THE LEFSCHETZ PROPERTIES OF ARTINIAN MONOMIAL ALGEBRAS ASSOCIATED TO PATHS AND CYCLES

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ABSTRACT. In this paper, we study the weak Lefschetz property of an artinian monomial algebra A_G defined by the sum of the edge ideal of a simple graph G and the square of the variables. We classify some important classes of graphs G where A_G has or fails the weak Lefschetz property, such as paths, cycles and several tadpole graphs.

1. INTRODUCTION

A graded algebra A satisfies the weak Lefschetz property (WLP) if there exists a linear form ℓ such that the multiplication map $\cdot\ell : A_i \longrightarrow A_{i+1}$ has maximal rank for all degree i, while A satisfies the strong Lefschetz property (SLP) if the multiplication map $\cdot\ell^j : A_i \longrightarrow A_{i+j}$ has maximal rank for all i and all j.

The Lefschetz properties of graded algebras have connections to several areas of mathematics. Due to this ubiquity, many classes of algebras have been studied with respect to the WLP and the SLP. At first glance, checking the WLP or the SLP might seem to be a simple problem of linear algebra. However, determining which graded algebras have the WLP or the SLP is notoriously difficult, and a number of natural families of algebras still simply remain uncharacterized. We refer the reader to [7] and [15] for an introduction to the Lefschetz properties.

In this paper, we study the SLP and/or WLP of artinian monimial algebras associated the edge ideals of graphs. More precisely, let G be a simple graph, i.e. $G = (V, E)$ is a pair where V is a set of elements called *vertices*, and E is a set of elements called *edges* which are unordered pairs of vertices from V. Suppose that $V = \{1, 2, \ldots, n\}$ and let k be a field and $R = k[x_1, \ldots, x_n]$. The edge ideal of G is the ideal

$$I_G = (x_i x_j \mid \{i, j\} \in E) \subset R.$$

Then, we say that

$$A_G = \frac{R}{(x_1^2, \ldots, x_n^2) + I_G}$$

is the *artinian monomial algebra associated to G*. We are interested the following problem.

Problem 1.1. Classify the simple graph G that A_G has or fails the WLP or SLP.

Date: September 6, 2022.

2020 *Mathematics Subject Classification.* 13F55, 13E10, 13F20, 05C31.

Key words and phrases. Artinian algebras; edge ideals; independence polynomials; weak Lefschetz property.

Note that A_G is an artinian algebra generated by quadratic monomials. The WLP of this algebra is also studied by Michałek and Miró-Roig [12]; Migliore, Nagel and Schenck [14]. In this paper, we will study the WLP for a larger class of these algebras.

2. Preliminaries

In this section we recall some standard terminology and notations from commutative algebra and combinatorial commutative algebra, as well as some results needed later on.

2.1. **The weak and strong Lefschetz properties.** We consider standard graded algebra $A = \oplus_{i\geq 0}[A]_i = R/I$, where $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k with all x_i's have degree 1 and $I \subset R$ is an artinian homogeneous ideal. Let us define the weak and strong Lefschetz properties for artinian algebras.

Definition 2.1. We say that A has the *weak Lefschetz property* (WLP) if there is a linear form $\ell \in [A]_1$ such that, for all integers j, the multiplication map

$$\cdot\ell : [A]_j \longrightarrow [A]_{j+1}$$

has maximal rank, i.e. it is injective or surjective. In this case the linear form ℓ is called a *Lefschetz element* of A. If for the general form $\ell \in [A]_1$ and for an integer number j the map $\cdot\ell : [A]_j \longrightarrow [A]_{j+1}$ does not have the maximal rank we will say that A *fails the WLP in degree* j.
We say that A has the *strong Lefschetz property* (SLP) if there is a linear form $\ell \in [A]_1$ such that, for all integers j and s, the multiplication map

$$\cdot\ell^s : [A]_j \longrightarrow [A]_{j+s}$$

has maximal rank.

In the case of one variable, the WLP and SLP trivially hold since all ideals are principal. The case of two variables there is a nice result in characteristic zero by Harima, Migliore, Nagel and Watanabe [8, Proposition 4.4].

Proposition 2.2. *Every artinian algebra $A = k[x, y]/I$, where k has characteristic zero, has the SLP (and consequently also the WLP).*

In a polynomial ring with more than two variables it is not true in general that every artinian monomial algebra has the SLP or WLP. The most general result in this case proved by Stanley in [16].

Theorem 2.3. *Let $R = k[x_1, \ldots, x_n]$, where k is of characteristic zero. Let I be an artinian monomial complete intersection, i.e. $I = (x_1^{d_1}, \ldots, x_n^{d_n})$. Then $A = R/I$ has the SLP.*

By using the action of a torus on monomial algebras, Migliore, Miró-Roig and Nagel proved the existence of the canonical Lefschetz element.

Proposition 2.4. [13, Proposition 2.2] *Let $I \subset R = k[x_1, \ldots, x_n]$ be an artinian monomial ideal. Then $A = R/I$ has the WLP if and only if $\ell = x_1 + x_2 + \cdots + x_n$ is a Lefschetz element for A.*

A necessary condition for the WLP and SLP of an artinian algebra A is the unimodality of the Hilbert series of A. To do it, we need some notations.

Definition 2.5. Let k be a filed and $A = \oplus_{j\geq 0}[A]_j$ be a standard graded k-algebra. The *Hilbert series* of A is the power series $\sum \dim_k[A]_i t^i$ and is denoted by $HS(A,t)$. The *Hilbert function* of A is the function $h_A : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $h_A(j) = \dim_k[A]_j$.
If A is an artinian graded algebra, then $[A]_i = 0$ for $i \gg 0$. We denote

$$D = \max\{i \mid [A]_i \neq 0\}.$$

The integer D is called the *socle degree* of A. In this case, the Hilbert series of A is a polynomial

$$HS(A,t) = 1 + h_1 t + \cdots + h_D t^D,$$

where $h_i = H_A(i) = \dim_k[A]_i > 0$. By definition, the degree of the Hilbert series for an artinian graded algebra A is equal to its socle degree $D := \max\{i \mid [A]_i \neq 0\}$. Since A is artinian and non-zero, this number also agrees with the *Castelnuovo-Mumford regularity* of A, so

$$\operatorname{reg}(A) = D = \deg(HS(A,t)).$$

Definition 2.6. A polynomial $\sum_{k=0}^{n} a_k x^k$ with non-negative coefficients is called *unimodal* if there is some m, such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

The *mode* of the unimodal polynomial $\sum_{k=0}^{n} a_k x^k$ defined by

$$\min\{k \mid a_{k-1} < a_k \geq a_{k+1} \geq \cdots \geq a_n\}.$$

Proposition 2.7. [7, Proposition 3.2] *If A has the WLP or SLP then the Hilbert series of A is unimodal.*

Finally, to study the failure of the WLP, the next lemma play a key role.

Lemma 2.8. [3, Lemma 7.8] *Let $A = A' \otimes_k A''$ be a tensor product of two graded artinian k-algebras A' and A''. Let $\ell' \in A'$ and $\ell'' \in A''$ be linear elements, and set $\ell = \ell' + \ell'' = \ell' \otimes 1 + 1 \otimes \ell'' \in A$. Then*

(a) *If the multiplication maps $\ell' : [A']_i \longrightarrow [A']_{i+1}$ and $\ell'' : [A'']_j \longrightarrow [A'']_{j+1}$ are both not surjective, then the multiplication map*

$$\ell : [A]_{i+j+1} \longrightarrow [A]_{i+j+2}$$

is not surjective.

(b) *If the multiplication maps $\ell' : [A']_i \longrightarrow [A']_{i+1}$ and $\ell'' : [A'']_j \longrightarrow [A'']_{j+1}$ are both not injective, then the multiplication map*

$$\ell : [A]_{i+j} \longrightarrow [A]_{i+j+1}$$

is not injective.

2.2. **Some definitions and results in graph theory.** From now on, a graph is a simple graph $G = (V, E)$, with V is the set of vertices and E is the set of edges. We start by recalling some basic definitions.

Definition 2.9. Let $G = (V, E)$ be a graph. The *disjoint union* of the graphs G_1, G_2 is a graph $G = G_1 \cup G_2$ having as vertex set the disjoint union of $V(G_1), V(G_2)$, and as edge set the disjoint union of $E(G_1), E(G_2)$. In particular, $\cup_n G$ denotes the disjoint union of $n > 1$ copies of the graph G.

Definition 2.10. Let $G = (V, E)$ be a graph.

(i) A subset X of V is called an *independent set* of G if for any $i, j \in X,\ \{i, j\} \notin E$, i.e., the vertices in X are pairwise non-adjacent. An independent set X is called *maximal* if for every vertices $v \in V \setminus X$, $X \cup \{v\}$ is not an independent set of G.
(ii) The *independence number* of a graph G is the maximum cardinality of an independent set in G. We denote this value as $\alpha(G)$.
(iii) A graph G is said to be *well-covered* if every maximal independent set of G has the same size and equal to $\alpha(G)$.

Definition 2.11. The independence polynomial of a graph G is the polynomial whose coefficient on x^k is given by the number of independent sets of order k in G. We denote this polynomial $I(G; t)$. So,

$$I(G;t) = \sum_{k=0}^{\alpha(G)} s_k(G)t^k,$$

where $s_k(G)$ is the number of independent sets of order k in G.

The independence polynomial of a graph was defined by I. Gutman and F. Harary in [5] as a generalization of the matching polynomial of a graph. The following equalities are very useful in calculating of the independence polynomial for various families of graphs (see, for instance, [5, 9]).

Proposition 2.12. *Let G_1, G_2, G be the graphs. Assume that $G = (V, E), w \in V, e = uv \in E$. Then the following equalities hold:*

(i) $I(G; t) = I(G - w; t) + t.I(G - N_w(G); t);$
(ii) $I(G; t) = I(G - e; t) - t^2.I(G - N_u(G) \cup N_v(G); t);$
(iii) $I(G_1 \cup G_2; t) = I(G_1; t).I(G_2; t);$

3. Artinian monomial algebras associated to edge ideals of graphs

Let $G = (V, E)$ be a graph, with $V = \{1, 2, \ldots, n\}$. Let $R = k[x_1, \ldots, x_n]$ be the standard polynomial ring over a field k. The edge ideal of G is the ideal

$$I_G = (x_i x_j \mid \{i, j\} \in E) \subset R.$$

Then, we say that

$$A_G = \frac{R}{(x_1^2, \ldots, x_n^2) + I_G}$$

is the *artinian monomial algebra associated to G*. The algebra A_G contains significant combinatorial information about G. It is easy to check the following.

Proposition 3.1. *The Hilbert series of A_G is equal to the independence polynomial of G, i.e.*

$$HS(A_G;t) = I(G;t) = \sum_{k=0}^{\alpha(G)} s_k(G)t^k.$$

As a consequence, $\operatorname{reg}(A_G) = \alpha(G)$ *and* A_G *is level if and only* G *is well-covered.*

Therefore, the WLP/SLP of A_G has strong consequences on the unimodality of the independence polynomial of G. Indeed, if $I(G;t)$ is not unimodal, then A_G fails the WLP by Proposition 2.7. Thus, to study the WLP/ SLP of A_G, it is enough to consider the graphs G such that their independence polynomial are unimodal. Concerning to the unimodality of the independence polynomial of graphs, we have the following famous conjecture.

Conjecture 3.2. [1] If G is a tree or forest, then the independence polynomial of G is unimodal.

The largest class of graphs for which the independence polynomials are known to be unimodal is the class of claw-free graphs [6]. However, Conjecture 3.2 is still very open.

Later, since a tree is a bipartite graph, Levit and Mandrescu [11] have gone further, conjecturing that for any bipartite graph G, $I(G;t)$ is unimodal. Unfortunately, a counterexample gave by A. Bhattacharyya and J. Kahn [2].

Example 3.3. Given positive integers m and $n > m$, let $G = (V, E)$ with $V = V_1 \cup V_2 \cup V_3$, where V_1, V_2, V_3 are disjoint; $|V_1| = n - m$ and $|V_2| = |V_3| = m$; E consists of a complete bipartite graph between V_1 and V_2 and a perfect matching between V_2 and V_3. Then G is a bipartite graph and for every $i \geq 0$, $s_i(G) = (2^i - 1)\binom{m}{i} + \binom{n}{i}$. Therefore, for $m \geq 95$ and $n = \lfloor m \log_2(3) \rfloor$. Then $I(G;t)$ is not unimodal. As a consequence, these graphs do not have the WLP.

The next examples are the simple graphs having the SLP.

Example 3.4.

(i) An empty graph is simply a graph with no edges. We denote the empty graph on n vertices by E_n. Then

$$A_{E_n} = R/(x_1^2, \ldots, x_n^2).$$

A seminar result of Stanley say that A_{E_n} has the SLP, see Theorem 2.3.

(ii) A complete graph on n vertices, denoted K_n, is the graph where every vertex is adjacent to every other vertex. It follows that

$$A_{K_n} = R/(x_1, \ldots, x_n)^2 \quad \text{and } I(K_n;t) = 1 + nt.$$

It is easy to see that A_{K_n} has the SLP.

If G has a small number of vertices, the we have a simple result.

Proposition 3.5. *Let $G = (V, E)$ be a graph. If $|V| \leq 3$, then A_G has the SLP.*

Proof. Since all artinian algebras in the polynomial ring with one or two variables have the SLP, it is enough to consider the case where $|V| = 3$. In this case, G is a empty graph; or a complete graph; or a path and one isolated vertex. A simple computation with `Macaulay2` [4] shows that A_G has the SLP. $\square$

Since the above proposition, we only need to consider the graphs having at least 4 vertices. Now, a simple family of graphs is considered, namely stars. Recall that a star of order n is a graph on $n+1$ vertices. This graph is formed by starting with a single vertex and adjoining n leaves. We denote this graph S_n. Let A_{S_n} be the artinian monomial algebra associated to a star S_n. Then $A_{S_n} = R/I$, where $R = k[x_0, x_1, \dots, x_n]$ and

$$I = (x_0^2, x_1^2, \dots, x_n^2) + (x_0x_1, \dots, x_0x_n).$$

It is easy to see that the independence polynomial of S_n is $I(S_n; t) = (1+t)^n + t$.

Proposition 3.6. *The algebra A_{S_n} has the WLP if and only if $n = 1, 2$. Moreover, if $n \geq 3$, then A_{S_n} fails the WLP in only one degree, namely it fails the injectivity from degree 1 to degree 2.*

Proof. Write $A_{S_n} = R/I$ as above and $\ell = x_0 + x_1 + \cdots + x_n$. Then the following exact sequence

$$0 \longrightarrow R/(I\colon x_0)(-1) \xrightarrow{\cdot x_0} R/I \longrightarrow R/(I, x_0) \longrightarrow 0$$

deduces the the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [R/(I\colon x_0)]_{j-1} & \xrightarrow{\cdot x_0} & [R/I]_j & \longrightarrow & [R/(I, x_0)]_j & \longrightarrow & 0\ , \\ & & \downarrow{\scriptstyle \cdot\ell} & & \downarrow{\scriptstyle \cdot\ell} & & \downarrow{\scriptstyle \cdot\ell} & & \\ 0 & \longrightarrow & [R/(I\colon x_0)]_j & \xrightarrow{\cdot x_0} & [R/I]_{j+1} & \longrightarrow & [R/(I, x_0)]_{j+1} & \longrightarrow & 0 \end{array} \tag{3.1}$$

with rows are exact, for all integer $j \geq 1$. Note that

$$(I, x_0) = (x_1^2, \dots, x_n^2, x_0)$$
$$I\colon x_0 = (x_0, x_1, \dots, x_n).$$

It follows that $R/(I, x_0) \cong S/J := k[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ an artinian monomial complete intersection, and hence it has the WLP by Theorem 2.3. Clearly, $R/(I\colon x_0) \cong k$. It follows from (3.1) that the multiplication map

$$\cdot\ell : [R/I]_j \longrightarrow [R : I]_{j+1}$$

has maximal rank for all j, except $j = 1$. In the later case, $\cdot\ell : [R/I]_1 \longrightarrow [R : I]_2$ is not injective. Thus R/I fails the WLP in degree 1, as desired. $\square$

We close this section by recall a recent result of J. Migliore, U. Nagel and H. Schenck.

Proposition 3.7. *Let A be the artinian monomial algebra associated to $G = \cup_{i=1}^r K_{n_i}$. Assume $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. Then A has the WLP if and only if one of the following holds:*

(1) *$n_2 = \cdots = n_r = 1$, i.e. G is the disjoint union of a complete graph K_{n_1} and an empty graph of order $r-1$.*
(2) *$n_3 = \cdots = n_r = 1$ and r is odd.*

In particular for $n \geq 2$, the disjoint of n complete graphs at least two vertices does not have the WLP.

Proof. Note that

$$A = \bigotimes_{i=1}^{r} k[x_{i,1}, \ldots, x_{i,n_i}]/(x_{i,1}^2, \ldots, x_{i,n_i}^2).$$

The above proposition is a result in [14, Theorem 4.8]. □

4. WLP for artinian monomial algebras associated to paths and cycles

In this section, we study the WLP for artinian monomial algebras associated to two common graphs, namely paths and cycles. From now on, we always denote by ℓ the sum of variables in a polynomial ring where we are considering.

4.1. Path on n-vertices.

Let P_n be a path on n vertices. Therefore, the artinian monomial algebra associated to P_n is

$$A_{P_n} = R/K,$$

where $K = (x_1^2, \ldots, x_n^2) + (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n) \subset R = k[x_1, \ldots, x_n]$. We have the following.

Proposition 4.1. *The independence polynomial of P_n is*

$$I(P_n; t) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} t^i.$$

Moreover, $I(P_n; t)$ is unimodal, with the mode $\lambda_n = \lceil \frac{5n+2-\sqrt{5n^2+20n+24}}{10} \rceil$.

Proof. In [10], G. Hopkins and W. Staton showed that

$$I(P_n; t) = F_{n+1}(t),$$

where $F_n(t),\ n \geq 0$, are the so-called Fibonacci polynomials, i.e., the polynomials defined recursively by

$$F_0(t) = 1; F_1(t) = 1; F_n(t) = F_{n-1}(t) + tF_{n-2}(t).$$

Based on this recurrence, one can deduce that

$$I(P_n; t) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} t^i.$$

The unimodality of the independence polynomial of P_n is implied from the fact that the independence polynomial of a claw-free graph (i.e., it has no induced subgraph isomorphic to $K_{1,3}$) is unimodal [6]. A simple computation shows that the mode of $I(P_n; t)$ is equal to $\lambda_n = \lceil \frac{5n+2-\sqrt{5n^2+20n+24}}{10} \rceil$. □

Lemma 4.2. *Let λ_n be the mode of P_n. For any $n \geq 2$, one has*

(i) $\lambda_{n+1} \geq \lambda_n$.
(ii) $\lambda_{n+3} - 1 \leq \lambda_n \leq \lambda_{n+4} - 1$.
(iii) $\lambda_{n+11} \geq \lambda_n + 3$.

Proof. Set $\alpha_n = \frac{5n+2-\sqrt{5n^2+20n+24}}{10}$. A straightforward computation shows that

$$\alpha_{n+1} \geq \alpha_n;\ \alpha_{n+3} - 1 \leq \alpha_n \leq \alpha_{n+4} - 1 \text{ and } \alpha_{n+11} \geq \alpha_n + 3.$$

The lemma implies from the basic property of the ceiling functions. □

Lemma 4.3. *Let A, A' and A'' be the artinian monomial algebras associated to P_n, P_{n-1} and P_{n-2}, respectively. Then, for every integer i, one has the following commutative diagrams*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [A'']_{i-1} & \longrightarrow & [A]_i & \longrightarrow & [A']_i & \longrightarrow & 0\ . \\ & & \downarrow \cdot\ell & & \downarrow \cdot\ell & & \downarrow \cdot\ell & & \\ 0 & \longrightarrow & [A'']_i & \longrightarrow & [A]_{i+1} & \longrightarrow & [A']_{i+1} & \longrightarrow & 0 \end{array}$$

Proof. Assume $A = R/K$ and let $I = K + (x_n)$ and $J = (K \colon x_n)$. Then $A' \cong R/I$ and $A'' \cong R/J$ and we have the following exact sequence

$$0 \longrightarrow R/J(-1) \xrightarrow{\cdot x_n} R/K \longrightarrow R/I \longrightarrow 0$$

that completes the proof of the above lemma. □

We now prove our main result.

Theorem 4.4. *Let A_n be the artinian monomial algebra associated to a path P_n. Then A_n has the WLP if and only if $n \in \{1, 2, \dots, 7, 9, 10, 13\}$.*

Proof. By using `Macaulay2` to compute the Hilbert series of A_n and $A_n/\ell A_n$ with $1 \leq n \leq 17$, it easy to see that A_n has the WLP for each $n \in \{1, 2, \dots, 7, 9, 10, 13\}$. Furthermore, for each $n \in \{8, 11, 14, 15, 17\}$, A_n only fails the surjectivity in one degree, this is the multiplication map by ℓ from degree λ_n to degree $\lambda_n + 1$. However, for integer $n \in \{12, 16\}$, A_n only fails the injectivity in one degree, this is the multiplication map by ℓ from degree $\lambda_n - 1$ to degree λ_n.

It remains to show the following assertion.

Claim: The multiplication map $\cdot\ell : [A_n]_{\lambda_n} \longrightarrow [A_n]_{\lambda_n+1}$ is not surjective for all $n \geq 17$.

We will prove the above claim by induction on n, having just shown the case $n = 17$. For $n \geq 18$, we consider the multiplication map $\cdot\ell : [A_n]_{\lambda_n} \longrightarrow [A_n]_{\lambda_n+1}$. To prove that this map is not surjective, we consider the following two cases.

Case 1: $\lambda_n = \lambda_{n-1}$. It is clear that the above claim holds by Lemma 4.3.

Case 2: $\lambda_n = \lambda_{n-1} + 1$. By Lemma 4.2, one has $\lambda_{n-1} = \lambda_{n-2} = \lambda_{n-3}$. In this case, we must have $n \geq 20$. Assume $A_n = R/K$ and set $I = K + (x_{n-2})$ and $J = (K \colon x_{n-2})$. Then we have the following exact sequence

$$0 \longrightarrow R/J(-1) \xrightarrow{\cdot x_{n-2}} R/K \longrightarrow R/I \longrightarrow 0\ ,$$

where $R/J \cong A_{n-4} \otimes_k k[x_n]/(x_n^2)$ and $R/I \cong A_{n-3} \otimes_k A_2$, with $A_2 = k[x_{n-1}, x_n]/(x_{n-1}, x_n)^2$. This exact sequence deduces the following diagram, with rows are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [R/J]_{\lambda_n - 1} & \longrightarrow & [A_n]_{\lambda_n} & \longrightarrow & [R/I]_{\lambda_n} & \longrightarrow & 0\ . \\ & & \downarrow{\cdot\ell} & & \downarrow{\cdot\ell} & & \downarrow{\cdot\ell} & & \\ 0 & \longrightarrow & [R/J]_{\lambda_n} & \longrightarrow & [A_n]_{\lambda_n+1} & \longrightarrow & [R/I]_{\lambda_n+1} & \longrightarrow & 0 \end{array}$$

The claim will be proven if we show the multiplication map $\cdot\ell : [R/I]_{\lambda_n} \longrightarrow [R/I]_{\lambda_n+1}$ is not surjective. By the inductive hypothesis, A_{n-3} fails the surjectivity from degree $\lambda_n - 1$ to degree λ_n, as $\lambda_{n-3} = \lambda_n - 1$. Clearly, the Hilbert function of A_2 is $(1, 2)$, and hence A_2 fails the surjectivity from degree 0 to degree 1. Then by Lemma 2.8, $R/I \cong A_{n-3} \otimes_k A_2$ fails the surjectivity from degree λ_n to degree $\lambda_n + 1$, as desired. □

The above theorem shows that A_n fails the WLP due to the failure the surjectivity for any $n \geq 17$. The next result also prove that A_n fails the injectivity for some cases.

Proposition 4.5. *Let A_n be the artinian monomial algebra associated to a path P_n and λ_n is the mode of the independence polynomial of $I(P_n; t)$. If $n \geq 12$ such that $\lambda_n = \lambda_{n-1} + 1$, then A_n fails the injectivity from degree $\lambda_n - 1$ to degree λ_n.*

Proof. We prove the above proposition by induction on $n \geq 12$. A computation with `Macaulay2` shows that the proposition holds for $n \in \{12, 16, 20\}$. Now consider $n \geq 21$ such that $\lambda_n = \lambda_{n-1} + 1$. Set

$$\begin{aligned} n_1 &= \max\{j \mid j < n \text{ and } \lambda_j = \lambda_{j-1} + 1\} \\ n_2 &= \max\{j \mid j < n_1 \text{ and } \lambda_j = \lambda_{j-1} + 1\} \\ m &= \max\{j \mid j < n_2 \text{ and } \lambda_j = \lambda_{j-1} + 1\}. \end{aligned}$$

Then, by Lemma 4.2(iii), $m \geq n - 11$. We have the following exact sequence

$$0 \longrightarrow A_m \otimes_k A_{n-m-3}(-1) \xrightarrow{\cdot x_{m+2}} A_n \longrightarrow A_{m+1} \otimes_k A_{n-m-2} \longrightarrow 0\ .$$

By using this exact sequence, it suffices to show that

$$\cdot\ell : [A_m \otimes_k A_{n-m-3}]_{\lambda_n - 2} \longrightarrow [A_m \otimes_k A_{n-m-3}]_{\lambda_n - 1}$$

is not injective. By the inductive hypothesis, A_m fails the injectivity from degree $\lambda_m - 1$ to λ_m. Observe that $\lambda_m = \lambda_n - 3$ and $n - m - 3 \leq 8$. Hence, $\lambda_{n-m-3} \leq 2$ and consequently, A_{n-m-3} fails the injectivity from degree 2 to degree 3. By Lemma 2.8, $A_m \otimes_k A_{n-m-3}$ fails the injectivity from degree $\lambda_n - 2$ to $\lambda_n - 1$, as desired. □

4.2. **Cycles on n-vertices.** Let C_n be a path on n vertices ($n \geq 3$). Therefore, the artinian monomial algebra associated to C_n is

$$A = R/K,$$

where $K = (x_1^2, \ldots, x_n^2) + (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1) \subset R = k[x_1, \ldots, x_n]$. We have the following.

Proposition 4.6. *The independence polynomial of C_n is*

$$I(C_n;t) = I(P_{n-1};t) + tI(P_{n-3};t)$$
$$= 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{i}\binom{n-i-1}{i-1}t^i.$$

Moreover, $I(C_n;t)$ is unimodal, with the mode $\rho_n = \lceil \frac{5n-4-\sqrt{5n^2-4}}{10} \rceil$.

Proof. In [10], G. Hopkins and W. Staton showed that

$$I(C_n;t) = 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{i}\binom{n-i-1}{i-1}t^i.$$

The unimodality of the independence polynomial of C_n is implied from the fact that the independence polynomial of a claw-free graph is unimodal [6]. A simple computation shows that the mode of $I(C_n;t)$ is equal to $\rho_n = \lceil \frac{5n-4-\sqrt{5n^2-4}}{10} \rceil$. □

Lemma 4.7. *For all $n \geq 5$, there are inequalities $\lambda_{n-1} \leq \rho_n \leq \lambda_{n-4} + 1 \leq \lambda_n$.*

Proof. By Lemma 4.2, $\lambda_{n-4} + 1 \leq \lambda_n$, hence it suffices to show that

$$\lambda_{n-1} \leq \rho_n \leq \lambda_{n-4} + 1.$$

For the inequality on the left, we have to show that

$$\frac{5(n-1)+2-\sqrt{5(n-1)^2+20(n-1)+24}}{10} \leq \frac{5n-4-\sqrt{5n^2-4}}{10}$$
$$\Leftrightarrow 5n-3-\sqrt{5n^2+10n+9} \leq 5n-4-\sqrt{5n^2-4}$$
$$\Leftrightarrow \sqrt{5n^2-4}+1 \leq \sqrt{5n^2+10n+9}$$
$$\Leftrightarrow 5n^2-3+2\sqrt{5n^2-4} \leq 5n^2+10n+9$$
$$\Leftrightarrow \sqrt{5n^2-4} \leq 5n+6 \Leftrightarrow (5n+6)^2-(5n^2-4) \geq 0$$
$$\Leftrightarrow 20n^2+60n+40 \geq 0,$$

which is clear.

For the inequality on the right, we have to show that

$$\frac{5n-4-\sqrt{5n^2-4}}{10} \leq \frac{5(n-4)+2-\sqrt{5(n-4)^2+20(n-4)+24}}{10}+1$$
$$\Leftrightarrow 5n-4-\sqrt{5n^2-4} \leq 5n-8-\sqrt{5n^2-20n+24}$$
$$\Leftrightarrow \sqrt{5n^2-20n+24}+4 \leq \sqrt{5n^2-4}$$
$$\Leftrightarrow 5n^2-20n+24+16+8\sqrt{5n^2-20n+24} \leq 5n^2-4 \quad \text{(by squaring)}$$
$$\Leftrightarrow 8\sqrt{5n^2-20n+24} \leq 20n-44$$
$$\Leftrightarrow 2\sqrt{5n^2-20n+24} \leq 5n-11$$
$$\Leftrightarrow 4(5n^2-20n+24) \leq (5n-11)^2$$
$$\Leftrightarrow 5n^2-30n+25 \geq 0 \Leftrightarrow 5(n-1)(n-5) \geq 0$$

which is true for all $n \geq 5$. The proof is completed. $\square$

One of the main results is the following.

Theorem 4.8. *Let B_n be the artinian monomial algebra associated to a cycle C_n. Then B_n has the WLP if and only if $n \in \{3, \dots, 11, 13, 14, 17\}$.*

Proof. Recall that ρ_n is the mode of the independence polynomial of C_n. By using `Macaulay2` to compute the Hilbert series of B_n and $B_n/\ell B_n$ with $3 \leq n \leq 20$, we can check that:

- B_n has the WLP for each $3 \leq n \leq 17$ and $n \notin \{12, 15, 16\}$;
- for $n \in \{12, 15, 18, 19\}$, then B_n fails the surjectivity from degree ρ_n to degree $\rho_n + 1$;
- for $n \in \{16, 20\}$, then B_n fails the injectivity from degree $\rho_n - 1$ to degree ρ_n.

Now assume that $n \geq 21$. By Lemmas 4.7 and 4.2, $\lambda_{n-1} \leq \rho_n \leq \lambda_{n-4} + 1 \leq \lambda_{n-1} + 1$. Recall that we will denote by A_n the artinian monomial algebra associated to P_n and by λ_n the mode of the independence polynomial of P_n. Consider the following two cases.

Case 1: $\rho_n = \lambda_{n-1}$. In this case, we will show that B_n fails the WLP by the failure of the surjectivity from degree ρ_n to degree $\rho_n + 1$. Indeed, we write $B_n = R/I$. Let $J = I + (x_n)$ and $K = (I \colon x_n)$. Then $A_{n-1} \cong R/J$ and $A_{n-3} \cong R/K$ and we have the following exact sequence

$$0 \longrightarrow R/K(-1) \xrightarrow{\cdot x_n} R/I \longrightarrow R/J \longrightarrow 0$$

that deduces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [A_{n-3}]_{\rho_n - 1} & \longrightarrow & [B_n]_{\rho_n} & \longrightarrow & [A_{n-1}]_{\rho_n} & \longrightarrow & 0\,. \\ & & \downarrow {\scriptstyle \cdot\ell} & & \downarrow {\scriptstyle \cdot\ell} & & \downarrow {\scriptstyle \cdot\ell} & & \\ 0 & \longrightarrow & [A_{n-3}]_{\rho_n} & \longrightarrow & [B_n]_{\rho_n + 1} & \longrightarrow & [A_{n-1}]_{\rho_n + 1} & \longrightarrow & 0 \end{array}$$

The proof of Theorem 4.4 shows that the multiplication map

$$\cdot\ell : [A_{n-1}]_{\rho_n} \longrightarrow [A_{n-1}]_{\rho_n + 1}$$

is not surjective for any $n \geq 18$.

Case 2: $\rho_n = \lambda_{n-1} + 1$. In this case, Lemma 4.7 yields $\lambda_{n-1} = \lambda_{n-4}$.

Denote $y_1 = x_{n-1}, y_2 = x_{n-2}$. We have the following diagram

$$\begin{array}{ccccc} [B_n]_{\rho_n} & \xrightarrow{/(x_n)} & [A_{n-1}]_{\lambda_{n-1}+1} & \xrightarrow{/(x_{n-3})} & \left[A_{n-4} \otimes \frac{k[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-4}+1} \\ \downarrow {\scriptstyle \cdot\ell} & & \downarrow {\scriptstyle \cdot\ell} & & \downarrow {\scriptstyle \cdot\ell} \\ [B_n]_{\rho_n+1} & \longrightarrow & [A_{n-1}]_{\lambda_{n-1}+2} & \longrightarrow & \left[A_{n-4} \otimes \frac{k[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-4}+2} \end{array}$$

By the proof of Theorem 4.4 and the fact that $n - 4 \geq 17$, the map $A_{n-4} \xrightarrow{\cdot\ell} A_{n-4}$ fails the surjectivity at degree λ_{n-4}. Since the map $k[y_1, y_2]/(y_1, y_2)^2 \xrightarrow{\cdot(y_1+y_2)} k[y_1, y_2]/(y_1, y_2)^2$ fails the surjectivity at degree 0, Lemma 2.8 yields that the third vertical map of the diagram fails the surjectivity at degree $\lambda_{n-4} + 1$.

By the surjectivity of the horizontal maps in the diagram, we conclude that first vertical map in the diagram fails the surjectivity at degree $\lambda_{n-4}+1=\rho_n$. Hence B_n does not have the WLP. This concludes the proof. □

5. WLP for artinian monomial algebras associated to tadpole graphs

An (m,n)-*tadpole graph*, also called a *dragon graph*, is the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge. We denote this graph by $T_{m,n}$. Note that $T_{m,n}$ is a graph on $m+n$ vertices and $m+n$ edges. In the case where $n=1$, $T(m,1)$ is called an *m-pan graph*.

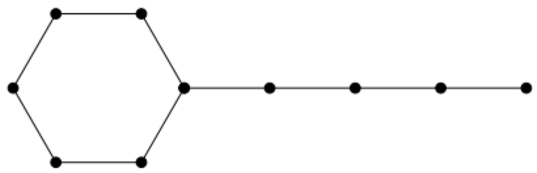

Figure 1. Tadpole graph $T_{6,4}$

By Proposition 2.12, the independence polynomial of $T_{m,n}$ is

$$\begin{aligned} I(T_{m,n};t) &= I(P_{m-1};t)I(P_n;t)+tI(P_{m-3};t)I(P_{n-1};t)\\ &= I(C_m;t)I(P_{n-1};t)+tI(P_{m-1};t)I(P_{n-2};t)\\ &= \sum_{i=0}^{\lfloor \frac{m}{2}\rfloor+\lfloor\frac{n+1}{2}\rfloor} s_i(T_{m,n})t^i. \end{aligned}$$

Recall that ρ_n and λ_n be the mode of $T(C_n;t)$ and $I(P_n;t)$, respectively. By Lemmas 4.2 and 4.7, it implies immediately the following.

Lemma 5.1. *If $i\geq \min\{\lambda_{m-1}+\lambda_n+1,\rho_m+\lambda_{n-1}+1\}$, then $s_i(T_{m,n})\geq s_{i+1}(T_{m,n})$ and if $i\leq \max\{\lambda_{m-1}+\lambda_n-1,\rho_m+\lambda_{n-1}-1\}$, then $s_{i-1}(T_{m,n})\leq s_i(T_{m,n})$.*

We now obtain the following.

Theorem 5.2. *Let A be the artinian monomial algebra associated to a tadpole graph $T_{m,n}$. Then A fails the WLP, provided when (m,n) is one of the following cases:*

(i) *$m=9,12,15,16$ or $m\geq 18$ and $n=8,11;14,15$ or $n\geq 17$.*
(ii) *$m=12,15,16,18,19$ or $m\geq 21$ and $n=9,12;15,16$ or $n\geq 18$.*

Proof. Assume that

$$A=\frac{k[x_1,\ldots,x_m,y_1,\ldots,y_n]}{(x_1^2,\ldots,x_m^2)+I_{C_m}+(y_1^2,\ldots,y_n^2)+I_{P_n}+(x_1y_1)}.$$

Recall that we denote by A_n and B_m the artinian monomial algebra associated to P_n and C_m, respectively. The first item is deduced by the exact sequence

$$A(-1)\xrightarrow{\cdot x_1} A\longrightarrow A_{m-1}\otimes_k A_n\longrightarrow 0\ ,$$

Lemma 2.8, Theorem 4.4. The second item is deduced by the exact sequence

$$A(-1) \xrightarrow{\cdot y_1} A \longrightarrow B_m \otimes_k A_{n-1} \longrightarrow 0 \,,$$

Lemma 2.8, Theorems 4.4 and 4.8. □

Next, we consider the $(3,n)$-tadpole graph $T_{3,n}$. Clearly, $T_{3,n}$ is a claw-free graph. Therefore, the independence polynomial of $T_{3,n}$ is unimodal [6]. By Proposition 2.12, we have

$$I(T_{3,n};t) = I(P_{n+2};t) + tI(P_n;t) = I(C_{n+3};t).$$

It follows that the mode of $I(T_{3,n};t)$ is equal to one of $I(C_{n+3};t)$, i.e.

$$\rho_{n+3} = \left\lceil \frac{5(n+3) - 4 - \sqrt{5(n+3)^2 - 4}}{10} \right\rceil.$$

Theorem 5.3. *Let D_n be the artinian monomial algebra associated to a tadpole graph $T_{3,n}$, $(n \geq 1)$. Then D_n has the WLP if and only if $n \in \{1, \dots, 8, 10, 11, 14\}$.*

Proof. The proof proceeds along the same lines as in the proof of Theorem 4.8 by replacing B_n by D_{n-3}. □

We now consider the $(n,1)$-tadpole graph $T_{n,1}$, i.e., the n-pan graph. We denote this graph by Pan_n. To study the WLP of Pan_n, we need to consider a family of graphs formed by adding an edge $\{1, n-1\}$ to the cycles C_n $(n \geq 4)$. We denote this graph by CE_n. Therefore, CE_n is a claw-free graph, and hence the independence polynomial of CE_n is unimodal [6]. By Proposition 2.12, we have

$$\begin{aligned} I(\mathrm{CE}_n;t) &= \sum_{i=0}^{\alpha(\mathrm{CE}_n)} s_i(\mathrm{CE}_n)t^i \\ &= I(P_{n-1};t) + tI(P_{n-4};t) \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-i}{i} + \binom{n-i-2}{i-1} \right] t^i. \end{aligned}$$

Then we have the following.

Lemma 5.4. *Let χ_n is the mode of $I(\mathrm{CE}_n;t)$ and λ_n be the mode of $I(P_n;t)$. For any $n \geq 4$, one has $\lambda_{n-1} \leq \chi_n \leq \lambda_{n-4} + 1$.*

Proof. Let $i \leq \lambda_{n-1}$. We need to show that

$$s_{i-1}(\mathrm{CE}_n) < s_i(\mathrm{CE}_n) \Longleftrightarrow \binom{n-i+1}{i-1} + \binom{n-i-1}{i-2} < \binom{n-i}{i} + \binom{n-i-2}{i-1}.$$

Since $i \le \lambda_{n-1}$, $\binom{n-i+1}{i-1} < \binom{n-i}{i}$. It suffices to show that

$$\begin{aligned}\binom{n-i-1}{i-2} \le \binom{n-i-2}{i-1} &\Leftrightarrow \frac{n-i-1}{(n-2i)(n-2i+1)} \le \frac{1}{i-1}\\ &\Leftrightarrow (n-i-1)(i-1) \le (n-2i)(n-2i+1)\\ &\Leftrightarrow 5i^2-(5n+2)i+n^2+2n-1 \ge 0\\ &\Leftrightarrow i \le \frac{5n+2-\sqrt{5n^2-20n+24}}{10} \text{ or } i \ge \frac{5n+2+\sqrt{5n^2-20n+24}}{10}.\end{aligned}$$

As $i \le \lambda_{n-1}$, it is enough to show that

$$\begin{aligned}&\frac{5(n-1)+2-\sqrt{5(n-1)^2+20(n-1)+24}}{10} \le \frac{5n+2-\sqrt{5n^2-20n+24}}{10}-1\\ &\Leftrightarrow 5n-3-\sqrt{5n^2+10n+9} \le 5n-8-\sqrt{5n^2-20n+24}\\ &\Leftrightarrow 5+\sqrt{5n^2-20n+24} \le \sqrt{5n^2+10n+9}\\ &\Leftrightarrow \sqrt{5n^2-20n+24} \le 3n-4\\ &\Leftrightarrow n^2-n-2 \ge 0\\ &\Leftrightarrow (n+1)(n-2) \ge 0,\end{aligned}$$

which is clear for any $n \ge 4$. It follows that $\lambda_{n-1} \le \chi_n$. It remains to show that if $i \ge \lambda_{n-4}+1$, then

$$s_i(\mathrm{CE}_n) \ge s_{i+1}(\mathrm{CE}_n) \Longleftrightarrow \binom{n-i}{i}+\binom{n-i-2}{i-1} \ge \binom{n-i-1}{i+1}+\binom{n-i-3}{i}.$$

By Lemma 4.2 $i \ge \lambda_{n-4}+1 \ge \lambda_{n-1}$, $\binom{n-i}{i} \ge \binom{n-i-1}{i+1}$. We have to show that

$$\begin{aligned}&\binom{n-i-2}{i-1} \ge \binom{n-i-3}{i}\\ &\qquad\Leftrightarrow \frac{n-i-2}{(n-2i-2)(n-2i-1)} \ge \frac{1}{i}\\ &\qquad\Leftrightarrow i(n-i-2) \ge (n-2i-2)(n-2i-1)\\ &\qquad\Leftrightarrow 5i^2-(5n-8)i+n^2-3n+2 \le 0\\ &\qquad\Leftrightarrow \frac{5n-8-\sqrt{5n^2-20n+24}}{10} \le i \le \frac{5n-8+\sqrt{5n^2-20n+24}}{10}\end{aligned}$$

Since $i \ge \lambda_{n-4}+1$, it is enough to show that

$$\begin{aligned}&\frac{5n-8-\sqrt{5n^2-20n+24}}{10} \le \frac{5(n-4)+2-\sqrt{5(n-4)^2+20(n-4)+24}}{10}+1\\ &\Leftrightarrow \frac{5n-8-\sqrt{5n^2-20n+24}}{10} \le \frac{5n-18+\sqrt{5n^2-20n+24}}{10}+1,\end{aligned}$$

which is clear. Thus $\chi_n \le \lambda_{n-4}+1$. □

Theorem 5.5. *With the above notations. Let A be the artinian monomial algebra associated to CE_n. Then A has the WLP if and only if $n \in \{4,\dots,8,10,11,14\}$.*

Proof. By using `Macaulay2` to compute the Hilbert series of A and $A/\ell A$ with $4 \le n \le 20$, we can check that:

- A has the WLP for each $4 \le n \le 14$ and $n \notin \{9, 12, 13\}$;
- for $n \in \{9, 12, 15, 16, 18, 19\}$, then A fails the surjectivity from degree χ_n to degree $\chi_n + 1$;
- for $n \in \{9, 13, 17, 20\}$, then A fails the injectivity from degree $\chi_n - 1$ to degree χ_n.

Now assume that $n \ge 21$. We will prove that A fails the surjectivity from degree χ_n to degree $\chi_n + 1$. The proof is completely similar as in the proof of Theorem 4.8. Recall that we will denote by A_n the artinian monomial algebra associated to P_n and by λ_n the mode of the independence polynomial of P_n. By Lemmas 4.2 and 5.4, $\lambda_{n-1} \le \chi_n \le \lambda_{n-4} + 1 \le \lambda_{n-1} + 1$. We consider the following two cases.

Case 1: $\chi_n = \lambda_{n-1}$. In this case, we will show that A fails the WLP by the failure of the surjectivity from degree χ_n to degree $\chi_n + 1$. It implies from the exact sequence

$$0 \longrightarrow A_{n-4}(-1) \xrightarrow{\cdot x_{n-1}} A \longrightarrow A_{n-1} \longrightarrow 0\ .$$

Case 2: $\chi_n = \lambda_{n-1} + 1$. In this case, Lemma 5.4 yields $\lambda_{n-1} = \lambda_{n-4}$. As in the proof of Theorem 4.8. Denote $y_1 = x_n, y_2 = x_{n-2}$, we have the following diagram

$$\begin{array}{ccccccc}
[A]_{\chi_n} & \xrightarrow{/(x_{n-1})} & [A_{n-1}]_{\lambda_{n-1}+1} & \xrightarrow{/(x_{n-3})} & \left[A_{n-4} \otimes \frac{k[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-4}+1} & \longrightarrow & 0 \\
\downarrow \cdot\ell & & \downarrow \cdot\ell & & \downarrow \cdot\ell & & \\
[A]_{\chi_n+1} & \longrightarrow & [A_{n-1}]_{\lambda_{n-1}+2} & \longrightarrow & \left[A_{n-4} \otimes \frac{k[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-4}+2} & \longrightarrow & 0
\end{array}$$

Since the third vertical map of the diagram fails the surjectivity at degree $\lambda_{n-4} + 1$, we conclude that first vertical map in the diagram fails the surjectivity at degree $\lambda_{n-4} + 1 = \chi_n$, as desired. $\square$

Now, we show the basic property of the mode of independence polynomial of Pan_n.

Lemma 5.6. *The independence polynomial $I(\mathrm{Pan}_n; t)$ of n-pan graph is unimodal. Let ζ_n, χ_n, ρ_n and λ_n be the mode of $I(\mathrm{Pan}_n; t), I(\mathrm{CE}_n; t), T(C_n; t)$ and $I(P_n; t)$, respectively. Then $\chi_{n+1} \le \zeta_n \le \rho_n + 1 \le \lambda_n + 1 \le \chi_{n+1} + 1$.*

Proof. By Proposition 2.12, we have

$$\begin{aligned}
I(\mathrm{Pan}_n; t) &= \sum_{i=0}^{\alpha(\mathrm{Pan}_n)} s_i(\mathrm{Pan}_n) t^i \\
&= I(C_n; t) + tI(P_{n-1}; t) \\
&= I(P_{n-1}; t) + t\big(I(P_{n-3}; t) + I(P_{n-1}; t)\big) \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor + 1} \left[\binom{n-i}{i} + \binom{n-i-1}{i-1} + \binom{n-i+1}{i-1}\right] t^i.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
s_i(\mathrm{Pan}_n) &= \binom{n-i}{i} + \binom{n-i-1}{i-1} + \binom{n-i+1}{i-1} \\
&= \binom{n-i+1}{i} + \binom{n-i-1}{i-1} + \binom{n-i}{i-2} \\
&= s_i(\mathrm{CE}_{n+1}) + \binom{n-i}{i-2}.
\end{aligned}$$

Now for any $i \leq \chi_{n+1}$, hence $s_{i-1}(\mathrm{CE}_{n+1}) < s_i(\mathrm{CE}_{n+1})$. We have to show $s_{i-1}(\mathrm{Pan}_n) < s_i(\mathrm{Pan}_n)$. It suffices to show that

$$\begin{aligned}
& \binom{n-i+1}{i-3} \leq \binom{n-i}{i-2} \\
\iff & \frac{n-i+1}{(n-2i+3)(n-2i+4)} \leq \frac{1}{i-2} \\
\iff & 5i^2 - (5n+17)i + n^2 + 9n + 14 \geq 0 \\
\iff & i \leq \frac{5n+17-\sqrt{5n^2-10n+9}}{10} \text{ or } i \geq \frac{5n+17+\sqrt{5n^2-10n+9}}{10}.
\end{aligned}$$

By Lemma 5.4, $i \leq \chi_{n+1} \leq \lambda_{n-3} + 1$, we need to show

$$\begin{aligned}
& \frac{5(n-3)+2-\sqrt{5(n-3)^2+20(n-3)+24}}{10} + 2 \leq \frac{5n+17-\sqrt{5n^2-10n+9}}{10} \\
\iff & \frac{5n+7-\sqrt{5n^2-10n+9}}{10} \leq \frac{5n+17-\sqrt{5n^2-10n+9}}{10},
\end{aligned}$$

which is clear. Thus $\chi_{n+1} \leq \zeta_n$.

Now for any $i \geq \rho_n + 1$. Since

$$\begin{aligned}
I(\mathrm{Pan}_n; t) &= \sum_{i=0}^{\alpha(\mathrm{Pan}_n)} s_i(\mathrm{Pan}_n) t^i \\
&= I(C_n; t) + tI(P_{n-1}; t),
\end{aligned}$$

we get $s_i(\mathrm{Pan}_n) = s_i(C_n) + s_{i-1}(P_{n-1})$. Since $i \geq \rho_n + 1$, we get $i - 1 \geq \rho_n \geq \lambda_{n-1}$. It follows that $s_i(C_n) \geq s_{i+1}(C_n)$ and $s_{i-1}(P_{n-1}) \geq s_i(P_{n-1})$. Thus $s_i(\mathrm{Pan}_n) \geq s_{i+1}(\mathrm{Pan}_n)$, which implies $\zeta_n \leq \rho_n + 1$. The two last inequalities are clear. □

Now we show the following theorem.

Theorem 5.7. *With the above notations. Let A be the artinian monomial algebra associated to* Pan_n *($n \geq 4$). Then A has the WLP if and only if $n \in \{4, \ldots, 10, 12, 13, 16\}$.*

Proof. By using `Macaulay2` to compute the Hilbert series of A and $A/\ell A$ with $4 \leq n \leq 20$, we can check that:

- A has the WLP for each $4 \leq n \leq 16$ and $n \notin \{11, 14, 15\}$;
- for $n \in \{11, 14, 17, 18, 20\}$, then A fails the surjectivity from degree ζ_n to degree $\zeta_n + 1$;

- for $n \in \{15, 19\}$, then A fails the injectivity from degree $\zeta_n - 1$ to degree ζ_n.

Now assume that $n \geq 21$. Recall that we denote by A_n the artinian monomial algebra associated to P_n and by λ_n the mode of the independence polynomial of P_n. By Lemmas 4.2 and 5.6, $\chi_{n+1} \leq \zeta_n \leq \rho_n + 1 \leq \lambda_n + 1 \leq \chi_{n+1} + 1$. We consider the following two cases.

Case 1: $\zeta_n = \chi_{n+1}$. In this case, A fails the surjectivity from degree ζ_n to degree $\zeta_n + 1$ by using the exact sequence

$$0 \longrightarrow A_{n-3}(-2) \xrightarrow{\cdot x_1 x_{n+1}} A \longrightarrow A_{\mathrm{CE}_{n+1}} \longrightarrow 0$$

and Theorem 5.5, where $A_{\mathrm{CE}_{n+1}}$ is the artinian monomial algebra associated to CE_{n+1}.

Case 2: $\zeta_n = \chi_{n+1} + 1$. In this case, Lemma 5.4 yields $\lambda_n = \rho_n = \zeta_n - 1$. Since $\lambda_n - \lambda_{n-3} \leq 1$, we consider the following two subcases:

Subcase 1: $\lambda_n = \lambda_{n-3}$. As in the proof of Theorem 4.8, denote $y_1 = x_n, y_2 = x_{n-2}$, we have the following diagram

$$\begin{array}{ccccccc}
[A]_{\zeta_n} & \xrightarrow{/(x_{n-1})} & [A_n]_{\lambda_n+1} & \xrightarrow{/(x_{n-2})} & \left[A_{n-3} \otimes \frac{k[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-3}+1} & \longrightarrow & 0 \\
\downarrow \cdot\ell & & \downarrow \cdot\ell & & \downarrow \cdot\ell & & \\
[A]_{\zeta_n+1} & \longrightarrow & [A_n]_{\lambda_n+2} & \longrightarrow & \left[A_{n-3} \otimes \frac{k[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-3}+2} & \longrightarrow & 0.
\end{array}$$

Since the third vertical map of the diagram fails the surjectivity at degree $\lambda_{n-3} + 1 = \zeta_n$, we conclude that first vertical map in the diagram fails the surjectivity at degree ζ_n.

Subcase 2: $\lambda_n = \lambda_{n-3} + 1$. Set

$$m = \max\{j \mid j \leq n \text{ and } \lambda_j = \lambda_{j-1} + 1\}.$$

Then $n - 2 \leq m \leq n$. Set

$$y = \begin{cases} x_{n-2} & \text{if } m = n - 2 \\ x_{n+1} & \text{if } m = n - 1. \end{cases}$$

Then we have the following diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & [A_m]_{\zeta_n-2} & \xrightarrow{\cdot y} & [A]_{\zeta_n-1} \\
 & & \downarrow \cdot\ell & & \downarrow \cdot\ell \\
0 & \longrightarrow & [A_m]_{\zeta_n-1} & \xrightarrow[\cdot y]{} & [A]_{\zeta_n}
\end{array}$$

Since $\zeta_n - 2 = \lambda_n - 1 = \lambda_m - 1$, we have the first vertical map of the diagram fails the injectivity at degree $\lambda_m - 1$ by Proposition 4.5. It follows that the second vertical map of the diagram fails the injectivity at degree $\zeta_n - 1$. To complete the proof of Theorem, we consider the case where $m = n$. In this case, one has $\rho_n = \lambda_n = \lambda_{n-1} + 1$. By Lemmas 4.2 and 4.7, $\lambda_{n-1} = \lambda_{n-4} = \lambda_{n-5} + 1$. Hence $\lambda_{n-4} = \zeta_n - 2$. Now we consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & [A_{n-4}]_{\zeta_n-3} & \xrightarrow{\cdot x_{n-4}x_{n-2}} & [A]_{\zeta_n-1} \\ & & \downarrow{\cdot \ell} & & \downarrow{\cdot \ell} \\ 0 & \longrightarrow & [A_{n-4}]_{\zeta_n-2} & \xrightarrow[\cdot x_{n-4}x_{n-2}]{} & [A]_{\zeta_n}. \end{array}$$

By Proposition 4.5, the first vertical map of the diagram fails the injectivity at degree $\lambda_{n-4} - 1 = \zeta_n - 3$. It follows that the second vertical map of the diagram fails the injectivity at degree $\zeta_n - 1$. Thus we complete the proof of Theorem. □

Acknowledgments

The second author was supported by the Vietnam Ministry of Education and Training under grant number B2022-DHH-01. The second author also acknowledges the partial support of the Core Research Program of Hue University, Grant No. NCM.DHH.2020.15. The paper was written while the authors visited the Vietnam Institute for Advanced Study in Mathematics (VIASM), they would like to thank VIASM for the very kind support and hospitality.

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