\ell(\phi)$ instead of $\ell((\phi(y), y))$ for simplicity.

$\langle \text{thm31} \rangle$

Theorem 3.1. *Define*

$$\mathcal{L}_g^+(f) := \max \left\{ \ell(\phi), \frac{m_j}{n_j} \mid \phi \in \mathcal{V}(fg), j = 1, \dots, k \right\} \quad \text{and} \quad \mathcal{L}_g^-(f) = \mathcal{L}_g^+(\bar{f}),$$

where $\bar{f}(x, y) := f(x, -y)$ and $\bar{g}(x, y) := g(x, -y)$. Then the Lojasiewicz exponent of g w.r.t f is given by

$$\mathcal{L}_g(f) = \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \}.$$

Proof. We first show that

$$\mathcal{L}_g(f) \geq \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \}. \quad (3) \quad \boxed{\text{one-side}}$$

By (2), it is obvious that

$$\mathcal{L}_g(f) \geq \max \{ \ell(\phi) \mid \phi \in \mathcal{V}(fg) \}.$$

Therefore we only need to show that

$$\mathcal{L}_g(f) \geq \frac{m_j}{n_j} \quad \text{for all } j = 1, \dots, k. \quad (4) \quad \boxed{>=\text{mul}}$$

To do this, fix $j \in \{1, \dots, k\}$ and consider the Newton polygons $\mathbb{P}(f, \beta_j)$ of f and $\mathbb{P}(g, \beta_j)$ of g relative to the arc β_j . Let $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ be respectively the vertices of $\mathbb{P}(f, \beta_j)$ and $\mathbb{P}(g, \beta_j)$ being closest to the y -axis. We will show that

$$x_1 = m_j \quad \text{and} \quad x_2 = n_j. \quad (5) \quad \boxed{\text{multiplicity}}$$

Indeed, in view of Weierstrass's preparation theorem and Puiseux's theorem (see, for example, [21, The Puiseux theorem, page 170] or [23, page 98]), we can write

$$f(x, y) = u(x, y)(x - \beta_j(y))^{m_j} \prod_{\phi} (x - \phi(y))^{\text{mult}_f(\phi)}, \quad (6) \text{ Puiseux-f}$$

where $u \in \mathbb{C}\{x, y\}$ with $u(0, 0) \neq 0$; $\text{mult}_f(\cdot)$ is the *multiplicity* of the corresponding Newton–Puiseux root of f ; and the product is taken over all Newton–Puiseux roots of f different from β_1, \dots, β_k . So

$$f(X + \beta_j(Y), Y) = u(X + \beta_j(Y), Y) X^{m_j} \prod_{\phi} (X + \beta_j(Y) - \phi(Y))^{\text{mult}_f(\phi)}.$$

This implies

$$(x_1, y_1) = \left(m_j, \text{ord} \prod_{\phi} (\beta_j(Y) - \phi(Y))^{\text{mult}_f(\phi)} \right). \quad (7) \text{ A1}$$

Similarly

$$(x_2, y_2) = \left(n_j, \text{ord} \prod_{\psi} (\beta_j(Y) - \psi(Y))^{\text{mult}_g(\psi)} \right),$$

where the product is taken over all Newton–Puiseux roots of g different from β_1, \dots, β_k . These yield (5) immediately. For each positive integer n , define the arc

$$x = \phi_n(y) = \beta_j(y) + y^n.$$

By (6), we have

$$\begin{aligned} f(\phi_n(y), y) &= u(\phi_n, y)(\phi_n(y) - \beta_j(y))^{m_j} \prod_{\phi} (\phi_n(y) - \phi(y))^{\text{mult}_f(\phi)} \\ &= u(\phi_n, y) y^{nx_1} \prod_{\phi} (\beta_j(y) - \phi(y) + y^n)^{\text{mult}_f(\phi)}. \end{aligned}$$

For n large enough, we have

$$\text{ord} (\beta_j(y) - \phi(y) + y^n) = \text{ord} (\beta_j(y) - \phi(y)).$$

So combining this with (7) yields $\text{ord} f(\phi_n(y), y) = nx_1 + y_1$. By the same way, we also have $\text{ord} g(\phi_n(y), y) = nx_2 + y_2$. Consequently

$$\ell(\phi_n) = \frac{nx_1 + y_1}{nx_2 + y_2}.$$

Note that

$$\lim_{n \rightarrow \infty} \ell(\phi_n) = \frac{x_1}{x_2} = \frac{m_j}{n_j},$$

so (4) follows from (2). Therefore, $\mathcal{L}_g(f) \geq \mathcal{L}_g^+(f)$. Similarly, one has $\mathcal{L}_g(f) \geq \mathcal{L}_g^-(f)$ and hence the inequality (3) holds. Now we need to show that the inequality in (3) is actually an equality.

Suppose for contradiction that

$$\mathcal{L}_g(f) > \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \}.$$

Then, by Lemma 3.1, there is a real analytic arc ϕ passing through the origin and not lying in the x -axis such that $g \circ \phi \not\equiv 0$ and

$$\ell(\phi) > \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \}.$$

Note that ϕ can be parametrized by either

$$(x = \phi(t), y = t) \quad \text{or} \quad (x = \phi(t), y = -t),$$

where $\phi(t)$ is an element in $\mathbb{R}\{t^{1/N}\}$ for some positive integer N with $\phi(0) = 0$. Without loss of generality we may assume that ϕ can be parametrized by $(x = \phi(t), y = t)$. Denote by E_1 and E_2 the highest Newton edges of $\mathbb{P}(f, \phi)$ and $\mathbb{P}(g, \phi)$ respectively. Let \mathcal{E}_i and θ_i be respectively its associated polynomial and Newton angle.

(Claim21) **Claim 3.1.** *We have $\tan \theta_1 = \tan \theta_2$.*

Proof. Assume for contradiction that, $\tan \theta_1 > \tan \theta_2$. Let ϕ_∞ be a final result of sliding ϕ along f . Write

$$\phi_\infty(y) = \phi(y) + \sum_{i \geq 1} a_i y^{\alpha_i},$$

where $a_i \in \mathbb{C} \setminus \{0\}$, $\tan \theta_1 = \alpha_1 < \alpha_2 < \dots$. We will show that $a_i \in \mathbb{R}$ for all $i \geq 1$. In fact, if this is not the case, for each $n \geq 0$, define the series

$$\phi_0(y) := \phi(y), \quad \phi_n(y) := \phi(y) + \sum_{i=1}^n a_i y^{\alpha_i} \quad \text{for } n \geq 1,$$

and let n_0 be the smallest index such that $a_{n_0} \notin \mathbb{R}$. Then $n_0 \geq 1$ and

$$\phi_{n_0}^{\mathbb{R}}(y) = \phi_{n_0-1}(y) + c y^{\alpha_{n_0}} + \text{higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. By applying Lemma 2.1, we obtain

$$\text{ord } f(\phi_{n_0}^{\mathbb{R}}(y), y) = \text{ord } f(\phi_{n_0-1}(y), y) > \dots > \text{ord } f(\phi(y), y)$$

and

$$\text{ord } g(\phi_{n_0}^{\mathbb{R}}(y), y) = \text{ord } g(\phi_{n_0-1}(y), y) = \dots = \text{ord } g(\phi(y), y).$$

So

$$\ell(\phi_{n_0}^{\mathbb{R}}) = \ell(\phi_{n_0-1}) > \dots > \ell(\phi) > \mathcal{L}_g^+(f),$$

a contradiction, since $\phi_{n_0}^{\mathbb{R}} \in \mathcal{V}_{\mathbb{R}}(f) \subset \mathcal{V}_{\mathbb{R}}(fg) \subset \mathcal{V}(fg)$. This shows that $a_n \in \mathbb{R}$ for all $n \geq 1$. But then this contradicts to the assumption that $\{f = 0\} \subset \{g = 0\}$ in \mathbb{R}^2 , hence

$$\tan \theta_1 \leq \tan \theta_2.$$

Now assume for contradiction that $\tan \theta_1 < \tan \theta_2$. Note that, θ_1 and θ_2 are Newton angles of $\mathbb{P}(fg, \phi)$. Then by Lemma 2.2(ii), there exists $\psi \in \mathcal{V}(fg)$ being of the form

$$\psi(y) = \phi(y) + cy^{\tan \theta_1} + \text{higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. It follows from Lemma 2.1(i) that

$$\text{ord } f(\psi(y), y) = \text{ord } f(\phi(y), y) \quad \text{and} \quad \text{ord } g(\psi(y), y) \leq \text{ord } g(\phi(y), y),$$

and hence $\ell(\psi) \geq \ell(\phi) > \mathcal{L}_g^+(f)$. This contradiction finishes the claim. \square

$\langle \text{Claim23} \rangle$ **Claim 3.2.** *The polynomial $\mathcal{E}_1\mathcal{E}_2$ has only one root.*

Proof. Assume for contradiction that $\mathcal{E}_1\mathcal{E}_2$ has two distinct roots. By Claim 3.1, $\theta_1 = \theta_2$, so $\mathcal{E}_1\mathcal{E}_2$ is the Newton polynomial associated to the highest Newton edge of $\mathbb{P}(fg, \phi)$ (with Newton angle θ_1). Then by Lemma 2.2(i), there exists $\psi \in \mathcal{V}(fg)$ being of the form

$$\psi(y) = \phi(y) + cy^{\tan \theta_1} + \text{higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. Then Lemma 2.1(ii) yields

$$\text{ord } f(\psi(y), y) = \text{ord } f(\phi(y), y) \quad \text{and} \quad \text{ord } g(\psi(y), y) = \text{ord } g(\phi(y), y).$$

Hence $\ell(\phi) = \ell(\psi)$ and so $\ell(\phi) \leq \mathcal{L}_g^+(f)$ which contradicts the assumption $\ell(\phi) > \mathcal{L}_g^+(f)$. \square

Let $a \in \mathbb{R}$ be the unique root of the polynomial $\mathcal{E}_1\mathcal{E}_2$ and let $\tilde{\phi}(y) := \phi(y) + ay^{\tan \theta_1}$. We denote by \tilde{E}_1 and \tilde{E}_2 the highest Newton edge of $\mathbb{P}(f, \tilde{\phi})$ and $\mathbb{P}(g, \tilde{\phi})$ respectively. For $i = 1, 2$, let $\tilde{\theta}_i$ and $\tilde{\mathcal{E}}_i$ be the Newton angle and the polynomial associated to \tilde{E}_i . Recall that E_1 and E_2 are respectively the highest Newton edges of $\mathbb{P}(f, \phi)$ and $\mathbb{P}(g, \phi)$. Let $B_i = (x_i, y_i)$ be the vertex of E_i which is not contained in the y -axis.

$\langle \text{Claim24} \rangle$ **Claim 3.3.** *If $\tilde{\phi}(y)$ is not a Newton–Puiseux root of f , then the following properties hold:*

- (i) B_i is a vertex of \tilde{E}_i , therefore $\deg \tilde{\mathcal{E}}_i = x_i = \deg \mathcal{E}_i$.
- (ii) $\tan \tilde{\theta}_1 = \tan \tilde{\theta}_2$.
- (iii) *The polynomial $\tilde{\mathcal{E}}_1\tilde{\mathcal{E}}_2$ has only one root.*
- (iv) $\ell(\tilde{\phi}) \geq \ell(\phi)$.

Proof. (i) Let us define the function

$$\mu(t) = \frac{tx_1 + y_1}{tx_2 + y_2}.$$

We first claim that $x_1y_2 \geq x_2y_1$. In fact, if this is not the case, i.e., $x_1y_2 < x_2y_1$, then the function μ is strictly decreasing and $y_1 > 0$. Since f is regular in x , there exists a Newton edge E of $\mathbb{P}(f, \phi)$ which is different from E_1 and has B_1 as a vertex. Let θ be the

Newton angle associated to E . Clearly $\theta < \theta_1 = \theta_2$, therefore, by Lemma 2.2(ii), there exists $\psi \in \mathcal{V}(f) \subset \mathcal{V}(fg)$ such that

$$\psi(y) = \phi(y) + cy^{\tan\theta} + \text{higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. We have

$$\text{ord } f(\phi(y), y) = x_1 \tan \theta_1 + y_1 \quad \text{and} \quad \text{ord } g(\phi(y), y) = x_2 \tan \theta_2 + y_2 = x_2 \tan \theta_1 + y_2,$$

Moreover, by Lemma 2.1(i) and the choice of the edge E ,

$$\text{ord } f(\psi(y), y) = x_1 \tan \theta + y_1 \quad \text{and} \quad \text{ord } g(\psi(y), y) \leq x_2 \tan \theta + y_2.$$

Hence

$$\ell(\phi) = \mu(\tan \theta_1) < \mu(\tan \theta) \leq \ell(\psi) \leq \mathcal{L}_g^+(f),$$

which is a contradiction. Hence we must have $x_1 y_2 \geq x_2 y_1$, i.e., the function μ is increasing.

Let \tilde{E} be the edge of $\mathbb{P}(f, \tilde{\phi})$ such that B_1 is the vertex having larger x -coordinate. Let $\tilde{\theta}$ be the Newton angle associated to \tilde{E} . Since $\tilde{\phi}(y)$ is not a Newton–Puiseux root of f , \tilde{E} is a compact edge and therefore $\theta_1 < \tilde{\theta} < \pi/2$. If \tilde{E} is not the highest Newton edge of $\mathbb{P}(f, \tilde{\phi})$, then by Lemma 2.2(ii), there exists $\varphi \in \mathcal{V}(f) \subset \mathcal{V}(fg)$ such that

$$\varphi(y) = \tilde{\phi}(y) + cy^{\tan\tilde{\theta}} + \text{higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. It follows from Lemma 2.1(i) that

$$\text{ord } f(\varphi(y), y) = x_1 \tan \tilde{\theta} + y_1 \quad \text{and} \quad \text{ord } g(\varphi(y), y) \leq x_2 \tan \tilde{\theta} + y_2.$$

Hence

$$\ell(\phi) = \mu(\tan \theta_1) \leq \mu(\tan \tilde{\theta}) \leq \ell(\varphi) \leq \mathcal{L}_g^+(f).$$

This contradiction yields $\tilde{E} \equiv \tilde{E}_1$, i.e., B_1 is a vertex of \tilde{E}_1 . Similarly we can show that B_2 is a vertex of \tilde{E}_2 and hence Item (i) follows.

(ii)–(iii) These can be proved by using completely the same argument as in Claims 3.1 and 3.2.

(iv) It follows from Items (ii) and (iii) that

$$\text{ord } f(\tilde{\phi}(y), y) = x_1 \tan \tilde{\theta}_1 + y_1 \quad \text{and} \quad \text{ord } g(\tilde{\phi}(y), y) = x_2 \tan \tilde{\theta}_2 + y_2 = x_2 \tan \tilde{\theta}_1 + y_1,$$

i.e.,

$$\ell(\tilde{\phi}) = \mu(\tan \tilde{\theta}_1) \geq \mu(\tan \theta_1) = \ell(\phi).$$

This implies (iv) and hence the claim follows. \square

We are now in position to complete the theorem. Applying Claim 3.3 (possibly infinitely) many times, we obtain ϕ_∞ as a final result of sliding of ϕ along f . This implies that, $x = \phi_\infty(y)$ is a common Newton–Puiseux root of f and g of multiplicities x_1 and x_2 respectively. Moreover, from the proof Claim 3.3, $x_1 y_2 \geq x_2 y_1$, it follows that

$$\ell(\phi) = \frac{x_1 \tan \theta_1 + y_1}{x_2 \tan \theta_1 + y_2} \leq \frac{x_1}{x_2} \leq \mathcal{L}_g^+(f).$$

This contradicts the assumption $\ell(\phi) > \mathcal{L}_g^+(f)$. The theorem is proved. \square

3.2. Second formula for the Lojasiewicz exponents. Recall that $\mathcal{V}_{\mathbb{R}}(f)$ is the set of real approximations of non-real Newton–Puiseux roots of f as defined in Definition 2.1. Let β_j , $j = 1, \dots, k$, be the common real Newton–Puiseux roots of f and g of multiplicity m_j and n_j respectively.

(thm32) **Theorem 3.2.** *Define*

$$\mathcal{L}_g^+(f) := \max \left\{ \ell(\gamma), \frac{m_j}{n_j} \mid \gamma \in \mathcal{V}_{\mathbb{R}}(f), j = 1, \dots, k \right\} \quad \text{and} \quad \mathcal{L}_g^-(f) = \mathcal{L}_{\bar{g}}^+(\bar{f}),$$

where $\bar{f}(x, y) := f(x, -y)$ and $\bar{g}(x, y) := g(x, -y)$. Then the Lojasiewicz exponent of g w.r.t f is given by

$$\mathcal{L}_g(f) = \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \}.$$

Proof. Since $\mathcal{V}_{\mathbb{R}}(f) \subset \mathcal{V}(fg)$ and $\mathcal{V}_{\mathbb{R}}(\bar{f}) \subset \mathcal{V}(\bar{f}\bar{g})$, by Theorem 3.1,

$$\mathcal{L}_g(f) = \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \} \geq \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \}.$$

Arguing by contradiction, we assume that

$$\mathcal{L}_g(f) > \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \}.$$

It follows from Theorem 3.1 and Lemma 3.1 that there is an analytic arc ϕ passing through the origin and not lying in the x -axis such that $g \circ \phi \not\equiv 0$ and

$$\mathcal{L}_g(f) = \ell(\phi) > \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \}.$$

Note that ϕ can be parametrized by either

$$(x = \phi(t), y = t) \quad \text{or} \quad (x = \phi(t), y = -t),$$

where $\phi(t)$ is an element in $\mathbb{R}\{t^{1/N}\}$ for some positive integer number N with $\phi(0) = 0$. Let \mathcal{E}_ϕ be the polynomial associated to the highest Newton edge of $\mathbb{P}(f, \phi)$. With no loss of generality, we can assume that ϕ has the following property:

For any analytic arc $\tilde{\phi}$ passing through the origin not lying in the x -axis and having the parametrization $(x = \tilde{\phi}(t), y = t)$ such that $g \circ \tilde{\phi} \not\equiv 0$ and

$$\mathcal{L}_g(f) = \ell(\tilde{\phi}) > \max \{ \mathcal{L}_g^+(f), \mathcal{L}_g^-(f) \},$$

if $\mathcal{E}_{\tilde{\phi}}$ is the polynomial associated to the highest Newton edge of $\mathbb{P}(f, \tilde{\phi})$, then $\deg \mathcal{E}_{\tilde{\phi}} \geq \deg \mathcal{E}_{\phi}$.

Indeed, if there is an analytic arc $\tilde{\phi}$ such that this property does not hold, i.e., $\deg \mathcal{E}_{\tilde{\phi}} < \deg \mathcal{E}_{\phi}$, then it is enough to replace ϕ by $\tilde{\phi}$ and repeat the process until the property is satisfied.

Let E_1 and E_2 be the highest Newton edges of $\mathbb{P}(f, \phi)$ and $\mathbb{P}(g, \phi)$ respectively. For each $i = 1, 2$, let \mathcal{E}_i and θ_i be the associated polynomial and the Newton angle of E_i respectively. Let $B_i = (x_i, y_i)$ be the vertex of E_i which is not contained in the y -axis. Then the following statement holds.

⟨Claim26⟩

Claim 3.4. *We have $\tan \theta_1 = \tan \theta_2$.*

Proof. Applying the same argument as in the proof of Claim 3.1, we get $\tan \theta_1 \leq \tan \theta_2$. Assume for contradiction that $\tan \theta_1 < \tan \theta_2$. Let

$$\psi(y) = \phi(y) + cy^{\tan \theta_1}$$

with a generic number c . It follows from Lemma 2.1 that

$$\text{ord } f(\psi(y), y) = x_1 \tan \theta_1 + y_1 = \text{ord } f(\phi(y), y)$$

and

$$\text{ord } g(\psi(y), y) \leq x_2 \tan \theta_1 + y_2 < x_2 \tan \theta_2 + y_2 = \text{ord } g(\phi(y), y).$$

These imply

$$\ell(\psi) > \ell(\phi) = \mathcal{L}_g(f),$$

which is a contradiction. □

?⟨Claim27⟩?

Claim 3.5. *The polynomial \mathcal{E}_1 has only real roots.*

Proof. Assume for contradiction that $a \notin \mathbb{R}$ is a root of \mathcal{E}_1 . It follows from Lemma 2.2(iii) that there exists $\psi \in \mathcal{V}_{\mathbb{R}}(f)$ of the form

$$\psi = \phi + cy^{\tan \theta_1} + \text{higher order terms}$$

with a generic real number c . Applying Lemma 2.1(i) we obtain

$$\text{ord } f(\psi(y), y) = \text{ord } f(\phi(y), y) \quad \text{and} \quad \text{ord } g(\psi(y), y) = \text{ord } g(\phi(y), y).$$

Therefore

$$\ell(\psi) = \ell(\phi) > \mathcal{L}_g^+(f),$$

which contradicts the definition of $\mathcal{L}_g^+(f)$. □

⟨Claim29⟩

Claim 3.6. *We have*

- (i) $x_1 y_2 = x_2 y_1$, and therefore $\ell(\phi) = \frac{x_1}{x_2} = \frac{t x_1 + y_1}{t x_2 + y_2}$ for all t .
- (ii) The polynomial $\mathcal{E}_1 \mathcal{E}_2$ has only one root.

Proof. (i) First of all, let us prove

$$x_1 y_2 \geq x_2 y_1. \quad (8) \quad \boxed{1221}$$

Indeed, if this is not the case, i.e., $x_1 y_2 < x_2 y_1$, then the function

$$\mu(t) := \frac{t x_1 + y_1}{t x_2 + y_2}$$

is strictly decreasing. Let $\psi(y) = \phi(y) + y^\rho$ with $0 < \rho < \tan \theta_1$ closed enough to $\theta_1 (= \theta_2$ by Claim 3.4) so that $\arctan \rho$ is larger than the other Newton angles of $\mathbb{P}(f, \phi)$ and $\mathbb{P}(g, \phi)$.

Then

$$\text{ord } f(\psi(y), y) = x_1 \rho + y_1 \quad \text{and} \quad \text{ord } g(\psi(y), y) = x_2 \rho + y_2.$$

So we get

$$\ell(\psi) = \mu(\rho) > \mu(\tan \theta_1) = \ell(\phi) = \mathcal{L}_g(f),$$

which contradicts the definition of $\mathcal{L}_g(f)$. Hence (8) must hold. Now let us prove that the sign “=” always occurs.

Let $c_j \in \mathbb{R}, j = 1, \dots, q$, be the roots of $\mathcal{E}_1(z)$ of multiplicity x_1^j with $x_1^j > 0$ and $q \geq 1$. We write

$$\mathcal{E}_2(z) = a(z) \prod_{j=1}^q (z - c_j)^{x_2^j}$$

with $x_2^j \geq 0$ and $a(c_j) \neq 0$. Observe that

$$\sum_{j=1}^q x_1^j = \deg \mathcal{E}_1 = x_1 \quad \text{and} \quad \sum_{j=1}^q x_2^j + \deg a(z) = \deg \mathcal{E}_2 = x_2. \quad (9) \quad \boxed{\text{deg}}$$

Let us denote by $A_i^j = (x_i^j, y_i^j)$ the intersection of the line $\{x = x_i^j\}$ with the edge E_i for each $i = 1, 2$. Set

$$\mu_j(t) := \frac{t x_1^j + y_1^j}{t x_2^j + y_2^j}.$$

Since $A_i^j \in E_i$, it follows that

$$\mu_j(\tan \theta_1) = \ell(\phi) \quad \text{and} \quad y_i^j = y_i + \tan \theta_1 (x_i - x_i^j), \quad \text{for all } i = 1, 2, j = 1, \dots, q. \quad (10) \quad \boxed{\text{equa22}}$$

We also notice that A_1^j is a vertex of the Newton polygon $\mathbb{P}(f, \tilde{\phi}_j)$ with $\tilde{\phi}_j(y) := \phi(y) + c_j y^{\theta_1}$. We shall show that

$$x_1^j y_2^j \leq x_2^j y_1^j \quad \text{for all } j = 1, \dots, q. \quad (11) \quad \boxed{\text{IE}}$$

In fact, by contradiction, assume that $x_1^j y_2^j > x_2^j y_1^j$, i.e., the function $\mu_j(t)$ is strictly increasing. From this and (10), for $\rho > \tan \theta_1$ sufficiently closed to $\tan \theta_1$, we have

$$\mathcal{L}_g(f) = \ell(\phi) = \mu_j(\tan \theta_1) < \mu_j(\rho) = \ell(\tilde{\phi}_j(y) + c_j y^\rho) \leq \mathcal{L}_g(f)$$

for every non-zero $c \in \mathbb{R}$, which is clearly a contradiction. Thus (11) must hold. Combining (10) and (11) yields

$$x_1^j [y_2 + \tan \theta_1 (x_2 - x_2^j)] \leq x_2^j [y_1 + \tan \theta_1 (x_1 - x_1^j)] \quad \text{for all } j = 1, \dots, q.$$

Summing up we obtain

$$(y_2 + x_2 \tan \theta_1) \sum_{j=1}^q x_1^j \leq (y_1 + x_1 \tan \theta_1) \sum_{j=1}^q x_2^j.$$

Combining this with (9), we get

$$(y_2 + x_2 \tan \theta_1) x_1 \leq (y_1 + x_1 \tan \theta_1) (x_2 - \deg a(z)) \leq (y_1 + x_1 \tan \theta_1) x_2.$$

Equivalently

$$x_1 y_2 \leq x_2 y_1.$$

By this and (8), we have $x_1 y_2 = x_2 y_1$ and Item (i) follows.

(ii) By Item (i), it follows that $x_1^j y_2^j = x_2^j y_1^j$ for all $j = 1, \dots, q$ and $\deg a(z) = 0$. Hence the function $\mu_j(t)$ is constant. Consider, for each j , the curve $\psi_j(y) = \tilde{\phi}_j + y^\rho$ for some $\rho > \tan \theta_1$ sufficiently closed to $\tan \theta_1$. Then

$$\ell(\psi_j) = \mu_j(\rho) = \mu_j(\tan \theta_1) = \ell(\phi).$$

Moreover, it is not hard to check that A_1^j is a vertex of the Newton polygon $\mathbb{P}(f, \psi_j)$. So $\deg \mathcal{E}_{\psi_j} = x_1^j$ where \mathcal{E}_{ψ_j} is the polynomial associated to the highest Newton edge of $\mathbb{P}(f, \psi_j)$. Then it follows from the choice of ϕ that $x_1^j \geq x_1$. This implies $q = 1$ and therefore, by the fact that $\deg a(z) = 0$, the polynomial $\mathcal{E}_1 \mathcal{E}_2$ must have only one root. The claim is proved. \square

Let $a \in \mathbb{R}$ be the unique root of the polynomial $\mathcal{E}_1 \mathcal{E}_2$ and let $\tilde{\phi}(y) := \phi(y) + ay^{\tan \theta_1}$. Let $\tilde{\mathbb{P}}_1 := \mathbb{P}(f, \tilde{\phi})$ and $\tilde{\mathbb{P}}_2 := \mathbb{P}(g, \tilde{\phi})$. We denote by \tilde{E}_i the Newton edge of $\tilde{\mathbb{P}}_i$ containing B_i as the vertex with the larger x -coordinate. Let $\tilde{\theta}_i$ and $\tilde{\mathcal{E}}_i$ be the Newton angle and the polynomial associated to \tilde{E}_i respectively.

(Claim32) **Claim 3.7.** *We have $\tan \tilde{\theta}_1 = \tan \tilde{\theta}_2$.*

Proof. Assume for contradiction that $\tan \tilde{\theta}_1 > \tan \tilde{\theta}_2$. Consider the curve

$$\psi(y) = \tilde{\phi} + cy^{\tan \tilde{\theta}_1}$$

with a generic number c . Then it follows from Lemma 2.1(i) that, for any $(u, v) \in \tilde{E}_2$ such that $(u, v) \neq B_2$, we have

$$\text{ord } f(\psi(y), y) = x_1 \tan \tilde{\theta}_1 + y_1$$

and

$$\text{ord } g(\psi(y), y) \leq u \tan \tilde{\theta}_1 + v < x_2 \tan \tilde{\theta}_1 + y_2.$$

Therefore

$$\ell(\psi) > \frac{x_1 \tan \tilde{\theta}_1 + y_1}{x_2 \tan \tilde{\theta}_1 + y_2} = \frac{x_1 \tan \theta_1 + y_1}{x_2 \tan \theta_1 + y_2} = \ell(\phi) = \mathcal{L}_g(f),$$

where the first equality follows from Claim 3.6(i). This is a contradiction.

Now, by contradiction, suppose that $\tan \tilde{\theta}_1 < \tan \tilde{\theta}_2$. Let us show that the polynomial $\tilde{\mathcal{E}}_1$ has only real root. In fact, if this is not the case, then by Lemma 2.2(iii), there exists $\psi \in \mathcal{V}_{\mathbb{R}}(f)$ of the form

$$\psi(y) = \tilde{\phi} + cy^{\tan \tilde{\theta}_1}$$

with a generic number c . It then follows from Lemma 2.1(i) that

$$\text{ord } f(\psi(y), y) = x_1 \tan \tilde{\theta}_1 + y_1 \quad \text{and} \quad \text{ord } g(\psi(y), y) \leq x_2 \tan \tilde{\theta}_1 + y_2.$$

Therefore, in view of Claim 3.6(i),

$$\ell(\psi) \geq \frac{x_1 \tan \tilde{\theta}_1 + y_1}{x_2 \tan \tilde{\theta}_1 + y_2} = \frac{x_1 \tan \theta_1 + y_1}{x_2 \tan \theta_1 + y_2} = \ell(\phi) = \mathcal{L}_g(f),$$

which is a contradiction, because $\psi \in \mathcal{V}_{\mathbb{R}}(f)$.

We now take $0 \neq a \in \mathbb{R}$ such that $\tilde{\mathcal{E}}_1(a) = 0$ and define $\gamma(y) = \tilde{\phi} + ay^{\tan \tilde{\theta}_1}$. Then it follows from Lemma 2.1(i) that

$$\text{ord } f(\gamma(y), y) > x_1 \tan \tilde{\theta}_1 + y_1 \quad \text{and} \quad \text{ord } g(\gamma(y), y) \leq x_2 \tan \tilde{\theta}_1 + y_2.$$

Therefore

$$\ell(\gamma) > \frac{x_1 \tan \tilde{\theta}_1 + y_1}{x_2 \tan \tilde{\theta}_1 + y_2} = \ell(\phi) = \mathcal{L}_g(f),$$

a contradiction. Hence $\tan \tilde{\theta}_1 = \tan \tilde{\theta}_2$. □

(Claim33)

Claim 3.8. *If $\tilde{\phi}(y)$ is not a Newton–Puiseux root of f , then it and the Newton polygons of f and g relative to it share the following properties with that of ϕ :*

- (i) $\tan \tilde{\theta}_1 = \tan \tilde{\theta}_2$.
- (ii) *The polynomial $\tilde{\mathcal{E}}_1 \tilde{\mathcal{E}}_2$ has only one root. In particular, for each $i = 1, 2$, \tilde{E}_i is the highest Newton edge of $\tilde{\mathbb{P}}_i$.*
- (iii) $\ell(\tilde{\phi}) = \frac{x_1}{x_2} = \ell(\phi)$.

Proof. It is clear that Item (i) follows from Claim 3.7. Furthermore, Items (ii) and (iii) can be proved by using the same argument as in the proof of Claim 3.6. □

We are now in position to complete the theorem. Applying Claim 3.8 (possibly infinitely) many times, we obtain a final result ϕ_{∞} of sliding of ϕ along f which is also that of g . This implies that, ϕ_{∞} is a common Newton–Puiseux root of f and g of multiplicities x_1 and x_2 respectively. Therefore,

$$\ell(\phi) = \frac{x_1}{x_2} \leq \mathcal{L}_g^+(f).$$

This contradicts to the assumption that $\ell(\phi) > \mathcal{L}_g^+(f)$. Hence the theorem follows. \square

Remark 3.1. Note that the supremum in (2) may not be attained, i.e., it is possible that there is no analytic arc $\phi \in \mathbb{R}^2$ at the origin such that $\mathcal{L}_g(f) = \ell(\phi)$. The following example is an illustration.

Let

$$f(x, y) = x^2 \quad \text{and} \quad g(x, y) = x(x^2 + y^2).$$

Then $f(x, y) \geq g^2(x, y)$ for (x, y) closed enough to the origin. So $\mathcal{L}_g(f) \leq 2$. On the other hand, for each positive integer k , let $\phi_k(t) = (t, t^{1/k})$, then we have

$$\ell(\phi_k) = \frac{2}{1 + \frac{2}{k}} \rightarrow 2 \quad \text{as } k \rightarrow +\infty.$$

Therefore $\mathcal{L}_g(f) = 2$.

Now, for any analytic arc $\phi(t) = (x(t), y(t))$ at the origin, we have

$$\ell(\phi) = \frac{2 \operatorname{ord} x(t)}{\operatorname{ord} x(t) + 2 \min\{\operatorname{ord} x(t), \operatorname{ord} y(t)\}} < \frac{2 \operatorname{ord} x(t)}{\operatorname{ord} x(t)} = 2,$$

i.e., $\mathcal{L}_g(f)$ is not attained for any analytic arc $\phi \in \mathbb{R}^2$ at the origin.

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