Solvability and stability of switched discrete-time linear singular systems under Lipschitz perturbations

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**ABSTRACT**
In this paper, the problem of solvability and stability for switched discrete-time linear singular (SDLS) systems under Lipschitz perturbations is studied. We first prove the unique existence of solution of SDLS systems under Lipschitz perturbations with different switching rules on two sides. The solution manifold is also described. Secondly, we derive some conditions for stability of these systems. Finally, some examples are given to illustrate the obtained results.

**KEYWORDS**
SDLS systems, index, solvability, stability, Lipschitz perturbation.

### 1. Introduction

In this paper we study solvability and stability of switched discrete-time linear singular (SDLS) systems of the form

\[ E_{\sigma(k+1)}x(k+1) = A_{\sigma(k)}x(k) + f_{\sigma(k)}(x(k)), \]  

(1.1)

where \( \sigma : \mathbb{N} \cup \{0\} \to \mathbb{N} := \{1, 2, \ldots, N\}, N \in \mathbb{N}, \) denotes the switching signal that determines which of the \( N \in \mathbb{N} \) modes is active at time \( k \).

Singular switched systems are models arising in diverse real-life applications such as power electronics and systems, air traffic and aircraft control, network control systems, robot manipulators, multibody systems, economic systems and so forth, (see, e.g. \cite{6, 12, 16, 18, 20}). These systems consist of a family of singular subsystems and a rule that controls the switching between them which in recent years have attracted a good deal of attention from researchers. On the other hand, the advent of many modern-day sampled-data control systems (or the dynamic Leontief system in economic) has necessitated a study of discrete-time singular systems because they can only change at discrete instants of time (see, e.g. \cite{5, 11, 13–15, 17, 19}). These lead switched discrete-time singular systems. They can also be obtained from switched continuous-time singular systems by some discretization methods.

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Recently, we have investigated solvability and stability of SDLS systems of the form
\[ E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \] in [2, 3], where the switching rules in matrices \( E \) and \( A \) are same. If the switching rules in matrices \( E \) and \( A \) are not same then it is more complicated. In [9], some first results on this case have considered for homogenous SDLS systems which have no perturbations. However, to the best of our knowledge, there are still no results about solvability and stability for SDLS systems of the form (1.1) under Lipschitz perturbations \( f \).

The purpose of the present paper is to fill this gap. We will consider SDLS systems of the form (1.1) with the different switching rules in matrices \( E \) and \( A \). The singularity of the leading coefficients make the analysis of system (1.1) difficult since computation of solutions is impossible at first sight. Even the solvability of the initial value problem is doubtful. Due to the fact that the dynamics of (1.1) are constrained and combined between singular systems, some extra difficulties appear in the analysis of solvability as well as stability characterized by index concepts of singular systems (see, e.g. [4, 8, 10]). Thus, in this paper, we will develop and modify the approach in [1, 3, 9] to investigate solvability and stability of SDLS systems under Lipschitz perturbations. The unique solution of (1.1) will be proved by using the contraction mapping principle. After that characterizations for stability of (1.1) will be derived by using methods of the Lyapunov functions and the solution evaluation.

The paper is organized as follows. In Section 2, we summarize some preliminary results of SDLS systems of index-1 and the discrete Gronwal inequality. In Section 3, we study solvability and a formula of solution of SDLS systems under Lipschitz perturbations. Section 4 deals with stability of these systems. The last section gives some conclusions.

2. Preliminary

For \( N \in \mathbb{N} \), denote \( \mathbb{N} = \{1, 2, \ldots, N\} \) and \( O \) by the zero matrix. Consider the homogeneous SDLS systems
\[ E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \] (2.1)
is of index-1 ([4], [9]), i.e., the following hypotheses are assumed to be fulfilled:

(i) \( \text{rank } E_i = r < n, \forall i \in \mathbb{N}, \)
(ii) \( S_{ij} \cap \ker E_i = \{0\}, \forall i, j \in \mathbb{N}, \) where \( S_{ij} = A_i^{-1}(\text{Im } E_j) = \{\xi \in \mathbb{R}^n : A_i \xi \in \text{Im } E_j\}. \)

It is proved that from hypothesis (ii) we have
\[ S_{ij} \oplus \ker E_i = \mathbb{R}^n, \forall i, j \in \mathbb{N}, \]
see, e.g. [3, 9]. Let the matrix \( V_{ij} = \{s_{ij}^{r}, \ldots, s_{ij}^{1}, h_{i}^{r+1}, \ldots, h_{i}^{n}\} \), whose columns form bases of \( S_{ij} \) and \( \ker E_i \), respectively, and \( Q = \text{diag}(O_r, I_{n-r}), \) \( P = I_n - Q. \) Here \( O_r \) is the \( r \times r \) zero matrix and \( I_{n-r} \) stands for the \( (n - r) \times (n - r) \) identity matrix. Then the matrix \( Q_{ij} := V_{ij}QV_{ij}^{-1} \) defines a projection onto \( \ker E_i \) along \( S_{ij} \) (i.e., \( Q_{ij}^2 = Q_{ij} \) and \( \text{Im } Q_{ij} = \ker E_i \)), and \( P_{ij} := I_n - Q_{ij} = V_{ij}PV_{ij}^{-1} \) is the projection onto \( S_{ij} \) along \( \ker E_i \). Further we define the so-called connecting operators \( Q_{ijm} := V_{ij}QV_{jm}^{-1}. \)

**Theorem 2.1.** ([9]). For switched discrete-time linear singular homogeneous system of index-1 (2.1), the following assertions hold:
\( G_{ijm} = E_j + A_i Q_{ijm} \) is non-singular;
\( E_j P_{jm} = E_j \);
\( P_{jm} = G_{ijm}^{-1} E_j \);
\( V_{jm}^{-1} G_{ijm}^{-1} A_i V_{ij} Q = Q \)
for all \( i, j, m \in \mathbb{N} \).

We need to use the following discrete Gronwall inequality to study the exponential
stability of SDLS systems in Section 4.

**Theorem 2.2.** ([7]) Assume that \( \{y_m\}, \{f_m\}, \{g_m\} \) are nonnegative sequences such
that
\[
    y_m \leq f_m + \sum_{0 \leq i < m} g_i y_i, \forall m \geq 0.
\]
Then
\[
    y_m \leq f_m + \sum_{0 \leq i < m} f_i g_i \prod_{i < j \leq m} (1 + g_j).
\]

3. Solvability

Consider a switched discrete-time singular system of the form:
\[
    E_{\sigma(k+1)} x(k+1) = A_{\sigma(k)} x(k) + f_{\sigma(k)}(x(k))
\]
where \( \sigma : \mathbb{N} \cup \{0\} \to \mathbb{N} \), is a switching signal taking values in the finite set \( \mathbb{N} \),
\( E_i, A_i \in \mathbb{R}^{n \times n} \) and \( f_i : \mathbb{R}^n \to \mathbb{R}^n, i \in \mathbb{N} \), are perturbations, \( x(k) \in \mathbb{R}^n \) is state vector
at time \( k \in \mathbb{N} \). Suppose that the matrices \( E_i \) are singular for all \( i \in \mathbb{N} \). Let us associate
system (3.1) with the initial condition
\[
    P_{\sigma(k_0)\sigma(k_0+1)} x(k_0) = P_{\sigma(k_0)\sigma(k_0+1)} \gamma,
\]
where \( \gamma \) is a given vector in \( \mathbb{R}^n \) and \( k_0 \) is a fixed nonnegative integer.

**Theorem 3.1.** Let \( f_{\sigma(k)}(x) \) be a Lipschitz continuous function with a sufficient small
Lipschitz coefficient, i.e.,
\[
    \|f_i(x) - f_i(\tilde{x})\| \leq L_i \|x - \tilde{x}\|, \forall x, \tilde{x} \in \mathbb{R}^n, i \in \mathbb{N},
\]
and
\[
    \omega_i := L_i \max\{\|Q_{ijm} G_{ijm}^{-1}\| : j, m \in \mathbb{N}\} < 1, \forall i \in \mathbb{N}.
\]
Then the IVP (3.1), (3.2) has a unique solution.

**Proof.** Multiplying on both sides of equation (3.1) from the left by
\[
    P_{\sigma(k+1)\sigma(k+2)} G_{\sigma(k)\sigma(k+1)\sigma(k+2)}^{-1}
\]
and \( Q_{\sigma(k+1)\sigma(k+2)} G_{\sigma(k)\sigma(k+1)\sigma(k+2)}^{-1} \),
respectively and observing that

\[ G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} E_{\sigma(k+1)} = P_{\sigma(k+1)\sigma(k+2)}, \]

\[ P_{\sigma(k+1)\sigma(k+2)} Q_{\sigma(k+1)\sigma(k+2)} = Q_{\sigma(k+1)\sigma(k+2)} P_{\sigma(k+1)\sigma(k+2) = O}, \]

we get

\[ P_{\sigma(k+1)\sigma(k+2)} x(k+1) = P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} x(k) \]
\[ + P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} f_{\sigma(k)}(x(k)), \]

(3.6)

\[ Q_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} x(k) = -Q_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} f_{\sigma(k)}(x(k)). \]

(3.5)

Let \( u(k) = P_{\sigma(k)\sigma(k+1)} x(k), v(k) = Q_{\sigma(k)\sigma(k+1)} x(k), (k \in \mathbb{N}) \) we get

\[ P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} v(k) \]
\[ = P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} Q_{\sigma(k)\sigma(k+1)} x(k) \]
\[ = P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} Q_{\sigma(k)\sigma(k+1)} x(k) \]
\[ = P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} \left( G_{\sigma(k)\sigma(k+1)\sigma(k+2)} x(k) - E_{\sigma(k+1)} V_{\sigma(k+1)\sigma(k+2)} Q_{\sigma(k)\sigma(k+1)} x(k) \right) \]
\[ = (P_{\sigma(k+1)\sigma(k+2)} - P_{\sigma(k+1)\sigma(k+2)} P_{\sigma(k+1)\sigma(k+2)} V_{\sigma(k+1)\sigma(k+2)} Q_{\sigma(k)\sigma(k+1)} x(k) \]
\[ = 0, \]

and from (3.5)

\[ u(k+1) = P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} (u(k) + v(k)) \]
\[ + P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} f_{\sigma(k)}(u(k) + v(k)) \]
\[ = P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} u(k) \]
\[ + P_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} f_{\sigma(k)}(u(k) + v(k)). \]

(3.7)

By item (iv) of Theorem 2.1,

\[ Q_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} Q_{\sigma(k)\sigma(k+1)} = V_{\sigma(k+1)\sigma(k+2)} Q_{\sigma(k)\sigma(k+1)} \]

Therefore, the left side of (3.6) can be expressed as

\[ Q_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} x(k) = Q_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} (u(k) + v(k)) \]
\[ = Q_{\sigma(k+1)\sigma(k+2)} G^{-1}_{\sigma(k)\sigma(k+1)\sigma(k+2)} A_{\sigma(k)} u(k) + V_{\sigma(k+1)\sigma(k+2)} Q_{\sigma(k)\sigma(k+1)} x(k). \]
Hence, it follows from (3.6) that
\[
V_{\sigma(k+1)\sigma(k+2)}Q_{\sigma(k)}^{-1}\sigma(k+1)\sigma(k+2)x(k)
\]
\[
= -Q_{\sigma(k+1)\sigma(k+2)}G_{\sigma(k)}^{-1}\sigma(k+1)\sigma(k+2)A_{\sigma(k)}u(k)
\]
\[
= -Q_{\sigma(k+1)\sigma(k+2)}G_{\sigma(k)}^{-1}\sigma(k+1)\sigma(k+2)A_{\sigma(k)}u(k)
\]
\[
= -Q_{\sigma(k+1)\sigma(k+2)}G_{\sigma(k)}^{-1}\sigma(k+1)\sigma(k+2)A_{\sigma(k)}u(k)
\]
Now multiplying on both sides of this relation by \(Q_{\sigma(k)}\sigma(k+1)\sigma(k+2)\) from the left we obtain
\[
v(k) = Q_{\sigma(k)}\sigma(k+1)x(k) = Q_{\sigma(k)}\sigma(k+1)\sigma(k+2)V_{\sigma(k)} \sigma(k+1)\sigma(k+2)x(k)
\]
\[
= -Q_{\sigma(k)}\sigma(k+1)\sigma(k+2)Q_{\sigma(k+1)}\sigma(k+2)G_{\sigma(k)}^{-1}\sigma(k+1)\sigma(k+2)A_{\sigma(k)}u(k)
\]
\[
= -Q_{\sigma(k)}\sigma(k+1)\sigma(k+2)G_{\sigma(k)}^{-1}\sigma(k+1)\sigma(k+2)A_{\sigma(k)}u(k)
\]
By equation (3.7), suppose that \(u := u(k)(k \geq k_0)\) is known, where
\[
u(k_0) = P_{\sigma(k_0)}\sigma(k_0+1)x(k_0) = P_{\sigma(k_0)}\sigma(k_0+1)\gamma
\]
is given. We consider an operator \(T_{ijm} : \text{Im}Q_{ij} \rightarrow \text{Im}Q_{ij}\) defined by
\[
T_{ijm}(v) := -Q_{ijm}G_{ijm}^{-1}[f_i(u + v) + A_iu].
\]
Since
\[
\|T_{ijm}(v) - T_{ijm}(\bar{v})\| = \|Q_{ijm}G_{ijm}^{-1}[f_i(u + v) - f_i(u + \bar{v})]\|
\]
\[
\leq \|Q_{ijm}G_{ijm}^{-1}\|\|f_i(u + v) - f_i(u + \bar{v})\|
\]
\[
\leq \|Q_{ijm}G_{ijm}^{-1}\|L_i\|v - \bar{v}\| \leq \omega_i\|v - \bar{v}\| < \|v - \bar{v}\|
\]
the operator \(T_{ijm}\) is contractive. Therefore equation (3.8) has a unique solution given by a mapping \(g_{\sigma(k)}\sigma(k+1) : \text{Im}P_{\sigma(k)}\sigma(k+1) \rightarrow \text{Im}Q_{\sigma(k)}\sigma(k+1), g_{\sigma(k)}\sigma(k+1)(u(k)) = v(k)\). Moreover, it is easy to show that \(g_{\sigma(k)}\sigma(k+1)\) is a Lipschitz continuous mapping having the Lipschitz constant
\[
K_{\sigma(k)} := \omega_{\sigma(k)}(L_{\sigma(k)} + \|A_{\sigma(k)}\|)L_{\sigma(k)}^{-1}(1 - \omega_{\sigma(k)})^{-1}.
\]
Thus, the IVP (3.1), (3.2) has a unique solution given by
\[
x(k) = u(k) + g_{\sigma(k)}\sigma(k+1)(u(k)),
\]
with \(u(k_0) = P_{\sigma(k_0)}\sigma(k_0+1)\gamma\). The proof is complete.
We define the Cauchy operator associated with system (3.1)

$$Φ_σ(k, h) = \prod_{l=h+1}^{k} P_{σ(l)σ(l+1)} G_{σ(l-1)σ(l+1)}^{-1} A_{σ(l-1)}$$ and $$Φ_σ(h, h) = P_{σ(h)σ(h+1)}.$$ \hfill (3.11)

Then, it is easy to see that $$Φ_σ(k, h)$$ satisfies the relation

$$Φ_σ(k, h) = Φ_σ(k, l)Φ_σ(l, h), \forall k \geq l \geq h.$$ 

Now, the variation of constants formula for the solution of system (3.1) is derived in the following corollary.

**Corollary 3.2.** The unique solution of system (3.1) with the initial conditions (3.2) satisfies the equation

$$x(k) = Φ_σ(k, k_0)P_{σ(k_0)σ(k_0+1)} γ + \sum_{i=k_0}^{k-1} Φ_σ(k, i+1)P_{σ(i+1)σ(i+2)} G_{σ(i)σ(i+1)}^{-1} A_{σ(i+1)} f_σ(i)(x(i))$$

$$- Q_{σ(k)σ(k+1)} G_{σ(k)σ(k+2)}^{-1} (f_σ(k)(x(k)) + A_{σ(k)} P_{σ(k)σ(k+1)} x(k)).$$ \hfill (3.12)

**Proof.** By equation (3.7), we imply that the solution $$u(k)$$ is given by the formula

$$u(k) = Φ_σ(k, k-1)P_{σ(k_0)σ(k_0+1)} γ + \sum_{i=k_0}^{k-1} Φ_σ(k, i+1)P_{σ(i+1)σ(i+2)} G_{σ(i)σ(i+1)}^{-1} A_{σ(i+1)} f_σ(i)(x(i))$$

and by equation (3.8), we have

$$v(k) = -Q_{σ(k)σ(k+1)} G_{σ(k)σ(k+2)}^{-1} (f_σ(k)(x(k)) + A_{σ(k)} P_{σ(k)σ(k+1)} x(k)).$$

Since $$x(k) = u(k) + v(k),$$ we obtain formula (3.12). \hfill \Box

In what follows, without loss of generality, assume that $$f_i(0) = 0, \forall i ∈ N.$$ This implies that $$g_{σ(k)σ(k+1)}(0) = 0$$ and equation (3.1) possesses a trivial solution $$x(k) ≡ 0.$$ It follows from (3.10) that each solution $$x(k)$$ of the IVP (3.1), (3.2) satisfies $$x(k) = P_{σ(k)σ(k+1)} x(k) + g_{σ(k)σ(k+1)}(P_{σ(k)σ(k+1)} x(k))$$ or equivalently,

$$Q_{σ(k)σ(k+1)} x(k) = -Q_{σ(k)σ(k+1)} G_{σ(k)σ(k+2)}^{-1} (f_σ(k)(x(k)) + A_{σ(k)} P_{σ(k)σ(k+1)} x(k)).$$

Let

$$Δ_i := \{ x ∈ \mathbb{R}^n : Q_{ij} x = -Q_{ijm} G_{i jm}^{-1} (f_i(x) + A_i P_{ij} x), \text{ for some } j, m ∈ N \}.$$ \hfill (3.13)

If $$x = x(k)$$ is any solution of the IVP (3.1), (3.2), then obviously, $$x(k) ∈ Δ_{σ(k)}(k ≥ k_0).$$ Conversely, for each $$θ ∈ Δ_i,$$ there exists a solution of (3.1) passing $$θ.$$ Indeed, let $$σ$$ be a switching signal satisfying $$σ(k) = i$$ and $$x(m, k; θ)(m ≥ k)$$ be a solution of (3.1)
satisfying the initial condition $P_{\sigma(k)\sigma(k+1)}x(k) = P_{\sigma(k)\sigma(k+1)}\theta$. Clearly,

$$x(k, k; \theta) = P_{\sigma(k)\sigma(k+1)}x(k) + g_{\sigma(k)\sigma(k+1)}(P_{\sigma(k)\sigma(k+1)}x(k))$$

$$= P_{\sigma(k)\sigma(k+1)}\theta + g_{\sigma(k)\sigma(k+1)}(P_{\sigma(k)\sigma(k+1)}\theta)$$

$$= P_{\sigma(k)\sigma(k+1)}\theta + Q_{\sigma(k)\sigma(k+1)}\theta = \theta. \quad (3.14)$$

We will prove that the set $\Delta_i$ does not depend on the choice of projections in the following proposition.

**Proposition 3.3.** Let the solution manifold $\Delta_i$ be defined in (3.13). Then, the following hold:

(i) $\Delta_i = \{ x \in \mathbb{R}^n : f_i(x) + A_i x \in \text{Im}E_j, \text{ for some } j \in \mathbb{N} \}$.

(ii) $\Delta_i \cap \ker E_i = \{ 0 \}$.

**Proof.** i) Letting $x \in \Delta_i$, then there exists $j, m \in \mathbb{N}$ such that

$$Q_{ij}x = -Q_{ijm}G_{ijm}^{-1}(f_i(x) + A_i P_{ij}x),$$

hence

$$x = P_{ij}x + Q_{ij}x = -Q_{ijm}G_{ijm}^{-1}f_i(x) + (I - Q_{ijm}G_{ijm}^{-1}A_i)P_{ij}x.$$ From this relation we have

$$f_i(x) + A_i x = (I - A_iQ_{ijm}G_{ijm}^{-1})f_i(x) + A_i(I - Q_{ijm}G_{ijm}^{-1}A_i)P_{ij}x.$$ Note that

$$A_i(I - Q_{ijm}G_{ijm}^{-1}A_i)P_{ij}x = (I - A_iQ_{ijm}G_{ijm}^{-1})A_iP_{ij}x.$$ Therefore

$$f_i(x) + A_i x = (I - A_iQ_{ijm}G_{ijm}^{-1})(f_i(x) + A_i P_{ij}x).$$ Since

$$A_iQ_{ijm}G_{ijm}^{-1} = (G_{ijm} - E_j)G_{ijm}^{-1} = I - E_jG_{ijm}^{-1},$$

it follows that

$$f_i(x) + A_i x = E_jG_{ijm}^{-1}\{f_i(x) + A_i P_{ij}x\} \in \text{Im}E_j.$$ Hence $x \in \Delta_i$.

Conversely, let $x \in \mathbb{R}^n$ such that $f_i(x) + A_i x \in \text{Im}E_j$ for some $j \in \mathbb{N}$. Then there exists $\xi \in \mathbb{R}^n, j \in \mathbb{N}$ such that $f_i(x) + A_i x = E_j\xi$. We will prove that for $m \in \mathbb{N}$,

$$Q_{ij}x = -Q_{ijm}G_{ijm}^{-1}(f_i(x) + A_i P_{ij}x),$$
or equivalent
\[ x = -Q_{ijm}G_{ijm}^{-1}(f_i(x) + A_i x) + Q_{ijm}G_{ijm}^{-1}A_i Q_{ijm} x + P_{ijm} x. \]

Denoting the right-hand side of this relation by \( w_{ij} \) and note that
\[
Q_{ijm}^{-1}G_{ijm}^{-1}(f_i(x) + A_i x) = Q_{ijm}^{-1}G_{ijm}^{-1}E_j x = Q_{ijm}^{-1}P_{jm} x
= V_{ij} QV_{jm}^{-1} V_{jm} PV_{jm}^{-1} x = V_{ij} QV_{jm}^{-1} x = 0,
\]
by Theorem 2.1 we get
\[
w_{ij} = Q_{ijm}^{-1}A_i Q_{ijm} x + P_{ijm} x
= Q_{ijm}^{-1} G_{ijm}^{-1} A_i V_{ij} QV_{jm}^{-1} V_{jm} QV_{ij} x + P_{ijm} x
= Q_{ijm} V_{jm} QV_{ij} x - Q_{ijm} G_{ijm}^{-1} E_j V_{jm} QV_{ij} x + P_{ijm} x
= V_{ij} QV_{jm}^{-1} V_{jm} QV_{ij} x - V_{ij} QV_{jm}^{-1} P_{jm} V_{jm} QV_{ij} x + P_{ijm} x
= V_{ij} QV_{ij}^{-1} x - V_{ij} QP QV_{ij}^{-1} x + P_{ijm} x
= Q_{ijm}^{-1} x + P_{ijm} x = x.
\]

Thus, \( x \in \Delta_i \) and the item (i) of Lemma 3.3 is proved.

(ii) Let \( x \in \Delta_i \cap \ker E_i \). Then we have \( x \in \Delta_i \) and \( P_{ijm} x = 0 \) for all \( j \in \mathbb{N} \). Since \( x \in \Delta_i \), it implies that
\[ Q_{ijm} x = g_{ijm}(P_{ijm} x) = 0 \]
and hence
\[ x = P_{ijm} x + Q_{ijm} x = 0. \]

The proof is complete. \( \square \)

Since \( G_{\sigma(k_0+2)\sigma(k_0)\sigma(k_0+1)} E_{\sigma(k_0)} = P_{\sigma(k_0)\sigma(k_0+1)} \), it is easy to see that the initial condition (3.2) is equivalent to the condition
\[ E_{\sigma(k_0)} x(k_0) = E_{\sigma(k_0)} \gamma, \forall k_0 \in \mathbb{N}. \tag{3.15} \]
which is independent of the choice of projections. Thus both initial conditions (3.2) and (3.15) are equivalent for all \( k_0 \in \mathbb{N} \). The unique solution of the IVP (3.1), (3.2) or (3.1), (3.15) will be denoted by \( x(k) = x(k, k_0; \gamma) \).

4. Stability

In this section the notions of stability of trivial solution are introduced and the necessary and sufficient conditions for stability of SDLS systems are established.

Definition 4.1. The trivial solution of (3.1) is said to be
(i) stable if for each \( \epsilon > 0 \), any \( k_0 \geq 0 \) and for all switching signals there exists a \\
\( \delta = \delta(\epsilon, k_0) \in (0, \epsilon) \) such that \( \| P_{\sigma(k_0)\sigma(k_0+1)} \gamma \| < \delta \) implies \( \| x(k, k_0; \gamma) \| < \epsilon \) for all \( k \geq k_0 \), uniformly stable if it is stable and \( \delta \) does not depend on \( k_0 \);
(ii) asymptotically stable if it is stable and for any \( k_0 \geq 0 \) and for all switching 
signals there exists a \( \delta = \delta(k_0) > 0 \) such that the inequality \( \| P_{\sigma(k_0)\sigma(k_0+1)} \gamma \| < \delta \) 
implies \( \| x(k, k_0; \gamma) \| \to 0 \) as \( k \to +\infty \);
(iii) exponentially stable if there exist \( M > 0, 0 < \lambda < 1 \) such that \( \| x(k, k_0; \gamma) \| \leq M \lambda^{k-k_0} \| P_{\sigma(k_0)\sigma(k_0+1)} \gamma \| \) for all \( k \geq k_0 \) and switching signals.

**Remark 4.2.** In the above definition, if replacing the initial condition \( P_{\sigma(k_0)\sigma(k_0+1)} \gamma \) by \( E_{\sigma(k_0)} \gamma \) then we get notions of \( E \)-stability, \( E \)-asymptotical stability and \( E \)-exponential stability (respectively). However, since the relation \( G_{i_m}^{-1} \gamma = P_{jm} \) and \( E_j P_{jm} = E_j \) for all \( i, j, k \in \mathbb{N} \), it is easy to show that they are equivalent to above notions (respectively).

Denote by \( \mathcal{K} \) the class of all increasing functions \( \psi \) from \( [0, \infty) \) into itself such that 
\( \psi(0) = 0, \psi(x) > 0 \) for \( x \neq 0 \) and \( \lim_{x \to 0^+} \psi(x) = 0 \).

**Lemma 4.3.** The trivial solution of (3.1) is stable if and only if there exists a function 
\( \psi \in \mathcal{K} \), such that for each nonnegative integer \( k_0 \) and for all switching signals, there 
holds the inequality

\[
\| x(k) \| \leq \psi(\| x(k_0) \|), \quad \forall k \geq k_0. \tag{4.1}
\]

**Proof.** Suppose first that for all switching signals and for each nonnegative integer 
\( k_0 \), there exists a function \( \psi \in \mathcal{K} \) satisfying condition (4.1). Since \( \psi \) is increasing and continuous at \( 0 \), for each positive \( \epsilon \) there exists \( \delta = \delta(\epsilon) \in (0, \epsilon) \) such that \( \psi(\delta) < \epsilon \). Let \( K = \max_{i \in \mathbb{N}} K_i \), where \( K_i \) is given by (3.9). If \( x(k) \) is an arbitrary solution of 
(3.1) satisfying \( |P_{\sigma(k_0)\sigma(k_0+1)} x(k_0)\| < \delta \) then

\[
\| x(k_0) \| = \| P_{\sigma(k_0)\sigma(k_0+1)} x(k_0) + g_{\sigma(k_0)\sigma(k_0+1)\sigma(k_0+2)} (P_{\sigma(k_0)\sigma(k_0+1)} x(k_0)) \| \\
\leq \| P_{\sigma(k_0)\sigma(k_0+1)} x(k_0) \| + \| g_{\sigma(k_0)\sigma(k_0+1)\sigma(k_0+2)} (P_{\sigma(k_0)\sigma(k_0+1)} x(k_0)) \| \\
\leq \| P_{\sigma(k_0)\sigma(k_0+1)} x(k_0) \| (1 + K_{\sigma(k_0)}) \leq \| P_{\sigma(k_0)\sigma(k_0+1)} x(k_0) \| (1 + K) < \delta. \tag{4.2}
\]

This implies that

\[
\| x(k) \| \leq \psi(\| x(k_0) \|) \leq \psi(\delta) < \epsilon, \quad \forall k \geq k_0, \forall \sigma,
\]

which implies that trivial solution of (3.1) is stable.

Conversely, suppose that the trivial solution of (3.1) is stable, i.e., for each positive 
\( \epsilon \) there exists a \( \delta = \delta(\epsilon) \in (0, \epsilon) \), such that if \( x(k) \) is any solution of (3.1) satisfying 
the inequality \( \| P_{\sigma(k_0)\sigma(k_0+1)} x(k_0)\| < \delta \) for all switching signals then \( \| x(k) \| < \epsilon \) for all 
\( k \geq k_0 \). Denote by \( \alpha(\epsilon) \) the supremum of such \( \delta(\epsilon) \). Clearly, if \( \| P_{\sigma(k_0)\sigma(k_0+1)} x(k_0)\| < \alpha(\epsilon) \) for some \( k_0 \) and for all \( \sigma \), then \( \| x(k) \| < \epsilon \) for all \( k \geq k_0 \). Further, the function 
\( \alpha(\epsilon) \) is positive and increasing and moreover, \( \alpha(\epsilon) \leq \epsilon \). Putting \( \beta(\epsilon) := \frac{\epsilon \alpha(\epsilon)}{(\epsilon + 1) H} \) for 
\( \epsilon \geq 0 \), where \( H := \max\{\| P_{ij} \| : i, j \in \mathbb{N} \} \). It is easy to see that \( 0 < \beta(\epsilon) < \frac{\alpha(\epsilon)}{H} \leq \frac{\epsilon}{H} \). 
\( \beta \) is strictly increasing and continuous at \( 0 \). Then there exists the strictly increasing
Necessity. Suppose that the trivial solution of (3.1) is a necessary and sufficient condition for the stability of the trivial solution of the SDLS system (3.1). We define the Lyapunov function \( \sigma \) satisfying the initial condition \( \sigma(0) = 0 \) continuous in the second variable at \( \sigma \) the function \( \psi \) theorem 4.5. The existence of the Lyapunov functions \( \sigma \) which is different from Lemma 3.3 in [1]. \[ \text{Theorem 4.5.} \text{ The existence of the Lyapunov functions } V_\sigma : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_+ \text{ being continuous in the second variable at } \gamma = 0 \text{ and the functions } a, \psi \in K, \text{ such that} \]

(i) \( a(\|y\|) \leq V_\sigma(k, y) \leq \psi_k(\|y\|), \forall k \geq 0, \forall y \in \Delta_{\sigma(k)}, \forall \sigma, \)

(ii) \( \Delta V_\sigma(k, y(k)) := V_\sigma(k+1, y(k+1)) - V_\sigma(k, y(k)) \leq 0, \forall k \geq 0, \forall \sigma, \text{ for any solution } y(k) \text{ of (3.1) corresponding } \sigma, \)

is a necessary and sufficient condition for the stability of the trivial solution of the SDLS system (3.1).

**Proof.** Necessity. Suppose that the trivial solution of (3.1) is stable. For each \( k_0, \) then according to Lemma 4.3, there exist functions \( \psi \) in \( K \) \( (k_0 \geq 0), \) such that for any solution \( x(k) \) of (3.1),

\[ \|x(k)\| \leq \psi_{k_0}(\|x(0)\|), \forall k \geq k_0, \forall \sigma. \]  

(4.3)

We define the Lyapunov function \[ V_\sigma(k_0, \gamma) := \sup_{m \in \mathbb{N}} \|x_\sigma(k_0 + m, k_0; \gamma)\|, \text{ for each } \gamma \in \mathbb{R}^n, k_0 \in \mathbb{N}, \]  

(4.4)

where \( x_\sigma(k_0 + m, k_0; \gamma) \) is the unique solution of (3.1) corresponding to switching signal \( \sigma \) satisfying the initial condition \( P_{\sigma(k_0)\sigma(k_0+1)}x_\sigma(k_0) = P_{\sigma(k_0)\sigma(k_0+1) \gamma} \). Inequality (4.3) ensures the correctness of definition (4.4). By (4.2), we have

\[ \|x_\sigma(k_0)\| \leq (K + 1)\|P_{\sigma(k_0)\sigma(k_0+1)}x_\sigma(k_0)\| = (K + 1)\|P_{\sigma(k_0)\sigma(k_0+1) \gamma}\| \leq (K + 1)H\|\gamma\|, \]

where the constants \( K, H \) are given Lemma 4.3. Define \( \hat{\psi}_{k_0}(t) := \psi_{k_0}((K + 1)Ht) \) for \( t \geq 0. \) Then we imply that

\[ V_\sigma(k_0, \gamma) \leq \hat{\psi}_{k_0}(\|x_\sigma(k_0)\|) \leq \psi_{k_0}((K + 1)H\|\gamma\|) = \hat{\psi}_{k_0}(\|\gamma\|), \forall k_0 \geq 0, \forall \gamma \in \mathbb{R}^n, \forall \sigma. \]
This implies that \( V_\sigma(k_0,0) = 0 \) and the continuity of the function \( V \) w.r.t. the second variable at \( \gamma = 0 \). For each \( y \in \Delta_\sigma(k_0) \), by (3.14), we have

\[
V_\sigma(k_0,y) = \sup_{l \in \mathbb{N}} \| x_\sigma(k_0 + l, k_0; y) \| \geq \| x_\sigma(k_0, k_0; y) \| = \| y \| = a(\| y \|) \tag{4.5}
\]

On the other hand, for each \( k_0 \geq 0 \) due to the unique solvability of (3.1)-(3.2), it is easy to see that

\[
\{ x_\sigma(k_0 + l, k_0; y(k_0)) : l \geq 0 \} = \{ y(k_0 + l) : l \geq 0 \} \\
\cup \{ y(k_0 + l) : l \geq 1 \} \cup \{ x_\sigma(k_0 + 1 + l, k_0 + 1; y(k_0 + 1)) : l \geq 0 \}, \tag{4.6}
\]

where \( \sigma_y(k) \) is the switching signal corresponding \( y(k) \). Thus

\[
V_\sigma(k_0 + 1, 1; y(k + 1)) = \sup_{l \geq 0} \| x_\sigma(k + 1 + l, k + 1; y(k + 1)) \| \\
\leq \sup_{l \geq 0} \| x_\sigma(k + l, k; y(k)) \| = V_\sigma(k, y(k)),
\]

which implies \( \Delta V_\sigma(k, y(k)) \leq 0 \). The necessity part is proved.

Sufficiency. We argue by contradiction by assuming that trivial solution of (3.1) is not stable, i.e., there exist a positive \( \epsilon_0 \), a nonnegative integer \( k_0 \) and a switching signal \( \sigma \), such that for all \( \delta \in (0, \epsilon_0) \), there exists a solution \( x_\sigma(k) \) of (3.1) satisfying the inequalities \( \| P_{\sigma(k_0)\sigma(k_0+1)} x_\sigma(k_0) \| < \delta \) and \( \| x_\sigma(k_1) \| \geq \epsilon_0 \) for some \( k_1 \geq k_0 \).

Since \( V_\sigma(k_0,0) = 0 \) and \( V_\sigma(k_0,\gamma) \) is continuous at \( \gamma = 0 \), there exists a \( \delta_0 = \delta_0(\epsilon, k_0) > 0 \), such that for all \( \xi \in \mathbb{R}^n \), \( \| \xi \| < \delta_0 \) and for all \( \sigma \) we have \( V_\sigma(k_0,\xi) < \epsilon_1 := a(\epsilon_0) \). Choosing \( \delta_0 \leq \{ \frac{\delta_0}{K+1}, \epsilon_0 \} \) we can find solution \( x_\sigma(k) \) of (3.1) satisfying \( \| P_{\sigma(k_0)\sigma(k_0+1)} x_\sigma(k_0) \| < \delta_0 \), however \( \| x_\sigma(k_1) \| \geq \epsilon_0 \) for some \( k_1 \geq k_0 \). Since \( \| P_{\sigma(k_0)\sigma(k_0+1)} x_\sigma(k_0) \| < \delta_0 \leq \frac{\delta_0}{K+1}, \| x_\sigma(k_0) \| < \delta_0 \) and one gets \( V_\sigma(k_0, x_\sigma(k_0)) < \epsilon_1 \).

On the other hand, using the properties of the function \( V \), we find

\[
V_\sigma(k_0, x_\sigma(k_0)) \geq V_\sigma(k_1, x_\sigma(k_1)) \geq a(\| x_\sigma(k_1) \|) \geq a(\epsilon_0) = \epsilon_1,
\]

which leads to a contradiction. The proof of Theorem 4.5 is complete. \( \square \)

If the trivial solution of (3.1) is uniformly stable then the function \( \psi_k \) in the above theorem can be chosen independently on \( k \). Therefore, a similar argument as in the above proof leads to the next result.

**Theorem 4.6.** The trivial solution of (3.1) is uniformly stable if and only if there exist two functions \( a, b \in \mathcal{K} \) and the Lyapunov functions \( V_\sigma : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_+ \), such that

(i) \( a(\| y \|) \leq V_\sigma(k, y) \leq b(\| y \|), \forall k \geq 0, \forall y \in \Delta_\sigma(k), \forall \sigma,\)

(ii) \( \Delta V_\sigma(k, y(k)) := V_\sigma(k + 1, y(k + 1)) - V_\sigma(k, y(k)) \leq 0, \forall k \geq 0, \forall \sigma, \) for any solution \( y(k) \) of (3.1) corresponding \( \sigma \).

Now, we derive a theorem on the asymptotical stability of the trivial solution of (3.1).

**Theorem 4.7.** Suppose that there exist the functions \( a, c, \psi_k \in \mathcal{K} \) and the Lyapunov functions \( V_\sigma : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \), such that
(i) \( a(\|y\|) \leq V_\sigma(k, y) \leq \psi_k(\|y\|), \forall k \geq 0, \forall y \in \Delta_\sigma(k), \forall \sigma, \)

(ii) \( \Delta V_\sigma(k, y(k)) := V_\sigma(k + 1, y(k + 1)) - V_\sigma(k, y(k)) \leq -c(\|y(k)\|), \forall k \geq 0, \forall \sigma, \) for any solution \( y(k) \) of (3.1) corresponding \( \sigma. \)

Then the trivial solution of (3.1) is asymptotically stable.

**Proof.** From Theorem 4.5, we have the trivial solution of (3.1) is stable. By item (ii), \( \{V_\sigma(k, y(k))\} \) is a decreasing sequence and is below bounded by 0. Therefore there exists the limit \( \lim_{k \to \infty} V_\sigma(k, y(k)). \) This implies that

\[
\lim_{k \to \infty} V_\sigma(k + 1, y(k + 1)) - V_\sigma(k, y(k)) = 0
\]

and hence \( \lim_{k \to \infty} c(\|y(k)\|) = 0. \) Since \( c \in K, \) it implies that \( \lim_{k \to \infty} \|y(k)\| = 0. \) Indeed, assume that \( \lim_{k \to \infty} \|y(k)\| \neq 0. \) Then for some \( \epsilon > 0, \) there exists a sequence \( \{k_m\} \subset \mathbb{N} \) such that \( k_m \to \infty \) and \( \|y(k_m)\| > \epsilon. \) This implies that \( c(\|y(k_m)\|) \geq c(\epsilon) > 0 \) which is a contradiction. The proof is complete. \( \square \)

We define

\[
\mu = \max\{L_i(1 + K_i)\|P^G_{ijm}\| : i, j, m \in \mathbb{N})\}.
\]

**Theorem 4.8.** Assume that there exist \( M > 0, 0 < \lambda < 1 \) such that

\[
\|\Phi_\sigma(k, h)\| \leq M\lambda^{k-h}, \forall k \geq k_0,
\]

and \( M\mu < 1 - \lambda. \) Then the trivial solution of (3.1) is exponentially stable.

**Proof.** From formula (3.7), we have

\[
u(k) = \Phi_\sigma(k, k_0)u(k_0) + \sum_{i=k_0}^{k-1} \Phi_\sigma(k, i+1)P_\sigma(i+1)\sigma(i+2)G^{-1}_{\sigma(i)\sigma(i+1)}f_\sigma(i)(u(i) + v(i)).
\]

This implies that

\[
\|\nu(k)\| = M\lambda^{k-k_0}\|u(k_0)\| + \sum_{i=k_0}^{k-1} M\lambda^{k-i-1}\|P_\sigma(i+1)\sigma(i+2)G^{-1}_{\sigma(i)\sigma(i+1)}\|L_\sigma(i)\|u(i) + v(i)\|
\]

\[
= M\lambda^{k-k_0}\|u(k_0)\| + \sum_{i=k_0}^{k-1} M\lambda^{k-i-1}\|P_\sigma(i+1)\sigma(i+2)G^{-1}_{\sigma(i)\sigma(i+1)}\|L_\sigma(i)(1 + K_\sigma(i))\|u(i)\|
\]

\[
\leq M\lambda^{k-k_0}\|u(k_0)\| + \sum_{i=k_0}^{k-1} M\lambda^{k-i-1}\mu\|u(i)\|.
\]

This is equivalent to

\[
\|\nu(k)\| \leq M\|\nu(k_0)\| + \sum_{i=k_0}^{k-1} \frac{M\mu}{\lambda^{k-i}}\mu\|u(i)\|, \forall k \geq k_0.
\]

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Therefore, if we put \( y_m = \frac{\|u_{m+k_0}\|}{\lambda^m}, f_m = M\|u(k_0)\|, g_m = \frac{M\mu}{\lambda} \) for all \( m \geq 0 \). Then we have

\[
y_m \leq f_m + \sum_{0 \leq i < m} g_i y_i, \forall m \geq 0.
\]

By Theorem 2.2, we get

\[
y_m \leq f_m + \sum_{0 \leq i < m} f_i g_i \prod_{i < j < m} (1 + g_j)
\]

\[
\leq M\|u(k_0)\| + \sum_{0 \leq i < m} M\|u(k_0)\| \frac{M\mu}{\lambda} \left(1 + \frac{M\mu}{\lambda}\right)^{m-i-1}.
\]

\[
= M\|u(k_0)\| + M\|u(k_0)\| \left(1 + \frac{M\mu}{\lambda}\right)^{m} - 1
\]

\[
= M\|u(k_0)\| \left(1 + \frac{M\mu}{\lambda}\right)^{m}.
\]

This implies that

\[
\|u(k)\| \leq M\|u(k_0)\| \left(1 + \frac{M\mu}{\lambda}\right)^{k-k_0} \lambda^{k-k_0} = M\|u(k_0)\| (\lambda + M\mu)^{k-k_0}, \forall k \geq k_0.
\]

Hence

\[
\|x(k)\| = \|u(k) + v(k)\| \leq (1 + K)\|u(k)\| \leq (1 + K)M\|u(k_0)\| (\lambda + M\mu)^{k-k_0}, \forall k \geq k_0.
\]

Since \( \lambda + M\mu < 1 \), the trivial solution of (3.1) is exponentially stable. The proof is complete.

**Example 4.9.** Consider the SDLS (3.1) with switching signal \( \sigma : \mathbb{N} \cup \{0\} \to \{1, 2, ..., N\} = I_N \) and

\[
E_i = \begin{pmatrix} 0 & i \\ 0 & i+1 \end{pmatrix}; \quad A_i = \begin{pmatrix} i+1 & 1 \\ -i-1 & 1 \end{pmatrix}
\]

and

\[
f_i(x) = \frac{\sin x_2}{i} (1, -1)^T; \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \quad i \in I_N.
\]

We have \( \ker E_i = \text{span}\{(1, 0)^T\}, \text{Im}E_i = \text{span}\{(0, 1)^T\} \) and \( S_{ij} = \text{span}\{(0, 1)^T\} \). Therefore, \( S_{ij} \cap \ker E_i = \{0\} \) and \( \text{rank} E_i = 1 < 2 \), hence the SDLS (3.1) is of index-1. Clearly,

\[
V_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \forall i, j \in I_N; \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
This implies that

\[ Q_{ij} = V_{ij}QV_{ij}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P; \quad P_{ij} = I_n - Q_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

A simple calculation shows that \( Q_{ijm} = V_{ij}QV_{ijm}^{-1} = Q_{ij} \forall i, j, m \in I_N \) and

\[ G_{ijm} = E_j + A_i Q_{ijm} = \begin{pmatrix} i + 1 \\ -i - 1 \end{pmatrix} j + 1 \] \( G_{ijm}^{-1} = \frac{1}{(i+1)(2j+1)} \begin{pmatrix} j + 1 & -j \\ i + 1 & i + 1 \end{pmatrix}. \]

Further, the function \( f_i(x) \) is Lipschitz continuous with the Lipschitz coefficient \( L_i = \frac{\sqrt{2}}{i} \). Indeed, we have

\[
\| f_i(x) - f_i(y) \| = \| \frac{\sin x_2}{i}(1,-1)^T - \frac{\sin y_2}{i}(1,-1)^T \| \\
\leq \frac{1}{i+1} |x_2 - y_2| \| (1,-1)^T \| = \frac{\sqrt{2}}{i} |x_2 - y_2| \\
\leq \frac{\sqrt{2}}{i} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \frac{\sqrt{2}}{i} \| x - y \|,
\]

where we use the Euclidean norms of vectors. Moreover, \( f_i(0) = 0 \) and

\[
\omega_i = L_i \max \{ \| Q_{ijm}G_{ijm}^{-1} \| : j, m \in N \} \\
= \max \{ \frac{\sqrt{4j^2 + 4j + 2}}{2j + 1} : j \in N \} \cdot \frac{1}{i(i+1)} \\
< \frac{\sqrt{10}}{3i(i+1)} < 1, \forall i \in I_N.
\]

According to Theorem 3.1, the SDLS (3.1), (3.2) has unique solution. From the definition of \( \Delta_i \), we have \( x \in \Delta_i \) if only if

\[ Q_{ij}x = -V_{ij}QV_{ij}^{-1}G_{ijm}^{-1}[f_i(x) + A_i P_{ij}x]. \]

This relation leads to \( x_1 = -\frac{\sin x_2}{i(i+1)} - \frac{x_2}{(i+1)(2j+1)}. \) Thus,

\[ \Delta_i = \left\{ x = (x_1, x_2)^T : x_1 = -\frac{\sin x_2}{i(i+1)} - \frac{x_2}{(i+1)(2j+1)}, j \in N \right\}. \]

Consider a function \( V_\sigma(k, \gamma) := 2\| P_{\sigma(k)\sigma(k+1)}y \| \) for all \( \gamma \in \mathbb{R}^2 \). We get for each \( y \in \Delta_i \),

\[
\| y \| = \sqrt{y_1^2 + y_2^2} = \sqrt{\left( \frac{\sin y_2}{i(i+1)} + \frac{y_2}{(i+1)(2j+1)} \right)^2 + y_2^2} \\
\leq \sqrt{\left( \frac{1}{i(i+1)} + \frac{1}{(i+1)(2j+1)} \right)^2 y_2^2 + y_2^2} \\
\leq \sqrt{2|y_2|^2} = \sqrt{2\| P_{\sigma(k)\sigma(k+1)}y \|} = V_\sigma(k, y).
\]
Moreover, $V_\sigma(k, y) = 2\|P_{\sigma(k}\sigma(k+1)}y\| = 2\|y\| \leq 2\|y\|$. Thus item (i) of Theorem 4.6 is satisfied. We suppose that $y(k)$ is a solution of (3.1) and putting $y(k) = u(k) + v(k)$, where $u(k) = P_{\sigma(k)\sigma(k+1)}y(k)$; $v(k) = Q_{\sigma(k)\sigma(k+1)}y(k)$, we have

$$\Delta V(k, y(k)) = 2(\|P_{\sigma(k)\sigma(k+2)}y(k+1)\| - \|P_{\sigma(k)\sigma(k+1)}y(k)\|) = 2(\|u(k+1)\| - \|u(k)\|)$$

Using equation (3.7) we find

$$u(k+1) = P_{jm}G_{ijm}^{-1}A_iu(k) + P_{jm}G_{ijm}^{-1}f_i(x(k)) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} u(k),$$

hence, $\|u(k+1)\| = \frac{2}{2j+1}\|u(k)\|$ and leading to $\|u(k+1)\| - \|u(k)\| \leq 0$. According to Theorem 4.6, the trivial solution of (3.1) is uniformly stable. Moreover, since

$$\|u(k+1)\| - \|u(k)\| \leq \frac{1-2j}{2j+1}\|u(k)\| \leq \frac{1-2j}{2(2j+1)}\|y(k)\| \leq \frac{-1}{2(2N+1)}\|y(k)\|.$$ 

Thus, by Theorem 4.7, the trivial solution of (3.1) is asymptotically stable.

**Example 4.10.** In this example we will use the infinity-norms of matrices. Consider the SDLS (3.1) with switching signal $\sigma : \mathbb{N} \cup \{0\} \to \{1, 2\} = \mathcal{N}$, and

$$E_1 = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad E_2 = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $f_i(x) = \frac{2x_i + 3\sin \frac{x_i}{4}}{3(i+1)(i+2)}(0, 0, 1)^T$, $x = (x_1, x_2)^T \in \mathbb{R}^2$, $i \in \mathcal{N}$. A simple computation shows that

$$\ker E_1 = \ker E_2 = \text{span}\{(0, 0, 1)^T\},$$

$$S_{11} = \text{span}\{(1, 0, 0)^T, (0, 1, 0)^T\}, S_{12} = \text{span}\{(3, 2, 0)^T, (0, 1, 0)^T\},$$

$$S_{21} = \text{span}\{(-1, 1, 0)^T, (0, 1, 0)^T\}, S_{22} = \text{span}\{(-1, 3, 0)^T, (0, 1, 0)^T\}.$$ 

Clearly $S_{ij} \cap \ker E_i = \{0\}, \forall i, j \in \mathcal{N}$ and rank $E_i = 2 < 3$, hence homogenous SDLS systems respectively with (3.1) with above data is of index-1. We have

$$V_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad V_{12} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad V_{21} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad V_{22} = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad Q_{ij} = Q; \quad P_{ij} = I_3 - Q_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad Q_{ijm} = Q, \forall i, j, m \in \mathcal{N}.
It is easy to compute that
\[
G_{11} = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad G_{11}^{-1} = \begin{pmatrix} 1/3 & 2/9 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\
G_{21} = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad G_{21}^{-1} = \begin{pmatrix} 2/7 & -1/7 & 0 \\ -1/21 & 4/21 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \forall i, m \in \mathbb{N}.
\]

Further, the function \( f_i(x) \) is Lipschitz continuous with the Lipschitz coefficient \( L_i = \frac{1}{12(i+1)(i+2)} \), \( i \in \mathbb{N} \). We calculate
\[
\omega_i := L_i \max \{ \| Q_{ijm} G_{ijm}^{-1} \| \} = L_i, K_i := \omega_i (L_i + \| A_i \|) L_i^{-1} (1 - \omega_i)^{-1},
\]

hence \( K_1 = \frac{25}{11}, K_2 = \frac{49}{25} \); \( \| P_{ijm} G_{ijm}^{-1} \| = \frac{5}{9} \), \( \| P_{ijm} G_{ijm}^{-1} \| = \frac{3}{7} \). Therefore \( \mu = \max \{ L_i (1 + K_i) \| P_{ijm} G_{ijm}^{-1} \| : i, j, m \in \mathbb{N} \} = \max \left\{ \frac{5}{18}, \frac{3}{14}, \frac{55}{414}, \frac{33}{322} \right\} = \frac{5}{18} \). Putting
\[
\Phi_{ijm} := P_{ijm} G_{ijm}^{-1} A_i = P_{\sigma(l) \sigma(l+1)} G_{\sigma(l-1) \sigma(l) \sigma(l+1)}^{-1} A_{\sigma(l-1)},
\]

we have
\[
\Phi_{11} = \begin{pmatrix} 1/3 & -1/9 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \Phi_{21} = \begin{pmatrix} -1/9 & 2/9 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
\Phi_{12} = \begin{pmatrix} 2/7 & -3/7 & 0 \\ -1/21 & 5/21 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \Phi_{22} = \begin{pmatrix} -3/7 & -1/7 & 0 \\ 5/21 & 4/21 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
\| \Phi_{11} \| = \frac{4}{9}; \quad \| \Phi_{21} \| = \frac{2}{3}; \quad \| \Phi_{12} \| = \frac{5}{7}; \quad \| \Phi_{22} \| = \frac{4}{7}.
\]

Thus if we choose \( \lambda = \max \{ \| \Phi_{ijm} \| : i, j, m \in \mathbb{N} \} = \frac{5}{7} \) and \( M = 1 \) then
\[
\| \Phi_{\sigma}(k, h) \| \leq \prod_{l=h+1}^{k} \| P_{\sigma(l) \sigma(l+1)} G_{\sigma(l-1) \sigma(l) \sigma(l+1)}^{-1} A_{\sigma(l-1)} \| \leq \left( \frac{5}{7} \right)^{k-h} = M \lambda^{k-h},
\]
for all \( k \geq h \geq k_0 \). Moreover we have
\[
M \mu = \frac{5}{18} < 1 - \lambda = \frac{2}{7}.
\]

Thus, by Theorem 4.8, the SDLS system with the above data \( \{(E_i, A_i, f_i)\}_{i=1,2} \) is exponentially stable.
5. Conclusion

In this paper, we have studied SDLS systems subject to Lipschitz perturbations. We derive solvability and establish a formula of solution for these equations. The stability of SDLS systems is investigated by using methods of the Lyapunov functions and the solution evaluation.

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References

