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ROBUST STABILITY FOR IMPLICIT DYNAMIC EQUATIONS WITH CAUSAL OPERATORS ON TIME SCALES

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Corresponding Author:	Nguyen Thu Ha Electric Power University VIET NAM
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Electric Power University
Corresponding Author's Secondary Institution:	
First Author:	Nguyen Thu Ha
First Author Secondary Information:	
Order of Authors:	Nguyen Thu Ha
Order of Authors Secondary Information:	
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Response to Reviewers:	We are very grateful to the referees for evaluating our manuscript and for providing us with precious comments and suggestions. In accordance with these comments and suggestions, we have carefully revised the paper. All the issues raised in the reports have been addressed. Also, we try to make revised manuscript smoother in English and to correct typo mistake, explaining the meaning of every steps in the presentation in hoping that the revised version responds all requirements of Referees

Statement of Revision

Robust stability for implicit dynamic equations with causal operators on time scales

Nguyen Thu Ha

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We are very grateful to the referees for evaluating our manuscript and for providing us with precious comments and suggestions. In accordance with these comments and suggestions, we have carefully revised the paper. All the issues raised in the reports have been addressed.

In what follows, we detail the changes made with respect to the referees' suggestions and concerns. For convenience, the comments and suggestions of the referees are printed in blue, whereas our statements of revision are printed in black.

Changes w.r.t. Comments and Suggestions of Reviewer 1

1. The paper is devoted to robust stability of implicit dynamic equations involving causal operators on time scales. Using the projector-based analysis, solvability of initial value problems is studied. Then, the preservation of stability under small nonlinear perturbations is investigated. Finally, a Bohl-Perron type stability theorem for implicit dynamic equations is proven. The results are extensions of some previous results for differential algebraic equations (continuous time scale) and implicit difference equations (discrete time scale). The new feature is the consideration of dynamic perturbations in term of causal linear and nonlinear operators.

   Thank you very much for your comments

2. This reviewer has two major comments:

   - First, though the analysis on time scales indeed unifies the continuous and the discrete time cases, researchers would be more interested in the most popular time scales that arise in real-life applications, that is $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.
     Thank you very much for your comments. In the revised version, we add to the introduction section some comments on the history of this problem for the most popular time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. In the body of text, we give some remarks to compare our obtained results with the previous one.
   - Second, the presentation of the results is very poor. It it not easy to follow the problem formulations as well as the main results. Furthermore, there are a plenty of typos, notation and English mistakes. The paper should be rewritten and restructured so that the problems and the results are clearly stated.
     Thank you very much for your comments. In the revised version, we try to improve the presentation by formulating more clearly notions, definitions, theorems; making smoother words and phrases... so that readers are easy to follow them. We try to

correct typos, notations and English mistakes.
Moreover, we also restructure the paper at several places so that the results are clearly presented.

- The author may split Section 3 into two sections: solvability analysis and robust stability.
  Thank a lot. We split Section 3 into the sections 3 and 4.
- The appendix should be removed or briefly presented in Sections 2 because it contains only two known auxiliary results without proofs.
  Yes, it is done. We remove the appendix to Section 2 (in revised version, it is Subsection 2.3) as the referee suggests.
- Finally, in this reviewer's opinion, a clear and concise presentation of results for differential-algebraic equations would be of greater interest. Then, an extension to dynamic equations on time scales seems to be straightforward.
  Thank you very much for your comment. To illustrate its meaning and strength, in the introduction of the revised version, we introduce briefly some results concerning with the robust stability for implicit difference equations or differential–algebraic one. Some our obtained results generalize one in implicit difference/differential equations as we mentioned in the remarks 3.2; 4.6 and 5.5. However, in order to avoid the complicated presentation, we are unable to compare all results.

Several typos and mistakes (but not all) are as follows:

3. Abstract, line 22: dynamical and dynamic are not the same.

   Thank you very much, we have corrected all these mistakes.

4. Page 1, line 32, "trivial cases ordinary differential/difference equations": wrong English usage; these cases are not trivial. 1

   Thank a lot. We have rewritten it.

5. Page 1, lines 35-42: too long sentence;

   Thank a lot. We have divided it into shorter sentences.

6. Page 1, line 41: dynamics and dynamic are not the same.

   Thank you, that was a typo mistake. We corrected it by changing "dynamics" to "dynamic".

7. Page 1, line 43: The abbreviation IDE should be explained at the first use.

   Thank a lot. We have corrected it and have explained for the first use of the notion IDE at the bottom of page 8.

8. Page 1, line 44: Another

   Thank a lot. We have corrected it.

9. Page 1, lines 51-55: This sentence should be rewritten because the English grammar is not correct. Furthermore, "the solution" (singular) means that the equation has a unique solution.

   Thank you for your comments. I we have corrected English grammar and have divided it into shorter sentences. We also use “the solutions” instead of “the solution” at suitable places.

10. Page 2, line 22: What does the author mean by “same gaps”?

    Yes, thank you very much, this is a senseless phrase. We have rewritten this sentence as follows: “Since the structure of points in time scales is rather divert and complicated, we need using some new techniques to prove main results in the paper.”

11. Page 3, line 30: Check the right-hand side of the equation.
    Yes, it is done. Further, we reformulate Theorem 2.2.

12. Page 4, Lemma 2.3: “see [13]” is misleading since the inequality in [13] is for the continuous time scale, not for general time scales.

    Thank a lot. In last version, we use Reference [13] ([14] in new version) where it is concerned with the Pachpatte type inequality of continuous time. To prove this inequality on time scale, it requires some long calculations that we do not want to present in this paper. Therefore, in the revised version, we cite the Gronwall inequality on time scale in [3].

13. Page 4, Lemma 2.4: Hardy’s inequality in [23] is on time scales?

    Thank for your remark. On [23] (now it is Reference [24]) one deals with Hardy’s inequality in continuous time. Therefore, to make thing clearer, we add the phrase “The following lemma deals with a version of Hardy inequality on time scales (see [24] for the continuous time version)” before formulating Lemma 2.4.

14. Page 4, Lemma 2.4: true for any real f.

    Thank you, that was a typo mistake. We have corrected it by changing into “true for any real functions f”.

15. Page 5, line 11: a reference for Hölder inequality on time scales should be cited.
    Yes, thank a lot. It is cited.

16. Page 5, line 24: a reference for Fubini inequality on time scales should be cited.
    Yes, thank you. It is cited.

17. Page 8, line 28: $\mathbb{R}^n \times \mathbb{R}^n$ is not correct..
    Yes, thank a lot. We have corrected as “valued in the set of $n \times n$–matrices” .

18. Page 8, line 30: Notation A_σ should be explained.
    Thank a lot. In the revised version, we add the phrase “For any function f defined on $\mathbb{T}$, we write f_σ for $f \circ \sigma$” to page 3 because this notation is used in some places.

19. Page 8, line 44: What does "argue the ways" mean?

    Thanks, that's an incorrect use of the word. I changed it to "deal the way"

20. Page 9, line 46: Here and later the author uses Picard approximation to prove that an integral equation has a unique solution. It is true, but only locally. In particular, if a nonlinear term arises as in (3.18), it is not sure that the local solution can be extended to the whole domain.

    Yes, thank you very much for your comments.To avoid a complicated presentation, we do not write in details the procedure of Picard approximation because it follows the well-known steps and (3.7) is only a linear integral equation. Further, the linearity of this integral equations ensures the global existence and uniqueness of the solution, which is defined on $\mathbb{T}_{t_0}$ for the initial value problem.

    For the non linear integral equation (4.6), we present in details the Picard approximation (see Theorem 4.3), which confirms the existence and uniqueness of solution defined on the whole domain.

21. Page 11, Definition 3.3: Why does "ω" come here.

    Thank a lot, that was a typo mistake. We have corrected it by changing "ω" to "α".

22. Page 11, line 38: "and or" is a typo.

    Thank you very much. In the revised version, we rewrite this sentence.

23. Page 11, Theorem 3.4: This is a statement. Therefore, avoid "is called".

    Thank a lot, we have deleted the words "is called".

24. Page 11, line 54: Let F be a causal nonlinear operator. An operator and a function are different.

    Thank you. We change to "causal nonlinear operator".

25. Page 11, line 12: Better to say "locally Lipschitz with a function m".

    Yes, thank you very much for your suggestion. We use "locally Lipschitz with a function m". Moreover, we add "In case m is a constant function, we say simply that Γ is m-locally Lipschitz continuous" for the convenience later.

26. Page 15, line 26: Gronwall-Bellman inequality on time-scales in Lemma 2.3.

    Thank a lot. This is a mistake. The inequality stated in Lemma 2.3 is Pachpatte type inequality. We have rewritten Lemma 2.3, where it is concerned with only Gronwall-Bellman inequality on time-scales.

27. Page 15, line 32: Let Assumption 1 hold.

    Thank you, we have changed it.

28. Page 15, line 58: What does γ-Lipschitz mean?

    In the new version, we add the notion γ-Lipschitz continuous (see the response in the item 25).

29. Page 19, line 20: Actually E has three eigenvalues.

    Yes, thank you very much, we have added another eigenvalue $\lambda = 1$ of the matrix E.

30. Page 19, line 51: What does k-Lipschitz mean?

    In the new version, we add the notion γ-Lipschitz continuous (see the item 25).

31. Page 20, line 20: dynamic equation

    Yes, thank a lot. It is a mistake. We have corrected it.

32. Page 20, line 23: dynamic equation

    Yes, thank a lot. It is a mistake. We have corrected it.

33. Page 23, line 21: dynamic equations

    Yes, thank a lot. It is a mistake. We have corrected it.

34. Page 25, lines 29-32: Better to avoid the itemize environment.

    Yes, thank you. We have changed it. Further, on the pages 3-5 we use Case1,Case 2... instead of using the bullets.

35. Page 28, line 8: The author

    Yes, it is done.

36. Page 28, line 14: supporting her work

    Thank you very much. In the revised version, we change this phrase.

37. References: please check carefully and write the author names with a consistency.

    Yes, thank a lot. We had checked them and have written the author names with a consistency.

In addition, we add a new reference [4] (for the Fubini Theorem on time scales) to the References.

In conclusion, we are very grateful to the referee's comments and suggestions. We try to make revised manuscript smoother in English and to correct typo mistake, explaining the meaning of every steps in the presentation in hoping that the revised version responds all requirements of Referees.

Changes w.r.t. Comments and Suggestions of Reviewer 2

1. The paper is devoted to the robust stability of implicit dynamic equations with causal operators on time scale. In the first part, the author investigates the solvability of these dynamic equations and then consider the preservation of stability under small perturbations. In the second part, an L_p version of Bohl-Perron Principle for implicit dynamical systems is studied. In my point of view, the main results are correct.

   Thank you very much for your comments.

2. If Lemma 2.4 has been given in [23] then the author does not need to prove in the manuscript.

   Thank a lot for your comment. We need to prove this lemma because in [23] (in the revised version, it is [24]) one deals with only the inequality of continuous time. In order to avoid confusions, we add the phrase "The following lemma deals with a version of Hardy inequality on time scales (see [24] for the continuous time version)".

3. Some comparisons, corollaries of results on solvability and robust stabilty for the linear implicit dynamic equations driven by a causal operator should be given to illustrate its meaning and strength.

   Thank you very much for your comment. To illustrate its meaning and strength, in the introduction of the revised version, we introduce briefly some results concerning with the robust stability for implicit difference equations or differential–algebraic one. Some our obtained results generalize one in implicit difference/differential equations as we mentioned in the remarks 3.2 and 4.6. However, in order to avoid the complicated presentation, we are unable to compare all results.

4. Some comparisons, corollaries and example of Bohl-Perron theorem for the linear implicit dynamic equations driven by a causal operator should be given to illustrate its meaning and strength.

   Thank you very much. We give the remark 5.5 to compare our obtained results Bohl-Perron theorem for the linear implicit dynamic equations. However, in order to avoid the complicated presentation, we are unable to compare every result.

5. Section Appendix is very short, it may move to Section Prelimilary

   Yes, it is done. We remove the appendix to Section 2 (in revised version, it is Subsection 2.3) as the referee suggests.

In addition, we add a new reference [4] (for Fubini Theorem on time scales) to the References.

In conclusion, we are very grateful to the referee's comments and suggestions. We try to make revised manuscript smoother in English and to correct typo mistake, explaining the meaning of every steps in the presentation in hoping that the revised version responds all requirements of Referees.

ROBUST STABILITY FOR IMPLICIT DYNAMIC EQUATIONS WITH CAUSAL OPERATORS ON TIME SCALES

NGUYEN THU HA

Abstract. In this paper, we study the robust stability of implicit dynamic equations with causal operators on time scales. First, we investigate the solvability of these dynamic equations and then consider the preservation of stability under small perturbations. An L_p version of Bohl-Perron Principle for implicit dynamic equations is also studied.

1. Introduction

The theory of analysis on time scales was introduced in 1988 by Stefan Hilger [17] in order to unify continuous and discrete calculus. Since then, there have been many works investigating the analysis on time scales. These works not only **unify** the cases of ordinary differential/difference equations but also **extend** to more complicated time scales.

One of important problems in analysis on time scales is the investigation of robust stability of dynamic equations. D. T. Son et al. in [13] consider the exponential stability of linear time-invariant systems on time scales via eigenvalues of matrices, Zhu et al. [26] consider the stability of delay dynamic systems. Du et al. in [11] and DaCunha J. in [8] study the robust stability for dynamic equations under the view of stability radii or for delay stochastic dynamic equations via Lyapunov functions. For differential algebra-equations, in [1], T. Berger studies exponential stability and its robustness for time-varying linear index-1 $A(t)\dot{x}(t) = B(t)x(t) + f(t)$ with class of allowable perturbations f. N.T. Ha in [16] studies the preservation of exponential stability of Volterra differential-algebra equation $A(t)x'(t) = B(t)x(t) + \Sigma x(t) + f(t),\ t \geqslant t_0$ where $\Sigma x(t) = \int_{t_0}^t K(t,s)x(s)ds$ is an integral operator. Mehrmann et al. in [23] analyse the stability of implicit difference equations under restricted perturbations. Ha et al. in [15] deals with the data dependence of exponential stability for linear implicit dynamic equations of arbitrary index $A_n x^{\Delta_n}(t) = B_n x(t)$. Some other works for the robustness of stability of time-varying implicit differential equations can be found in [5, 6, 10, 19, 20].

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Key words and phrases. Robust stability, Causal operator, Differential-algebraic equation, Input-output operator, Bohl-Perron theorem.

Another aspect to consider the stability of a system is the Bohl-Perron type theorem. This problem considers the relationship between the Lyapunov stability in initial values and the boundedness of input-output system. The earliest work in this topic belongs to Perron [25](1930). He proved his celebrated theorem which says that if the solutions of the linear differential equation $\dot{x}(t) + A(t)x(t) = f(t), t \geqslant 0$ are "good" for every "rather good" function $f(\cdot)$, then the solutions of the corresponding homogeneous equation $\dot{x}(t) + A(t)x(t) = 0, t \geqslant 0$ are bounded or exponentially stable. The study of Bohl-Perron theorem concerned with delay differential/difference equations can be founded in [2, 9, 24, 25];

In this paper, we want to go further in studying robust stability and Bohl-Perron theorem of so-called implicit dynamic equations on time scales. More precisely, we deal with the robust stability of the linear dynamic equations where the leading term may be degenerate and the right hand term is driven by a causal operator (or "aftereffect operators", see [7]). This class of dynamic equations is very important both in practice and theory because it is generalised from differential/difference equations, integral equations, delay or functional differential equations, which often are used to describe mathematical models in economy, industry, eco-systems.

Since the structure of points on time scales is rather divert and complicated, we need using some new techniques to prove main results in the paper.

The paper is organised as follows. In next section, we introduce some notions of analysis on time scales and prove the Hardy inequality. Section 3 is concerned with the solvability of implicit linear dynamic equations. In Section 4, we studies the L_p–stability or the preservation of exponential stability for implicit dynamic equation under small perturbation. Section 5 deals with the famous Bohl-Perron theorem for implicit dynamic equations. We introduce some weighted spaces and show that the exponential stability is equivalent to the fact that the solutions of these equations are elements of such spaces.

2. Preliminary

2.1. **Time scales.** A time scale is an arbitrary, nonempty, closed subset of the set of real numbers $\mathbb{R}$ equipped by the topology inherited from the standard topology on $\mathbb{R}$.

Consider a time scale $\mathbb{T}$. We define the *forward operator* $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and *graininess function* $\mu(t) = \sigma(t) - t, t \in \mathbb{T}$; the *backward operator* $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ and $\nu(t) = t - \rho(t)$. We supplement $\sup \emptyset = \inf \mathbb{T}, \inf \emptyset = \sup \mathbb{T}$.

For any $x, y \in \mathbb{T}$, the addition (*circle plus*) $\oplus$ and subtraction (*circle minus*) $\ominus$ of x, y are defined:

(1) $x \oplus y := x + y + \mu(t)xy;$

(2) $x \ominus y := \dfrac{x-y}{1+\mu(t)y}$.

A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered.

A function $f : \mathbb{T} \to \mathbb{R}$ is *regulated* if there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point.

A regulated function f is called *rd-continuous* if it is continuous at every right-dense point, and *ld-continuous* if it is continuous at every left-dense point. It is easy to see that a function is continuous if and only if it is both rd-continuous and ld-continuous. The set of rd-continuous functions defined on the interval $J \subset \mathbb{T}$, valued in X, will be denoted by $\mathrm{C}_{\mathrm{rd}}(J, X)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive* (resp., *positively regressive*) if for every $t \in \mathbb{T}$, we have $1 + \mu(t)f(t) \neq 0$ (resp., $1 + \mu(t)f(t) > 0$). Denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ (resp., $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R})$) the set of (resp., positively regressive) regressive functions, and $\mathrm{C}_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$ (resp., $\mathrm{C}_{\mathrm{rd}}\mathcal{R}^+(\mathbb{T}, \mathbb{R})$) the set of rd-continuous (resp., positively regressive) regressive functions from $\mathbb{T}$ to $\mathbb{R}$. For any function f defined on $\mathbb{T}$, we write f_σ for $f(\sigma)$.

Definition 2.1 (Delta Derivative)**.** A function $f : \mathbb{T} \to \mathbb{R}^d$ is called *delta differentiable* at t if there exists a vector $f^\Delta(t)$ such that for all $\varepsilon > 0$,

$$\|f_\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)\| \leqslant \varepsilon|\sigma(t) - s|,$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$ and for some $\delta > 0$. The vector $f^\Delta(t)$ is called the *delta derivative* of f at t.

Theorem 2.2 ([3])**.** *Let p be regressive and $t_0 \in \mathbb{T}$. Then, there exists a unique solution of the initial value problem*

$$y^\Delta(t) = p(t)y(t),\ y(t_0) = 1.$$

This solution, namely $e_p(t, t_0)$, is called the exponential function (at t_0) on the time scales $\mathbb{T}$.

The exponential functions in time scales have the following properties

(i) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,
(ii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$, in particular $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$,
(iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$,
(iv) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$.

Let $\mathbb{T}$ be a time scale. For any $a, b \in \mathbb{T}$, the notation $[a, b]$ or (a, b) means the segment on $\mathbb{T}$, that is $[a, b] = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$ or $(a, b) = \{t \in \mathbb{T} : a < t < b\}$ and $\mathbb{T}_a = \{t \geqslant a : t \in \mathbb{T}\}$. We can define a measure $\Delta_{\mathbb{T}}$ on $\mathbb{T}$ by considering the Caratheodory construction of measures when we put $\Delta_{\mathbb{T}}[a, b) = b - a$. The Lebesgue integral of a measurable function f with respect to $\Delta_{\mathbb{T}}$ is denoted by $\int_a^b f(s)\Delta_{\mathbb{T}} s$ or $\int_a^b f(s)\Delta s$. For more details of analysis on time scales we can refer to [3].

In this paper, we suppose that the time scales $\mathbb{T}$ is unbounded above, i.e., $\sup \mathbb{T} = \infty$ and its graininess is bounded, i.e., $\mu^* = \sup\{\mu(t) : t \in \mathbb{T}\} < \infty$.

Let $s, t \in \mathbb{T}$ and $L_p\big([s,t];\mathbb{R}^n\big)$ be the space of all p–integrable functions $f : [s,t] \to \mathbb{R}^n$ equipped with the norm

$$\|f\|_{L_p([s,t];\mathbb{R}^n)} = \left(\int_s^t \|f(\tau)\|^p \Delta\tau\right)^{\frac{1}{p}}.$$

Denote by $L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)$ the space of all $f : \mathbb{T}_{t_0} \to \mathbb{R}^n$ such that $f\big|_{[s,t]} \in L_p\big([s,t];\mathbb{R}^n\big)$ for all $t > s \geqslant t_0$ and by $C_b(\mathbb{T}_{t_0};\mathbb{R}^n)$ the set of the continuous functions, bounded on $\mathbb{T}_{t_0}$, valued in $\mathbb{R}^n$.

The truncated operators π_k at $k \in \mathbb{T}_{t_0}$ and $[\,\cdot\,]_k$ on $L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)$ are defined by

$$\pi_k(x)(t) = \begin{cases} x(t), & \text{if } t \in [t_0,k] \\ 0, & \text{if } t \in (k,\infty), \end{cases} \quad \text{and} \quad [x(t)]_k = \begin{cases} 0 & \text{if } t \in [t_0,k) \\ x(t) & \text{if } t \in [k,\infty). \end{cases}$$

for every $x \in L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)$. It is clear that

$$\pi_k + [\ ]_k = I.$$

Let

$$\mathcal{L} = \mathcal{L}\big(L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right), L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)\big),$$

be the space of the linear operators Σ from $L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)$ to $L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)$ such that $\pi_t\Sigma\pi_t$ maps continuously from $L_p([t_0,t];\mathbb{R}^n)$ to $L_p([t_0,t];\mathbb{R}^n)$.

Every element $f \in L_p\big([s,t];\mathbb{R}^n\big)$ can be considered as an element $\overline{f} \in L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)$ by putting $\overline{f} = \pi_t[f]_s$. Similarly, if $f \in L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)$, we can restrict the definition domain of f to obtain an element $\overline{f} \in L_p\big([s,t];\mathbb{R}^n\big)$. In the following, we identify f and $\overline{f}$ if there is no confusion. Similarly, every continuous operator $\Sigma \in \mathcal{L}\big(L_p([s,t];\mathbb{R}^n), L_p([s,t];\mathbb{R}^n)\big)$ can be considered as

$$\overline{\Sigma} \in \mathcal{L}\big(L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right), L_p^{\text{loc}}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)\big),$$

by setting $\overline{\Sigma}(\cdot) = \pi_t\Sigma\pi_t[\cdot]_s$.

An operator $\Sigma \in \mathcal{L}$ is said to be *causal* if for every $t \in \mathbb{T}_{t_0}$

$$\pi_t\Sigma\pi_t = \pi_t\Sigma. \tag{2.1}$$

To simplify notations, in the following, we write $L_p[s,t]; C[s,t]$ for $L_p([s,t];\mathbb{R}^n)$, $C_b([s,t];\mathbb{R}^n)$ respectively.

2.2. Some inequalities on time scales.

Lemma 2.3 (Gronwall inequality on time scales (see [3, Theorem 6.1]))**.** *Let $f(\cdot), k(\cdot)$ be two non negative continuous functions defined on $\mathbb{T}_{t_0}$. Assume that $f(t)$ satisfies the inequality*

$$f(t) \leqslant f_0 + \int_{t_0}^t k(s)f(s)\Delta s, \quad \textit{for all } t \in \mathbb{T}_{t_0}.$$

Then, the relation $f(t) \leqslant f_0 e_k(t,t_0)$ holds for all $t \in \mathbb{T}_{t_0}$.

The following lemma deals with a version of Hardy inequality on time scales (see [24] for the continuous time version).

Lemma 2.4 (Hardy inequality on time scales)**.** *Let $1 \leqslant p < \infty$, $t_0 \in \mathbb{T}$ and $U(x), V(x)$ are positive functions. Then,*

$$\Big[\int_{t_0}^{\infty} \Big|U(x)\int_{t_0}^{x} f(t)\Delta t\Big|^p \Delta x\Big]^{\frac{1}{p}} \leqslant p^{\frac{1}{p}} q^{\frac{1}{q}} B \Big[\int_{t_0}^{\infty} \big|V(x)f(x)\big|^p \Delta x\Big]^{\frac{1}{p}} \tag{2.2}$$

is true for all real functions f, where

$$B = \sup_{r>0} \Big[\int_{r}^{\infty} |U(x)|^p \Delta x\Big]^{\frac{1}{p}} \Big[\int_{t_0}^{r} |V(x)|^{-q} \Delta x\Big]^{\frac{1}{q}},$$

and $\frac{1}{p} + \frac{1}{q} = 1$ (with the convention $0^\infty = \infty^0 = 1$).

Proof.
Case 1: $1 < p < \infty$.

Let $h(x) = \int_{t_0}^{x} V^{-q}(t)\Delta t$ and $g(x) = h(x)^{\frac{1}{pq}}$. Using Hölder inequality (see [3, Theorem 6.13]) gets

$$\begin{aligned}\int_{t_0}^{\infty}\Big|U(x)\int_{t_0}^{x} f(t)\Delta t\Big|^p \Delta x &= \int_{t_0}^{\infty} |U(x)|^p \left|\int_{t_0}^{x} V(t)f(t)g_\sigma(t)V^{-1}(t)g_\sigma^{-1}(t)\Delta t\right|^p \Delta x \\ &\leqslant \int_{t_0}^{\infty} |U(x)|^p \left(\Big[\int_{t_0}^{x} V^{-q}(t)g_\sigma^{-q}(t)\Delta t\Big]^{\frac{p}{q}} \int_{t_0}^{x} [V(t)f(t)g_\sigma(t)]^p \Delta t\right) \Delta x.\end{aligned}$$

It follows from Fubini theorem (see [4, Theorem 2.15]) that

$$\begin{aligned}&\int_{t_0}^{\infty} \Big|U(x)\int_{t_0}^{x} f(t)\Delta t\Big|^p \Delta x \\ &\leqslant \int_{t_0}^{\infty} [V(t)f(t)g_\sigma(t)]^p \left(\int_{\sigma(t)}^{\infty} |U(x)|^p \Big[\int_{t_0}^{x} V^{-q}(s)g_\sigma^{-q}(s)\Delta s\Big]^{\frac{p}{q}} \Delta x\right) \Delta t \\ &= \int_{t_0}^{\infty} |V(t)f(t)|^p| \left(|g_\sigma(t)|^p \int_{\sigma(t)}^{\infty} |U(x)|^p \Big[\int_{t_0}^{x} V^{-q}(s)g_\sigma^{-q}(s)\Delta s\Big]^{\frac{p}{q}} \Delta x\right) \Delta t.\end{aligned}$$

Therefore, if x is right-dense then $(g^p(x))^\Delta = \frac{1}{q}V^{-q}(x)g^{-q}(x)$. In case x is right-scattered, by using the finite-increments formula we have

$$\begin{aligned}\big(g^p(x)\big)^\Delta = \big(h^{\frac{1}{q}}(x)\big)^\Delta &= \frac{h^{\frac{1}{q}}(\sigma(x)) - h^{\frac{1}{q}}(x)}{\mu(x)} = \frac{\big[h(x) + \mu(x)V^{-q}(x)\big]^{\frac{1}{q}} - h^{\frac{1}{q}}(x)}{\mu(x)} \\ &\geqslant \frac{1}{q}V^{-q}(x)\left(\int_{t_0}^{\sigma(x)} V^{-q}(s)\Delta s\right)^{-\frac{1}{p}} = \frac{1}{q}V^{-q}(x)g_\sigma^{-q}(x),\end{aligned}$$

which implies that $\displaystyle\int_{t_0}^{x} V^{-q}(s)g_\sigma^{-q}(s)\Delta s = qg^p(x) = q\left(\int_{t_0}^{x} V^{-q}(t)\Delta t\right)^{\frac{1}{q}}.$

Further,

$$\begin{aligned}
&\int_{t_0}^{\infty}\Big|U(x)\int_{t_0}^{x} f(t)\Delta t\Big|^p \Delta x\\
&\quad\leqslant q^{\frac{p}{q}}\int_{t_0}^{\infty}|V(t)f(t)|^p\left(g_\sigma^p(t)\int_{\sigma(t)}^{\infty}|U(x)|^p\left[\int_{t_0}^{x}V^{-q}(s)\Delta s\right]^{\frac{p}{q^2}}\Delta x\right)\Delta t\\
&\quad\leqslant q^{\frac{p}{q}}B^{\frac{p}{q}}\int_{t_0}^{\infty}|V(t)f(t)|^p\left(g_\sigma^p(t)\int_{\sigma(t)}^{\infty}|U(x)|^p\Big(\int_{x}^{\infty}|U(s)|^p\Delta s\Big)^{-\frac{1}{q}}\Delta x\right)\Delta t\\
&\quad\leqslant q^{\frac{p}{q}}B^{\frac{p}{q}}\int_{t_0}^{\infty}|V(t)f(t)|^p\left(g_\sigma(t)^p\int_{\sigma(t)}^{\infty}|U(x)|^pG(x)^{-\frac{1}{q}}\Delta x\right)\Delta t,
\end{aligned}$$

where $G(x)=\int_x^\infty |U(s)|^p\Delta s$. When x is right-scattered, by using the finite-increments formula again we have

$$\begin{aligned}
(G^{\frac{1}{p}}(x))^{\Delta} &= \frac{G^{\frac{1}{p}}(\sigma(x))-G^{\frac{1}{p}}(x)}{\mu(x)} = \frac{G^{\frac{1}{p}}\big(x+\mu(x)\big)-G^{\frac{1}{p}}(x)}{\mu(x)}\\
&= \frac{\big[G(x)-\mu(x)U^p(x)\big]^{\frac{1}{p}}-G(x)^{\frac{1}{p}}}{\mu(x)} \leqslant -\frac{1}{p}U(x)^pG(x)^{-\frac{1}{q}}.
\end{aligned}$$

If x is right-dense then

$$\big(G^{\frac{1}{p}}(x)\big)^{\Delta} = -\frac{1}{p}U(x)^pG^{-\frac{1}{q}}(x).$$

Hence, $\displaystyle\int_{\sigma(t)}^{\infty}|U(x)|^pG^{-\frac{1}{q}}(x)\Delta x \leqslant p\big(G_\sigma(t)\big)^{\frac{1}{p}} = p\left(\int_{\sigma(t)}^{\infty}|U(s)|^p\Delta s\right)^{\frac{1}{p}}.$

Summing up we have

$$\int_{t_0}^{\infty}\Big|U(x)\int_{t_0}^{x}f(t)\Delta t\Big|^p\Delta x \leqslant pq^{\frac{p}{q}}B^{\frac{p}{q}}\int_{t_0}^{\infty}\big|V(t)f(t)\big|^p\Big(g_\sigma(t)^p\int_{\sigma(t)}^{\infty}|U(s)|^p\Delta s\Big)^{\frac{1}{p}}\Delta t.$$

Since $\displaystyle g_\sigma^p(t)\Big(\int_{\sigma(t)}^{\infty}|U(s)|^p\Delta s\Big)^{\frac{1}{p}} = \Big(\int_{t_0}^{\sigma(t)}V^{-q}(s)\Delta s\Big)^{\frac{1}{q}}\Big(\int_{\sigma(t)}^{\infty}|U(s)|^p\Delta s\Big)^{\frac{1}{p}} \leqslant B,$

$$\int_{t_0}^{\infty}\Big|U(x)\int_{t_0}^{x}f(t)\Delta t\Big|^p\Delta x \leqslant q^{\frac{p}{q}}pB^p\int_{t_0}^{\infty}\big|V(t)f(t)\big|^p\Delta t.$$

This means that

$$\left(\int_{t_0}^{\infty}\Big|U(x)\int_{t_0}^{x}f(t)\Delta t\Big|^p\Delta x\right)^{\frac{1}{p}} \leqslant q^{\frac{1}{q}}p^{\frac{1}{p}}B\left(\int_{t_0}^{\infty}\big|V(t)f(t)\big|^p\Delta t\right)^{\frac{1}{p}}.$$

Case 2: When $p=1$, (2.2) becomes

$$\int_{t_0}^{\infty}\Big|U(x)\int_{t_0}^{x}f(t)\Delta t\Big|\Delta x \leqslant B\int_{t_0}^{\infty}\big|V(x)f(x)\big|\Delta x. \qquad (2.3)$$

Using Fubini theorem gets

$$\int_{t_0}^{\infty}\Big|U(x)\int_{t_0}^{x} f(t)\Delta t\Big|\Delta x = \int_{t_0}^{\infty}\Big(|f(t)|\int_{\sigma(t)}^{\infty}|U(x)|\Delta x\Big)\Delta t$$

$$= \int_{t_0}^{\infty}\Big(|V(t)f(t)|\frac{1}{|V(t)|}\int_{\sigma(t)}^{\infty}|U(x)|\Delta x\Big)\Delta t$$

$$\leqslant \int_{t_0}^{\infty}\Big(|V(t)f(t)|\sup_{0\leqslant s\leqslant\sigma(t)}\frac{1}{|V(s)|}\int_{\sigma(s)}^{\infty}|U(x)|\Delta x\Big)\Delta t \leqslant B\int_{t_0}^{\infty}|V(t)f(t)|\Delta t.$$

Case 3: $p=\infty$. It can be proved by a similar way. Lemma is proved. $\square$

Remark 2.5. Let $\alpha(\cdot)$ be an rd–continuous function defined on $\mathbb{T}$ satisfying

$$0<\alpha_1=\min_{t\in\mathbb{T}}\alpha(t)\leqslant\max_{t\in\mathbb{T}}\alpha(t)=\alpha_2<\infty.$$

Let $U(t)=V(t)=e_{\ominus\alpha}(t,t_0)$. Then,

$$B=\sup_{r\in\mathbb{T}_{t_0}}\Big[\int_r^{\infty}(e_{\ominus\alpha}(s,t_0))^p\Delta s\Big]^{\frac{1}{p}}\Big[\int_{t_0}^{r}(e_{\ominus\alpha}(s,t_0))^{-q}\Delta s\Big]^{\frac{1}{q}}\leqslant\frac{1}{\eta_\alpha}, \tag{2.4}$$

where $\eta_\alpha=\frac{\alpha_1}{1+\alpha_2\mu^*}$. Moreover,

$$\Big[\int_{t_0}^{\infty}\Big|e_{\ominus\alpha}(t,t_0)\int_{t_0}^{t}f(\tau)\Delta\tau\Big|^p\Delta t\Big]^{\frac{1}{p}}\leqslant\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}}{\eta_\alpha}\Big[\int_{t_0}^{\infty}\big|e_{\ominus\alpha}(t,t_0)f(t)\big|^p\Delta t\Big]^{\frac{1}{p}}. \tag{2.5}$$

Indeed, it is easy to prove that $1\leqslant\frac{(1+x)^p-1}{x}\leqslant p$ for $x\in[-1,0)$. Since $-1\leqslant\ominus\alpha(s)\mu(s)\leqslant 0$ for all $s\in\mathbb{T}$,

$$\ominus\alpha(s)p\leqslant\frac{(1+\ominus\alpha(s)\mu(s))^p-1}{\mu(s)}\leqslant\ominus\alpha(s)\leqslant-\frac{\alpha_1}{1+\alpha_2\mu^*}=-\eta_\alpha.$$

By definition when $\mu(s)>0$ we have

$$(e^p_{\ominus\alpha}(s,t_0))^{\Delta}=\frac{\big[1+\ominus\alpha(s)\mu(s)\big]^p-1}{\mu(s)}e^p_{\ominus\alpha}(s,t_0)\leqslant-\eta_\alpha e^p_{\ominus\alpha}(s,t_0).$$

If $\mu(s)=0$ then $(e^p_{\ominus\alpha}(s,t_0))^{\Delta}=\ominus\alpha(s)pe^p_{\ominus\alpha}(s,t_0)$. Therefore

$$-e^p_{\ominus\alpha}(r,t_0)=\int_r^{\infty}(e^p_{\ominus\alpha}(s,t_0))^{\Delta}\Delta s\leqslant\int_r^{\infty}-\eta_\alpha e^p_{\ominus\alpha}(s,t_0)\Delta s.$$

Thus,

$$\int_r^{\infty}e^p_{\ominus\alpha}(s,t_0)\Delta s \;\leqslant\; \frac{e^p_{\ominus\alpha}(r,t_0)}{\eta_\alpha}. \tag{2.6}$$

Now we recall the inequality $-1\geqslant\frac{\left(1+x\right)^{-q}-1}{x}\geqslant-q$ for $x\in[-1,0)$, which implies that

$$-\ominus\alpha(s)q\geqslant\frac{\big[1+\ominus\alpha(s)\mu(s)\big]^{-q}-1}{\mu(s)}\geqslant-\ominus\alpha(s)\geqslant\frac{\alpha_1}{1+\alpha_2\mu^*}=\eta_\alpha.$$

Therefore, when $\mu(s) > 0$,

$$(e^{-q}_{\ominus\alpha}(s,t_0))^{\Delta} = \frac{\big[1+\ominus\alpha(t)\mu(s)\big]^{-q}-1}{\mu(s)} e^{-q}_{\ominus\alpha}(s,t_0) \geqslant \eta_\alpha e^{-q}_{\ominus\alpha}(s,t_0).$$

If $\mu(s)=0$ we have $(e^{-q}_{\ominus\alpha}(s,t_0))^{\Delta} = -\ominus\alpha(s) q e^{-q}_{\ominus\alpha}(s,t_0)$, which implies that

$$e^{-q}_{\ominus\alpha}(r,t_0) - 1 = \int_{t_0}^{r} (e^{-q}_{\ominus\alpha}(s,t_0))^{\Delta}\Delta s \geqslant \eta_\alpha \int_{t_0}^{r} e^{-q}_{\ominus\alpha}(s,t_0)\Delta s.$$

Thus, $$\int_{t_0}^{r} e^{-q}_{\ominus\alpha}(s,t_0)\Delta s \leqslant \frac{e^{-q}_{\ominus\alpha}(r,t_0)-1}{\eta_\alpha} \leqslant \frac{e^{-q}_{\ominus\alpha}(r,t_0)}{\eta_\alpha}. \tag{2.7}$$

Combining (2.6) and (2.7) obtains

$$\sup_{r\in\mathbb{T}_{t_0}} \Big[\int_{r}^{\infty} e^{p}_{\ominus\alpha}(s,t_0)\Delta s\Big]^{\frac{1}{p}} \Big[\int_{t_0}^{r} e^{-q}_{\ominus\alpha}(s,t_0)\Delta s\Big]^{\frac{1}{q}} \leqslant \frac{1}{\eta_\alpha}.$$

This means $B \leqslant \frac{1}{\eta_\alpha}$ and we obtain the estimation (2.5).

Lemma 2.6. *Let $\mathbb{U}: X \to Y$, $\mathbb{V}: Y \to X$ be the bounded linear operators in Banach spaces X, Y. Then the operator $I - \mathbb{U}\mathbb{V}$ is invertible if and only if $I - \mathbb{V}\mathbb{U}$ is invertible. Furthermore,*

$$(I-\mathbb{V}\mathbb{U})^{-1} = I + \mathbb{V}(I-\mathbb{U}\mathbb{V})^{-1}\mathbb{U}.$$

Proof. See [18]. □

2.3. **Some surveys on linear algebra.**

Lemma 2.7. *Let $\overline{A}$ and $\overline{B}$ be given $n\times n-$ matrices, and $\overline{Q}$ be a projector onto $\ker \overline{A}$, i.e., $\overline{Q}^2 = \overline{Q}, \operatorname{im} \overline{Q} = \ker \overline{A}$. Denote $S = \{x : \overline{B}x \in \operatorname{im} \overline{A}_\sigma\}$. Let T be a continuous function defined on $\mathbb{T}_a$, taking values in $Gl(\mathbb{R}^n)$ such that $T|\ker E_\sigma$ is an isomorphism between $\ker E_\sigma$ and $\ker E$. The following assertions are equivalent*

a) $S \cap \ker \overline{A} = \{0\}$.
b) *the matrix* $\overline{G} = \overline{A}_\sigma - \overline{B}T\,\overline{Q}_\sigma$ *is nonsingular.*
c) $\mathbb{R}^n = S \oplus \ker \overline{A}$.

Proof. The proof of this lemma can be found in [22], Appendix 1, Lemma A1, p.329. □

Lemma 2.8. *$\overline{A}, \overline{B}$, $\overline{Q}, \overline{G}$ mentioned in Lemma 2.7 and suppose that the matrix $\overline{G}$ is nonsingular. Then, there hold the following relations:*

a) $\overline{P}_\sigma = \overline{G}^{-1}\,\overline{A}_\sigma$ *where* $\overline{P}_\sigma = I - \overline{Q}_\sigma$.
b) $-\overline{G}^{-1}\,\overline{B}\,T\overline{Q}_\sigma = \overline{Q}_\sigma$.
c) $\widehat{\overline{Q}} := -T\overline{Q}_\sigma\,\overline{G}^{-1}\overline{B}$, *called canonical projector, is the projector onto* $\ker \overline{A}$ *along* S.
d) $T\overline{Q}_\sigma\,\overline{G}^{-1}$ *does not depend on the choice of* T *and* $\overline{Q}$.

Proof. The results in this lemma are proved in [22], p.319. □

3. Solvability of implicit dynamic equations

Let $A(\cdot), B(\cdot)$ be two continuous functions defined on $\mathbb{T}_{t_0}$, valued in the set of $n \times n$–matrices ($\mathbb{R}^{n\times n}$ for brief), $f \in L_p^{loc}\big(\mathbb{T}_{t_0}; \mathbb{R}^n\big)$ and $\Sigma \in \mathcal{L}$ be a causal operator. Consider the linear implicit dynamic equations on time scales (IDE for short)

$$A_\sigma(t)x^\Delta(t) = B(t)x(t) + \big(\Sigma x(\cdot)\big)(t) + f(t),\ t \in \mathbb{T}_{t_0}. \tag{3.1}$$

To solve this equation, we suppose that ker $A(\cdot)$ is smooth in the sense there exists a continuously Δ–differentiable projector $Q(t)$ onto ker $A(t)$, i.e., Q is continuously differentiable and $Q^2 = Q$, im $Q(t) = \ker A(t)$ for all $t \in \mathbb{T}_{t_0}$. By setting $P = I - Q$ we can rewrite the equation (3.1) as

$$A_\sigma(t)\big(Px\big)^\Delta(t) = \overline{B}(t)x(t) + \big(\Sigma x(\cdot)\big)(t) + f(t),\ t \in \mathbb{T}_{t_0}, \tag{3.2}$$

where $\overline{B} := B + A_\sigma P^\Delta$.

It is seen that the solution $x(\cdot)$ of the equation (3.2), if it exists, is not necessarily differentiable but it is required that the component $Px(\cdot)$ is Δ–differentiable almost everywhere on $\mathbb{T}_{t_0}$.

Based on this remark, we deal with the way we solve the equations (3.1) by splitting the solution in a Δ–differentiable component and an algebraic relation. First, we introduce the so-called index-1 concept. Define the linear operators

$$G := A_\sigma - \overline{B}TQ_\sigma \quad \text{and} \quad \widehat{G} := A_\sigma - \big(\overline{B} + \Sigma\big)TQ_\sigma = \big(I - \Sigma TQ_\sigma G^{-1}\big)G.$$

It is clear that $G \in L_\infty^{loc}\big(\mathbb{T}_{t_0}; \mathbb{R}^{n\times n}\big)$, $\widehat{G} \in \mathcal{L}\big(L^{\text{loc}}(\mathbb{T}_{t_0}; \mathbb{R}^n),\ L^{\text{loc}}(\mathbb{T}_{t_0}; \mathbb{R}^n)\big)$.

Definition 3.1. The IDE (3.1) is said to be index-1 if $G(t)$ and $\widehat{G}$ are invertible for all $t \in \mathbb{T}_{t_0}$.

Because $G(t)$ is invertible for all $t \in \mathbb{T}_{t_0}$, we see that $\widehat{G}$ is invertible if and only if $I - \Sigma TQ_\sigma G^{-1}$ is invertible, which is equivalent to the invertibility of $H := I - TQ_\sigma G^{-1}\Sigma$ by Lemma 2.6.

In the following we assume that the equation (3.1) is index-1. Multiplying both sides of (3.2) with $P_\sigma G^{-1}$, $Q_\sigma G^{-1}$ and using Lemmas 2.7 and 2.8 we get

$$u^\Delta = \big(P^\Delta + P_\sigma G^{-1}\overline{B}\big)u + P_\sigma G^{-1}\Sigma x + P_\sigma G^{-1}f, \tag{3.3}$$

$$v = TQ_\sigma G^{-1}\overline{B}u + TQ_\sigma G^{-1}\Sigma(u+v) + TQ_\sigma G^{-1}f, \tag{3.4}$$

where $u = Px, v = Qx$.

Since $H = I - TQ_\sigma G^{-1}\Sigma$ is invertible and $H^{-1}TQ_\sigma G^{-1}\Sigma = H^{-1} - H^{-1}(I - TQ_\sigma G^{-1}\Sigma) = H^{-1} - I$, it follows from (3.4) that

$$\begin{aligned} v &= H^{-1}TQ_\sigma G^{-1}\big(\overline{B} + \Sigma\big)u + H^{-1}TQ_\sigma G^{-1}f \\ &= -u + H^{-1}\widehat{P}u + H^{-1}TQ_\sigma G^{-1}f, \end{aligned}$$

where $\widehat{Q} = I - \widehat{P} = -TQ_\sigma G^{-1}\overline{B}$ is the canonical projection onto $\ker A$ (see Lemma 2.8). Hence,

$$x = H^{-1}\widehat{P}u + H^{-1}TQ_\sigma G^{-1}f. \tag{3.5}$$

Substituting this relation into (3.3) obtains

$$\begin{aligned} u^\Delta &= \big(P^\Delta + P_\sigma G^{-1}\overline{B}\big)u + P_\sigma G^{-1}\Sigma H^{-1}\big(\widehat{P}u + TQ_\sigma G^{-1}f\big) + P_\sigma G^{-1}f \\ &= \big(P^\Delta + P_\sigma G^{-1}\overline{B}\big)u + P_\sigma G^{-1}\Sigma H^{-1}\widehat{P}u + P_\sigma G^{-1}\mathbb{S}f, \end{aligned} \tag{3.6}$$

with $\mathbb{S} := I + \Sigma H^{-1}TQ_\sigma G^{-1}$.

The equation (3.6) is called inherit dynamic equation of (3.2).

For any $x_0 \in \mathbb{R}^n$, there exists uniquely a solution u of (3.6) with the initial condition $u(t_0) = P(t_0)x_0$. Indeed, u is a solution of (3.6) if and only if

$$\begin{aligned} u(t) = u(t_0) &+ \int_{t_0}^t \big(P^\Delta + P_\sigma G^{-1}\overline{B}\big)u(s)\Delta s \\ &+ \int_{t_0}^t \Big(P_\sigma G^{-1}\Sigma H^{-1}\widehat{P}u + P_\sigma G^{-1}\mathbb{S}f(s)\Big)\Delta s. \end{aligned} \tag{3.7}$$

We can use the Picard approximation to prove that that (3.7) has a unique solution $u(t), t \in \mathbb{T}_{t_0}$ with the initial condition $u(t_0) = P(t_0)x(t_0)$.

Hence, we can get the solution $x(\cdot)$ of the dynamic equation (3.2) by the formula (3.5).

Remark 1. As the argument mentioned above, the solution of (3.2) is deduced from the solution of the inherit dynamic equation (3.6), which has the initial condition $u(t_0) = P(t_0)x_0$. Therefore, we can formulate the initial condition of (3.2) as

$$P(t_0)(x(t_0) - x_0) = 0,$$

but we do not require $x(t_0) = x_0$ as in ordinary dynamic equations.

Remark 3.2. If $\Sigma = 0$, we obtain the solvability of (3.2) in [12]. When $\mathbb{T} = \mathbb{R}$ and $\Sigma x(t) = \int_0^t H(t,s)x(s)ds$ is an integral operator, we obtain this result in [16].

For the implicit dynamic equation (3.2) we can establish a so-called variation of constants formula. First, consider the homogeneous equation corresponding to (3.2)

$$A_\sigma(t)\big(Py\big)^\Delta(t) = \overline{B}(t)y(t) + \big(\Sigma[y(\cdot)]_s\big)(t),\ t \geqslant s. \tag{3.8}$$

where $s \in \mathbb{T}_{t_0}$.

Denote by $y(t,s,y_0), t \geqslant s, y_0 \in \mathbb{R}^n$ a unique solution of the homogeneous equation (3.8) with initial value condition $P(s)(y(s) - y_0) = 0$. In the following we write simply $y(t,s)$ or $y(t)$ for $y(t,s,y_0)$ if there is no confusion.

Let $\Phi(t,s), t \geqslant s \geqslant t_0$ be the Cauchy matrix generated by (3.8). It is defined as the solution of the matrix dynamic equation

$$A(t)\Phi^\Delta(t,s) = B(t)\Phi(t,s) + \Sigma\big([\Phi(\cdot,s)]_s\big)(t), \tag{3.9}$$

$$P(s)\big(\Phi(s,s)-I\big)=0,\ t\geqslant s.$$

Following (3.6), the inherit matrix equation of (3.9) has the form

$$\Phi_0^{\Delta}(t,s)=\big(P^{\Delta}+P_\sigma G^{-1}\overline{B}\big)\Phi_0(t,s)+P_\sigma G^{-1}\Sigma H^{-1}\big(\widehat{P}[\Phi_0(t,s)]_s. \quad (3.10)$$

This equation always exists a unique solution with the initial condition $\Phi_0(s,s)=I$. Therefore, from (3.5) we see that

$$\Phi(t,s)=\big(H^{-1}\widehat{P}\,[\Phi_0(\cdot,s)]_sP(s)\big)(t),\ t\geqslant s. \quad (3.11)$$

The variation of constants formula for the solution of (3.2) now can be formulated as

Theorem 3.3. *The unique solution $x(\cdot)$ of the equation* (3.2) *with the initial condition $P(t_0)(x(t_0)-x_0)=0$ can be expressed as*

$$\begin{aligned} x(t)=\ &\Phi(t,t_0)P(t_0)x_0\ +\int_{t_0}^{t}\Phi(t,\sigma(\tau))P_\sigma G^{-1}\left(\mathbb{S}f\right)(\tau)\Delta\tau \\ &+\big(H^{-1}TQ_\sigma G^{-1}f\big)(t), \end{aligned} \quad (3.12)$$

for all $t\in\mathbb{T}_{t_0}$.

Proof. First, we deal with the variation of constants formula for the solution of (3.6). Let $u(\cdot)$ be the the solution of (3.6) with the initial condition $u(t_0)=P(t_0)x_0$, we show that

$$u(t)=\Phi_0(t,t_0)P(t_0)x_0+\int_{t_0}^{t}\Phi_0(t,\sigma(\tau))P_\sigma G^{-1}\big(\mathbb{S}f\big)(\tau)\Delta\tau. \quad (3.13)$$

Indeed, differentiating both sides of (3.13) obtains

$$\begin{aligned} u^{\Delta}(t)&=\Phi_0^{\Delta}(t,t_0)P(t_0)x_0+\int_{t_0}^{t}\Phi_0^{\Delta}(t,\sigma(\tau))P_\sigma G^{-1}\mathbb{S}f(\tau)\Delta\tau+P_\sigma G^{-1}\mathbb{S}f(t)\\ &=\Big[\big(P^{\Delta}+P_\sigma G^{-1}\overline{B}\big)\Phi_0(t,t_0)+\Sigma^*\widehat{P}\Phi_0(t,t_0)\Big]P(t_0)x_0+P_\sigma G^{-1}\mathbb{S}f(t)\\ &\quad+\int_{t_0}^{t}\Big[\big(P^{\Delta}+P_\sigma G^{-1}\overline{B}\big)\Phi_0(t,\sigma(\tau))+\Sigma^*\widehat{P}[\Phi_0(t,\sigma(\cdot))]_{\sigma(\tau)}\Big]P_\sigma G^{-1}\mathbb{S}f(\tau)\Delta\tau\\ &=\big(P^{\Delta}+P_\sigma G^{-1}\overline{B}\big)\Big[\Phi_0(t,t_0)+\int_{t_0}^{t}\Phi_0(t,\sigma(\tau))P_\sigma G^{-1}\mathbb{S}f(\tau)\Delta\tau\Big]\\ &\quad+\Sigma^*\Big(\widehat{P}\Big[\Phi_0(t,t_0)P(t_0)x_0+\int_{t_0}^{t}[\Phi_0(t,\sigma(\cdot))]_{\sigma(\tau)}P_\sigma G^{-1}\mathbb{S}f(\tau)\Delta\tau\Big]\Big)\\ &\quad+P_\sigma G^{-1}\mathbb{S}f(t)=\big(P^{\Delta}+P_\sigma G^{-1}\overline{B}\big)u(t)+\Sigma^*\widehat{P}u(t)+P_\sigma G^{-1}\mathbb{S}f(t). \end{aligned}$$

where $\Sigma^*=P_\sigma G^{-1}\Sigma H^{-1}$.

This relation says that $u(\cdot)$ given by (3.13) is the solution of (3.6).

Next, we can act $H^{-1}\widehat{P}$ to both sides of (3.13) and use the expression (3.5) to see that the unique solution $x(\cdot)$ of (3.2) can be given by the variation of constants formula (3.12). The proof is complete. □

4. Robust stability of implicit dynamic equation under small perturbation

We are now in position to consider the robust stability of (3.8) under small perturbations.

Assumption 1. There exists a differentiable projector $Q(\cdot)$ onto $\ker A(\cdot)$ such that $P = I - Q$ is bounded on $\mathbb{T}_{t_0}$ by constant K_0.

Definition 4.1. The IDE (3.8) is said to be α-exponentially stable if there exists positive numbers M, α such that

$$\|y(t, s, y_0)\| \leqslant M \, \|P(s)y_0\| \, e_{\ominus\alpha}(t, s), \quad t \geqslant s \geqslant t_0.$$

Remark 2. There are some definitions of exponential stability for dynamic equations on time scales. One can either use the exponential function $e^{-\alpha(t-s)}$ or $e_{-\alpha}(t, s)$. All these definitions are equivalent. Here we use the exponential function $e_{\ominus\alpha}(t, s)$ since we do not need to assume that $-\alpha \in \mathcal{R}^+$.

It is known that under Assumption 1, the uniform stability (resp. exponential stability) is equivalent to the boundedness (resp. exponential stability) of the Cauchy matrix Φ. More precisely, we have

Theorem 4.2. *The IDE* (3.8) *is* α*-exponentially stable if there exists positive numbers* M, α *such that*

$$\|\Phi(t, s)\| \leqslant M e_{\ominus\alpha}(t, s), \;\; t \geqslant s \geqslant t_0. \tag{4.1}$$

From (3.11), it is seen that

$$P(t)\Phi(t, s) = P(t)\big(H^{-1}\widehat{P}\,\Phi_0(\cdot, s)P(s)\big)(t) = \Phi_0(t, s)P(s), \;\; t \geqslant s. \tag{4.2}$$

Therefore, if the relation (4.1) and Assumption 1 hold then

$$\|\Phi_0(t, s)P(s)\| = \|P(t)\Phi(t, s)\| \leqslant K_0 M e_{\alpha\omega}(t, s), \;\; t \geqslant s. \tag{4.3}$$

We are to consider the preservation of stability for (3.8) under small perturbations.

Let F be a causal nonlinear operator from $L_p^{\mathrm{loc}}\big(\mathbb{T}_{t_0}; \mathbb{R}^n\big)$ to $L_p^{\mathrm{loc}}\big(\mathbb{T}_{t_0}; \mathbb{R}^n\big)$, i.e., the condition (2.1) for F is satisfied

$$\pi_t F \pi_t = \pi_t F, \qquad \text{for all } t \in \mathbb{T}_{t_0}.$$

Suppose further that $F(\theta) = \theta$, where $\theta(t) = 0$, for all $t \in \mathbb{T}_{t_0}$.

Consider the semi linear implicit dynamic equation with causal operators

$$A_\sigma(t)x^\Delta(t) = B(t)x(t) + \Sigma x(\cdot)(t) + F\big(x(\cdot)\big)(t), \;\; t \in \mathbb{T}_{t_0}. \tag{4.4}$$

Since $F(\theta) = \theta$, the equation (4.4) has the trivial solution $x = \theta$.

In order to study the solvability of (4.4), we need some further assumptions on the operator F. A causal nonlinear operator $\Gamma : \mathbb{T}_{t_0} \times L_p^{\mathrm{loc}}\big(\mathbb{T}_{t_0}; \mathbb{R}^n\big) \to L_p^{\mathrm{loc}}\big(\mathbb{T}_{t_0}; \mathbb{R}^n\big)$ is called a locally Lipschitz with the function m if

- $\pi_t\Gamma(t, u) = \pi_t\Gamma(t, \pi_t u)$ for every $u \in L_p^{\mathrm{loc}}\,\big(\mathbb{T}_{t_0}; \mathbb{R}^n\big)$.

- there exists a positive continuous function $m_\cdot$ such that

$$\|\Gamma(t,x)-\Gamma(t,y)\|_{L_p[t_0,t]} \leqslant m_t\|x-y\|_{L_p[t_0,t]},$$

for all $t\in\mathbb{T}_{t_0}$ and $x,y\in L_p^{\mathrm{loc}}(\mathbb{T}_{t_0};\mathbb{R}^n)$.

In case m is a constant function, we say simply that Γ is m-locally Lipschitz continuous.

Theorem 4.3. *Suppose that*

a. *the function $P_\sigma G^{-1}\mathbb{S}F(x)$ is locally Lipschitz continuous in x with the function k.*
b. *the function $H^{-1}TQ_\sigma G^{-1}F(x)$ is γ–locally Lipschitz continuous in x with $\gamma<1$.*

Then the equation (4.4) *is solvable on $\mathbb{T}_{t_0}$. Moreover, for any $T>0$, there is a constant M_T such that*

$$\|x(\cdot)\|_{L_p[t_0,t]} \leqslant M_T\|P(t_0)x_0\|, \text{ for all } t_0\leqslant t\leqslant T, \tag{4.5}$$

where $x(\cdot)$ is the solution of (4.4) *with the initial condition $P(t_0)(x(t_0)-x_0)=0$.*

Proof. Note that if x is the solution of the equation(4.4) then due to (3.12) it can be expressed by

$$x(t)=\Phi(t,t_0)P(t_0)x_0+\int_{t_0}^t\Phi(t,\sigma(\rho))P_\sigma G^{-1}\mathbb{S}F(x(\cdot))(\rho)\Delta\rho \\ +H^{-1}TQ_\sigma G^{-1}F(x)(t). \tag{4.6}$$

To prove the solvability of the equation (4.4), we show that the integral equation (4.6) has a unique solution on every interval $[t_0,T]$ for fixed $T>t_0$. Construct by induction a sequence (x_n)

$$x_0(t)=P(t_0)x_0, t\in[t_0,T],$$
$$x_{n+1}(t)=\Phi(t,t_0)P(t_0)x_0+\int_{t_0}^t\Phi(t,\sigma(\rho))P_\sigma G^{-1}\mathbb{S}F(x_n)(\rho)\Delta\rho \\ +H^{-1}TQ_\sigma G^{-1}F\big(x_n\big)(t),\quad t\in[t_0,T],\ n\geqslant 1.$$

Denote $N_T=\sup_{t_0\leqslant s\leqslant t\leqslant T}\|\Phi(t,s)\|$. We choose two positive constants ξ_1,ξ_2 such that $\gamma_1:=\xi_2\gamma<1$ and $|x+y|^p\leqslant\xi_1|x|^p+\xi_2|y|^p, \forall\, x,y\in\mathbb{R}$. It is seen that

$$\big\|x_{n+1}-x_n\big\|^p_{L_p[t_0,t]}\leqslant\xi_1\int_{t_0}^t\bigg\|\int_{t_0}^s\Phi(s,\sigma(\rho))P_\sigma G^{-1}\mathbb{S}\big(F(x_n)-F(x_{n-1})\big)(\rho)\Delta\rho\bigg\|^p\Delta s \\ +\xi_2\big\|H^{-1}TQ_\sigma G^{-1}\big(F(x_n)-F(x_{n-1})\big)\big\|^p_{L_p[t_0,t]}$$
$$\leqslant\xi_1N_T^p\int_{t_0}^t\bigg\|\int_{t_0}^sP_\sigma G^{-1}\mathbb{S}\big(F(x_n)-F(x_{n-1})\big)(\rho)\Delta\rho\bigg\|^p\Delta s+\xi_2\gamma\left\|x_n-x_{n-1}\right\|^p_{L_p[t_0,t]}$$
$$\leqslant\xi_1N_T^pT^{\frac{p}{q}}\int_{t_0}^t\int_{t_0}^s\big\|P_\sigma G^{-1}\mathbb{S}\big(F(x_n)-F(x_{n-1})\big)(\rho)\big\|^p\Delta\rho\Delta s+\gamma_1^p\|x_n-x_{n-1}\|_{L_p[t_0,t]}$$

$$\leqslant \xi_1 N_T^p T^{\frac{p}{q}} \int_{t_0}^{t} \left\| P_\sigma G^{-1}\mathbb{S}\big(F(x_n)-F(x_{n-1})\big)\right\|^p_{L_p[t_0,s]}\Delta s + \gamma_1^p \|x_n - x_{n-1}\|^p_{L_p[t_0,t]}$$
$$\leqslant \xi_1 N_T^p T^{\frac{p}{q}} k_T^p \int_{t_0}^{t} \|x_n - x_{n-1}\|^p_{L_p[t_0,s]}\Delta s + \gamma_1^p \|x_n - x_{n-1}\|^p_{L_p[t_0,t]}$$
$$= \overline{N}\int_{t_0}^{t} \|x_n - x_{n-1}\|^p_{L_p[t_0,s]}\Delta s + \gamma_1^p \|x_n - x_{n-1}\|^p_{L_p[t_0,t]},$$

with $\overline{N} = \xi_1 N_T^p T^{\frac{p}{q}}$. Hence, by induction we get

$$\|x_{n+1} - x_n\|^p_{L_p[t_0,T]} \leqslant \|x_1 - x_0\|^p_{L_p[t_0,t]} \sum_{k=1}^{n} \overline{N}^k h_k(t,t_0)\gamma_1^{p(n-k)},$$

where $h_0(t,s) = 1, h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau$ for all $t \in [t_0,T]$. Using the inequality (see [21, Theorem 4.1])

$$h_k(t,s) \leqslant \frac{(t-s)^k}{k!} \leqslant \frac{T^k}{k!},$$

gets

$$\|x_{n+1} - x_n\|^p_{L_p[t_0,T]} \leqslant \|x_1 - x_0\|^p_{L_p[t_0,T]} \sum_{k=1}^{n} \overline{N}^k \frac{T^k}{k!}\gamma_1^{p(n-k)}.$$

Hence, $\|x_{n+1}(\cdot) - x_n(\cdot)\|_{L_p[t_0,T]} \leqslant \|x_1 - x_0\|_{L_p[t_0,T]} \sum_{k=1}^{n} \left(\frac{\overline{N}T}{k!}\right)^{\frac{k}{p}} \gamma_1^{n-k}.$

It note that $\sum_{k=0}^{n} \left(\frac{\overline{N}T}{k!}\right)^{\frac{k}{p}} \gamma_1^{n-k}$ is the general term of the product of two series $\sum_{n=0}^{\infty} \gamma_1^k$ and $\sum_{n=0}^{\infty} \left(\frac{\overline{N}T}{k!}\right)^{\frac{k}{p}}$. Therefore, (x_n) a Cauchy sequence in $L_p[t_0,T]$, which converges in $L_p[t_0,T]$ to a function x. By letting $n \to \infty$ in (4.8), we obtain

$$\begin{aligned} x(t) &= \Phi(t,t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi(t,\sigma(\rho))P_\sigma G^{-1}\mathbb{S}F(x(\cdot))(\rho)\Delta\rho \\ &\quad + H^{-1}TQ_\sigma G^{-1}F\big(x(\cdot)\big)(t). \end{aligned}$$

This means that $x(\cdot)$ is the solution of (4.4) with the initial condition $P(t_0)(x(t_0) - x_0) = 0$.

We prove (4.5). It is seen that

$$\begin{aligned} \|x\|_{L_p[t_0,t]} &\leqslant \|\Phi(\cdot,t_0)P(t_0)x_0\|_{L_p[t_0,t]} + \left\|H^{-1}TQ_\sigma G^{-1}F\big(x(\cdot)\big)\right\|_{L_p[t_0,t]} \\ &\quad + \left\|\int_{t_0}^{\cdot} \Phi(\cdot,\sigma(\rho))P_\sigma G^{-1}\mathbb{S}F(x(\cdot))(\rho)\Delta\rho\right\|_{L_p[t_0,t]} \\ &\leqslant N_T T^{1/p}\|P(t_0)x_0\| + \gamma\|x\|_{L_p[t_0,t]} \\ &\quad + \Big(\int_{t_0}^{t}\left\|\int_{t_0}^{\tau} \Phi(\tau,\sigma(\rho))P_\sigma G^{-1}\mathbb{S}F(x))(\rho)\Delta\rho\right\|^p \Delta\tau\Big)^{1/p}. \end{aligned}$$

Therefore,

$$\begin{aligned}\|x\|_{L_p[t_0,t]} \leqslant & \frac{N_T}{1-\gamma}\Big[T^{\frac{1}{p}}\left\|P(t_0)x_0\right\| + \Big(\int_{t_0}^{t}\Big\|\int_{t_0}^{\tau} P_\sigma G^{-1}\mathbb{S}F(x))(\rho)\Delta\rho\Big\|^p\Delta\tau\Big)^{\frac{1}{p}}\Big]\\ \overset{\text{Hölder}}{\leqslant} & \frac{N_T}{1-\gamma}\Big[T^{\frac{1}{p}}\left\|P(t_0)x_0\right\| + \Big(\int_{t_0}^{t}\tau^{\frac{p}{q}}\left\|P_\sigma G^{-1}\mathbb{S}F(x)\right\|^p_{L_p[t_0,\tau]}\Delta\tau\Big)^{\frac{1}{p}}\Big]\\ \leqslant & \frac{N_T T^{\frac{1}{p}}}{1-\gamma}\Big[\left\|P(t_0)x_0\right\| + \Big(\int_{t_0}^{t}k_\tau^p\left\|x\right\|^p_{L_p[t_0,\tau]}\Delta\tau\Big)^{\frac{1}{p}}\Big].\end{aligned}$$

By using the inequality Gronwall in Lemma 2.3, we have

$$\|x(\cdot)\|_{L_p[t_0,t]} \leqslant M_T\left\|P(t_0)x_0\right\|,\ t\in[t_0,T],$$

for a certain $M_T > 0$. The proof is complete. □

In the following, let Assumptions 1 hold.

Definition 4.4 (See [18])**.** The implicit dynamic equation (4.4) is said to be L_p-stable if there exist constants $M_1, M_2 > 0$ such that

$$\|P(t)x(t;t_0,x_0)\|_{\mathbb{R}^n} \leqslant M_1\left\|P(t_0)x_0\right\|_{\mathbb{R}^n},\ t\in\mathbb{T}_{t_0}, \tag{4.7}$$

$$\|x(\cdot;t_0,x_0)\|_{L_p(\mathbb{T}_{t_0})} \leqslant M_2\left\|P(t_0)x_0\right\|_{\mathbb{R}^n}. \tag{4.8}$$

Theorem 4.5. *Assume that the IDE* (3.8) *is index-1, $\alpha-$exponentially stable and $H^{-1}TQ_\sigma G^{-1}F$ is $\gamma-$locally Lipschitz continuous with $\gamma < 1$. Further,*

(1) *There exists a continuous function $m : \mathbb{T}_{t_0} \to \mathbb{R}_+$ such that*

$$\left\|P_\sigma G^{-1}\left[\mathbb{S}F(x)\right]_t - P_\sigma G^{-1}\left[\mathbb{S}F(y)\right]_t\right\|_{L_p[t_0,T]} \leqslant m_t\|x-y\|_{L_p[t_0,T]}, \tag{4.9}$$

for $t_0 < t \leqslant T < \infty$ and $x, y \in L_p^{loc}\left(\mathbb{T}_{t_0};\mathbb{R}^n\right)$.

(2) *There holds*

$$\limsup_{t\to\infty} m_t < \frac{1-\gamma}{N_\alpha}, \tag{4.10}$$

where $N_\alpha = \frac{M(1+\mu^*\alpha)}{\eta_\alpha}p^{\frac{1}{p}}q^{\frac{1}{q}}$ with α, M to be defined in Theorem 4.2 and η_α given in Remark 2.5.

Then, the solution of the perturbed dynamic equation (4.4) *is L_p–stable.*

Proof. By (4.9) and the γ–locally Lipschitz continuity of $H^{-1}TQ_\sigma G^{-1}F(x)$ in x with $\gamma < 1$, it is seen that for any $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(\cdot)$ of (4.4), defined on $\mathbb{T}_{t_0}$, with the initial condition $P(t_0)(x(t_0) - x_0) = 0$. From (4.10), there is a $\xi > t_0$ such that

$$m_t < \frac{1-\gamma}{N_\alpha}, \quad \text{for all } t \geqslant \xi. \tag{4.11}$$

Using (4.6) gets

$$x(t) = \Gamma_0(t) + \int_{\xi}^{t}\Phi(t,\sigma(\rho))P_\sigma G^{-1}\left[\mathbb{S}F\big(x\big)\right]_\xi(\rho)\Delta\rho + H^{-1}TQ_\sigma G^{-1}F(x)(t), \tag{4.12}$$

where

$$\Gamma_0(t)=\Phi(t,t_0)P(t_0)x_0+\int_{t_0}^{t}\Phi(t,\sigma(\rho))P_\sigma G^{-1}(\pi_\xi\mathbb{S}F(x))(\rho)\Delta\rho.$$

From the exponential stability of (3.8) and the inequality (2.6) it deduces

$$\begin{aligned}\left\|\Phi(\cdot,t_0)P(t_0)x_0\right\|_{L_p(\mathbb{T}_{t_0})}&=\Big(\int_{t_0}^{\infty}\|\Phi(t,t_0)P(t_0)x_0\|^p\,\Delta t\Big)^{\frac{1}{p}}\\&\leqslant M\|P(t_0)x_0\|\Big(\int_{t_0}^{\infty}e^p_{\ominus\alpha}(t,t_0)\Delta t\Big)^{\frac{1}{p}}\leqslant M\eta_\alpha^{-\frac{1}{p}}\|P(t_0)x_0\|.\end{aligned}$$

Further,

$$\begin{aligned}&\Big\|\int_{t_0}^{\cdot}\Phi(\cdot,\sigma(\rho))P_\sigma G^{-1}(\pi_\xi\mathbb{S}F(x))(\rho)\Delta\rho\Big\|_{L_p(\mathbb{T}_{t_0})}\\&\quad=\Big[\int_{t_0}^{\infty}\Big(\int_{t_0}^{t}\Phi(t,\sigma(\rho))P_\sigma G^{-1}(\pi_\xi\mathbb{S}F(x))(\rho)\Delta\rho\Big)^p\Delta t\Big]^{\frac{1}{p}}\\&\quad\leqslant M\Big[\int_{t_0}^{\infty}\Big(\int_{t_0}^{t}e_{\ominus\alpha}(t,\sigma(\rho))\left\|P_\sigma G^{-1}(\pi_\xi\mathbb{S}F(x))(\rho)\right\|\Delta\rho\Big)^p\Delta t\Big]^{\frac{1}{p}}\\&\quad\leqslant M\Big[\int_{t_0}^{\infty}\Big(e_{\ominus\alpha}(t,t_0)\int_{t_0}^{t}\frac{1+\alpha\mu^*}{e_{\ominus\alpha}(\rho,t_0)}\left\|P_\sigma G^{-1}\big(\pi_\xi\mathbb{S}F(x)\big)(\rho)\right\|\Delta\rho\Big)^p\Delta t\Big]^{\frac{1}{p}}.\end{aligned}$$

By using Hardy inequality in Lemma 2.4 with the weight functions $U(t)=V(t)=e_{\ominus\alpha}(t,t_0)$ we have

$$\begin{aligned}&\Big\|\int_{t_0}^{\cdot}\Phi(\cdot,\sigma(\rho))P_\sigma G^{-1}\big(\pi_\xi\mathbb{S}F(x)\big)(\rho)\Delta\rho\Big\|_{L_p(\mathbb{T}_{t_0})}\\&\leqslant\frac{M(1+\alpha\mu^*)}{\eta_\alpha}p^{\frac{1}{p}}q^{\frac{1}{q}}\Big[\int_{t_0}^{\xi}\left\|P_\sigma G^{-1}\mathbb{S}F(x)(\rho)\right\|^p\Delta\rho\Big]^{\frac{1}{p}}=N_\alpha\left\|P_\sigma G^{-1}\mathbb{S}F(x)\right\|_{L_p[t_0,\xi]}.\end{aligned}$$

By virtue of the property (4.9) of $P_\sigma G^{-1}\big(\mathbb{S}F\big)(x)$, it is seen that

$$\begin{aligned}&\Big\|\int_{t_0}^{\cdot}\Phi(\cdot,\sigma(\rho))P_\sigma G^{-1}\big(\pi_\xi\mathbb{S}F(x)\big)(\rho)\Delta\rho\Big\|_{L_p(\mathbb{T}_{t_0})}\\&\qquad\leqslant m_{t_0}N_\alpha\left\|x(\cdot)\right\|_{L_p[t_0,\xi]}\leqslant m_{t_0}M_\xi N_\alpha\|P(t_0)x_0)\|.\end{aligned}\tag{4.13}$$

Thus,

$$\left\|\Gamma_0(\cdot)\right\|_{L_p(\mathbb{T}_{t_0})}\leqslant L_1\left\|P(t_0)x_0\right\|,\tag{4.14}$$

where $L_1=M\eta_\alpha^{-\frac{1}{p}}+m_{t_0}M_\xi N_\alpha$ and M_ξ is defined in (4.5).

On the other hand, from (4.12) it follows that

$$\begin{aligned}\left\|x(\cdot)\right\|_{L_p[\xi,t]}&\leqslant\left\|\Gamma_0(\cdot)\right\|_{L_p[\xi,t]}+\Big\|\int_{\xi}^{\cdot}\Phi(\cdot,\sigma(\rho))P_\sigma G^{-1}\left[\mathbb{S}F\big(x\big)\right]_\xi(\rho)\Delta\rho\Big\|_{L_p[\xi,t]}\\&\quad+\left\|H^{-1}TQ_\sigma G^{-1}F(x)(t)\right\|_{L_p[\xi,t]}\leqslant\left\|\Gamma_0(\cdot)\right\|_{L_p(\mathbb{T}_{t_0})}\end{aligned}$$

$$+\Big\|\int_{\xi}^{\cdot}\Phi(\cdot,\sigma(\rho))P_\sigma G^{-1}\big[\mathbb{S}F(x)\big]_\xi(\rho)\Delta\rho\Big\|_{L_p[\xi,t]}+\gamma\|x(\cdot)\|_{L_p[\xi,t]}.$$

Similary as in (4.13), using the Hardy inequality comes to

$$\Big\|\int_{\xi}^{\cdot}\Phi(\cdot,\sigma(\rho))P_\sigma G^{-1}\big[\mathbb{S}F(x)\big]_\xi(\rho)\Delta\rho\Big\|_{L_p[\xi,t]} \leqslant N_\alpha\left\|P_\sigma G^{-1}[\mathbb{S}F(x)]_\xi\right\|_{L_p[\xi,t]}\leqslant N_\alpha m_\xi\left\|x(\cdot)\right\|_{L_p[\xi,t]}. \quad (4.15)$$

Combining (4.11), (4.14) and (4.15) yields

$$(1-\gamma-N_\alpha m_\xi)\|x(\cdot)\|_{L_p[\xi,t]}\leqslant L_1\left\|P(t_0)x(t_0)\right\|.$$

Thus,

$$\|x(\cdot)\|_{L_p[\xi,t]}\leqslant\frac{L_1}{1-\gamma-N_\alpha m_\xi}\left\|P(t_0)x(t_0)\right\|,\ \ t\in\mathbb{T}_{t_0}.$$

From this inequality it is seen that

$$\|x(\cdot)\|_{L_p(\mathbb{T}_{t_0})}\leqslant M_2\left\|P(t_0)x_0\right\|_{\mathbb{R}^n},$$

where $M_2=M_\xi+\frac{L_1}{1-\gamma-N_\alpha m_\xi}$. Thus, we get (4.8).

We now prove the boundedness of $u=Px$ on $\mathbb{T}_{t_0}$. Using Hölder inequality and (2.7), (4.3) obtains

$$\begin{aligned}
\|u(t)\| &= \Big\|\Phi_0(t,t_0)P(t_0)x_0+\int_{t_0}^{t}\Phi_0(t,\sigma(\rho))P_\sigma G^{-1}\mathbb{S}F(x)(\rho)\Delta\rho\Big\|\\
&\leqslant MK_0\|u(t_0)\|+MK_0\int_{t_0}^{t}e_{\ominus\alpha}(t,\sigma(\rho))\big\|P_\sigma G^{-1}\mathbb{S}F(x)(\rho)\big\|\Delta\rho\\
&\leqslant MK_0\|u(t_0)\|+MK_0\left[\int_{t_0}^{t}e^q_{\ominus\alpha}(t,\sigma(\rho))\Delta\rho\right]^{\frac1q}\left[\int_{t_0}^{t}\big\|P_\sigma G^{-1}\mathbb{S}F(x)(\rho)\big\|^p\Delta\rho\right]^{\frac1p}\\
&\leqslant MK_0\Big[\|u(t_0)\|+m_{t_0}(1+\alpha\mu^*)\eta_\alpha^{-\frac1q}\|x(\cdot)\|_{L_p(t_0,t)}\Big]\leqslant M_1\|u(t_0)\|, \quad (4.16)
\end{aligned}$$

for all $t\in\mathbb{T}_{t_0}$, where $M_1=MK_0\Big[1+m_{t_0}M_2(1+\alpha\mu^*)\eta_\alpha^{-\frac1q}\Big]$. This means that we have (4.7). The proof is complete. □

Remark 4.6. If $\Sigma=0$, we obtain Theorem 3.3 in [12] from Theorem 4.5.

Example 4.7. Let

$$\mathbb{T}=\bigcup_{n=0}^{\infty}\left[\frac{3n}{2},\frac{3n}{2}+1\right],$$

and q be a continuous function defined on $\mathbb{T}$ such that $0\leqslant q\leqslant 1$. We see that $\mu(t)=0$ if $t\neq\frac{3n}{2}+1$ and $\mu(\frac{3n}{2}+1)=\frac12$, for all $n\geqslant 0$. Consider the implicit dynamic equation (3.1) with $x=(x_1,x_2,x_3)^\top$ and

$$A=\begin{pmatrix}1&1&0\\0&0&0\\0&1&0\end{pmatrix},\quad B=\begin{pmatrix}-1&-1&1\\1&0&1\\1&-1&0\end{pmatrix},\quad \Sigma=\begin{pmatrix}0&0&0\\0&0&-h(\cdot)\\0&0&0\end{pmatrix}, \quad (4.17)$$

where

$$h(y)(t) = \int_{t_0}^{t} q(s)e_{-1\ominus q}(t,s)y(s)\Delta s,\ t \geqslant 0.$$

It is easy to see that the equation (4.17) is index-1 and

$$G^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \ PG^{-1}\Sigma = \begin{pmatrix} 0 & 0 & h \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \ QG^{-1}\Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h \end{pmatrix},$$

$$QG^{-1}B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix},\qquad PG^{-1}B = \begin{pmatrix} -3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Further,

$$H(y_1, y_2, y_3)(t) = \big(y_1(t), y_2(t), y_3(t) - h(y_3)(t)\big)^{\top},$$

which follows that

$$H^{-1}(y_1, y_2, y_3)(t) = \big(y_1(t), y_2(t), -y_3(t) + g(y_3)(t)\big),$$

where $g(z) = \int_{t_0}^{t} \ominus q(s)e_{-1}(t,s)z(s)\Delta s$. Thus,

$$H^{-1}QG^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1-g & 0 \end{pmatrix},\ PG^{-1}\mathbb{S} = \begin{pmatrix} 1 & -h(-1+g)-1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We prove that (3.8) is exponentially stable. Indeed, from (3.4) (with $f = 0$) we gets

$$y_3(t) - \int_{s}^{t} q(\tau)e_{-1\ominus q}(t,\tau)y_3(\tau)\Delta\tau = -y_1,\ t \geqslant s,$$

which implies

$$y_3(t) = -y_1(t) + \int_{s}^{t} \ominus q(\tau)e_{-1}(t,\tau)y_1(\tau)\Delta\tau,\ t \geqslant s. \tag{4.18}$$

Substituting it into the equation of (3.3) obtains

$$\begin{cases} y_1^{\Delta}(t) = -3y_1(t) + \int_{s}^{t} \ominus q(\tau)e_{-1}(t,\tau)y_1(\tau)\Delta\tau, \\ y_2^{\Delta}(t) = y_1(t) - y_2(t). \end{cases} \tag{4.19}$$

Put

$$z(t) = \int_{s}^{t} \ominus q(\tau)e_{-1}(t,\tau)y_1(\tau)\Delta\tau,$$

we get

$$\begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix}^{\Delta} = \begin{pmatrix} -3 & 0 & 1 \\ 1 & -1 & 0 \\ \ominus q(t).(1-\mu(t)) & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix} := D(t) \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix}.$$

Denote

$$E = \begin{pmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1.6 \end{pmatrix} \quad \text{and} \quad \overline{y} = (y_1, y_2, z)^\top.$$

The matrix E has three eigenvalues $\lambda_1 \approx 1.62$, $\lambda_2 = 1$ and $\lambda_3 \approx 0.98$. Consider the Lyapunov function $V(y_1, y_2, z) = y_1^2 + y_2^2 + 0.2y_1 z + 1.6z^2$. By direct calculation we have (see [8])

$$\begin{aligned} V^\Delta(y_1, y_2, z) &= \overline{y}^\top (D(t)E + ED^\top(t) + \mu(t)D(t)ED^\top(t))\overline{y} \\ &< -0.49(y_1^2 + y_2^2 + z^2) < -\frac{0.49}{1.62} V(y_1, y_2, z). \end{aligned}$$

Therefore, the dynamic equation (4.19) is α–exponentially stable with $\alpha \approx \sqrt{\frac{0.49}{1.62}}$ and $M = \sqrt{\frac{1}{0.98}}$. Further, from (4.18) we see that

$$\begin{aligned} |y_3(t)| &\leqslant |y_1(t)| + \int_s^t | \ominus q(\tau) e_{-1}(t, \tau) y_1(\tau)| \Delta\tau \\ &\leqslant M e_{\ominus\alpha}(t, s) + M \int_s^t | \ominus q(\tau)| e_{-1}(t, \tau) e_{\ominus\alpha}(\tau, s) \Delta\tau \\ &\leqslant M e_{\ominus\alpha}(t, s) \Big(1 + \int_s^t | \ominus q(\tau)| e_{-1\oplus\alpha}(t, \tau) \Delta\tau \Big). \end{aligned}$$

Since $0 \leqslant q \leqslant 1$ on $\mathbb{T}$ and $\alpha < 1$,

$$M_1 := M \Big(1 + \sup_{t_0 \leqslant s \leqslant t < \infty} \int_s^t | \ominus q(\tau)| e_{-1\oplus\alpha}(t, \tau) \Delta\tau \Big) < \infty.$$

Hence,

$$\|y(t)\| \leqslant M_1 e_{\ominus\alpha}(t, s) \|P(s)y(s)\|, t \geqslant s.$$

Thus, (3.8) is α–exponentially stable.

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a k–Lipschitz function. Put $C = p^{\frac{1}{p}} q^{\frac{1}{q}} B$ with B given by (2.4). It is easy to see that $H^{-1}QG^{-1}F$ is γ–Lipschitz continuous with $\gamma = k(C + 1)$ and $PG^{-1}\mathbb{S}F$ satisfies (4.9) with $m = k\big(3 + C(1 + C)\big)$. Therefore, if the real number k satisfies

$$k\big(3 + C(1 + C)\big) \leqslant \frac{1 - k(C + 1)}{N_\alpha},$$

then (4.4) is L_p–stable by Theorem 4.5.

We now deal with the preservation the exponential stability under small perturbation. For any $\lambda \in \mathbb{R}$ and $t_0 \leqslant s < t \leqslant \infty$, on the space $L_p[s, t]$, introduce a new norm $\| \cdot \|_{L_p^\lambda[s,t]}$ given by

$$\|z\|^p_{L_p^\lambda[s,t]} = \int_s^t \|e_\lambda(\tau, t_0) z(\tau)\|^p d\tau.$$

Denote by $L_p^\lambda(\mathbb{T}_{t_0})$ the subset of $L^{\mathrm{loc}}(\mathbb{T}_{t_0})$ consisting of all continuous functions z with $\|z\|_{L_p^\lambda(\mathbb{T}_{t_0})} < \infty$ and by $B^\lambda(\mathbb{T}_{t_0})$ the subset $z \in L_p^{\mathrm{loc}}(\mathbb{T}_{t_0})$ such that $\sup_{t\in\mathbb{T}_{t_0}} e_\lambda(t,t_0)\,\|z(t)\| < \infty$ with the norm $\|z\| = \sup_{t\in\mathbb{T}_{t_0}} e_\lambda(t,t_0)\,\|z(t)\|$.

For $s \geqslant t_0$, consider the semi linear implicit dynamic equation

$$A_\sigma(t)x^\Delta(t) = B(t)x(t) + \Sigma[x(\cdot)(t)]_s + F\big([x(\cdot)]_s\big)(t),\ t \in \mathbb{T}_s. \tag{4.20}$$

Definition 4.8. Let $\alpha > 0$. The implicit dynamic equation of (4.20) is called α–exponentially stable if there exists a positive constant K such that

$$\|x(t)\| \leqslant K e_{\ominus\alpha}(t,s)\,\|P(s)x(s)\|\,, \qquad t \geqslant s,$$

for all solution x of (4.20).

Theorem 4.9. *Suppose that the IDE* (3.8) *is index-1, α–exponentially stable and there exists $0 < \lambda < \alpha$ such that $H^{-1}TQ_\sigma G^{-1}(x)$ is locally γ–Lipschitz continuous in x with $\gamma < 1$ in norm $\|\cdot\|_{L_p^\lambda[t_0,t]}$ and the conditions* (4.9), (4.10) *are satisfied. Suppose further that for any $s \geqslant t_0$, the function $(I - H^{-1}Q_\sigma G^{-1}F)^{-1}$ acts continuously from $B^\lambda(\mathbb{T}_{t_0})$ to $B^\lambda(\mathbb{T}_{t_0})$.*

Then, the implicit dynamic equation (4.20) *is exponentially stable.*

Proof. Let $x(t)$ be a solution of (4.20). For any $t \geqslant s$, by putting $z(t) = e_\lambda(t,s)x(t)$ we obtain

$$\begin{aligned}
A_\sigma(t)(Pz)^\Delta(t) &= A_\sigma(t)\Big(e_\lambda(\sigma(t),s)(Px)^\Delta(t) + \lambda e_\lambda(t,s)Px(t)\Big)\\
&= e_\lambda(\sigma(t),s)\big[\overline{B}x + \Sigma[x]_s + F[x]_s\big](t) + \lambda A_\sigma(t)e_\lambda(t,s)Px(t)\\
&= \big(\lambda A_\sigma + (1+\lambda\mu)\overline{B}\big)(t)z(t) + \widetilde{\Sigma}[z]_s(t) + \widetilde{F}[z]_s(t),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\Sigma}[z(\cdot)]_s(t) &= e_\lambda(\sigma(t),s)\Sigma\big(e_{\ominus\lambda}(\cdot,s)[z(\cdot)]_s\big)(t),\\
\widetilde{F}[z(\cdot)]_s(t) &= e_\lambda(\sigma(t),s)F\big(e_{\ominus\lambda}(\cdot,s)[z(\cdot)]_s\big)(t).
\end{aligned} \tag{4.21}$$

Thus, z is the solution of the implicit dynamic equation

$$A_\sigma(t)(Pz)^\Delta(t) = \big(\lambda A_\sigma + (1+\lambda\mu)\overline{B}\big)(t)z(t) + \widetilde{\Sigma}[z]_s(t) + \widetilde{F}[z]_s(t). \tag{4.22}$$

On the other hand,

$$\begin{aligned}
\widetilde{G} &= A_\sigma - \big(\lambda A_\sigma + (1+\lambda\mu(t))\overline{B}\big)TQ_\sigma = G - \lambda\mu BTQ_\sigma\\
&= G\big(I - \lambda\mu G^{-1}BTQ_\sigma\big) = G(I + \lambda\mu Q_\sigma).
\end{aligned}$$

We see that $(I+\lambda\mu Q_\sigma)^{-1} = \big(\widehat{P} + (1+\lambda\mu)Q_\sigma\big)^{-1} = P_\sigma + \dfrac{1}{1+\lambda\mu}Q_\sigma$, which implies $\widetilde{G}$ is invertible, and $\widetilde{G}^{-1} = \big(P_\sigma + \dfrac{1}{1+\lambda\mu}Q_\sigma\big)G^{-1}$. Furthermore,

$$P_\sigma\widetilde{G}^{-1} = P_\sigma G^{-1};\ \ Q_\sigma\widetilde{G}^{-1} = \frac{1}{1+\lambda\mu}Q_\sigma G^{-1}. \tag{4.23}$$

Let $\widetilde{H} = I - TQ_\sigma \widetilde{G}^{-1}\widetilde{\Sigma}$. We have

$$\begin{aligned}\widetilde{H} &= I - e_\lambda(\sigma(\cdot), s)TQ_\sigma\big(P_\sigma + \frac{1}{1+\lambda\mu}Q_\sigma\big)G^{-1}\Sigma e_{\ominus\lambda}(\cdot, s)\\ &= I + e_\lambda(\cdot, s)\big(H - I\big)e_{\ominus\lambda}(\cdot, s) = e_\lambda(\cdot, s)He_{\ominus\lambda}(\cdot, s).\end{aligned}$$

Therefore, $\widetilde{H}$ is invertible and for any $z \in L_p^{loc}(\mathbb{T}_{t_0})$,

$$\widetilde{H}^{-1}z = e_\lambda(\cdot, s)H^{-1}e_{\ominus\lambda}(\cdot, s)z. \tag{4.24}$$

This means that the equation (4.22) is index-1. The homogeneous equation corresponding to (4.22) is

$$A_\sigma(t)(Py)^\Delta(t) = \big(\lambda A_\sigma + (1+\lambda\mu)\overline{B}\big)(t)y(t) + \widetilde{\Sigma}[y]_s(t). \tag{4.25}$$

Assume that $\Psi(t, s)$ is the Cauchy matrix of (4.25). It is clear that $\Psi(t, s) = e_\lambda(t, s)\Phi(t, s)$. Therefore, by the α–exponentially stable property of (3.8), we have

$$\|\Psi(t, s)\| \leqslant Me_{\ominus(\alpha\ominus\lambda)}(t, s), \ \ t \geqslant s \geqslant t_0.$$

Further, the variation of constant formula of solutions of (4.22) holds

$$\begin{aligned}z(t) = \Psi(t, s)P(s)z(s) &+ \int_s^t \Psi(t, \sigma(\rho))P_\sigma\widetilde{G}^{-1}\widetilde{\mathbb{S}}\widetilde{F}[z(\cdot)]_s(\tau)\Delta\tau\\ &+ \widetilde{H}^{-1}TQ_\sigma\widetilde{G}^{-1}\widetilde{F}[z(\cdot)]_s(t), \ \ t \geqslant s,\end{aligned} \tag{4.26}$$

where $\widetilde{\mathbb{S}} := I + \widetilde{\Sigma}\widetilde{H}^{-1}TQ_\sigma\widetilde{G}^{-1}$.

By the Lipschitz condition of $H^{-1}TQ_\sigma G^{-1}F(\cdot)$ in the norm $\|\cdot\|_{L_p^\lambda[t_0,t]}$, for all $z_1, z_2 \in L_p^{loc}(\mathbb{T}_{t_0})$, we have

$$\begin{aligned}&\big\|\widetilde{H}^{-1}TQ_\sigma\widetilde{G}^{-1}\widetilde{F}(z_1) - \widetilde{H}^{-1}TQ_\sigma\widetilde{G}^{-1}\widetilde{F}(z_2)\big\|_{L_p[t_0,t]}\\ &= \big\|e_\lambda(\cdot, s)H^{-1}e_{\ominus\lambda}(\cdot, s)TQ_\sigma G^{-1}e_\lambda(\cdot, s)\big[F(e_{\ominus\lambda}(\cdot, s)z_1) - F(e_{\ominus\lambda}(\cdot, s)z_2)\big]\big\|_{L_p[t_0,t]}\\ &= \big\|H^{-1}TQ_\sigma G^{-1}\big[F(e_{\ominus\lambda}(\cdot, s)z_1) - F(e_{\ominus\lambda}(\cdot, s)z_2)\big]\big\|_{L_p^\lambda[t_0,t]} \leqslant \gamma\,\|z_1 - z_2\|_{L_p[t_0,t]}\,.\end{aligned}$$

In addition, by combining with (4.21), (4.23) and (4.24) we get

$$\begin{aligned}P_\sigma\widetilde{G}^{-1}\big[\widetilde{\mathbb{S}}\widetilde{F}(z_1)\big]_t &= P_\sigma G^{-1}\big[\big(I + \widetilde{\Sigma}\widetilde{H}^{-1}Q_\sigma\widetilde{G}^{-1}\big)e_\lambda(\sigma(\cdot), s)F\big(e_{\ominus\lambda}(\cdot, s)z_1\big)\big]_t\\ &= e_\lambda(\sigma(t), s)P_\sigma G^{-1}\big[\big(I + \Sigma H^{-1}Q_\sigma G^{-1}\big)F\big(e_{\ominus\lambda}(\cdot, s)z_1\big)\big]_t\\ &= \big(1+\lambda\mu\big)e_\lambda(\cdot, s)P_\sigma G^{-1}[\mathbb{S}F\big(e_{\ominus\lambda}(\cdot, s)z_1\big)]_t.\end{aligned}$$

Therefore, from (4.9), it yields

$$\begin{aligned}&\Big\|P_\sigma\widetilde{G}^{-1}\big[\widetilde{\mathbb{S}}\widetilde{F}\big(z_1\big)\big]_t - P_\sigma\widetilde{G}^{-1}\big[\widetilde{\mathbb{S}}\widetilde{F}\big(z_2\big)\big]_t\Big\|^p_{L_p[t_0,T]}\\ &= \Big\|\big(1+\lambda\mu\big)e_\lambda(\cdot, s)P_\sigma G^{-1}\Big[\mathbb{S}\Big(F\big(e_{\ominus\lambda}(\cdot, s)z_1\big) - F\big(e_{\ominus\lambda}(\cdot, s)z_2\big)\Big)\Big]_t\Big\|^p_{L_p[t_0,T]}\\ &\leqslant \big(1+\lambda\mu^*\big)\Big\|P_\sigma G^{-1}\Big[\mathbb{S}\Big(F\big(e_{\ominus\lambda}(\cdot, s)z_1\big) - F\big(e_{\ominus\lambda}(\cdot, s)z_2\big)\Big)\Big]_t\Big\|^p_{L_p^\lambda[t_0,T]}\\ &\leqslant m_t\big(1+\lambda\mu^*\big)\,\|z_1 - z_2\|_{L_p[t_0,T]}\,,\end{aligned}$$

for all $0 \leqslant t \leqslant T$ and $z_1, z_2 \in L_p^{\mathrm{loc}}(\mathbb{T}_{t_0})$. In particular,

$$\left\| P_\sigma \widetilde{G}^{-1}[\widetilde{\mathbb{S}}\widetilde{F}(z)]_t \right\| \leqslant m_t\big(1+\mu^*\lambda\big) \left\| z \right\|_{L_p[t_0,T]}.$$

By virtue of these estimates, it is seen that all assumptions of Theorem 4.5 are satisfied, which implies there is $L_2 > 0$ such that

$$\left\| [z]_s(\cdot) \right\|_{L_p(\mathbb{T}_{t_0})} \leqslant L_2 \|P(s)z(s)\|.$$

Let $h(t) = \Psi(t,s)P(s)z(s) + \int_s^t \Psi(t,\sigma(\tau))P\widetilde{G}^{-1}\widetilde{\mathbb{S}}\widetilde{F}[z]_s(\tau)\Delta\tau, t \geqslant s$. Since

$$\frac{\alpha-\lambda}{1+\lambda\mu^*} \leqslant \min_{t\in\mathbb{T}_0} \alpha \ominus \lambda(t) \leqslant \max_{t\in\mathbb{T}_0} \alpha\ominus\lambda(t) \leqslant \alpha-\lambda,$$

by the same way as in (4.16) we can use Hardy inequality to obtain

$$\begin{aligned}
\|h(t)\| &\leqslant \big\|\Psi(t,s)P(s)z(s)\big\| + \left\| \int_s^t \Psi(t,\sigma(\tau))P\widetilde{G}^{-1}\widetilde{\mathbb{S}}\widetilde{F}[z]_s(\tau)\Delta\tau \right\| \\
&\leqslant \; MK_0\|P(s)z(s)\| + MK_0(1+\alpha\mu^*)\eta_{\alpha\ominus\lambda}^{-\frac{1}{q}} \left\| P\widetilde{G}^{-1}\widetilde{\mathbb{S}}\widetilde{F}[z]_s(\cdot) \right\|_{L_p(t_0,t)} \\
&\leqslant \; MK_0\|P(s)z(s)\| + MK_0(1+\alpha\mu^*)^2\eta_{\alpha\ominus\lambda}^{-\frac{1}{q}} m_s \left\| [z]_s(\cdot) \right\|_{L_p(t_0,t)} \\
&\leqslant \; MK_0\Big[1 + (1+\alpha\mu^*)^2\eta_{\alpha\ominus\lambda}^{-\frac{1}{q}} m_{t_0}L_2\Big] \left\| P(s)z(s) \right\| := L_3 \left\| P(s)z(s) \right\|.
\end{aligned}$$

This means that $[h]_s \in B^0(\mathbb{T}_{t_0})$. Moreover, from (4.26), it is clear that

$$\begin{aligned}
h(t) &= z(t) - \widetilde{H}^{-1}Q_\sigma\widetilde{G}^{-1}\widetilde{F}[z]_s(t) \\
&= z(t) - e_\lambda(t,s)H^{-1}Q_\sigma G^{-1}F\big(e_{\ominus\lambda}(\cdot,s)[z(\cdot)]_s\big)(t) \\
\Leftrightarrow e_{\ominus\lambda}(t,s)h(t) &= e_{\ominus\lambda}(t,s)z(t) - H^{-1}Q_\sigma G^{-1}F\big(e_{\ominus\lambda}(\cdot,s)[z(\cdot)]_s\big)(t) \\
\Leftrightarrow e_{\ominus\lambda}(t,s)h(t) &= (I - H^{-1}Q_\sigma G^{-1}F)(e_{\ominus\lambda}(\cdot,s)[z(\cdot)]_s)(t).
\end{aligned}$$

Since $H^{-1}QG^{-1}F$ is γ–locally Lipschitz continuous with $\gamma < 1$ in $\|\cdot\|_{L_p^\lambda[t_0,t]}$, $I - H^{-1}QG^{-1}F$ is invertible and

$$z(t) = e_\lambda(t,s)\big(I - H^{-1}QG^{-1}F\big)^{-1}\big(e_{\ominus\lambda}(\cdot,s)[h]_s(\cdot)\big)(t).$$

On the other hand, $[h]_s \in B^0(\mathbb{T}_{t_0})$ implies that $e_{\ominus\lambda}(\cdot,s)[h]_s \in B^\lambda(\mathbb{T}_{t_0})$. Therefore, by notting that $(I - H^{-1}QG^{-1}F)^{-1}$ acts continuously from $B^\lambda(\mathbb{T}_{t_0})$ to $B^\lambda(\mathbb{T}_{t_0})$, there exists C_1 such that

$$\begin{aligned}
\sup_{t\geqslant s}\|z(t)\| &= \sup_{t\geqslant s}\left\| e_\lambda(t,s)\big(I - H^{-1}QG^{-1}F\big)^{-1}\big(e_{\ominus\lambda}(\cdot,s)[h]_s(\cdot)\big)(t) \right\| \\
&= \left\| \big(I - H^{-1}QG^{-1}F\big)^{-1}\big(e_{\ominus\lambda}(\cdot,s)[h]_s(\cdot)\big) \right\|_{B^\lambda(\mathbb{T}_{t_0})} \\
&\leqslant C_1 \left\| e_{\ominus\lambda}(\cdot,s)[h]_s(\cdot) \right\|_{B^\lambda(\mathbb{T}_{t_0})} \\
&= C_1 \left\| h \right\|_{B^0(\mathbb{T}_{t_0})} \leqslant C_1L_3\|P(s)z(s)\| := K\|P(s)z(s)\|.
\end{aligned}$$

Thus,

$$\|x(t,s)\| \leqslant Ke_{\ominus\lambda}(t,s)\|P(s)x(s)\|, \;\; t \geqslant s \geqslant t_0.$$

The proof is complete. $\square$

5. Bohl-Perron type theorem

We now pass to the study of the Bohl-Perron's Theorem for implicit dynamic equations with causal operators. That is, we investigate the relation between the exponential stability of homogeneous equation (3.8) and the boundedness of solutions of non homogeneous equation (3.2). We keep Assumption 1 to this section. For any $\beta \geqslant 0$, define the weight space

$$\mathcal{L}_p^\beta(t_0) = \left\{ \begin{array}{l} q \in L_p^{loc}(\mathbb{T}_{t_0};\mathbb{R}^n) : \int_{t_0}^\infty e_\beta^p(t,t_0)\|P_\sigma G^{-1}\mathbb{S}q(t)\|^p\Delta t < \infty \\ \text{and } \int_{t_0}^\infty e_\beta^p(t,t_0)\|TQ_\sigma G^{-1}\mathbb{S}q(t)\|^p\Delta t < \infty \end{array} \right\}$$

with the norm

$$\|q\|_{\mathcal{L}_p^\beta(t_0)} = \Big(\int_{t_0}^\infty e_\beta(t,t_0)\big(\|P_\sigma G^{-1}\mathbb{S}q(t)\|^p + \|TQ_\sigma G^{-1}\mathbb{S}q(t)\|^p\big)\Delta t\Big)^{\frac{1}{p}}.$$

Lemma 5.1. *$\mathcal{L}_p^\beta(t_0)$ is a Banach space.*

Proof. We need only to prove the positive definiteness of the norm $\|.\|_{\mathcal{L}_p^\beta(t_0)}$. By noting $I+H^{-1}TQ_\sigma G^{-1}\Sigma = H^{-1}$, it follows that $\mathbb{S} := I+\Sigma H^{-1}TQ_\sigma G^{-1}$ is invertible by Lemma 2.6. If $q \in \mathcal{L}_p^\beta(t_0)$ with $\|q\|_{\mathcal{L}_p^\beta(t_0)} = 0$ then

$$P_\sigma G^{-1}\mathbb{S}q(t) = 0 \quad \text{and} \quad TQ_\sigma G^{-1}\mathbb{S}q(t) = 0,$$

for almost everywhere $t \in \mathbb{T}_{t_0}$. Since G, T and $\mathbb{S}$ are invertible, $q = 0$. The proof is complete. $\square$

Denote also

$$L_p^\beta(t_0) = \left\{ y \in L_p^{loc}(\mathbb{T}_{t_0};\mathbb{R}^n) : \int_{t_0}^\infty e_\beta^p(t,t_0)\|y(t)\|^p\Delta t < \infty \right\}.$$

Theorem 5.2. *Suppose that $\beta < \alpha$. If the system (3.8) is α-exponentially stable, then for every $f \in \mathcal{L}_p^\beta(t_0)$, the solution of (3.2) with the initial $P(t_0)x(t_0) = 0$ is in $L_p^\beta(t_0)$.*

Proof. Following (3.12), the solution of (3.2) with the initial $P(t_0)x(t_0) = 0$ is presented by

$$x(t) = \int_{t_0}^t \Phi(t,s)P_\sigma G^{-1}\mathbb{S}f(s)\,\Delta s + H^{-1}TQ_\sigma G^{-1}f(t).$$

Further, from the exponential stability of (3.8) we have

$$\begin{aligned}
&\Big\| \int_{t_0}^{\cdot} \Phi(\cdot,s)P_\sigma G^{-1}\mathbb{S}f(s)\,\Delta s\Big\|_{L_p^\beta(t_0)} \\
&\qquad = \Big(\int_{t_0}^\infty e_\beta^p(t,t_0)\Big\|\int_{t_0}^t \Phi(t,s)P_\sigma G^{-1}\mathbb{S}f(s)\,\Delta s\Big\|^p\Delta t\Big)^{\frac{1}{p}} \\
&\qquad = M\left(\int_{t_0}^\infty \Big[e_\beta(t,t_0)\int_{t_0}^t e_{\ominus\alpha}(t,s)\,\big\|P_\sigma G^{-1}\mathbb{S}f(s)\big\|\,\Delta s\Big]^p\Delta t\right)^{\frac{1}{p}}
\end{aligned}$$

$$= M\left(\int_{t_0}^{\infty}\Big[e_{\beta\ominus\alpha}(t,t_0)\int_{t_0}^{t} e_\alpha(s,t_0)\left\|P_\sigma G^{-1}\mathbb{S}f(s)\right\|\,\Delta s\Big]^p\Delta t\right)^{\frac{1}{p}}.$$

By using again Hardy inequality with $U(t)=V(t)=e_{\beta\ominus\alpha}(t,t_0)$ we obtain

$$\left\|\int_{t_0}^{\cdot}\Phi(\cdot,s)PG^{-1}\mathbb{S}f(s)\,\Delta s\right\|_{L_p^\beta(\mathbb{T}_{t_0})}\leqslant\frac{Mp^{\frac{1}{p}}q^{\frac{1}{q}}}{\eta_{\alpha\ominus\beta}}\left(\int_{t_0}^{\infty}e_\beta^p(s,t_0)\left\|P_\sigma G^{-1}\mathbb{S}f(s)\right\|^p\Delta s\right)^{\frac{1}{p}}<\infty.$$

Moreover,

$$\begin{aligned}TQ_\sigma G^{-1}\mathbb{S}&=TQ_\sigma G^{-1}\big(I+\Sigma H^{-1}TQ_\sigma G^{-1}\big)\\&=TQ_\sigma G^{-1}-(I-TQ_\sigma G^{-1}\Sigma)H^{-1}TQ_\sigma G^{-1}+H^{-1}TQ_\sigma G^{-1}\\&=H^{-1}TQ_\sigma G^{-1},\end{aligned}$$

which implies that

$$\left\|H^{-1}TQ_\sigma G^{-1}f\right\|_{L_p^\beta(t_0)}=\Big(\int_{t_0}^{\infty}e_\beta^p(t,t_0)\left\|TQ_\sigma G^{-1}\mathbb{S}f(t)\right\|^p\Delta t\Big)^{\frac{1}{p}}<\infty.$$

Thus, $x\in L_p^\beta(t_0)$. The proof is complete. $\square$

To prove the inverse relation, consider the operator $\mathcal{F}$ defined by

$$\mathcal{F}f(t)=\int_{t_0}^{t}\Phi\big(t,\sigma(\tau)\big)P_\sigma G^{-1}\mathbb{S}f(\tau)\,\Delta\tau+\big(H^{-1}TQ_\sigma G^{-1}f\big)(t),\qquad(5.1)$$

with $t\in\mathbb{T}_{t_0}$ and $f\in\mathcal{L}_p^\beta\big(t_0\big)$. By formula (3.12), it is seen that $\mathcal{F}f(\cdot)$ is the solution of the dynamic equation (3.2) with the initial $P(t_0)x(t_0)=0$.

Lemma 5.3. *Let $\gamma,\beta\geqslant 0$ and suppose that the operator $\mathcal{F}$ acts $\mathcal{L}_1^\gamma(t_0)$ to $L_p^\beta(t_0)$. Then, $\mathcal{F}$ is continuous. This means that*

$$\|\mathcal{F}\|_{\mathcal{L}}:=k<\infty.\qquad(5.2)$$

Proof. By assumption, for any $f\in\mathcal{L}_1^\gamma(t_0)$, the solution $x(t)$ associated to f of (3.2) with the initial condition $P(t_0)x(t_0)=0$ is in $L_p^\beta(t_0)$. We define a family of operators $\{V_t\}_{t\in\mathbb{T}_{t_0}}$ as following:

$$\begin{aligned}V_t:\mathcal{L}_1^\gamma(t_0)&\longrightarrow L_p^\beta(t_0)\\ f\ &\longmapsto V_t(f)=\pi_t\mathcal{F}(f).\end{aligned}$$

From the assumption of Lemma, we have

$$\begin{aligned}\sup_{t\in\mathbb{T}_{t_0}}\|V_tf\|_{L_p^\beta(t_0)}&=\sup_{t\in\mathbb{T}_{t_0}}\Big(\int_{t_0}^{t}e_\beta^p(\tau,t_0)\,\|\mathcal{F}f(\tau)\|^p\,\Delta\tau\Big)^{\frac{1}{p}}\\&=\Big(\int_{t_0}^{\infty}e_\beta^p(t,t_0)\,\|\mathcal{F}f(t)\|^p\,\Delta t\Big)^{\frac{1}{p}}<\infty,\end{aligned}$$

for any $f\in\mathcal{L}_1^\gamma(t_0)$. Using Uniform Boundedness Principle gets

$$\|\mathcal{F}\|_{\mathcal{L}}=\sup_{t\in\mathbb{T}_{t_0}}\|V_t\|:=k<\infty.\qquad(5.3)$$

The proof Lemma 5.3 is complete. □

Theorem 5.4. *Suppose that for $\gamma < \beta$, the following assumptions hold*
1. The unique solution of the Cauchy problem (3.2) *with the initial condition $P(t_0)x(t_0) = 0$, associated with every $f \in \mathcal{L}_1^{\gamma}(t_0)$ is in $L_p^{\beta}(t_0)$.*
2. The operator $QH^{-1}\widehat{P}$ acts continuously on $B^{\gamma}(\mathbb{T}_{t_0}, \mathbb{R}^n)$.
3. The operator $P_\sigma G^{-1}\Sigma$, acting $L_p^{loc}(\mathbb{T}_{t_0})$ to $L_p^{loc}(\mathbb{T}_{t_0})$ and there exists a positive number K_1 such that for all $z \in L_p^{loc}(\mathbb{T}_{t_0})$ and $t \in \mathbb{T}_0$,

$$a.\quad \left\| e_{\ominus\varepsilon}(\cdot, t_0) P_\sigma G^{-1}\Sigma H^{-1}\widehat{P} z(\cdot)\right\|_{L_p^{\beta}[t_0,t]} \leqslant K_1 \left\| z(\cdot)\right\|_{L_p^{\beta}[t_0,t]}, \tag{5.4}$$

$$b.\quad \operatorname*{esssup}_{t\in\mathbb{T}_{t_0}} e_{\ominus\varepsilon}(t, t_0) \left\| \big(P^{\Delta} + P_\sigma G^{-1}\overline{B}\big)(t)\right\| \leqslant K_1, \tag{5.5}$$

where $\beta \ominus \gamma \ominus \varepsilon < 0$.

Then, the index-1 IDE (3.8) *is exponentially stable.*

Proof. First, we know that $\Phi_0(t, s), t \geqslant s \geqslant t_0$ is the solution of the inherit dynamic equation (3.10). Then, it is rd–continuous in (t, s). Let $s \geqslant t_0$, for any $h > s$ and $v \in \mathbb{R}^n$, put

$$f(t) = e_{\ominus\gamma}(t, t_0) A_\sigma(t)\mathbf{1}_{[s,h]}(t) P_\sigma(s)v,\ \ t \in \mathbb{T}_{t_0}.$$

It is clear that $\mathbb{S}f = f$, $Q_\sigma G^{-1}\mathbb{S}f = 0$ and

$$\begin{aligned}\|f\|_{\mathcal{L}_1^{\gamma}(t_0)} &= \int_{t_0}^{\infty} e_\gamma(t, t_0)\left\| \big[e_{\ominus\gamma}(\cdot, t_0) P_\sigma G^{-1}\mathbb{S} A_\sigma(\cdot)\mathbf{1}_{[s,h]}(\cdot) P_\sigma(s)v\big](t)\right\|\Delta t\\ &= \int_s^h \left\|P_\sigma(t)P_\sigma(s)v\right\|\Delta t \leqslant (h-s)K_0\left\|P_\sigma(s)v\right\|_{\mathbb{R}^n}.\end{aligned}$$

This means that $f \in \mathcal{L}_1^{\gamma}(t_0)$. Let $x(t)$ be the solution of (3.2) associated to f with the initial condition $P(t_0)x(t_0) = 0$ and $\overline{u}(t) = P(t)x(t), t \in \mathbb{T}_{t_0}$. Since $f(t) = 0$ for $t_0 \leqslant t \leqslant s$, it follows from (3.13) that

$$\overline{u}(t) = \int_s^t \Phi_0(t, \sigma(\tau))P_\sigma G^{-1}\mathbb{S}f(\tau)\Delta\tau = P(t)\mathcal{F}f(t),$$

for all $t \geqslant s$ and $\overline{u}(t) = 0$ for $t_0 \leqslant t \leqslant s$. Hence,

$$\|\overline{u}(\cdot)\|_{L_p^{\beta}(t_0)} = \|P(\cdot)\mathcal{F}f(\cdot)\|_{L_p^{\beta}(t_0)} \leqslant kK_0\|f(\cdot)\|_{\mathcal{L}_1^{\gamma}(t_0)} \leqslant (h-s)kK_0^2\left\|P_\sigma(s)v\right\|_{\mathbb{R}^n}.$$

On the other hand, by noting that

$$\overline{u}(t) = \int_s^{h\wedge t} e_{\ominus\gamma}(\tau, t_0)\Phi_0(t, \sigma(\tau))P_\sigma(\tau)P_\sigma(s)v\Delta\tau$$

we come to

$$\begin{aligned}&\left(\int_s^{\infty} e_\beta^p(t, t_0)\left\|\int_s^{h\wedge t} e_{\ominus\gamma}(\tau, t_0)\Phi_0(t, \sigma(\tau))P_\sigma(\tau)P_\sigma(s)v\Delta\tau\right\|^p\Delta t\right)^{\frac{1}{p}}\\ &\qquad\leqslant k(h-s)K_0^2\left\|P_\sigma(s)v\right\|.\end{aligned}$$

Dividing both sides of this inequality by $h-s$ and letting $h\to\sigma(s)$ obtain

$$\left(\int_s^\infty e_\beta^p(t,t_0)\left\|\Phi_0(t,\sigma(s))P_\sigma(s)v\right\|^p\Delta t\right)^{\frac{1}{p}}\leqslant kK_0^2e_\gamma(s,t_0)\|P_\sigma(s)v\|,$$

for arbitrary $s\geqslant t_0$. Hence, for arbitrary $s>t_0$,

$$\left(\int_s^\infty e_\beta^p(t,t_0)\left\|\Phi_0(t,s)P(s)v\right\|^p\Delta t\right)^{\frac{1}{p}}\leqslant kK_0^2e_\gamma(s,t_0)\|P(s)v\|. \tag{5.6}$$

Let $s>t_0$, denote by $y(\cdot)$ the solution of the equation (3.8) with the initial condition $P(s)(y(s)-v)=0$. Put $u=Py$ then $y(t)=\Phi(t,s)P(s)v$ and $u(\cdot)=\Phi_0(\cdot,s)P(s)v$ for $t\geqslant s$. We have the following estimates

+ From (5.6)

$$\begin{aligned}\|[u]_s\|_{L_p^\beta(t_0)}&=\|[\Phi_0(\cdot,s)P(s)v]_s\|_{L_p^\beta(t_0)}=\left(\int_s^\infty e_\beta^p(t,t_0)\left\|\Phi_0(t,s)P(s)v\right\|^p\Delta t\right)^{\frac{1}{p}}\\&\leqslant kK_0^2e_\gamma(s,t_0)\left\|P(s)v\right\|.\end{aligned} \tag{5.7}$$

+ By virtue of (3.6)

$$\left\|e_{\ominus\varepsilon}(\cdot,t_0)[u]_s^\Delta\right\|_{L_p^\beta(t_0)}=\left\|e_{\ominus\varepsilon}(\cdot,t_0)\Big[\big(P^\Delta+P_\sigma G^{-1}\overline{B}\big)[u]_s+P_\sigma G^{-1}\Sigma H^{-1}\widehat{P}[u]_s\Big]\right\|_{L_p^\beta(t_0)}.$$

Further,

$$\begin{aligned}&\left\|e_{\ominus\varepsilon}(\cdot,t_0)\big(P^\Delta+P_\sigma G^{-1}\overline{B}\big)[u]_s\right\|_{L_p^\beta(t_0)}\\&\quad=\left(\int_s^\infty e_\beta^p(t,t_0)\left\|e_{\ominus\varepsilon}(\tau,t_0)\big(P^\Delta+P_\sigma G^{-1}\overline{B}\big)u(\tau)\right\|^p\Delta\tau\right)^{\frac{1}{p}}\\&\quad\leqslant K_1\left(\int_s^\infty e_\beta^p(\tau,t_0)\left\|u(\tau)\right\|^p d\tau\right)^{\frac{1}{p}}\leqslant kK_0^2K_1e_\gamma(s,t_0)\|P(s)v\|,\end{aligned}$$

and

$$\left\|(e_{\ominus\varepsilon}(\cdot,t_0)P_\sigma G^{-1}\Sigma H^{-1}\widehat{P}[u]_s\right\|_{L_p^\beta(t_0)}\leqslant K_1\left\|[u(\cdot)]_s\right\|_{L_p^\beta(t_0)}$$

$$=K_1\left(\int_s^\infty\left\|e_\beta^p(\tau,s)\Phi_0(\tau,s)P(s)v(\tau)\right\|^p\Delta\tau\right)^{\frac{1}{p}}\leqslant kK_0^2K_1e_\gamma(s,t_0)\left\|P(s)v\right\|.$$

+ Thus,

$$\left\|e_{\ominus\varepsilon}(\cdot,t_0)[u]_s^\Delta\right\|_{L_p^\beta(t_0)}\leqslant K_2e_\gamma(s,t_0)\left\|P(s)v\right\|, \tag{5.8}$$

where $K_2=2kK_0^2K_1$.

On the other hand,

$$\left\|u^\Delta(t)\right\|=\left\|\lim_{s\to t}\frac{u(s)-u(\sigma(t))}{s-\sigma(t)}\right\|\geqslant\left|\lim_{s\to t}\frac{\|u(s)\|-\|u(\sigma(t))\|}{s-\sigma(t)}\right|=\left|\|u(t)\|^\Delta\right|.$$

By direct calculation, it yields

$$e_\gamma(t,t_0)\left\|u(t)\right\|=e_\gamma(s,t_0)\left\|u(s)\right\|+\int_s^t\big(e_\gamma(\tau,t_0)\left\|u(\tau)\right\|\big)^\Delta\Delta\tau$$

$$
\begin{aligned}
=&e_\gamma(s,t_0)\left\|u(s)\right\| + \int_s^t \Big(\gamma e_\gamma(\tau,t_0)\left\|u(\tau)\right\| + \int_s^t e_\gamma(\sigma(\tau),t_0)\left\|u(\tau)\right\|^\Delta\Big)\Delta\tau\\
\leqslant& e_\gamma(s,t_0)\left\|u(s)\right\| + \gamma\int_s^t e_\gamma(\tau,t_0)\left\|u(\tau)\right\|\Delta\tau + (1+\gamma\mu^*)\int_s^t e_\gamma(\tau,t_0)\left\|u^\Delta(\tau)\right\|\Delta\tau\\
=&e_\gamma(s,t_0)\left\|u(s)\right\| + \gamma\int_s^t e_{\gamma\ominus\beta}(\tau,t_0)\left\|e_\beta(\tau,t_0)u(\tau)\right\|\Delta\tau\\
&+(1+\gamma\mu^*)\int_s^t e_{\gamma\ominus\beta\oplus\varepsilon}(\tau,t_0)e_\beta(\tau,t_0)e_{\ominus\varepsilon}(\tau,t_0)\left\|u^\Delta(\tau)\right\|\Delta\tau\\
\leqslant& e_\gamma(s,t_0)\left\|u(s)\right\| + \gamma\left(\int_s^\infty e^q_{\gamma\ominus\beta}(\tau,t_0)\Delta\tau\right)^{\frac1q}\left(\int_s^\infty \left\|e_\beta(\tau,t_0)u(\tau)\right\|^p\Delta\tau\right)^{\frac1p}\\
&+(1+\gamma\mu^*)\left(\int_s^\infty e^q_{\gamma\ominus\beta\oplus\varepsilon}(\tau,t_0)\Delta\tau\right)^{\frac1q}\left(\int_s^\infty e^p_\beta(\tau,t_0)\left\|e_{\ominus\varepsilon}(\tau,t_0)u^\Delta(\tau)\right\|^p\Delta\tau\right)^{\frac1p}.
\end{aligned}
$$

Combining with (2.6), (5.7) and (5.8), we can find $N_1, N_2 > 0$, such that

$$
\begin{aligned}
e_\gamma(t,t_0)\left\|u(t)\right\| &\leqslant \Big(1+\gamma N_1 kK_0^2 + (1+\gamma\mu^*)N_2K_2\Big)e_\gamma(s,t_0)\left\|P(s)v\right\|\\
&= K_3 e_\gamma(s,t_0)\left\|u(s)\right\|,
\end{aligned}
$$

where $K_3 = 1 + k\gamma N_1 K_0^2 + (1+\gamma\mu^*)N_2K_2$. Thus

$$\left\|u(t)\right\| \leqslant K_3 e_{\ominus\gamma}(t,s)\left\|u(s)\right\|, \text{ for all } t\geqslant s. \tag{5.9}$$

On the other hand, by assumption, the operator $QH^{-1}\widehat{P}$ acts continuously on $B^\gamma(\mathbb{T}_{t_0})$. Then, with $v(t) = Qy(t) = QH^{-1}\widehat{P}[u(\cdot)]_s(t)$ we have

$$\sup_{t\geqslant s}\left\|e_\gamma(t,s)v(t)\right\| \leqslant \big\|QH^{-1}\widehat{P}\big\|\sup_{t\geqslant s}\left\|e_\gamma(t,s)u(t)\right\| \leqslant K_3\big\|QH^{-1}\widehat{P}\big\|\left\|u(s)\right\|.$$

Combining with (5.9) obtains

$$\left\|y(t)\right\| \leqslant K_4 e_{\ominus\gamma}(t,s)\left\|P(s)v\right\|, \text{ for all } t\geqslant s,$$

where $K_4 = K_3\big(\|QH^{-1}\widehat{P}\| + 1\big)$. The proof is complete. $\square$

Remark 5.5. When $\mathbb{T} = \mathbb{R}$ and $\Sigma x(t) = \int_0^t H(t,s)x(s)ds$ is an integral operator, we see that Theorem 5.4 is generalized from Theorem 4.2 in [16].

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References

[1] T. Berger, Robustness of stability of time-varying index-1 DAEs, *Math. Control Signals Syst.*, **26**(2014), 403-433.

[2] E. Braverman; I. M. Karabash, Bohl-Perron-type stability theorems for linear difference equations with infinite delay, *J. Difference Equ. Appl.* **18** (2012), no. 5, 909-939.
[3] M. Bohner and A. Peterson, *Dynamic equations on time scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
[4] M. Bohner and G. Sh. Guseinov, Double integral calculus of variations on time scales. *Computers & Mathematics with Applications*, **54**(2007), Issue 1, 45-57.
[5] M. Bracke, *On stability radii of parametrized linear differential-algebraic systems*. Ph.D thesis, University of Kaiserslautern, 2000.
[6] C. J. Chyan; N.H. Du and V.H. Linh, On data dependence of exponential stability and stability radii for linear time-varying differential-algebraic systems, *Journal of Differential Equations*, **245**(2008), 2078-2102.
[7] C. Corduneanu, *Functional equations with causal operators*. Taylor & Francis e-Library, 2005.
[8] J.J. DaCunha, Stability for time varying linear dynamic systems on time scales, *Journal of Computational and Applied Mathematics*. **176** (2005), 381-410.
[9] L. Yu, Daleckii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*. Amer. Math. Soc., Providence, RI 1971.
[10] N. H. Du and V. H. Linh, Stability radii for linear time-invarying differential-equations with respect to dynamic perturbations, *J. Differential Equations*, **230**(2006), 579-599.
[11] N.H. Du; N.T. Dieu and L.A. Tuan, Stability of stochastic dynamic equations with time-varying delay on time scales, *Stoch. Anal. Appl.* **38** (2020), no. 5, 909-928.
[12] D.D. Thuan; K.C. Nguyen; N.T. Ha and P.V. Quoc, On stability, Bohl exponent and BohlPerron theorem for implicit dynamic equations, *International Journal of Control*, **94**(2011), no. 12, 3520-3532.
[13] T. S. Doan; A. Kalauch and S. Siegmund, Exponential stability of linear time-invariant systems on time scales, *Nonlinear Dyn. Syst. Theory* **9** (2009), no. 1, 37-50.
[14] A. Filatov and L. Sarova, *Integralnye neravenstva i teorija nelineinyh kolebanii*. Moskva, 1976.
[15] N.T. Ha; N.H. Du and D.D. Thuan, On data dependence of stability domains, exponential stability and stability radii for implicit linear dynamic equations, *Math. Control Signals Sys.*, **28(2)**(2016), 1-28.
[16] N.T. Ha, On the robust stability of Volterra differential-algebraic equations, *Systems Control Lett.*, **149**(2021), paper No. 104883, 11 pp.
[17] S. Hilger, *Analysis on measure chains - a unified approach to continuous and discrete calculus*, Results. Math., **18**(1990), 18-56.
[18] B. Jacob, A formula for the stability radius of time-varying systems, *J. Differential Equations*, **142**(1998), 167-187.
[19] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations, Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.
[20] D.G. Luenberger, Control of linear dynamic market systems, *J. Eco. Dyn. Control*, **10**(1986), 339-351.
[21] M. Bohner and D.A. Lutz, Asymptotic expansions and analytic dynamic equations, *Z. Angew. Math. Mech.*, **86**(2006), no. 1, 37-45.
[22] R. März, "Extra-ordinary differential equation attempts to an analysis of differential algebraic system", *Progress in Mathematics*, **168**(1998), 313-334.
[23] V. Mehrmann and D.D. Thuan, Stability analysis of implicit difference equations under restricted perturbations, *SIAM J. Matrix Anal. Appl.*, **36**(2015), 178-202.
[24] B.G. Pachpatte, *Mathematical Inequalities*, vol. 67, North-Holland Mathematical Library 2005.
[25] O. Perron, Die Stabilitatsfrage bei Differentialgleichungen, *Math. Z.*, 32(1930), 703-728.
[26] Zhu Zhi-Qiang and Wang Qi-Ru, Stability and periodicity of solutions for delay dynamic systems on time scales, *Electron. J. Differential Equations*, **100**(2014), 11 pp.

Department of Science, Electric Power University, 235 Hoang Quoc Viet Str., Hanoi, Vietnam.

E-mail address: thuha@epu.edu.vn