Solvability and stability of switched discrete-time linear singular systems under Lipschitz perturbations

Do Duc Thuan[a] and Ninh Thi Thu[b]

[a]School of Applied Mathematics and Informatics, Hanoi University for Science and Technology, Hanoi, Vietnam; [b]Department of Mathematics, Mechanics and Informatics, Vietnam National University, Hanoi, Vietnam.

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ABSTRACT
In this paper, the problem of solvability and stability for switched discrete-time linear singular (SDLS) systems under Lipschitz perturbations is studied. We first prove the unique existence of solution of SDLS systems under Lipschitz perturbations with different switching rules on two sides. The solution manifold is also described. Secondly, we derive some conditions for stability of these systems. Finally, some examples are given to illustrate the obtained results.

KEYWORDS
SDLS systems, index, solvability, stability, Lipschitz perturbation.

1. Introduction

In this paper we study solvability and stability of switched discrete-time linear singular (SDLS) systems of the form

$$E_{\sigma(k+1)}x(k+1) = A_{\sigma(k)}x(k) + f_{\sigma(k)}(x(k)), \tag{1.1}$$

where $\sigma : \mathbb{N} \cup \{0\} \to \underline{N} := \{1, 2, \ldots, N\}, N \in \mathbb{N}$, denotes the switching signal that determines which of the $N \in \mathbb{N}$ modes is active at time k.

Singular switched systems are models arising in diverse real-life applications such as power electronics and systems, air traffic and aircraft control, network control systems, robot manipulators, multibody systems, economic systems and so forth, (see, e.g. [6, 12, 16, 18, 20]). These systems consist of a family of singular subsystems and a rule that controls the switching between them which in recent years have attracted a good deal of attention from researchers. On the other hand, the advent of many modern-day sampled-data control systems (or the dynamic Leontief system in economic) has necessitated a study of discrete-time singular systems because they can only change at discrete instants of time (see, e.g. [5, 11, 13–15, 17, 19]). These lead switched discrete-time singular systems. They can also be obtained from switched continuous-time singular systems by some discretization methods.

CONTACT Corresponding Author: D.D. Thuan, email: thuan.doduc@hust.edu.vn, ducthuank7@gmail.com

Recently, we have investigated solvability and stability of SDLS systems of the form $E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k)$ in [2, 3], where the switching rules in matrices E and A are same. If the switching rules in matrices E and A are not same then it is more complicated. In [9], some first results on this case have considered for homogenous SDLS systems which have no perturbations. However, to the best of our knowledge, there are still no results about solvability and stability for SDLS systems of the form (1.1) under Lipschitz perturbations f.

The purpose of the present paper is to fill this gap. We will consider SDLS systems of the form (1.1) with the different switching rules in matrices E and A. The singularity of the leading coefficients make the analysis of system (1.1) difficult since computation of solutions is impossible at first sight. Even the solvability of the initial value problem is doubtful. Due to the fact that the dynamics of (1.1) are constrained and combined between singular systems, some extra difficulties appear in the analysis of solvability as well as stability characterized by index concepts of singular systems (see, e.g. [4, 8, 10]). Thus, in this paper, we will develop and modify the approach in [1, 3, 9] to investigate solvability and stability of SDLS systems under Lipschitz perturbations. The unique existence of solution of (1.1) will be proved by using the contraction mapping principle. After that characterizations for stability of (1.1) will be derived by using methods of the Lyapunov functions and the solution evaluation.

The paper is organized as follows. In Section 2, we summarize some preliminary results of SDLS systems of index-1 and the discrete Gronwal inequality. In Section 3, we study solvability and a formula of solution of SDLS systems under Lipschitz perturbations. Section 4 deals with stability of these systems. The last section gives some conclusions.

2. Preliminary

For $N \in \mathbb{N}$, denote $\underline{N} = \{1, 2, \dots, N\}$ and O by the zero matrix. Consider the homogeneous SDLS systems

$$E_{\sigma(k+1)}x(k+1) = A_{\sigma(k)}x(k) \tag{2.1}$$

is of index-1 ([4], [9]), i.e., the folllowing hypotheses are assumed to be fulfilled:

(i) $\operatorname{rank} E_i = r < n, \forall i \in \underline{N}$,

(ii) $S_{ij} \cap \ker E_i = \{0\}, \forall i, j \in \underline{N}$, where $S_{ij} = A_i^{-1}(\mathrm{ImE_j}) = \{\xi \in \mathbb{R}^n : A_i\xi \in \mathrm{ImE_j}\}$.

It is proved that from hypothesis (ii) we have

$$S_{ij} \oplus \ker E_i = \mathbb{R}^n, \forall i, j \in \underline{N},$$

see, e.g. [3, 9]. Let the matrix $V_{ij} = \{s_{ij}^1, \dots, s_{ij}^r, h_i^{r+1}, \dots, h_i^n\}$, whose columns form bases of S_{ij} and $\ker E_i$, respectively, and $Q = \operatorname{diag}(O_r, I_{n-r}), P = I_n - Q$. Here O_r is the $r \times r$ zero matrix and I_{n-r} stands for the $(n-r) \times (n-r)$ identity matrix. Then the matrix $Q_{ij} := V_{ij}QV_{ij}^{-1}$ defines a projection onto $\ker E_i$ along S_{ij} (i.e., $Q_{ij}^2 = Q_{ij}$ and $\mathrm{ImQ_{ij}} = \ker \mathrm{E_i}$), and $P_{ij} := I_n - Q_{ij} = V_{ij}PV_{ij}^{-1}$ is the projection onto S_{ij} along $\ker E_i$. Further we define the so-called connecting operators $Q_{ijm} := V_{ij}QV_{jm}^{-1}$.

Theorem 2.1. *([9]). For switched discrete-time linear singular homogeneous system of index-1* (2.1)*, the following assertions hold:*

(i) $G_{ijm} = E_j + A_i Q_{ijm}$ *is non-singular ;*
(ii) $E_j P_{jm} = E_j$*;*
(iii) $P_{jm} = G_{ijm}^{-1} E_j$*;*
(iv) $V_{jm}^{-1} G_{ijm}^{-1} A_i V_{ij} Q = Q$*.*

for all $i, j, m \in \underline{N}$*.*

We need to use the following discrete Gronwall inequality to study the exponential stability of SDLS systems in Section 4.

Theorem 2.2. *([7]) Assume that* $\{y_m\}, \{f_m\}, \{g_m\}$ *are nonnegative sequences such that*

$$y_m \leq f_m + \sum_{0 \leq i < m} g_i y_i, \forall m \geq 0.$$

Then

$$y_m \leq f_m + \sum_{0 \leq i < m} f_i g_i \prod_{i < j < m} (1 + g_j).$$

3. Solvability

Consider a switched discrete-time singular system of the form:

$$E_{\sigma(k+1)} x(k+1) = A_{\sigma(k)} x(k) + f_{\sigma(k)}(x(k)) \tag{3.1}$$

where $\sigma : \mathbb{N} \cup \{0\} \to \underline{N}$, is a switching signal taking values in the finite set $\underline{N}$, $E_i, A_i \in \mathbb{R}^{n \times n}$ and $f_i : \mathbb{R}^n \to \mathbb{R}^n, i \in \underline{N}$, are perturbations, $x(k) \in \mathbb{R}^n$ is state vector at time $k \in \mathbb{N}$. Suppose that the matrices E_i are singular for all $i \in \underline{N}$. Let us associate system (3.1) with the initial condition

$$P_{\sigma(k_0)\sigma(k_0+1)} x(k_0) = P_{\sigma(k_0)\sigma(k_0+1)} \gamma, \tag{3.2}$$

where γ is a given vector in $\mathbb{R}^n$ and k_0 is a fixed nonnegative integer.

Theorem 3.1. *Let* $f_{\sigma(k)}(x)$ *be a Lipschitz continuous function with a sufficient small Lipschitz coefficient, i.e.,*

$$\|f_i(x) - f_i(\tilde{x})\| \leq L_i \|x - \tilde{x}\|, \forall x, \tilde{x} \in \mathbb{R}^n, i \in \underline{N}, \tag{3.3}$$

and

$$\omega_i := L_i \max\{\|Q_{ijm} G_{ijm}^{-1}\| : j, m \in \underline{N}\} < 1, \forall i \in \underline{N}. \tag{3.4}$$

Then the IVP (3.1), (3.2) *has a unique solution.*

Proof. Multiplying on both sides of equation (3.1) from the left by

$$P_{\sigma(k+1)\sigma(k+2)} G_{\sigma(k)\sigma(k+1)\sigma(k+2)}^{-1} \text{ and } Q_{\sigma(k+1)\sigma(k+2)} G_{\sigma(k)\sigma(k+1)\sigma(k+2)}^{-1},$$

respectively and observing that

$$\begin{aligned}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}E_{\sigma(k+1)} &= P_{\sigma(k+1)\sigma(k+2)},\\ P_{\sigma(k+1)\sigma(k+2)}Q_{\sigma(k+1)\sigma(k+2)} &= Q_{\sigma(k+1)\sigma(k+2)}P_{\sigma(k+1)\sigma(k+2)} = O,\end{aligned}$$

we get

$$\begin{aligned}P_{\sigma(k+1)\sigma(k+2)}x(k+1) = P_{\sigma(k+1)\sigma(k+2)}&G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}x(k)\\ &+ P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}f_{\sigma(k)}(x(k)),\end{aligned} \tag{3.5}$$

$$Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}x(k) = -Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}f_{\sigma(k)}(x(k)). \tag{3.6}$$

Let $u(k) = P_{\sigma(k)\sigma(k+1)}x(k), v(k) = Q_{\sigma(k)\sigma(k+1)}x(k), (k \in \mathbb{N})$ we get

$$\begin{aligned}&P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}v(k)\\ &= P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}Q_{\sigma(k)\sigma(k+1)}x(k)\\ &= P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}Q_{\sigma(k)\sigma(k+1)\sigma(k+2)}V_{\sigma(k+1)\sigma(k+2)}QV^{-1}_{\sigma(k)\sigma(k+1)}x(k)\\ &= P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}(G_{\sigma(k)\sigma(k+1)\sigma(k+2)} - E_{\sigma(k+1)})V_{\sigma(k+1)\sigma(k+2)}QV^{-1}_{\sigma(k)\sigma(k+1)}x(k)\\ &= (P_{\sigma(k+1)\sigma(k+2)} - P_{\sigma(k+1)\sigma(k+2)}P_{\sigma(k+1)\sigma(k+2)})V_{\sigma(k+1)\sigma(k+2)}QV^{-1}_{\sigma(k)\sigma(k+1)}x(k)\\ &= 0,\end{aligned}$$

and from (3.5)

$$\begin{aligned}u(k+1) &= P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}(u(k)+v(k))\\ &\qquad + P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}f_{\sigma(k)}(u(k)+v(k))\\ &= P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}u(k)\\ &\qquad + P_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}f_{\sigma(k)}(u(k)+v(k)).\end{aligned} \tag{3.7}$$

By item (iv) of Theorem 2.1,

$$Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}Q_{\sigma(k)\sigma(k+1)} = V_{\sigma(k+1)\sigma(k+2)}QV^{-1}_{\sigma(k)\sigma(k+1)}.$$

Therefore, the left side of (3.6) can be expressed as

$$\begin{aligned}&Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}x(k) = Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}(u(k)+v(k))\\ &= Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}u(k) + V_{\sigma(k+1)\sigma(k+2)}QV^{-1}_{\sigma(k)\sigma(k+1)}x(k).\end{aligned}$$

Hence, it follows from (3.6) that

$$V_{\sigma(k+1)\sigma(k+2)}QV^{-1}_{\sigma(k)\sigma(k+1)}x(k)$$
$$= -Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}u(k) - Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}f_{\sigma(k)}(x(k)).$$

Now multiplying on both sides of this relation by $Q_{\sigma(k)\sigma(k+1)\sigma(k+2)}$ from the left we obtain

$$\begin{aligned}
v(k) = Q_{\sigma(k)\sigma(k+1)}x(k) &= Q_{\sigma(k)\sigma(k+1)\sigma(k+2)}V_{\sigma(k+1)\sigma(k+2)}QV^{-1}_{\sigma(k)\sigma(k+1)}x(k) \\
&= -Q_{\sigma(k)\sigma(k+1)\sigma(k+2)}Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}u(k) \\
&\qquad - Q_{\sigma(k)\sigma(k+1)\sigma(k+2)}Q_{\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}f_{\sigma(k)}(x(k)) \\
&= -Q_{\sigma(k)\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}A_{\sigma(k)}u(k) \\
&\qquad - Q_{\sigma(k)\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}f_{\sigma(k)}(x(k)) \\
&= -Q_{\sigma(k)\sigma(k+1)\sigma(k+2)}G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)}(f_{\sigma(k)}(u(k)+v(k)) + A_{\sigma(k)}u(k)). \qquad (3.8)
\end{aligned}$$

By equation (3.7), suppose that $u := u(k)(k \geq k_0)$ is known, where

$$u(k_0) = P_{\sigma(k_0)\sigma(k_0+1)}x(k_0) = P_{\sigma(k_0)\sigma(k_0+1)}\gamma$$

is given. We consider an operator $T_{ijm} : \mathrm{Im}Q_{ij} \to \mathrm{Im}Q_{ij}$ defined by

$$T_{ijm}(v) := -Q_{ijm}G^{-1}_{ijm}[f_i(u+v) + A_i u].$$

Since

$$\begin{aligned}
\|T_{ijm}(v) - T_{ijm}(\tilde{v})\| &= \|Q_{ijm}G^{-1}_{ijm}[f_i(u+v) - f_i(u+\tilde{v})\| \\
&\leq \|Q_{ijm}G^{-1}_{ijm}\|\|f_i(u+v) - f_i(u+\tilde{v})\| \\
&\leq \|Q_{ijm}G^{-1}_{ijm}\|L_i\|v-\tilde{v}\| \leq \omega_i\|v-\tilde{v}\| < \|v-\tilde{v}\|,
\end{aligned}$$

the operator T_{ijm} is contractive. Therefore equation (3.8) has a unique solution given by a mapping $g_{\sigma(k)\sigma(k+1)} : \mathrm{Im}P_{\sigma(k)\sigma(k+1)} \to \mathrm{Im}Q_{\sigma(k)\sigma(k+1)}, g_{\sigma(k)\sigma(k+1)}(u(k)) = v(k)$. Moreover, it is easy to show that $g_{\sigma(k)\sigma(k+1)}$ is a Lipschitz continuous mapping having the Lipschitz constant

$$K_{\sigma(k)} := \omega_{\sigma(k)}(L_{\sigma(k)} + \|A_{\sigma(k)}\|)L^{-1}_{\sigma(k)}(1-\omega_{\sigma(k)})^{-1}. \qquad (3.9)$$

Thus, the IVP (3.1), (3.2) has a unique solution given by

$$x(k) = u(k) + g_{\sigma(k)\sigma(k+1)}(u(k)), \qquad (3.10)$$

with $u(k_0) = P_{\sigma(k_0)\sigma(k_0+1)}\gamma$. The proof is complete. □

We define the Cauchy operator associated with system (3.1)

$$\Phi_\sigma(k,h) = \prod_{l=h+1}^{k} P_{\sigma(l)\sigma(l+1)} G^{-1}_{\sigma(l-1)\sigma(l)\sigma(l+1)} A_{\sigma(l-1)} \text{ and } \Phi_\sigma(h,h) = P_{\sigma(h)\sigma(h+1)}. \tag{3.11}$$

Then, it is easy to see that $\Phi_\sigma(k,h)$ satisfies the relation

$$\Phi_\sigma(k,h) = \Phi_\sigma(k,l)\Phi_\sigma(l,h), \ \forall k \geq l \geq h.$$

Now, the variation of constants formula for the solution of system (3.1) is derived in the following corollary.

Corollary 3.2. *The unique solution of system* (3.1) *with the initial conditions* (3.2) *satisfies the equation*

$$\begin{aligned} x(k) =& \Phi_\sigma(k,k_0) P_{\sigma(k_0)\sigma(k_0+1)}\gamma + \sum_{i=k_0}^{k-1} \Phi_\sigma(k,i+1) P_{\sigma(i+1)\sigma(i+2)} G^{-1}_{\sigma(i)\sigma(i+1)\sigma(i+2)} f_{\sigma(i)}(x(i)) \\ &- Q_{\sigma(k)\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} (f_{\sigma(k)}(x(k)) + A_{\sigma(k)} P_{\sigma(k)\sigma(k+1)} x(k)). \end{aligned} \tag{3.12}$$

Proof. By equation (3.7), we imply that the solution $u(k)$ is given by the formula

$$u(k) = \Phi_\sigma(k,k-1) P_{\sigma(k_0)\sigma(k_0+1)}\gamma + \sum_{i=k_0}^{k-1} \Phi_\sigma(k,i+1) P_{\sigma(i+1)\sigma(i+2)} G^{-1}_{\sigma(i)\sigma(i+1)\sigma(i+2)} f_{\sigma(i)}(x(i))$$

and by equation (3.8), we have

$$v(k) = -Q_{\sigma(k)\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} (f_{\sigma(k)}(x(k)) + A_{\sigma(k)} P_{\sigma(k)\sigma(k+1)} x(k)).$$

Since $x(k) = u(k) + v(k)$, we obtain formula (3.12). □

In what follows, without loss of generality, assume that $f_i(0) = 0, \forall i \in \underline{N}$. This implies that $g_{\sigma(k)\sigma(k+1)}(0) = 0$ and equation (3.1) possesses a trivial solution $x(k) \equiv 0$. It follows from (3.10) that each solution $x(k)$ of the IVP (3.1), (3.2) satisfies $x(k) = P_{\sigma(k)\sigma(k+1)} x(k) + g_{\sigma(k)\sigma(k+1)}(P_{\sigma(k)\sigma(k+1)} x(k))$ or equivalently,

$$Q_{\sigma(k)\sigma(k+1)} x(k) = -Q_{\sigma(k)\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} (f_{\sigma(k)}(x(k)) + A_{\sigma(k)} P_{\sigma(k)\sigma(k+1)} x(k)).$$

Let

$$\Delta_i := \{x \in \mathbb{R}^n : Q_{ij}x = -Q_{ijm} G^{-1}_{ijm}(f_i(x) + A_i P_{ij} x), \text{ for some } j,m \in \underline{N}\}. \tag{3.13}$$

If $x = x(k)$ is any solution of the IVP (3.1), (3.2), then obviously, $x(k) \in \Delta_{\sigma(k)} (k \geq k_0)$. Conversely, for each $\theta \in \Delta_i$, there exists a solution of (3.1) passing θ. Indeed, let σ be a switching signal satisfying $\sigma(k) = i$ and $x(m,k;\theta)(m \geq k)$ be a solution of (3.1) satisfying the initial condition $P_{\sigma(k)\sigma(k+1)}x(k) = P_{\sigma(k)\sigma(k+1)}\theta$. Clearly,

$$\begin{aligned} x(k,k;\theta) &= P_{\sigma(k)\sigma(k+1)}x(k) + g_{\sigma(k)\sigma(k+1)}(P_{\sigma(k)\sigma(k+1)}x(k)) \\ &= P_{\sigma(k)\sigma(k+1)}\theta + g_{\sigma(k)\sigma(k+1)}(P_{\sigma(k)\sigma(k+1)}\theta) \\ &= P_{\sigma(k)\sigma(k+1)}\theta + Q_{\sigma(k)\sigma(k+1)}\theta = \theta. \end{aligned} \tag{3.14}$$

We will prove that the set Δ_i does not depend on the choice of projections in the following proposition.

Proposition 3.3. *Let the solution manifold Δ_i be defined in* (3.13). *Then, the following hold:*

(i) $\Delta_i = \{x \in \mathbb{R}^n : f_i(x) + A_i x \in \mathrm{Im} E_j,\ \textit{for some } j \in \underline{N}\}$.
(ii) $\Delta_i \cap \ker E_i = \{0\}$.

Proof. i) Letting $x \in \Delta_i$, then there exists $j, m \in \underline{N}$ such that

$$Q_{ij}x = -Q_{ijm}G_{ijm}^{-1}(f_i(x) + A_i P_{ij}x),$$

hence

$$x = P_{ij}x + Q_{ij}x = -Q_{ijm}G_{ijm}^{-1}f_i(x) + (I - Q_{ijm}G_{ijm}^{-1}A_i)P_{ij}x.$$

From this relation we have

$$f_i(x) + A_i x = (I - A_i Q_{ijm}G_{ijm}^{-1})f_i(x) + A_i(I - Q_{ijm}G_{ijm}^{-1}A_i)P_{ij}x.$$

Note that

$$A_i(I - Q_{ijm}G_{ijm}^{-1}A_i)P_{ij}x = (I - A_i Q_{ijm}G_{ijm}^{-1})A_i P_{ij}x.$$

Therefore

$$f_i(x) + A_i x = (I - A_i Q_{ijm}G_{ijm}^{-1})(f_i(x) + A_i P_{ij}x).$$

Since

$$A_i Q_{ijm}G_{ijm}^{-1} = (G_{ijm} - E_j)G_{ijm}^{-1} = I - E_j G_{ijm}^{-1},$$

it follows that

$$f_i(x) + A_i x = E_j G_{ijm}^{-1}\{f_i(x) + A_i P_{ij}x\} \in \mathrm{Im}E_j.$$

Hence $x \in \Delta_i$.

Conversely, let $x \in \mathbb{R}^n$ such that $f_i(x) + A_i x \in \mathrm{Im}E_j$ for some $j \in \underline{N}$. Then there exists $\xi \in \mathbb{R}^n, j \in \underline{N}$ such that $f_i(x) + A_i x = E_j \xi$. We will prove that for $m \in \underline{N}$,

$$Q_{ij}x = -Q_{ijm}G_{ijm}^{-1}(f_i(x) + A_i P_{ij}x),$$

or equivalent

$$x = -\,Q_{ijm}G_{ijm}^{-1}(f_i(x)+A_ix)+Q_{ijm}G_{ijm}^{-1}A_iQ_{ij}x+P_{ij}x.$$

Denoting the right-hand side of this relation by w_{ij} and note that

$$\begin{aligned} Q_{ijm}^{-1}G_{ijm}^{-1}(f_i(x)+A_ix) &= Q_{ijm}^{-1}G_{ijm}^{-1}E_j\xi = Q_{ijm}^{-1}P_{jm}\xi \\ &= V_{ij}QV_{jm}^{-1}V_{jm}PV_{jm}^{-1}\xi = V_{ij}QPV_{jm}^{-1}\xi = 0, \end{aligned}$$

by Theorem 2.1 we get

$$\begin{aligned} w_{ij} &= Q_{ijm}G_{ijm}^{-1}A_iQ_{ij}x+P_{ij}x \\ &= Q_{ijm}^{-1}G_{ijm}^{-1}A_iV_{ij}QV_{jm}^{-1}V_{jm}QV_{ij}^{-1}x+P_{ij}x \\ &= Q_{ijm}G_{ijm}^{-1}(G_{ijm}-E_j)V_{jm}QV_{ij}^{-1}x+P_{ij}x \\ &= Q_{ijm}V_{jm}QV_{ij}^{-1}x-Q_{ijm}G_{ijm}^{-1}E_jV_{jm}QV_{ij}^{-1}x+P_{ij}x \\ &= V_{ij}QV_{jm}^{-1}V_{jm}QV_{ij}^{-1}x-V_{ij}QV_{jm}^{-1}P_{jm}V_{jm}QV_{ij}^{-1}x+P_{ij}x \\ &= V_{ij}QQV_{ij}^{-1}x-V_{ij}QPQV_{ij}^{-1}x+P_{ij}x \\ &= Q_{ij}x+P_{ij}x = x. \end{aligned}$$

Thus, $x \in \Delta_i$ and the item (i) of Lemma 3.3 is proved.

(ii) Let $x \in \Delta_i \cap \ker E_i$. Then we have $x \in \Delta_i$ and $P_{ij}x = 0$ for all $j \in \underline{N}$. Since $x \in \Delta_i$, it implies that

$$Q_{ij}x = g_{ij}(P_{ij}x) = 0$$

and hence

$$x = P_{ij}x + Q_{ij}x = 0.$$

The proof is complete. □

Since $G_{\sigma(k_0+2)\sigma(k_0)\sigma(k_0+1)}^{-1}E_{\sigma(k_0)} = P_{\sigma(k_0)\sigma(k_0+1)}$, it is easy to see that the initial condition (3.2) is equivalent to the condition

$$E_{\sigma(k_0)}x(k_0) = E_{\sigma(k_0)}\gamma, \forall k_0 \in \mathbb{N}. \tag{3.15}$$

which is independent of the choice of projections. Thus both initial conditions (3.2) and (3.15) are equivalent for all $k_0 \in N$. The unique solution of the IVP (3.1), (3.2) or (3.1), (3.15) will be denoted by $x(k) = x(k, k_0; \gamma)$.

4. Stability

In this section the notions of stability of trivial solution are introduced and the necessary and sufficient conditions for stability of SDLS systems are established.

Definition 4.1. *The trivial solution of* (3.1) *is said to be* *(i) stable if for each $\epsilon > 0$, any $k_0 \geq 0$ and for all switching signals there exists a $\delta = \delta(\epsilon, k_0) \in (0, \epsilon]$ such that $\|P_{\sigma(k_0)\sigma(k_0+1)}\gamma\| < \delta$ implies $\|x(k, k_0; \gamma)\| < \epsilon$ for all $k \geq k_0$, uniformly stable if it is stable and δ does not depend on k_0;*

(ii) asymptotically stable if it is stable and for any $k_0 \geq 0$ and for all switching signals there exists a $\delta = \delta(k_0) > 0$ such that the inequality $\|P_{\sigma(k_0)\sigma(k_0+1)}\gamma\| < \delta$ implies $\|x(k, k_0; \gamma)\| \to 0$ as $k \to +\infty$);

(iii) exponentially stable if there exist $M > 0, 0 < \lambda < 1$ such that $\|x(k, k_0; \gamma)\| \leq M\lambda^{k-k_0}\|P_{\sigma(k_0)\sigma(k_0+1)}\gamma\|$ for all $k \geq k_0$ and switching signals.

Remark 4.2. *In the above definition, if replacing the initial condition $P_{\sigma(k_0)\sigma(k_0+1)}\gamma$ by $E_{\sigma(k_0)}\gamma$ then we get notions of E-stability, E-asymptotical stability and E-exponential stability (respectively). However, since the relation $G^{-1}_{ijm}E_j = P_{jm}$ and $E_jP_{jm} = E_j$ for all $i, j, k \in \underline{N}$, it is easy to show that they are equivalent to above notions (respectively).*

Denote by $\mathcal{K}$ the class of all increasing functions ψ from $[0, \infty)$ into itself such that $\psi(0) = 0, \psi(x) > 0$ for $x \neq 0$ and $\lim\limits_{x \to 0^+} \psi(x) = 0$.

Lemma 4.3. *The trivial solution of* (3.1) *is stable if and only if there exists a function $\psi \in \mathcal{K}$, such that for each nonnegative integer k_0 and for all switching signals, there holds the inequality*

$$\|x(k)\| \leq \psi(\|x(k_0)\|), \quad \forall k \geq k_0. \tag{4.1}$$

Proof. Suppose first that for all switching signals and for each nonnegative integer k_0, there exists a function $\psi \in \mathcal{K}$ satisfying condition (4.1). Since ψ is increasing and continuous at 0, for each positive ϵ there exists $\delta = \delta(\epsilon) \in (0, \epsilon]$ such that $\psi(\delta) < \epsilon$. Let $K = \max_{i \in \underline{N}} K_i$, where K_i is given by (3.9). If $x(k)$ is an arbitrary solution of (3.1) satisfying $|P_{\sigma(k_0)\sigma(k_0+1)}x(k_0)\| < \delta_1 := \dfrac{\delta}{K+1}$ then

$$\begin{aligned}\|x(k_0)\| &= \|P_{\sigma(k_0)\sigma(k_0+1)}x(k_0) + g_{\sigma(k_0)\sigma(k_0+1)\sigma(k_0+2)}(P_{\sigma(k_0)\sigma(k_0+1)}x(k_0))\| \\ &\leq \|P_{\sigma(k_0)\sigma(k_0+1)}x(k_0)\| + \|g_{\sigma(k_0)\sigma(k_0+1)\sigma(k_0+2)}(P_{\sigma(k_0)\sigma(k_0+1)}x(k_0))\| \\ &\leq \|P_{\sigma(k_0)\sigma(k_0+1)}x(k_0)\|(1 + K_{\sigma(k_0)}) \leq \|P_{\sigma(k_0)\sigma(k_0+1)}x(k_0)\|(1 + K) < \delta.\end{aligned} \tag{4.2}$$

This implies that

$$\|x(k)\| \leq \psi(\|x(k_0)\|) \leq \psi(\delta) < \epsilon, \quad \forall k \geq k_0, \forall \sigma,$$

which implies that trivial solution of (3.1) is stable.

Conversely, suppose that the trivial solution of (3.1) is stable, i.e., for each positive ϵ there exists a $\delta = \delta(\epsilon) \in (0, \epsilon]$, such that if $x(k)$ is any solution of (3.1) satisfying the inequality $\|P_{\sigma(k_0)\sigma(k_0+1)}x(k_0)\| < \delta$ for all switching signals then $\|x(k)\| < \epsilon$ for all $k \geq k_0$. Denote by $\alpha(\epsilon)$ the supremum of such $\delta(\epsilon)$. Clearly, if $\|P_{\sigma(k_0)\sigma(k_0+1)}x(k_0)\| < \alpha(\epsilon)$ for some k_0 and for all σ, then $\|x(k)\| < \epsilon$ for all $k \geq k_0$. Further, the function $\alpha(\epsilon)$ is positive and increasing and moreover, $\alpha(\epsilon) \leq \epsilon$. Putting $\beta(\epsilon) := \dfrac{\epsilon\alpha(\epsilon)}{(\epsilon+1)H}$ for $\epsilon \geq 0$, where $H := \max\{\|P_{ij}\| : i, j \in \underline{N}\}$. It is easy to see that $0 < \beta(\epsilon) < \dfrac{\alpha(\epsilon)}{H} \leq \dfrac{\epsilon}{H}$, β is strictly increasing and continuous at 0. Then there exists the strictly increasing inverse of β from $\operatorname{Im}\beta$ to $[0,\infty)$ which can be expanded to $\psi \in \mathcal{K}$. Let $x(k)$ be a solution of (3.1) and k_0 be a fixed nonegative integer. Set $\epsilon_k := \|x(k)\|$ and consider two possibilities. If $\|x(k)\| = 0$ then $\|x(k)\| = 0 \le \psi(\|x(k_0)\|)$ since ψ is nonnegative. Now suppose that $\epsilon_k := \|x(k)\| > 0$. If $\|x(k_0)\| < \beta(\epsilon_k)$ then

$$\|P_{\sigma(k_0)\sigma(k_0+1)}x(k_0)\| \le H\beta(\epsilon_k) < \alpha(\epsilon_k).$$

This implies that $\|x(k)\| < \epsilon_k = \|x(k)\|, \forall k \ge k_0$, which is contradiction. Therefore $\|x(k_0)\| \ge \beta(\epsilon_k)$ which is equivalent to

$$\|x(k)\| = \epsilon_k \le \beta^{-1}(\|x(k_0)\|) = \psi(\|x(k_0)\|).$$

The proof is complete. □

Remark 4.4. *The above lemma is developed and modified from Lemma 3.3 in [1]. Here, ψ is a function of $\|x(k_0)\|$ which doesn't depend of the choice of projections and $\psi \in \mathcal{K}$ containing the class of all continuous and strictly increasing functions $\hat{\psi}$ from $[0,\infty)$ into itself, such that $\hat{\psi}(0) = 0$. Moreover, to prove converse, we have constructed the function ψ which is different from Lemma 3.3 in [1].*

Theorem 4.5. *The existence of the Lyapunov functions $V_\sigma : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_+$ being continuous in the second variable at $\gamma = 0$ and the functions $a, \psi_k \in \mathcal{K}$, such that*

(i) $a(\|y\|) \le V_\sigma(k,y) \le \psi_k(\|y\|),\ \forall k \ge 0, \forall y \in \Delta_{\sigma(k)}, \forall\sigma,$

(ii) $\Delta V_\sigma(k,y(k)) := V_\sigma(k+1,y(k+1)) - V_\sigma(k,y(k)) \le 0, \forall k \ge 0, \forall \sigma$, *for any solution* $y(k)$ *of* (3.1) *corresponding* σ,

is a necessary and sufficient condition for the stability of the trivial solution of the SDLS system (3.1).

Proof. *Necessity. Suppose that the trivial solution of* (3.1) *is stable. For each k_0, then according to Lemma 4.3, there exist functions $\psi_{k_0} \in \mathcal{K}$ $(k_0 \ge 0)$, such that for any solution $x(k)$ of* (3.1),

$$\|x(k)\| \le \psi_{k_0}(\|x(k_0)\|),\ \forall k \ge k_0, \forall \sigma. \tag{4.3}$$

We define the Lyapunov function

$$V_\sigma(k_0,\gamma) := \sup_{m\in\mathbb{N}} \|x_\sigma(k_0+m,k_0;\gamma)\|,\ \textit{for each } \gamma \in \mathbb{R}^n, k_0 \in \mathbb{N}, \tag{4.4}$$

where $x_\sigma(k_0+m,k_0;\gamma)$ is the unique solution of (3.1) *corresponding to switching signal σ satisfying the initial condition $P_{\sigma(k_0)\sigma(k_0+1)}x_\sigma(k_0) = P_{\sigma(k_0)\sigma(k_0+1)}\gamma$. Inequality* (4.3) *ensures the correctness of definition* (4.4). *By* (4.2), *we have*

$$\|x_\sigma(k_0)\| \le (K+1)\|P_{\sigma(k_0)\sigma(k_0+1)}x_\sigma(k_0)\| = (K+1)\|P_{\sigma(k_0)\sigma(k_0+1)}\gamma\| \le (K+1)H\|\gamma\|,$$

where the constants K, H are given Lemma 4.3. Define $\hat{\psi}_{k_0}(t) := \psi_{k_0}((K+1)Ht)$ for $t \ge 0$. Then we imply that

$$V_\sigma(k_0,\gamma) \le \psi_{k_0}(\|x_\sigma(k_0)\|) \le \psi_{k_0}((K+1)H\|\gamma\|) = \hat{\psi}_{k_0}(\|\gamma\|), \forall k_0 \ge 0, \forall \gamma \in \mathbb{R}^n, \forall \sigma.$$

This implies that $V_\sigma(k_0, 0) = 0$ and the continuity of the function V w.r.t the second variable at $\gamma = 0$. For each $y \in \Delta_{\sigma(k_0)}$, by (3.14), we have

$$V_\sigma(k_0, y) = \sup_{l \in \mathbb{N}} \|x_\sigma(k_0 + l, k_0; y)\| \geq \|x_\sigma(k_0, k_0; y)\| = \|y\| := a(\|y\|). \tag{4.5}$$

On the other hand, for each $k_0 \geq 0$ due to the unique solvability of (3.1)-(3.2), it is easy to see that

$$\begin{aligned} \{x_\sigma(k_0 + l, k_0; y(k_0)) : l \geq 0\} &= \{y(k_0 + l) : l \geq 0)\} \\ \supset \{y(k_0 + l) : l \geq 1)\} &\supset \{x_\sigma(k_0 + 1 + l, k_0 + 1; y(k_0 + 1)) : l \geq 0\}, \end{aligned} \tag{4.6}$$

where $\sigma_y(k)$ is the switching signal corresponding $y(k)$. Thus

$$\begin{aligned} V_\sigma(k+1, y(k+1)) &= \sup_{l \geq 0} \|x_\sigma(k + 1 + l, k + 1; y(k + 1))\| \\ &\leq \sup_{l \geq 0} \|x_\sigma(k + l, k; y(k))\| = V_\sigma(k, y(k)), \end{aligned}$$

which implies $\Delta V_\sigma(k, y(k)) \leq 0$. The necessity part is proved.

Sufficiency. We argue by contradiction by assuming that trivial solution of (3.1) is not stable, i.e., there exist a positive ϵ_0, a nonnegative integer k_0 and a switching signal σ, such that for all $\delta \in (0, \epsilon_0]$, there exists a solution $x_\sigma(k)$ of (3.1) satisfying the inequalities $\|P_{\sigma(k_0)\sigma(k_0+1)} x_\sigma(k_0)\| < \delta$ and $\|x_\sigma(k_1)\| \geq \epsilon_0$ for some $k_1 \geq k_0$.

Since $V_\sigma(k_0, 0) = 0$ and $V_\sigma(k_0, \gamma)$ is continuous at $\gamma = 0$, there exists a $\delta_0' = \delta_0'(\epsilon, k_0) > 0$, such that for all $\xi \in \mathbb{R}^n$, $\|\xi\| < \delta_0'$ and for all σ we have $V_\sigma(k_0, \xi) < \epsilon_1 := a(\epsilon_0)$. Choosing $\delta_0 \leq \{\frac{\delta_0'}{K+1}, \epsilon_0\}$ we can find solution $x_\sigma(k)$ of (3.1) satisfying $\|P_{\sigma(k_0)\sigma(k_0+1)} x_\sigma(k_0)\| < \delta_0$, however $\|x_\sigma(k_1)\| \geq \epsilon_0$ for some $k_1 \geq k_0$. Since $\|P_{\sigma(k_0)\sigma(k_0+1)} x_\sigma(k_0)\| < \delta_0 \leq \frac{\delta_0'}{K+1}$, $\|x_\sigma(k_0)\| < \delta_0'$ and one gets $V_\sigma(k_0, x_\sigma(k_0)) < \epsilon_1$. On the other hand, using the properties of the function V, we find

$$V_\sigma(k_0, x_\sigma(k_0)) \geq V_\sigma(k_1, x_\sigma(k_1)) \geq a(\|x_\sigma(k_1)\|) \geq a(\epsilon_0) = \epsilon_1,$$

which leads to a contradiction. The proof of Theorem 4.5 is complete. □

If the trivial solution of (3.1) is uniformly stable then the function ψ_k in the above theorem can be chosen independently on k. Therefore, a similar argument as in the above proof leads to the next result.

Theorem 4.6. *The trivial solution of (3.1) is uniformly stable if and only if there exist two functions $a, b \in \mathcal{K}$ and the Lyapunov functions $V_\sigma : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_+$, such that*

(i) $a(\|y\|) \leq V_\sigma(k, y) \leq b(\|y\|)$, $\forall k \geq 0, \forall y \in \Delta_{\sigma(k)}, \forall \sigma$,
(ii) $\Delta V_\sigma(k, y(k)) := V_\sigma(k+1, y(k+1)) - V_\sigma(k, y(k)) \leq 0, \forall k \geq 0, \forall \sigma$, *for any solution $y(k)$ of (3.1) corresponding σ.*

Now, we derive a theorem on the asymptotical stability of the trivial solution of (3.1).

Theorem 4.7. *Suppose that there exist the functions $a, c, \psi_k \in \mathcal{K}$ and the Lyapunov functions $V_\sigma : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, such that* *(i)* $a(\|y\|) \le V_\sigma(k, y) \le \psi_k(\|y\|),\ \forall k \ge 0, \forall y \in \Delta_{\sigma(k)}, \forall \sigma,$

(ii) $\Delta V_\sigma(k, y(k)) := V_\sigma(k+1, y(k+1)) - V_\sigma(k, y(k)) \le -c(\|y(k)\|), \forall k \ge 0, \forall \sigma,$ *for any solution* $y(k)$ *of* (3.1) *corresponding* σ.

Then the trivial solution of (3.1) *is asymptotically stable.*

Proof. From Theorem 4.5, we have the trivial solution of (3.1) is stable. By item *(ii)*, $\{V_\sigma(k, y(k))\}$ is a decreasing sequence and is below bounded by 0. Therefore there exists the limit $\lim_{k\to\infty} V_\sigma(k, y(k))$. This implies that

$$\lim_{k\to\infty} V_\sigma(k+1, y(k+1)) - V_\sigma(k, y(k)) = 0$$

and hence $\lim_{k\to\infty} c(\|y(k)\|) = 0$. Since $c \in \mathcal{K}$, it implies that $\lim_{k\to\infty} \|y(k)\| = 0$. Indeed, assume that $\lim_{k\to\infty} \|y(k)\| \ne 0$. Then for some $\epsilon > 0$, there exists a sequence $\{k_m\} \subset \mathbb{N}$ such that $k_m \to \infty$ and $\|y(k_m)\| > \epsilon$. This implies that $c(\|y(k_m)\|) \ge c(\epsilon) > 0$ which is a contradiction. The proof is complete. $\square$

We define

$$\mu = \max\{L_i(1+K_i)\|P_{jm}G^{-1}_{ijm}\| : i, j, m \in \underline{N}\}.$$

Theorem 4.8. *Assume that there exist* $M > 0, 0 < \lambda < 1$ *such that*

$$\|\Phi_\sigma(k, h)\| \le M\lambda^{k-h},\ \forall k \ge h \ge k_0,$$

and $M\mu < 1 - \lambda$. *Then the trivial solution of* (3.1) *is exponentially stable.*

Proof. From formula (3.7), we have

$$u(k) = \Phi_\sigma(k, k_0)u(k_0) + \sum_{i=k_0}^{k-1} \Phi_\sigma(k, i+1)P_{\sigma(i+1)\sigma(i+2)}G^{-1}_{\sigma(i)\sigma(i+1)\sigma(i+2)}f_{\sigma(i)}(u(i) + v(i)).$$

This implies that

$$\|u(k)\| = M\lambda^{k-k_0}\|u(k_0)\| + \sum_{i=k_0}^{k-1} M\lambda^{k-i-1}\|P_{\sigma(i+1)\sigma(i+2)}G^{-1}_{\sigma(i)\sigma(i+1)\sigma(i+2)}\|L_{\sigma(i)}\|u(i) + v(i)\|$$

$$= M\lambda^{k-k_0}\|u(k_0)\| + \sum_{i=k_0}^{k-1} M\lambda^{k-i-1}\|P_{\sigma(i+1)\sigma(i+2)}G^{-1}_{\sigma(i)\sigma(i+1)\sigma(i+2)}\|L_{\sigma(i)}(1 + K_{\sigma(i)})\|u(i)\|$$

$$\le M\lambda^{k-k_0}\|u(k_0)\| + \sum_{i=k_0}^{k-1} M\lambda^{k-i-1}\mu\|u(i)\|.$$

This is equivalent to

$$\frac{\|u(k)\|}{\lambda^{k-k_0}} \le M\|u(k_0)\| + \sum_{i=k_0}^{k-1} \frac{M\mu}{\lambda}\frac{\|u(i)\|}{\lambda^{i-k_0}}, \forall k \ge k_0.$$

Therefore, if we put $y_m = \dfrac{\|u_{m+k_0}\|}{\lambda^m}$, $f_m = M\|u(k_0)\|, g_m = \dfrac{M\mu}{\lambda}$ for all $m \geq 0$. Then we have

$$y_m \leq f_m + \sum_{0 \leq i < m} g_i y_i, \forall m \geq 0.$$

By Theorem 2.2, we get

$$\begin{aligned}
y_m &\leq f_m + \sum_{0 \leq i < m} f_i g_i \prod_{i<j<m} (1+g_j) \\
&\leq M\|u(k_0)\| + \sum_{0 \leq i < m} M\|u(k_0)\| \frac{M\mu}{\lambda} \left(1 + \frac{M\mu}{\lambda}\right)^{m-i-1} \\
&= M\|u(k_0)\| + M\|u(k_0)\| \left(\left(1 + \frac{M\mu}{\lambda}\right)^m - 1 \right) \\
&= M\|u(k_0)\| \left(1 + \frac{M\mu}{\lambda}\right)^m .
\end{aligned} \tag{4.7}$$

This implies that

$$\|u(k)\| \leq M\|u(k_0)\| \left(1 + \frac{M\mu}{\lambda}\right)^{k-k_0} \lambda^{k-k_0} = M\|u(k_0)\|(\lambda + M\mu)^{k-k_0}, \forall k \geq k_0.$$

Hence

$$\|x(k)\| = \|u(k) + v(k)\| \leq (1+K)\|u(k)\| \leq (1+K)M\|u(k_0)\|(\lambda + M\mu)^{k-k_0}, \forall k \geq k_0.$$

Since $\lambda + M\mu < 1$, the trivial solution of (3.1) is exponentially stable. The proof is complete. □

Example 4.9. Consider the SDLS (3.1) with switching signal $\sigma : \mathbb{N} \cup \{0\} \to \{1, 2, ..., N\} = I_N$ and

$$E_i = \begin{pmatrix} 0 & i \\ 0 & i+1 \end{pmatrix}; \qquad A_i = \begin{pmatrix} i+1 & 1 \\ -i-1 & 1 \end{pmatrix}$$

and

$$f_i(x) = \frac{\sin x_2}{i}(1, -1)^T; \qquad x = (x_1, x_2)^T \in \mathbb{R}^2, \quad i \in I_N.$$

We have $\ker E_i = \operatorname{span}\{(1,0)^{\mathrm{T}}\}, \mathrm{ImE_i} = \operatorname{span}\{(0,1)^{\mathrm{T}}\}$ and $S_{ij} = \operatorname{span}\{(0,1)^{\mathrm{T}}\}$. Therefore, $S_{ij} \cap \ker E_i = \{0\}$ and $\operatorname{rank} E_i = 1 < 2$, hence the SDLS (3.1) is of index-1. Clearly,

$$V_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \forall i, j \in I_N; \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that

$$Q_{ij} = V_{ij} Q V_{ij}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P; \quad P_{ij} = I_n - Q_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

A simple calculation shows that $Q_{ijm} = V_{ij} Q V_{jm}^{-1} = Q_{ij} = P, \forall i, j, m \in I_N$ and

$$G_{ijm} = E_j + A_i Q_{ijm} = \begin{pmatrix} i+1 & j \\ -i-1 & j+1 \end{pmatrix}; \quad G_{ijm}^{-1} = \frac{1}{(i+1)(2j+1)} \begin{pmatrix} j+1 & -j \\ i+1 & i+1 \end{pmatrix}.$$

Further, the function $f_i(x)$ is Lipschitz continuous with the Lipschitz coefficient $L_i = \dfrac{\sqrt{2}}{i}$. Indeed, we have

$$\begin{aligned} \|f_i(x) - f_i(y)\| &= \|\frac{\sin x_2}{i}(1,-1)^T - \frac{\sin y_2}{i}(1,-1)^T\| \\ &\le \frac{1}{i+1}|x_2 - y_2| \|(1,-1)^T\| = \frac{\sqrt{2}}{i}|x_2 - y_2| \\ &\le \frac{\sqrt{2}}{i}\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} = \frac{\sqrt{2}}{i}\|x - y\|, \end{aligned}$$

where we use the Euclidean norms of vectors. Moreover, $f_i(0) = 0$ and

$$\begin{aligned} \omega_i &= L_i \max\{\|Q_{ijm} G_{ijm}^{-1}\| : j, m \in \underline{N}\} \\ &= \max\{\frac{\sqrt{4j^2 + 4j + 2}}{2j+1} : j \in \underline{N}\} \frac{1}{i(i+1)} \\ &< \frac{\sqrt{10}}{3i(i+1)} < 1, \forall i \in I_N. \end{aligned}$$

According to Theorem 3.1, the SDLS (3.1), (3.2) has unique solution. From the definition of Δ_i, we have $x \in \Delta_i$ if only if

$$Q_{ij} x = -V_{ij} Q V_{jm}^{-1} G_{ijm}^{-1} [f_i(x) + A_i P_{ij} x].$$

This relation leads to $x_1 = -\dfrac{\sin x_2}{i(i+1)} - \dfrac{x_2}{(i+1)(2j+1)}$. Thus,

$$\Delta_i = \Big\{ x = (x_1, x_2)^T : x_1 = -\frac{\sin x_2}{i(i+1)} - \frac{x_2}{(i+1)(2j+1)}, j \in \underline{N} \Big\}.$$

Consider a function $V_\sigma(k, \gamma) := 2\|P_{\sigma(k)\sigma(k+1)}\gamma\|$ for all $\gamma \in \mathbb{R}^2$. We get for each $y \in \Delta_i$,

$$\begin{aligned} \|y\| = \sqrt{y_1^2 + y_2^2} &= \sqrt{\left(\frac{\sin y_2}{i(i+1)} + \frac{y_2}{(i+1)(2j+1)}\right)^2 + y_2^2} \\ &\le \sqrt{\left(\frac{1}{i(i+1)} + \frac{1}{(i+1)(2j+1)}\right)^2 y_2^2 + y_2^2} \\ &\le 2|y_2| = 2\|P_{\sigma(k)\sigma(k+1)} y\| = V_\sigma(k, y). \end{aligned}$$

Moreover, $V_\sigma(k,y) = 2\|P_{\sigma(k)\sigma(k+1)}y\| = 2|y_2| \le 2\|y\|$. Thus item (i) of Theorem 4.6 is satisfied. We suppose that $y(k)$ is a solution of (3.1) and putting $y(k) = u(k) + v(k)$, where $u(k) = P_{\sigma(k)\sigma(k+1)}y(k)$; $v(k) = Q_{\sigma(k)\sigma(k+1)}y(k)$, we have

$$\Delta V(k, y(k)) = 2(\|P_{\sigma(k+1)\sigma(k+2)}y(k+1)\| - \|P_{\sigma(k)\sigma(k+1)}y(k)\|) = 2(\|u(k+1)\| - \|u(k)\|)$$

Using equation (3.7) we find

$$u(k+1) = P_{jm}G_{ijm}^{-1}A_iu(k) + P_{jm}G_{ijm}^{-1}f_i(x(k)) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{2j+1} \end{pmatrix} u(k),$$

hence, $\|u(k+1)\| = \dfrac{2}{2j+1}\|u(k)\|$ and leading to $\|u(k+1)\| - \|u(k)\| \le 0$. According to Theorem 4.6, the trivial solution of (3.1) is uniformly stable. Moreover, since

$$\|u(k+1)\| - \|u(k)\| \le \frac{1-2j}{2j+1}\|u(k)\| \le \frac{1-2j}{2(2j+1)}\|y(k)\| \le \frac{-1}{2(2N+1)}\|y(k)\|.$$

Thus, by Theorem 4.7, the trivial solution of (3.1) is asymptotically stable.

Example 4.10. In this example we will use the infinity-norms of matrices. Consider the SDLS (3.1) with switching signal $\sigma : \mathbb{N} \cup \{0\} \to \{1, 2\} = \underline{N}$, and

$$E_1 = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad E_2 = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $f_i(x) = \dfrac{2x_1 + 3\sin\frac{x_2}{4}}{3(i+1)(i+2)}(0,0,1)^T$, $\quad x = (x_1, x_2)^T \in \mathbb{R}^2$, $i \in \underline{N}$. A simple computation shows that

$$\begin{aligned}
&\ker E_1 = \ker E_2 = \operatorname{span}\{(0,0,1)^{\mathrm{T}}\}, \\
&\quad S_{11} = \operatorname{span}\{(1,0,0)^{\mathrm{T}}, (0,1,0)^{\mathrm{T}}\}, \mathrm{S}_{12} = \operatorname{span}\{(3,2,0)^{\mathrm{T}}, (0,1,0)^{\mathrm{T}}\}, \\
&\quad S_{21} = \operatorname{span}\{(-1,1,0)^{\mathrm{T}}, (0,1,0)^{\mathrm{T}}\}, \mathrm{S}_{22} = \operatorname{span}\{(-1,3,0)^{\mathrm{T}}, (0,1,0)^{\mathrm{T}}\}.
\end{aligned}$$

Clearly $S_{ij} \cap \ker E_i = \{0\}, \forall i, j \in \underline{N}$ and $\operatorname{rank} E_i = 2 < 3$, hence homogenous SDLS systems respectively with (3.1) with above data is of index-1. We have

$$V_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad V_{12} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad V_{21} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad V_{22} = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad Q_{ij} = Q; \quad P_{ij} = I_3 - Q_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; Q_{ijm} = Q, \forall i, j, m \in \underline{N}.$$

It is easy to compute that

$$G_{i1m} = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad G_{i1m}^{-1} = \begin{pmatrix} 1/3 & 2/9 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$
$$G_{i2m} = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad G_{i2m}^{-1} = \begin{pmatrix} 2/7 & -1/7 & 0 \\ -1/21 & 4/21 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \forall i, m \in \underline{N}.$$

Futher, the function $f_i(x)$ is Lipschitz continuos with the Lipschitz coefficient $L_i = \frac{11}{12(i+1)(i+2)}$, $i \in \underline{N}$. We calculate

$$\omega_i := L_i \max\{\|Q_{ijm}G_{ijm}^{-1}\|\} = L_i, K_i := \omega_i(L_i + \|A_i\|)L_i^{-1}(1-\omega_i)^{-1},$$

hence $K_1 = \frac{25}{11}$, $K_2 = \frac{49}{23}$; $\|P_{1m}G_{i1m}^{-1}\| = \frac{5}{9}$, $\|P_{2m}G_{i2m}^{-1}\| = \frac{3}{7}$. Therefore $\mu = \max\{L_i(1+K_i)\|P_{jm}G_{ijm}^{-1}\| : i, j, m \in \underline{N}\} = \max\left\{\frac{5}{18}; \frac{3}{14}; \frac{55}{414}; \frac{33}{322}\right\} = \frac{5}{18}$. Putting

$$\Phi_{ijm} := P_{jm}G_{ijm}^{-1}A_i = P_{\sigma(l)\sigma(l+1)}.G_{\sigma(l-1)\sigma(l)\sigma(l+1)}^{-1}A_{\sigma(l-1)},$$

we have

$$\Phi_{11m} = \begin{pmatrix} 1/3 & -1/9 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \Phi_{21m} = \begin{pmatrix} -1/9 & 2/9 & 0 \\ 1/3 & 1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$
$$\Phi_{12m} = \begin{pmatrix} 2/7 & -3/7 & 0 \\ -1/21 & 5/21 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \Phi_{22m} = \begin{pmatrix} -3/7 & -1/7 & 0 \\ 5/21 & 4/21 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$
$$\|\Phi_{11m}\| = \frac{4}{9}; \quad \|\Phi_{21m}\| = \frac{2}{3}; \quad \|\Phi_{12m}\| = \frac{5}{7}; \quad \|\Phi_{22m}\| = \frac{4}{7}.$$

Thus if we choose $\lambda = \max\{\|\Phi_{ijm}\| : i, j, m \in \underline{N}\} = \frac{5}{7}$ and $M = 1$ then

$$\|\Phi_\sigma(k,h)\| \le \prod_{l=h+1}^{k} \|P_{\sigma(l)\sigma(l+1)}G_{\sigma(l-1)\sigma(l)\sigma(l+1)}^{-1}A_{\sigma(l-1)}\| \le \left(\frac{5}{7}\right)^{k-h} = M\lambda^{k-h},$$

for all $k \ge h \ge k_0$. Moreover we have

$$M\mu = \frac{5}{18} < 1 - \lambda = \frac{2}{7}.$$

Thus, by Theorem 4.8, the SDLS system with the above data $\{(E_i, A_i, f_i)\}_{i=1,2}$ is exponentially stable.

5. Conclusion

In this paper, we have studied SDLS systems subject to Lipschitz perturbations. We derive solvability and establish a formula of solution for these equations. The stability of SDLS systems is investigated by using methods of the Lyapunov functions and the solution evaluation.

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