Sparse-grid sampling recovery and numerical integration of functions having mixed smoothness

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Abstract

We give a short survey of recent results on sparse-grid linear algorithms of approximate recovery and integration of functions possessing a unweighted or weighted Sobolev mixed smoothness based on their sampled values at a certain finite set. Some of them are extended to more general cases.

Keywords and Phrases: Sampling recovery; sampling widths; Numerical weighted integration; Quadrature; Unweighted and weighted Sobolev spaces of mixed smoothness; Sparse grids; Hyperbolic crosses in the function domain; Asymptotic order.

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Contents

1 Introduction

In recent decades, there has been increasing interest in solving approximation and numerical problems that involve functions depending on a large number d of variables. Without further assumptions the computation time typically grows exponentially in d, and the problems become intractable even for mild dimensions d. This is the so-called curse of conditionality coined by Bellman [2]. In sampling recovery and numerical integration, a classical model in attempt to overcome it which has been widely studied, is to impose certain mixed smoothness or more general anisotropic smoothness conditions on the function to be approximated, and to employ sparse grids for construction of approximation algorithms for sampling recovery or numerical integration.

In this survey paper, we discuss sparse-grid linear algorithms of approximate reconstruction and integration of functions possessing a Sobolev mixed smoothness, based on their sampling values at a certain finite set. The problem of sampling recovery is treated in both unweighted setting for functions defined on a compact subset in $\mathbb{R}^d$, and weighted setting for functions defined on $\mathbb{R}^d$. The optimality of sampling recovery is considered in terms of linear sampling widths. The problem of numerical integration and optimal quadrature is treated in weighted setting for functions defined on the whole $\mathbb{R}^d$. We focus our attention on linear sampling algorithms and quadratures based on sparse grids. Some known results are extended to more general cases. We do not consider the problem of dependence on dimension and postpone discussion on this problem to other surveys.

We begin with some notions related to linear sampling recovery and numerical integration of functions based on their values. Let Ω be a domain in $\mathbb{R}^d$, X a normed space of functions on Ω and $f \in X$. Given $\boldsymbol{X}_k = \{\boldsymbol{x}_i\}_{i=1}^k \subset \Omega$ and a collection $\boldsymbol{\Phi}_k := \{\varphi_i\}_{i=1}^k$ of k functions in X, to approximately recover f from the sampled values $\{f(\boldsymbol{x}_i)\}_{i=1}^k$ we can use a linear sampling algorithm defined by

$$S_k(f) := S_k(\boldsymbol{X}_k, \boldsymbol{\Phi}_k, f) := \sum_{i=1}^k f(\boldsymbol{x}_i)\varphi_i \tag{1.1}$$

For convenience, we assume that some points from $\boldsymbol{X}_k$ and some functions from $\boldsymbol{\Phi}_k$ may coincide.

If $\boldsymbol{W}$ is a compact set in X, for $n \in \mathbb{N}$ we define the linear sampling n-width of the set $\boldsymbol{W}$ in X as

$$\varrho_n(\boldsymbol{W}, X) := \inf_{\boldsymbol{X}_k, \boldsymbol{\Phi}_k, \ k \le n} \ \sup_{f \in \boldsymbol{W}} \|f - S_k(\boldsymbol{X}_k, \boldsymbol{\Phi}_k, f)\|_X.$$

This quantity characterizes the optimal sampling recovery for the set $\boldsymbol{W}$ by linear algorithms from n sampled values. Some different concepts of optimal sampling recovery are discussed in Subsections 2.3 and 2.5.

Let v be a nonnegative Lebesgue measurable function on Ω. Denote by μ_v the measure on Ω defined via the density function v. We are interested in numerical approximation of weighted integrals

$$\int_\Omega f(\boldsymbol{x})\mu_v(\mathrm{d}\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{1.2}$$

Given $\boldsymbol{X}_k = \{\boldsymbol{x}_i\}_{i=1}^k \subset \Omega$ and a collection $\boldsymbol{\Lambda}_k := \{\lambda_i\}_{i=1}^k$ of integration weights, to approximate them we use quadratures of the form

$$Q_k(f) := \sum_{i=1}^k \lambda_i f(\boldsymbol{x}_i) \tag{1.3}$$

with the convention $Q_0(f) := 0$. For convenience, we assume that some of the integration nodes may coincide.

Let $\boldsymbol{W}$ be a set of continuous functions on Ω. Denote by $\mathcal{Q}_n$ the family of all quadratures Q_k of the form (1.3) with $k \le n$. The optimality of quadratures from $\mathcal{Q}_n$ for $f \in \boldsymbol{W}$ is characterized by the quantity

$$\mathrm{Int}_n(\boldsymbol{W}) := \inf_{Q_n \in \mathcal{Q}_n} \ \sup_{f \in \boldsymbol{W}} \left| \int_\Omega f(\boldsymbol{x})\, \mu_v(\mathrm{d}\boldsymbol{x}) - Q_n(f) \right|. \tag{1.4}$$

For sampling recovery and numerical integration problems considered in the present paper, the set $\boldsymbol{W}$ is usually the unit ball of a normed space W of functions on Ω, which is also called function class.

The unweighted sampling recovery of functions on a compact set having a mixed smoothness, is a classical topic in multivariate approximation. There is a large number of works devoted to this topic. However, here are still many open problems on the right asymptotic order of linear sampling n-widths and asymptotically optimal sampling algorithms of sampling recovery. We refer the reader to the book [17, Section 5] for a detailed survey and bibliography on this research direction. The present paper updates it with some recent results mainly on linear sampling constructive algorithms, in particular, on Smolyak sparse grids for periodic functions having a mixed Sobolev smoothness. We also discuss the right asymptotic order of linear sampling n-widths of the unit ball of unweighted Sobolev spaces (unweighted Sobolev function class) of a mixed smoothness, and asymptotic optimality of sampling algorithms in terms of sampling n-widths.

Concerning weighted sampling recovery, for the Gaussian-weighted Sobolev space $W^\alpha_p(\mathbb{R}^d, \gamma)$ of mixed smoothness $r \in \mathbb{N}$ for $1 < p < \infty$, in [16], we proved the asymptotic order of sampling n-widths in the Gaussian-weighted space $L_q(\mathbb{R}^d, \gamma)$ of the unit ball of $W^r_p(\mathbb{R}^d, \gamma)$ (Gaussian-weighted Sobolev function class) for $1 \le q < p < \infty$ and $q = p = 2$, and propose a novel method for constructing asymptotically optimal linear sampling algorithms. In [16], we investigated the numerical approximation of integrals over $\mathbb{R}^d$ equipped with the standard Gaussian measure γ for integrands belonging to the Gaussian-weighted Sobolev spaces $W^r_p(\mathbb{R}^d, \gamma)$ of mixed smoothness $r \in \mathbb{N}$ for $1 < p < \infty$. We proved the the right asymptotic order of the quantity of optimal quadrature and proposed a novel method for constructing asymptotically optimal quadratures. In the present paper, we extend these results on sampling recovery and numerical integration to weighted Sobolev spaces $W^r_p(\mathbb{R}^d, \mu)$, where μ is the measure having the tensor product of Freud-type weights as the density function.

In [15], we studied the numerical approximation of weighted integrals over $\mathbb{R}^d$ for integrands from weighted Sobolev spaces of mixed smoothness $W^r_1(\mathbb{R}^d, \mu)$, where μ is the measure with the density function as the tensor product of a Freud-type weight. We proved upper and lower bounds of the asymptotic order of optimal quadrature for functions from these spaces. In the one-dimensional case ($d = 1$), we obtained the right asymptotic order the quantity of optimal quadrature. For $d \ge 2$, the upper bound is performed by sparse-grid quadratures with integration nodes on step hyperbolic crosses in the function domain $\mathbb{R}^d$. We will give a brief description of the results and their proofs from [15].

We briefly describe the structure and content of the present paper.

In Section 2, we consider the problem of sampling recovery of functions from a certain set by linear algorithms using its values at a certain finite set of points, and their asymptotic optimality in terms of linear sampling n-widths. We focus our attention on functions mainly from unweighted periodic Sobolev spaces $W^r_p(\mathbb{T}^d)$ of a mixed smoothness, and lessly from unweighted periodic Hölder-Nikol'skii spaces $H^r_p(\mathbb{T}^d)$ of a mixed smoothness. As a related problem we also treat linear approximations of functions from these spaces. In Subsection 2.1, we introduce notions of Kolmogorov and linear n-widths – the well-known characterizations of optimal linear approximation; definitions of Sobolev spaces $W^r_p(\mathbb{T}^d)$ and Besov spaces $B^r_{p,\theta}(\mathbb{T}^d)$ of periodic functions having a mixed smoothness. Subsection 2.2 presents Littlewood-Paley-type theorems on B-spline quasi-interpolation sampling representation for spaces $W^r_p(\mathbb{T}^d)$ and $B^r_{p,\theta}(\mathbb{T}^d)$ which play a central role in linear sampling recovery by sparse-grid Smolyak algorithms of functions from these spaces.

Subsection 2.3 is devoted to the sampling recovery by Smolyak sparse-grid algorithms for functions from unweighted periodic Sobolev spaces $W^r_p(\mathbb{T}^d)$ and Besov spaces $B^r_{p,\theta}(\mathbb{T}^d)$ of a mixed smoothness, and their asymptotic optimality in terms of Smolyak and linear sampling n-widths. In Subsection 2.4, we present some recent results of [26] on inequality between the linear sampling and Kolmogorov n-widths and as consequences some new results on right asymptotic order of linear sampling n-widths for the unweighted Sobolev function class $\boldsymbol{W}^r_p(\mathbb{T}^d)$ and Hölder-Nikol'skii function class $\boldsymbol{H}^r_p(\mathbb{T}^d)$ Subsection 2.5 discusses some different concepts of optimality in sampling recovery and their relations for unweighted Sobolev spaces of a mixed smoothness.

In Section 3, we consider the problems of linear sampling recovery and approximation of functions in weighted Sobolev spaces. The optimality of linear sampling recovery and approximation is treated in terms of linear sampling, linear and Kolmogorov n-widths. In Subsection 3.1, we introduce weighted Sobolev spaces of a mixed smoothness, and recall previous results of [16] on right asymptotic order of sampling, Kolmogorov and linear n-widths of the Gaussian-weighted Sobolev function classes $\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma)$ of a mixed smoothness in the Gaussian-weighted space $L_q(\mathbb{R}^d;\gamma)$. In Subsection 3.2, we extend these results to the weighted Sobolev function class $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ and weighted space $L_q(\mathbb{R}^d;\mu)$ for the measure μ associated with a Freud-type weight as the density function for the case $1 \le q < p < \infty$.

In Section 4, we present and extend some recent results of [16] on numerical weighted integration over $\mathbb{R}^d$ for functions from weighted Sobolev spaces $W^r_p(\mathbb{R}^d;\mu)$ of mixed smoothness $r \in \mathbb{N}$ for $1 < p < \infty$ and the measure μ associated with a Freud-type weight as the density function. In Subsection 4.1, we recall the previous results from [16] on numerical weighted integration over $\mathbb{R}^d$ for functions from Gaussian-weighted Sobolev spaces $W^r_p(\mathbb{R}^d;\gamma)$ of mixed smoothness $r \in \mathbb{N}$ for $1 < p < \infty$. In Subsection 4.2, based on a quadrature on the d-cube for numerical unweighted integration of functions from classical Sobolev spaces of mixed smoothness r, by assembling we construct a quadrature on $\mathbb{R}^d$ for numerical integration of functions from weighted Sobolev spaces $W^r_p(\mathbb{R}^d;\mu)$ which preserves the convergence rate. In Subsection 4.3, we prove the right asymptotic order of optimal quadrature for the weighted Sobolev function class $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ for $1 < p < \infty$.

In Section 5, we present some recent results of [15] on numerical weighted integration over $\mathbb{R}^d$ for functions from weighted Sobolev spaces $W^r_1(\mathbb{R}^d;\mu)$ of mixed smoothness $r \in \mathbb{N}$, in particular, upper and lower bounds of the quantity of optimal quadrature $\mathrm{Int}_n\big(\boldsymbol{W}^r_1(\mathbb{R}^d;\mu)\big)$ for the measure μ associated with Freud-type weights as the density function. In Subsection 5.1, we briefly describe these results and then give comments on related works. In Subsection 5.2, for one-dimensional numerical integration, we present the right asymptotic order of the quantity of optimal quadrature $\mathrm{Int}_n\big(\boldsymbol{W}^r_1(\mathbb{R};\mu)\big)$ and its shortened proof from [15]. In Subsection 5.3, for multivariate numerical integration, we present some results and their shortened proofs from [15] on upper and lower bounds of the quantity of optimal quadrature $\mathrm{Int}_n\big(\boldsymbol{W}^r_1(\mathbb{R}^d;\mu)\big)$, and a construction of quadratures based on step-hyperbolic-cross grids of integration nodes which gives the upper bounds. In Subsection 5.4, we extend the results of the previous subsection to Markov-Sonin weights.

Notation. Denote $\mathbf{1} := (1,...,1) \in \mathbb{R}^d$; for $\boldsymbol{x} \in \mathbb{R}^d$, $\boldsymbol{x} =: (x_1,...,x_d)$; $|\boldsymbol{x}|_p :=$

$\left(\sum_{j=1}^d |x_j|^p\right)^{1/p}$ $(1 \le p < \infty)$ and $|\boldsymbol{x}|_\infty := \max_{1\le j\le d} |x_j|$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, the inequality $\boldsymbol{x} \le \boldsymbol{y}$ means $x_i \le y_i$ for every $i = 1, ..., d$. For $x \in \mathbb{R}$, denote $\mathrm{sign(x)} := 1$ if $x \ge 0$, and $\mathrm{sign(x)} := -1$ if $x < 0$. We use letters C and K to denote general positive constants which may take different values. For the quantities $A_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$, we write $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant $C > 0$ such that $A_n(f, \boldsymbol{k}) \le C B_n(f, \boldsymbol{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ (the notation $A_n(f, \boldsymbol{k}) \gg B_n(f, \boldsymbol{k})$ has the obvious opposite meaning), and $A_n(f, \boldsymbol{k}) \asymp B_n(f, \boldsymbol{k})$ if $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k}) \ll A_n(f, \boldsymbol{k})$. Denote by $|G|$ the cardinality of the set G. For a normed space X, denote by the boldface $\boldsymbol{X}$ the unit ball in X.

2 Unweighted sampling recovery

2.1 Introducing remarks

In this section, we consider one of basic problems in approximation theory. Namely, we are interested in sampling recovery, i.e., approximate reconstruction of a function by sparse-grid linear algorithms using its values at a certain finite set of points. We mainly focus our attention on functions from unweighted Sobolev spaces of a mixed smoothness. It is related to the well-known linear and Kolmogorov n-widths in linear approximation. Let us recall them.

Let $n \in \mathbb{N}$ and let X be a normed space and $\boldsymbol{W}$ a central symmetric compact set in X. Then the Kolmogorov n-width of $\boldsymbol{W}$ is defined by

$$d_n(\boldsymbol{W}, X) = \inf_{L_n} \sup_{f\in\boldsymbol{W}} \inf_{g\in L_n} \|f - g\|_X,$$

where the left-most infimum is taken over all subspaces L_n of dimension at most n in X. The linear n-width of the set F which is defined by

$$\lambda_n(\boldsymbol{W}, X) := \inf_{A_n} \sup_{f\in\boldsymbol{W}} \|f - A_n(f)\|_X,$$

where the infimum is taken over all linear operators A_n in X with rank $A_n \le n$.

The concepts of Kolmogorov n-widths and linear n-widths are related to linear approximation. Namely, $d_n(\boldsymbol{W}, X)$ characterizes the optimal approximation of elements from X by linear subspaces of dimension at most n, and $\lambda_n(\boldsymbol{W}, X)$ by linear methods of rank at most n.

In the present paper, we pay our attention mainly to the problem of linear sampling recovery and its optimality in terms of linear sampling n-widths for Sobolev classes of a mixed smoothness. However, it is useful and convenient to consider this problem together with the problem of linear and Kolmogorov n-widths.

Obviously, we have the inequalities

$$d_n(\boldsymbol{W}, X) \le \lambda_n(\boldsymbol{W}, X) \le \varrho_n(\boldsymbol{W}, X). \tag{2.1}$$

Notice that if X is a Hilbert space, then

$$\lambda_n(\boldsymbol{W}, X) = d_n(\boldsymbol{W}, X).$$

We are interested in linear sampling recovery and approximation of functions having a mixed smoothness on $\mathbb{R}^d$ which are 1-periodic at each variable. It is convenient to consider them as functions defined in the d-torus $\mathbb{T}^d = [0,1]^d$ which is defined as the Castrian product of d copies of the interval $[0,1]$ with the identification of the end points. To avoid confusion, we use the notation $\mathbb{I}^d$ to denote the standard unit d-cube $[0,1]^d$. Notice that all the results presented in this section for functions on $\mathbb{T}^d$ also hold true for functions on $\mathbb{I}^d$ in appropriate forms.

Let us give a notion of periodic Sobolev space of mixed smoothness. We define the univariate Bernoulli kernel

$$F_r(x) := \ 1 + 2\sum_{k=1}^\infty k^{-r} \cos(\pi k x - r\pi/2), \quad x \in \mathbb{T},$$

and the multivariate Bernoulli kernels as the corresponding tensor products

$$F_r(\boldsymbol{x}) := \ \prod_{j=1}^d F_r(x_j), \quad \boldsymbol{x} \in \mathbb{T}^d. \tag{2.2}$$

Let $r > 0$ and $1 \le p \le \infty$. Denote by $L_p(\mathbb{T}^d)$ the normed space of functions on $\mathbb{T}^d$ with the pth integral norm $\|\cdot\|_p$ for $1 \le p < \infty$, and the ess sup-norm $\|\cdot\|_p$ for $p = \infty$, where we make use of the abbreviation $\|\cdot\|_p := \|\cdot\|_{L_p(\mathbb{T}^d)}$. If $r > 0$ and $1 \le p \le \infty$, we define the Sobolev space $W^r_p(\mathbb{T}^d)$ of mixed smoothness r by

$$W^r_p(\mathbb{T}^d) := \ \Big\{f \in L_p(\mathbb{T}^d) : \ f \ = \ F_r * \varphi := \int_{\mathbb{T}^d} F_r(\boldsymbol{x} - \boldsymbol{y})\varphi(\boldsymbol{y}) d\boldsymbol{y}, \quad \|\varphi\|_p < \infty\Big\}, \tag{2.3}$$

and the norm in this space by

$$\|f\|_{W^r_p(\mathbb{T}^d)} := \|\varphi\|_p$$

for f represented as in (2.3). For $1 < p < \infty$, the space $W^r_p(\mathbb{T}^d)$ coincides with the set of all $f \in L_p(\mathbb{T}^d)$ such that the norm

$$\Big\|\sum_{\boldsymbol{s}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{s})(1+|s_1|^2)^{r/2} \, \dots \, (1+|s_d|^2)^{r/2} \, e^{\pi i(\boldsymbol{s},\cdot)}\Big\|_p$$

is finite, where $\hat{f}(\boldsymbol{s})$ denotes the usual $\boldsymbol{s}$th Fourier coefficient of f.

We next give a definition of Besov spaces of mixed smoothness r. For univariate functions f on $\mathbb{T}$ the ℓth difference operator Δ^ℓ_h is defined by

$$\Delta^\ell_h(f, x) := \ \sum_{j=0}^\ell (-1)^{\ell-j}\binom{\ell}{j} f(x + jh). \tag{2.4}$$

Denote by $[d]$ the set of all natural numbers from 1 to d. If u is any subset of $[d]$, for multivariate functions on $\mathbb{T}^d$ the mixed (ℓ, u)th difference operator $\Delta_{\boldsymbol{h}}^{\ell,u}$ is defined by

$$\Delta_{\boldsymbol{h}}^{\ell,u} := \prod_{i \in u} \Delta_{h_i}^{\ell}, \quad \Delta_{\boldsymbol{h}}^{\ell,\varnothing} := I,$$

where the univariate operator $\Delta_{h_i}^{\ell}$ is applied to the univariate function f by considering f as a function of variable x_i with the other variables held fixed, and $I(f) := f$ for functions f on $\mathbb{T}^d$. We also use the abbreviation $\Delta_{\boldsymbol{h}}^{\ell} := \Delta_{\boldsymbol{h}}^{\ell,[d]}$.

For $u \subset [d]$, let

$$\omega_\ell^u(f, \boldsymbol{t})_p := \sup_{h_i < t_i, i \in u} \|\Delta_{\boldsymbol{h}}^{\ell,u}(f)\|_p, \ \boldsymbol{t} \in \mathbb{I}^d,$$

be the mixed (ℓ, u)th modulus of smoothness of f (in particular, $\omega_l^{\varnothing}(f, t)_p = \|f\|_p$).

If $0 < p, \theta \le \infty$, $r > 0$ and $\ell > r$, we introduce the quasi-semi-norm $|f|_{B_{p,\theta}^{r,u}}$ for functions $f \in L_p(\mathbb{T}^d)$ by

$$|f|_{B_{p,\theta}^{r,u}(\mathbb{T}^d)} := \begin{cases} \left(\int_{\mathbb{I}^d} \{\prod_{i \in u} t_i^{-r} \omega_\ell^u(f, \boldsymbol{t})_p\}^\theta \prod_{i \in u} t_i^{-1} d\boldsymbol{t}\right)^{1/\theta}, & \theta < \infty, \\ \sup_{\boldsymbol{t} \in \mathbb{I}^d} \prod_{i \in u} t_i^{-r} \omega_\ell^u(f, \boldsymbol{t})_p, & \theta = \infty \end{cases}$$

(in particular, $|f|_{B_{p,\theta}^{r,\emptyset}(\mathbb{T}^d)} = \|f\|_p$).

For $0 < p, \theta \le \infty$ and $0 < r < l$, the Besov space $B_{p,\theta}^r(\mathbb{T}^d)$ is defined as the set of functions $f \in L_p(\mathbb{T}^d)$ for which the Besov quasi-norm $\|f\|_{B_{p,\theta}^r(\mathbb{T}^d)}$ is finite. The Besov quasi-norm is defined by

$$\|f\|_{B_{p,\theta}^r(\mathbb{T}^d)} := \sum_{u \subset [d]} |f|_{B_{p,\theta}^{r,u}(\mathbb{T}^d)}.$$

We also call the particular space $H_p^r(\mathbb{T}^d) := B_{p,\infty}^r(\mathbb{T}^d)$ Hölder-Nikol'skii space.

2.2 B-spline quasi-interpolation representations

A central role in sparse-grid linear sampling recovery on Smolyak sparse grids of functions having a mixed smoothness play sampling representations which are based on dyadic scaled B-splines with integer knots or trigonometric kernels and constructed from function values at dyadic lattices. These representations are in the form of a B-spline or trigonometric polynomial series provided with discrete equivalent norm for functions in Sobolev and Besov spaces of a mixed smoothness. By employing them we can construct sampling algorithms for recovery on Smolyak sparse grids of functions from the corresponding spaces which in some cases give the asymptotically optimal rate of the approximation error. We refer the reader to [17, Section 5] for detail and bibliography on this topic. In this subsection, we present a Littlewood-Paley-type theorem from [14] on B-spline quasi-interpolation sampling representation for periodic unweighted Sobolev and Besov spaces of a mixed smoothness.

In order to construct B-spline quasi-interpolation sampling representations on $\mathbb{T}^d$, we auxiliarily introduce quasi-interpolation operators for functions on $\mathbb{R}^d$. For a given natural number ℓ, denote by M_ℓ the cardinal B-spline of order ℓ with support $[0,\ell]$ and knots at the points $0,1,...,\ell$. We fixed an even number $\ell \in \mathbb{N}$ and take the cardinal B-spline $M = M_\ell$ of order ℓ. Let $\Lambda = \{\lambda(j)\}_{|j|\le\mu}$ be a given finite even sequence, i.e., $\lambda(-j) = \lambda(j)$ for some $\mu \ge \frac{\ell}{2} - 1$. We define the linear operator Q for functions f on $\mathbb{R}$ by

$$Q(f,x) := \sum_{s\in\mathbb{Z}} \Lambda(f,s) M(x-s), \tag{2.5}$$

where

$$\Lambda(f,s) := \sum_{|j|\le\mu} \lambda(j) f(s-j+\ell/2). \tag{2.6}$$

The operator Q is local and bounded in $C(\mathbb{R})$ (see [7, p. 100–109]). An operator Q of the form (2.5)–(2.6) is called a *quasi-interpolation operator in* $C(\mathbb{R})$ if it reproduces $\mathcal{P}_{\ell-1}$, i.e., $Q(f) = f$ for every $f \in \mathcal{P}_{\ell-1}$, where $\mathcal{P}_{\ell-1}$ denotes the set of d-variate polynomials of degree at most $\ell - 1$ in each variable.

We present some well-known examples of quasi-interpolation operators. A piecewise linear quasi-interpolation operator is defined as

$$Q(f,x) := \sum_{s\in\mathbb{Z}} f(s) M(x-s),$$

where M is the symmetric piecewise linear B-spline with support $[-1,1]$ and knots at the integer points $-1,0,1$. It is related to the classical Faber-Schauder basis of the hat functions (see, e.g., [12, 49], for details). A quadric quasi-interpolation operator is defined by

$$Q(f,x) := \sum_{s\in\mathbb{Z}} \frac{1}{8}\{-f(s-1) + 10f(s) - f(s+1)\} M(x-s),$$

where M is the symmetric quadric B-spline with support $[-3/2,3/2]$ and knots at the half integer points $-3/2,-1/2,1/2,3/2$. Another example is the cubic quasi-interpolation operator

$$Q(f,x) := \sum_{s\in\mathbb{Z}} \frac{1}{6}\{-f(s-1) + 8f(s) - f(s+1)\} M(x-s),$$

where M is the symmetric cubic B-spline with support $[-2,2]$ and knots at the integer points $-2,-1,0,1,2$.

If Q is a quasi-interpolation operator of the form (2.5)–(2.6), for $h > 0$ and a function f on $\mathbb{R}$, we define the operator $Q(\cdot;h)$ by

$$Q(f;h) := \sigma_h \circ Q \circ \sigma_{1/h}(f),$$

where $\sigma_h(f,x) = f(x/h)$. Let Q be a quasi-interpolation operator of the form (2.5)–(2.6) in $C(\mathbb{R})$. If $k \in \mathbb{N}_0$, we introduce the operator Q_k by

$$Q_k(f,x) := Q(f,x;h^{(k)}), \ x \in \mathbb{R}, \quad h^{(k)} := \ell^{-1}2^{-k}.$$

We define the integer translated dilation $M_{k,s}$ of M by

$$M_{k,s}(x) := M(\ell 2^k x - s), \ k \in \mathbb{Z}_+, \ s \in \mathbb{Z}.$$

Then we have for $k \in \mathbb{Z}_+$,

$$Q_k(f)(x) \ = \ \sum_{s \in \mathbb{Z}} a_{k,s}(f) M_{k,s}(x), \ \forall x \in \mathbb{R},$$

where the coefficient functional $a_{k,s}$ is defined by

$$a_{k,s}(f) := \ \Lambda(f, s; h^{(k)}) = \sum_{|j| \le \mu} \lambda(j) f(h^{(k)}(s - j + r)). \tag{2.7}$$

For $\boldsymbol{k} \in \mathbb{N}_0^d$, let the mixed operator $Q_{\boldsymbol{k}}$ be defined by

$$Q_{\boldsymbol{k}} := \prod_{i=1}^d Q_{k_i}, \tag{2.8}$$

where the univariate operator Q_{k_i} is applied to the univariate function f by considering f as a function of variable x_i with the other variables held fixed. We define the d-variable B-spline $M_{k,s}$ by

$$M_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{x}) := \ \prod_{i=1}^d M_{k_i,s_i}(x_i), \ \boldsymbol{k} \in \mathbb{N}_0^d, \ \boldsymbol{s} \in \mathbb{Z}^d, \tag{2.9}$$

where $\mathbb{N}_0^d := \{\boldsymbol{s} \in \mathbb{Z}^d : s_i \ge 0, \ i \in [d]\}$. Then we have

$$Q_{\boldsymbol{k}}(f, \boldsymbol{x}) \ = \ \sum_{s \in \mathbb{Z}^d} a_{\boldsymbol{k},\boldsymbol{s}}(f) M_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^d,$$

where $M_{\boldsymbol{k},\boldsymbol{s}}$ is the mixed B-spline defined in (2.9), and

$$a_{\boldsymbol{k},\boldsymbol{s}}(f) \ = \ \left(\prod_{j=1}^d a_{k_j,s_j} \right)(f), \tag{2.10}$$

and the univariate coefficient functional a_{k_i,s_i} is applied to the univariate function f by considering f as a function of variable x_i with the other variables held fixed.

Since $M(\ell\, 2^k x) = 0$ for every $k \in \mathbb{N}_0$ and $x \notin (0, 1)$, we can extend the restriction to the interval $[0, 1]$ of the B-spline $M(\ell\, 2^k \cdot)$ to an 1-periodic function on the whole $\mathbb{R}$. Denote this periodic extension by N_k and define

$$N_{k,s}(x) := \ N_k(x - h^{(k)} s), \ k \in \mathbb{Z}_+, \ s \in I(k),$$

where

$$I(k) := \{0, 1, ..., \ell 2^k - 1\}.$$

We define the d-variable B-spline $N_{\boldsymbol{k},\boldsymbol{s}}$ by

$$N_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{x}) := \ \bigotimes_{i=1}^d N_{k_i,s_i}(x_i), \ \boldsymbol{k} \in \mathbb{N}_0^d, \ \boldsymbol{s} \in I(\boldsymbol{k}),$$

where

$$I(\boldsymbol{k}) := \prod_{i=1}^{d} I(k_i).$$

Then we have for functions f on $\mathbb{T}^d$,

$$Q_{\boldsymbol{k}}(f, \boldsymbol{x}) \ = \ \sum_{\boldsymbol{s} \in I(\boldsymbol{k})} a_{\boldsymbol{k},\boldsymbol{s}}(f) N_{\boldsymbol{k},\boldsymbol{s}}(x), \quad \forall x \in \mathbb{T}^d. \tag{2.11}$$

There holds the inequality

$$\|Q_{\boldsymbol{k}}(f)\|_{C(\mathbb{T}^d)} \ \ll \ \|f\|_{C(\mathbb{T})}, \quad f \in C(\mathbb{T}^d).$$

For $\boldsymbol{k} \in \mathbb{N}_0^d$, we write $\boldsymbol{k} \to \infty$ if $k_i \to \infty$ for $i \in [d]$). We have the convergence for every $f \in C(\mathbb{T}^d)$,

$$\|f - Q_{\boldsymbol{k}}(f)\|_{C(\mathbb{T}^d)} \to 0, \ \boldsymbol{k} \to \infty. \tag{2.12}$$

For convenience we define the univariate operator Q_{-1} by putting $Q_{-1}(f) = 0$ for all f on $\mathbb{I}$. Let the operators $q_{\boldsymbol{k}}$ be defined in the manner of the definition (2.8) by

$$q_{\boldsymbol{k}} := \ \prod_{i=1}^{d} \Big(Q_{k_i} - Q_{k_i - 1} \Big), \ \boldsymbol{k} \in \mathbb{N}_0^d.$$

From the equation

$$Q_{\boldsymbol{k}} = \sum_{\boldsymbol{k}' \le \boldsymbol{k}} q_{\boldsymbol{k}'}$$

and (2.12) it is easy to see that a continuous function f has the decomposition

$$f \ = \ \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} q_{\boldsymbol{k}}(f)$$

with the convergence in the norm of $C(\mathbb{T}^d)$. From the refinement equation for the B-spline M, in the univariate case, we can represent the component functions $q_{\boldsymbol{k}}(f)$ as

$$q_{\boldsymbol{k}}(f) = \sum_{\boldsymbol{s} \in I(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f) N_{\boldsymbol{k},\boldsymbol{s}}, \tag{2.13}$$

where $c_{\boldsymbol{k},\boldsymbol{s}}$ are certain coefficient functionals of f. In the multivariate case, the representation (2.13) holds true with the $c_{k,s}$ which are defined in the manner of the definition (2.10) by

$$c_{\boldsymbol{k},\boldsymbol{s}}(f) \ = \ \left(\prod_{j=1}^{d} c_{k_j, s_j} \right)(f).$$

The following periodic B-spline quasi-interpolation representation for continuous functions on $\mathbb{T}^d$ was proven in [14, Lemma 2.1].

Lemma 2.1 *Every continuous function f on $\mathbb{T}^d$ is represented as B-spline series*

$$f \ = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} q_{\boldsymbol{k}}(f) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \sum_{\boldsymbol{s}\in I(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f) N_{\boldsymbol{k},\boldsymbol{s}}, \tag{2.14}$$

converging in the norm of $C(\mathbb{T}^d)$, where the coefficient functionals $c_{\boldsymbol{k},\boldsymbol{s}}(f)$ are explicitly constructed as linear combinations of at most m_0 of function values of f for some $m_0 \in \mathbb{N}$ which is independent of $\boldsymbol{k}, \boldsymbol{s}$ and f.

Theorem 2.2 *Let $1 < p < \infty$ and $1/p < r < \ell$. Then every function $f \in W_p^r(\mathbb{T}^d)$ can be represented as the series* (2.14) *converging in the norm of $W_p^r(\mathbb{T}^d)$, and there holds the inequality*

Theorem 2.3 *Let $1 < p < \infty$ and $0 < r < \ell - 1$. Then for every function f on $\mathbb{T}^d$ represented as a B-spline series*

$$f \ = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} q_{\boldsymbol{k}} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \sum_{\boldsymbol{s}\in I(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}} N_{\boldsymbol{k},\boldsymbol{s}}, \tag{2.15}$$

we have $f \in W_p^r(\mathbb{T}^d)$ and

$$\|f\|_{W_p^r(\mathbb{T}^d)} \ \ll \ \left\| \left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \left| 2^{r|\boldsymbol{k}|_1} q_{\boldsymbol{k}} \right|^2 \right)^{1/2} \right\|_p,$$

whenever the right hand side is finite.

Theorem 2.4 *Let $1 < p < \infty$ and $1/p < r < \ell - 1$. Then we have*

$$\left\| \left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \left| 2^{r|\boldsymbol{k}|_1} q_{\boldsymbol{k}}(f) \right|^2 \right)^{1/2} \right\|_p \ \asymp \ \|f\|_{W_p^r(\mathbb{T}^d)}, \quad \forall f \in W_p^r(\mathbb{T}^d).$$

Theorems 2.2– 2.4 were proven in [14, Theorems 3.1–3.3].

Theorems on B-spline quasi-interpolation sampling representations with discrete equivalent quasi-norm in terms of coefficient functionals have been proved in [11]–[13], [18] for various non-periodic Besov spaces and periodic Besov spaces (see also [17, Section 5] for comments and bibliography). We recall the periodic version from [14, Corollary 3.1].

Theorem 2.5 *Let $0 < p, \theta \le \infty$ and $1/p < r < \min\{2\ell, 2\ell - 1 + 1/p\}$. Then a periodic function $f \in B_{p,\theta}^r$ can be represented by the B-spline series* (2.14) *satisfying the relation*

$$\left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} 2^{r|\boldsymbol{k}|_1\theta} \|q_{\boldsymbol{k}}(f)\|_p^\theta \right)^{1/\theta} \ \asymp \ \|f\|_{B_{p,\theta}^r}$$

with the sum over $\boldsymbol{k}$ changing to the supremum when $\theta = \infty$.

2.3 Sampling recovery by Smolyak sparse-grid algorithms

Based on the B-spline quasi-interpolation representation (2.14), we construct linear sampling algorithms on Smolyak sparse grids induced by partial sums of the series in (2.14) as follows. For $m \in \mathbb{N}$, the well known periodic Smolyak grid of points $G^d(m) \subset \mathbb{T}^d$ is defined as

$$G^d(m) := \{\boldsymbol{x} = 2^{-\boldsymbol{k}}\boldsymbol{s} : \boldsymbol{k} \in \mathbb{N}^d, \ |\boldsymbol{k}|_1 = m, \ \boldsymbol{s} \in I(\boldsymbol{k})\}.$$

Here and in what follows, we use the notations: $\boldsymbol{xy} := (x_1y_1, ..., x_dy_d)$; $2^{\boldsymbol{x}} := (2^{x_1}, ..., 2^{x_d})$; $|\boldsymbol{x}|_1 := \sum_{i=1}^d |x_i|$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$; x_i denotes the ith coordinate of $\boldsymbol{x} \in \mathbb{R}^d$, i.e., $\boldsymbol{x} =: (x_1, ..., x_d)$.

For $m \in \mathbb{N}_0$, we define the operator R_m by

$$R_m(f) := \sum_{|\boldsymbol{k}|_1 \le m} q_{\boldsymbol{k}}(f) \ = \ \sum_{|\boldsymbol{k}|_1 \le m} \ \sum_{\boldsymbol{s} \in I(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f)\, N_{\boldsymbol{k},\boldsymbol{s}}.$$

For functions f on $\mathbb{T}^d$, R_m defines the linear sampling algorithm on the Smolyak grid $G^d(m)$

$$R_m(f) \ = \ S_n(\boldsymbol{X}_n, \boldsymbol{\Phi}_n, f) \ = \ \sum_{\boldsymbol{y} \in G^d(m)} f(\boldsymbol{x})\psi_{\boldsymbol{x}},$$

where $n := \ |G^d(m)|$, $\boldsymbol{X}_n := \{\boldsymbol{x} \in G^d(m)\}$, $\boldsymbol{\Phi}_n := \{\varphi_{\boldsymbol{x}}\}_{\boldsymbol{x} \in G^d(m)}$ and for $\boldsymbol{x} = 2^{-\boldsymbol{k}}\boldsymbol{s}$, $\varphi_{\boldsymbol{x}}$ are explicitly constructed as linear combinations of at most at most m_0 B-splines $N_{\boldsymbol{k},\boldsymbol{j}}$ for some $m_0 \in \mathbb{N}$ which is independent of $\boldsymbol{k}, \boldsymbol{s}, m$ and f. The operator R_m is also called Smolyak (sparse-gird) sampling algorithm initiated and used by him [40] for quadrature and interpolation for functions having mixed smoothness. It plays an important role in sampling recovery of multivariate functions and its applications (see [4], [17] for comments and bibliography).

Theorem 2.6 *Let $1 < p, q < \infty$ and $1/p < r < \ell$. Then we have*

$$\big\|f - R_m(f)\big\|_q \ \ll \ \|f\|_{W_p^r(\mathbb{T}^d)} \times \begin{cases} 2^{-rm} m^{(d-1)/2}, & p \ge q, \\ 2^{-(r-1/p+1/q)m} & p < q, \end{cases} \quad \forall f \in W_p^r(\mathbb{T}^d).$$

Proof. This theorem was proven in [14, Theorem 4.1]. To understand the basic role of Theorems 2.2–2.4 playing in its proof, let us recall that short proof. Let f be a function in $W_p^r(\mathbb{T}^d)$. Since $r > 1/p$, f is continuous on $\mathbb{T}^d$ and consequently, we obtain by Lemma 2.1

$$f - R_m(f) \ = \sum_{|\boldsymbol{k}|_1 > m} q_{\boldsymbol{k}}(f) \tag{2.16}$$

with uniform convergence.

We first consider the case $p \ge q$. Due to the inequality $\|f\|_q \le \|f\|_p$, it is sufficient prove this case of the theorem for $p = q$. From (2.16) and Theorem 2.2 we have

$$\begin{aligned}\big\|f - R_m(f)\big\|_p &= \Big\|\sum_{|\boldsymbol{k}|_1 > m} q_{\boldsymbol{k}}(f)\Big\|_p \le \Bigg\|\bigg(\sum_{|\boldsymbol{k}|_1 > m} 2^{-2r|\boldsymbol{k}|_1}\bigg)^{1/2}\bigg(\sum_{|\boldsymbol{k}|_1 > m} \big|2^{r|\boldsymbol{k}|_1} q_{\boldsymbol{k}}(f)\big|^2\bigg)^{1/2}\Bigg\|_p \\ &\le \bigg(\sum_{|\boldsymbol{k}|_1 > m} 2^{-2r|\boldsymbol{k}|_1}\bigg)^{1/2}\Bigg\|\bigg(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \big|2^{r|\boldsymbol{k}|_1} q_{\boldsymbol{k}}(f)\big|^2\bigg)^{1/2}\Bigg\|_p \\ &\ll 2^{-rm} m^{(d-1)/2}\, \|f\|_{W_p^r(\mathbb{T}^d)}.\end{aligned}$$

We next consider the case $p < q$. From [3, Lemma 3] one can prove the inequality

$$\|f\|_q \ll \|f\|_{W_p^{1/p-1/q}}(\mathbb{T}^d), \quad \forall f \in W_p^{1/p-1/q}(\mathbb{T}^d).$$

Hence, by (2.16), Theorems 2.3 and 2.2 we derive that

$$\begin{aligned}\big\|f - R_m(f)\big\|_q &\ll \Big\|\sum_{|\boldsymbol{k}|_1 > m} q_{\boldsymbol{k}}(f)\Big\|_{W_p^{1/p-1/q}(\mathbb{T}^d)} \ll \Bigg\|\bigg(\sum_{|\boldsymbol{k}|_1 > m} \big|2^{(1/p-1/q)|\boldsymbol{k}|_1} q_{\boldsymbol{k}}(f)\big|^2\bigg)^{1/2}\Bigg\|_p \\ &\le 2^{-(r-1/p+1/q)m}\Bigg\|\bigg(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \big|2^{r|\boldsymbol{k}|_1} q_{\boldsymbol{k}}(f)\big|^2\bigg)^{1/2}\Bigg\|_p \\ &\ll 2^{-(r-1/p+1/q)m}\, \|f\|_{W_p^r(\mathbb{T}^d)}.\end{aligned}$$

The theorem is completely proven. □

Throughout the present paper, for a normed space X, we denote by boldface letter $\boldsymbol{X}$ in unit ball in X.

From Theorem 2.6 and the lower bounds in Theorem 2.9 below we also obtain

Corollary 2.7 *Let $1 < p, q < \infty$ and $1/p < r < \ell$. Then we have*

$$\sup_{f\in \boldsymbol{W}_p^r(\mathbb{T}^d)} \big\|f - R_m(f)\big\|_q \asymp \begin{cases} 2^{-rm} m^{(d-1)/2}, & p \ge q, \\ 2^{-(r-1/p+1/q)m} & p < q. \end{cases}$$

If the approximation error is measured in the norm of $L_\infty(\mathbb{T}^d)$, we have [14, Theorem 4.3]

Theorem 2.8 *Let $1 < p < \infty$ and $1/p < r < \ell$. Then we have*

$$\sup_{f\in \boldsymbol{W}_p^r(\mathbb{T}^d)} \big\|f - R_m(f)\big\|_\infty \asymp 2^{-(r-1/p)m} m^{(d-1)(1-1/p)}.$$

It worths to notice the following. For approximation of functions from $\boldsymbol{W}_p^r(\mathbb{T}^d)$, we could take the sampling operator

$$R_m^{\square}(f) := \sum_{|\boldsymbol{k}|_\infty \le m} q_{\boldsymbol{k}}(f)$$

on the traditional standard grid

$$G^d_\square(m) := \big\{2^{-\boldsymbol{k}}\boldsymbol{s} : |\boldsymbol{k}|_\infty = m,\ \boldsymbol{s} \in I(\boldsymbol{k})\big\}.$$

However, it is easy to verify that the error of approximation of $f \in \boldsymbol{W}^r_p(\mathbb{T}^d)$ by $R^\square_m(f)$ is the same as by $R_m(f)$. On the other hand, the sparsity of the grid $G^d(m)$ in the operator $R_m(f)$, is much higher than the sparsity of the grids $G^d_\square(m)$ comparing $|G^d(m)| \asymp 2^m m^{d-1}$ with $|G^d_\square(m)| \asymp 2^{dm}$.

A natural question arises are there sampling algorithms that use the Smolyak sparse grids $G^d(m)$ which give bounds better than $R_m(f)$ for the Sobolev class $\boldsymbol{W}^r_p(\mathbb{T}^d)$. To answer this question, let us introduce the Smolyak sampling n-width $\varrho^s_n(\boldsymbol{W})_q$ characterizing optimality of sampling recovery on Smolyak grids $G^d(m)$ with respect to the function class $\boldsymbol{W}$ by

$$\varrho^{\rm s}_n(\boldsymbol{W}, L_q(\mathbb{T}^d)) \ := \ \inf_{|G^d(m)|\le n,\, \Phi_m} \ \sup_{f\in \boldsymbol{W}} \|f - S_m(\boldsymbol{\Phi}_m, f)\|_q,$$

where for a family $\boldsymbol{\Phi}_m = \{\varphi_{\boldsymbol{x}}\}_{\boldsymbol{x}\in G^d(m)}$ of functions we define the linear sampling algorithm $S_m(\boldsymbol{\Phi}_m, \cdot)$ on Smolyak grids $G^d(m)$ by

$$S_m(\boldsymbol{\Phi}_m, f) \ = \ \sum_{\boldsymbol{y}\in G^d(m)} f(\boldsymbol{x})\varphi_{\boldsymbol{x}}.$$

The upper letter s indicates that we restrict to Smolyak grids here. From the definition it follows that

$$\varrho_n(\boldsymbol{W}, L_q(\mathbb{T}^d)) \ \le \ \varrho^{\rm s}_n(\boldsymbol{W}, L_q(\mathbb{T}^d)). \tag{2.17}$$

The following theorem [14, Theorem 4.2] confirms the asymptotic optimality of the Smolyak sampling algorithms R_m for the sampling recovery by using the sparse Smolyak grids $G^d(m)$ of the Sobolev class $\boldsymbol{W}^r_p(\mathbb{T}^d)$.

Theorem 2.9 *Let $1 < p, q < \infty$ and $r > 1/p$. For $n \in \mathbb{N}$, let m_n be the largest integer number such that $|G^d(m_n)| \le n$. Then we have*

$$\sup_{f\in \boldsymbol{W}^r_p(\mathbb{T}^d)} \big\|f - R_{m_n}(f)\big\|_q \asymp \varrho^{\rm s}_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \begin{cases} \left(\frac{(\log n)^{d-1}}{n}\right)^r (\log n)^{(d-1)/2}, & p \ge q, \\ \left(\frac{(\log n)^{d-1}}{n}\right)^{(r-1/p+1/q)}, & p < q. \end{cases} \tag{2.18}$$

Another interesting question is when the Smolyak sampling algorithms R_{m_n} given as in Theorem 2.9, are asymptotically optimal for the linear sampling n-widths $\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d))$. So far there are known only a few cases when the asymptotic optimality of the Smolyak sampling algorithms R_{m_n} is confirmed as in the following theorem. Moreover, to our knowledge, excepting Smolyak sparse-grid sampling algorithms by using B-spine quasi-interpolation or de la Vallée Poussin kernels, there is no asymptotically optimal linear sampling algorithms of another type for $\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d))$ with $1 \le p, q \le \infty$.

Theorem 2.10 *Let $r > 1/p$ and for $n \in \mathbb{N}$, let m_n be the largest integer number such that $|G^d(m_n)| \le n$. Then we have that*

(i) *for $1 < p < q \le 2$ or $2 \le p < q < \infty$,*

$$\sup_{f \in \boldsymbol{W}^r_p(\mathbb{T}^d)} \big\|f - R_{m_n}(f)\big\|_q \asymp \varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{(r-1/p+1/q)}, \text{ and} \tag{2.19}$$

(ii)

$$\sup_{f \in \boldsymbol{W}^r_2(\mathbb{T}^d)} \big\|f - R_{m_n}(f)\big\|_\infty \asymp \varrho_n(\boldsymbol{W}^r_2(\mathbb{T}^d), L_\infty(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{(r-1/2)} (\log n)^{(d-1)/2}.$$

In this theorem, the claim(i) was proven in [14] for $r > \max\{1/p, 1/2\}$ and in [6] for $r > 1/p$. The special case $p = 2 < q$ was proven in [5]. The claim (ii) was proven in [45].

For the Hölder-Nikol'skii function classes $\boldsymbol{H}^r_p(\mathbb{T}^d)$, we have the following right asymptotic of order of sampling n-widths.

Theorem 2.11 *Let $r > 1/p$ and for $n \in \mathbb{N}$, let m_n be the largest integer number such that $|G^d(m_n)| \le n$. Then we have that*

(i) *for $1 < p < q \le 2$,*

$$\sup_{f \in \boldsymbol{H}^r_p(\mathbb{T}^d)} \big\|f - R_{m_n}(f)\big\|_q \asymp \varrho_n(\boldsymbol{H}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{(r-1/p+1/q)} (\log n)^{(d-1)/q}, \text{ and}$$

(ii)

$$\sup_{f \in \boldsymbol{H}^r_\infty(\mathbb{T}^2)} \big\|f - R_{m_n}(f)\big\|_\infty \asymp \varrho_n(\boldsymbol{H}^r_\infty(\mathbb{T}^2), L_\infty(\mathbb{T}^2)) \asymp \left(\frac{\log n}{n}\right)^{r} (\log n).$$

In this theorem, the claim (i) was proven in [10], and the claim (ii) in [46]. Moreover, the claim (i) was the first result on the wright asymptotic order of sampling n-widths for classes of functions having a mixed smoothness. We refer the reader to [17, Section 5] for results on sampling recovery and sampling n-widths for functions from Besov spaces $B^r_{p,\theta}(\mathbb{T}^d)$.

2.4 Sampling recovery in reproducing kernel Hilbert spaces

In the previous section, we presented various aspects of sampling recovery by using Smolyak algorithms $R_m(f)$ on sparse grids $G^d(m)$, in particular, their asymptotic optimalities in terms of linear sampling n-widths $\varrho^{\mathrm{s}}_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d))$ and $\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d))$. The right asymptotic order of $\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d))$ $(r > 1/p)$ can be achieved by the

Smolyak sampling algorithms $R_{m_n}(f)$ in the cases $1 < p < q \le 2$ or $2 \le p < q < \infty$ or $p = 2,\ q = \infty$. It is interesting to notice that all these cases are restricted by the strict inequality $p < q$.

It is a dilemma that the problem of right asymptotic order of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_p(\mathbb{T}^d))$ for $1 \le p \le \infty$ has been open for long time, see Open Problem 4.1 in [17]. From very recent results of [26] on inequality between the linear sampling and Kolmogorov n-widths of the unit ball of a reproducing kernel Hilbert subspace of the space $L_2(\Omega;\nu)$ one can immediately deduce the right asymptotic order of $\varrho_n(\boldsymbol{W}^r_2, L_2(\mathbb{T}^d))$ which solved the outstanding Open Problem 4.1 in [17] for the particular case $p = 2$. This open problem is still not solved for the case $p \ne 2$. Unfortunately, even in the solved case $p = 2$, we do not know any explicit asymptotically optimal linear sampling algorithm since its proof is based on an inequality between the linear sampling and Kolmogorov n-widths. The problem of construction of asymptotically optimal linear sampling algorithms for this case is still open.

In this section, we present some results of [26] and their consequences on right asymptotic order of $\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d))$ for the case $1 < q \le 2 \le p < \infty$.

Let H be a separable reproducing kernel (RK) Hilbert space on a Ω and ν a positive measure on Ω such that H is compactly embedded into the space $L_2(\Omega;\nu)$. We say that H satisfy the finite trace assumption if there holds the condition

$$\int_\Omega K(\boldsymbol{x},\boldsymbol{x})\nu(\mathrm{d}\boldsymbol{x}) \ < \ \infty, \tag{2.20}$$

where $K(\cdot,\cdot)$ is the RK of H.

Recall that $\boldsymbol{H}$ denotes the unit ball of H. From the definitions we already know the inequality $\varrho_n(\boldsymbol{H}, L_2(\Omega;\nu)) \ge d_n(\boldsymbol{H}, L_2(\Omega;\nu))$. The following theorem claims an inverse inequality [26, Theorem 1].

Theorem 2.12 *Let H be a separable RK Hilbert space on the domain Ω and μ a positive measure such that H is compactly embedded into $L_2(\Omega;\nu)$. Assume that H satisfy the finite trace assumption* (2.20). *Then there is a constant $\lambda \in \mathbb{N}$ such that*

$$\varrho_{\lambda n}(\boldsymbol{H}, L_2(\Omega;\nu))^2 \ \le \ \frac{1}{n}\sum_{m\ge n} d_m(\boldsymbol{H}, L_2(\Omega;\nu))^2. \tag{2.21}$$

From this theorem we can deduce an important consequence [26, Corollary 2].

Corollary 2.13 *Under the assumptions of Theorem 2.12 assume that*

$$d_n(\boldsymbol{H}, L_2(\Omega;\nu)) \ \ll \ n^{-\alpha} log^{-\beta} n \tag{2.22}$$

for some $\alpha \ge 1/2$ and $\beta \in \mathbb{R}$. Then

$$\varrho_n(\boldsymbol{H}, L_2(\Omega;\nu)) \ \ll \ \begin{cases} n^{-\alpha} log^{-\beta} n & \text{if } \ \alpha > 1/2, \\ n^{-\alpha} log^{-\beta+1/2} n & \text{if } \ \alpha = 1/2 \text{ and } \beta > 1/2. \end{cases} \tag{2.23}$$

Moreover, there exists $\boldsymbol{H}$ such that these bounds are sharp.

Theorem 2.14 *Let $r > 1/2$. Then*

$$\varrho_n(\boldsymbol{W}^r_2(\mathbb{T}^d), L_2(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{-r}. \tag{2.24}$$

Proof. This theorem can be considered a particular case of Theorem 2.17 below. It is a consequence of Corollary 2.13. However, for completeness, let us give an independent proof based on Corollary 2.13. Notice that $W^r_2(\mathbb{T}^d)$ is an RK Hilbert space. The RK of $W^r_2(\mathbb{T}^d)$ is the Bernoulli kernel $K = F_r$ defined as in (2.2). For $r > 1/2$, the space $W^r_2(\mathbb{T}^d)$ satisfies the finite trace assumption (2.20). Hence, this corollary directly follows from Corollary 2.13 and the known result

$$d_n(\boldsymbol{W}^r_2(\mathbb{T}^d), L_2(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{-r}, \tag{2.25}$$

see, e.g., [17, Theorem 4.3.1]. ◻

Theorem 2.12 can be extended to another situation for $\varrho_n(\boldsymbol{W}, L_2(\Omega;\nu))$ when $\boldsymbol{W}$ is not the unit ball of an RK Hilbert space. This extension allows to obtain a result more general than Theorem 2.14.

Assumption A. Let $\boldsymbol{W}$ be a class of complex-valued functions on the set Ω. We say that $\boldsymbol{F}$ satisfies Assumption A, if there is a metric on $\boldsymbol{W}$ such that $\boldsymbol{W}$ is continuously embedded into $L_2(\Omega;\nu)$, separable, and for each $\boldsymbol{x} \in \Omega$, the function evaluation $f \mapsto f(\boldsymbol{x})$ is continuous on $\boldsymbol{W}$.

The following theorem and corollary have been proven also in [26, Theorem 3 and Corollary 4] (see also [33, Proposition 11] for a slight improvement).

Theorem 2.15 *Assume that F satisfies Assumption A. Then for every $0 < \tau < 2$, there is a constant $\lambda \in \mathbb{N}$ such that*

$$\varrho_{\lambda n}(\boldsymbol{W}, L_2(\Omega;\nu))^\tau \le \frac{1}{n}\sum_{m\ge n} d_m(\boldsymbol{W}, L_2(\Omega;\nu))^\tau. \tag{2.26}$$

Corollary 2.16 *Assume that $\boldsymbol{W}$ satisfies Assumption A and that*

$$d_n(\boldsymbol{W}, L_2(\Omega;\nu)) \ll n^{-\alpha} log^{-\beta} n \tag{2.27}$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Then

$$\varrho_n(\boldsymbol{W}, L_2(\Omega;\nu)) \ll \begin{cases} n^{-\alpha} log^{-\beta} n & \text{if } \alpha > 1/2, \\ n^{-\alpha} log^{-\beta+1/2} n & \text{if } \alpha = 1/2 \text{ and } \beta > 1, \\ 1 & \text{otherwise.} \end{cases} \tag{2.28}$$

Moreover, there exists H such that these bounds are sharp.

From these results we obtain

Theorem 2.17 *Let $r > 1/2$ and $1 < q \le 2 \le p < \infty$. Then*

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{-r}. \tag{2.29}$$

Proof. Notice that for $r > 1/2$ and $q = 2$, the set $\boldsymbol{W}^r_p(\mathbb{T}^d)$ satisfies Assumption A. Hence, the asymptotic order (2.29) for $q = 2$ directly follows from Corollary 2.16 and the known result

$$d_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{-r} \tag{2.30}$$

for $1 < q \le p < \infty$, see, e.g., [17, Theorem 4.3.1]. The case $1 < q < 2$ is implied from the case $q = 2$, (2.30) and the inequalities

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \ll \varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_2(\mathbb{T}^d)),$$

and

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \gg d_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)).$$

□

We have a similar result for the Hölder-Nikol'skii classes.

Theorem 2.18 *Let $r > 1/2$ and $1 < q \le 2 \le p < \infty$. Then*

$$\varrho_n(\boldsymbol{H}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{-r} (\log n)^{(d-1)/2}. \tag{2.31}$$

Proof. This corollary can be proven in a way similar to the proof of Theorem 2.17 with a certain modification, based on Corollary 2.16. In particular, (2.30) is replaced with

$$d_n(\boldsymbol{H}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{-r} (\log n)^{(d-1)/2}$$

for $1 < q \le 2 \le p < \infty$, see, e.g., [17, Theorem 4.3.10]. □

We conjecture that the right asymptotic orders (2.29) and (2.31) are still hold true for $r > 1/p$ and $1 < q \le p < \infty$.

2.5 Different concepts of optimality in sampling recovery

Optimality in sampling recovery can be understood in different ways in terms of sampling widths. It depends on the restriction of sampling algorithms we consider. In the previous sections, optimalities of sampling recovery based on n function values, are treated in terms of linear sampling n-widths $\varrho_n(\boldsymbol{W}, X)$ and Smolyak sampling n-widths $\varrho^{\mathrm{s}}_n(\boldsymbol{W}, X)$. For the first sampling n-widths, we are restricted with linear sampling algorithms, and for the second n-widths, with linear sampling algorithms on the Smolyak grids. Theorems 2.10, 2.17 and 2.9 show that for the Sobolev class of mixed smoothness $\boldsymbol{W}^r_p(\mathbb{T}^d)$, the asymptotic orders of these sampling n-widths coincide in some cases and differ in other cases. In this subsection, we consider two more sampling widths which characterize other optimalities of sampling recovery.

The Kolmogorov sampling n-width of the set $\boldsymbol{W}$ in X as

$$\varrho_n^{\mathrm{k}}(\boldsymbol{W}, X) := \inf_{\substack{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\in\Omega,\\ A_n:\mathbb{R}^n\to L_n,\ \dim L_n\le n}} \sup_{f\in\boldsymbol{W}} \|f - A_n(f(\boldsymbol{x}_1),\ldots,f(\boldsymbol{x}_n))\|_X,$$

where the inf is taken over all collections $(\boldsymbol{x}_k)_{k=1}^n$ of n points in Ω, all mappings $A_n : \mathbb{R}^n \to L_n$ and all linear subspaces $L_n \subset X$ of dimension at most n.

The absolute sampling n-width of the set $\boldsymbol{W}$ in X as

$$\varrho_n^{\mathrm{abs}}(\boldsymbol{W}, X) := \inf_{\substack{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\in\Omega,\\ A_n:\mathbb{R}^n\to X}} \sup_{f\in\boldsymbol{W}} \|f - A_n(f(\boldsymbol{x}_1),\ldots,f(\boldsymbol{x}_n))\|_X,$$

where the inf is taken over all collections $(\boldsymbol{x}_k)_{k=1}^n$ of n points in Ω, and all mappings $A_n : \mathbb{R}^n \to X$.

The sampling n-widths ϱ_n^{k} and ϱ_n are inspired by the concepts of the Kolmogorov n-widths and the linear n-widths. The sampling n-widths ϱ_n^{k} was introduced in [9], ϱ_n in [45], ϱ_n^{abs} in [48], and ϱ_n^{s} in [18].

From the definitions we can see that

$$\varrho_n^{\mathrm{abs}}(\boldsymbol{W}, X) \le \varrho_n^{\mathrm{k}}(\boldsymbol{W}, X) \le \varrho_n(\boldsymbol{W}, X) \le \varrho_n^{\mathrm{s}}(\boldsymbol{W}, X). \tag{2.32}$$

For given d, r and p, q, we temporarily use the abbreviation:

$$\rho_n := \rho_n(\boldsymbol{W}_p^r(\mathbb{T}^d), L_q(\mathbb{T}^d)),$$

where ρ_n denotes one of ϱ_n^{abs}, ϱ_n^{k}, ϱ_n and ϱ_n^{s}.

For the univariate Sobolev class $\boldsymbol{W}_p^r(\mathbb{T})$, it is known that if $d = 1$, $1 \le p, q \le \infty$ and $r > 1/p$, then we have that

$$\varrho_n^{\mathrm{abs}} \asymp \varrho_n^{\mathrm{k}} \asymp \varrho_n \asymp \varrho_n^{\mathrm{s}} \asymp n^{-r-(1/p-1/q)_+},$$

showing these sampling n-widths have the same asymptotic order (see, e.g., [17], [47], [38]). The picture may be changed for the multivariate Sobolev class of mixed smoothness $\boldsymbol{W}_p^r(\mathbb{T}^d)$ with $d > 1$. We list some known particular cases when the asymptotic orders of some of these sampling n-widths coincide.

Theorem 2.19 *Let $d > 1$. Then we have that*

(i) *for $r > 1/p$ and $1 < p < q \le 2$,*

$$\varrho_n^{\mathrm{k}} \asymp \varrho_n \asymp \varrho_n^{\mathrm{s}} \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{(r-1/p+1/q)}, \tag{2.33}$$

(ii) *for* $r > 1/2$ *and* $p = 2$ *and* $q = \infty$,

$$\varrho_n^{\rm abs} \ \asymp \ \varrho_n^{\rm k} \asymp \ \varrho_n \asymp \varrho_n^{\rm s} \ \asymp \ \left(\frac{(\log n)^{d-1}}{n}\right)^{(r-1/2)} (\log n)^{(d-1)/2},$$

(iii) *for* $r > 1/2$ *and* $1 < q \le 2 \le p < \infty$,

$$\varrho_n^{\rm abs} \ \asymp \ \varrho_n^{\rm k} \asymp \ \varrho_n \asymp \left(\frac{(\log n)^{d-1}}{n}\right)^{-r}.$$

Proof. The equality $\varrho_n^{\rm abs} = \varrho_n$ in the claim (ii) follow from [8, Proposition 13], and the inequality $\varrho_n \ll \varrho_n^{\rm abs}$ for $r > 1/2$ and $1 < q \le 2 \le p < \infty$ from [8, Proposition 14]. Hence this theorem immediately follows from the inequalities (2.1) and (2.32), Theorem 2.9 and known results on Kolmogorov n-widths $d_n(\boldsymbol{W}_p^r(\mathbb{T}^d), L_q(\mathbb{T}^d))$ (see [17, Section 4.3]). ∎

Results of the very recent papers [31, 19] show that in some cases the absolute sampling n-widths may decay faster than the Kolmogorov sampling n-widths if the dimension d is large sufficiently. In particular, it has been proven [19] the following result.

Theorem 2.20 *Let* $d > 1$, $r > 1/p$ *and* $1 < p < 2$, $q = 2$. *Then we have that*

$$\varrho_n^{\rm abs} \ \ll \ \left(\frac{(\log n)^{d-1}}{n}\right)^{(r-1/p+1/2)} (\log n)^{-(d-1)(1/p-1/2)+3r}. \tag{2.34}$$

If $d > \frac{3r}{1/p-1/2} + 1$, then from (2.33) and (2.34), one can see that

$$\varrho_n^{\rm abs} \ \ll \ (\log n)^{-\alpha} \varrho_n^{\rm k}$$

with $\alpha := (d-1)(1/p - 1/2) + 3r > 0$.

Excepting the cases in the claims (ii) and (iii) of Theorem 2.19, the problem of right asymptotic order of the absolute sampling n-widths $\varrho_n^{\rm abs}$ is still open for $d > 1$, $1 \le p, q \le \infty$ and $r > 1/p$.

3 Weighted sampling recovery

3.1 Introducing remarks

In the previous section, we considered the problem of linear sampling recovery for functions in unweighted Sobolev spaces $W_p^r(\mathbb{T}^d)$. In this section, we consider the problems of linear sampling recovery and linear approximation of functions in weighted Sobolev spaces $W_p^r(\mathbb{R}^d;\mu)$. The approximation error is measured by the norm of the weighted space $L_q(\mathbb{R}^d;\mu)$. The optimality of of sampling recovery and linear approximation is treated in terms of linear sampling, linear and Kolmogorov n-widths.

We first introduce weighted Sobolev spaces of mixed smoothness. Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Let v be a nonnegative Lebesgue measurable function on Ω. Denote by μ_v the measure on Ω defined via the density function v, i.e., for every Lebesgue measurable set $A \subset \Omega$,

$$\mu_v(A) = \int_A v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

For $1 \leq p < \infty$, let $L_p(\Omega; \mu_v)$ be the weighted space of all Lebesgue measurable functions f on Ω such that the norm

$$\|f\|_{L_p(\Omega;\mu_v)} := \left(\int_\Omega |f(\boldsymbol{x})|^p \mu_v(\mathrm{d}\boldsymbol{x}) \right)^{1/p} = \left(\int_\Omega |f(\boldsymbol{x})|^p v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{1/p} \tag{3.1}$$

is finite. Due to (3.1), the density function v is called also a weight and the space $L_p(\Omega; \mu_v)$ a weighted space. For $r \in \mathbb{N}$, we define the weighted Sobolev space $W^r_p(\Omega; \mu_v)$ of mixed smoothness r as the normed space of all functions $f \in L_p(\Omega; \mu_v)$ such that the weak (generalized) partial derivative $D^{\boldsymbol{k}} f$ belongs to $L_p(\Omega; \mu_v)$ for every $\boldsymbol{k} \in \mathbb{N}^d_0$ satisfying the inequality $|\boldsymbol{k}|_\infty \leq r$. The norm of a function f in this space is defined by

$$\|f\|_{W^r_p(\Omega;\mu_v)} := \left(\sum_{|\boldsymbol{k}|_\infty \leq r} \|D^{\boldsymbol{k}} f\|^p_{L_p(\Omega;\mu_v)} \right)^{1/p}. \tag{3.2}$$

We write $L_p(\Omega) := L_p(\Omega; \mu_v)$ and $W^r_p(\Omega) := W^r_p(\Omega; \mu_v)$ if $v(\boldsymbol{x}) = 1$ for every $\boldsymbol{x} \in \Omega$.

In this section, we are interested in approximation of functions from $W^r_p(\Omega; \mu_w)$, where the density function or equivalently, the weight

$$w(\boldsymbol{x}) := w_{\lambda,a,b}(\boldsymbol{x}) := \prod_{i=1}^d w(x_i), \tag{3.3}$$

is the tensor product of univariate Freud-type weights

$$w(x) := w_{\lambda,a,b}(x) := \exp(-a|x|^\lambda + b), \quad \lambda > 1, \quad a > 0, \quad b \in \mathbb{R}. \tag{3.4}$$

The approximation error is measured by the norm of the weighted space $L_q(\Omega; \mu_w)$ for $1 \leq q < \infty$. The most important parameter in the weight $w_{\lambda,a,b}$ is λ. The parameter b which produces only a possitive constant in the weight $w_{\lambda,a,b}$ is introduced for a certain normalization for instance, for the standard Gaussian weight (see (3.5) below) which is one of the most important weights. In what follows, we fix the parameters λ, a, b and for simplicity, drop them from the notation. As the weight $w = w_{\lambda,a,b}$ is fixed, we also use the abbreviations:

$$L_p(\Omega; \mu) := L_p(\Omega; \mu_w), \quad W^r_p(\Omega; \mu) := W^r_p(\Omega; \mu_w).$$

The standard d-dimensional Gaussian measure γ with the density function

$$g(\boldsymbol{x}) = (2\pi)^{-d/2} \exp(-|\boldsymbol{x}|_2^2/2). \tag{3.5}$$

is a particular case of measure μ_w. The well-known spaces $L_p(\mathbb{R}^d;\gamma)$ and $W^r_p(\mathbb{R}^d;\gamma)$ are used in many applications.

Notice that any function $f \in W^r_p(\mathbb{R}^d;\mu)$ is equivalent in the sense of the Lesbegue measure to a continuous (not necessarily bounded) function on $\mathbb{R}^d$ (see [15, Lemma 3.1]). Hence in what follows we always assume that the functions $f \in W^r_p(\mathbb{R}^d;\mu)$ are continuous. We need this assumption for well defining sampling algorithms and quadratures for functions $f \in W^r_p(\mathbb{R}^d;\mu)$.

The problems of linear sampling recovery and linear approximation of functions in Gaussian-weighted Sobolev spaces $W^r_p(\mathbb{R}^d;\gamma)$ have been studied in [16]. For the set $\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma)$, $r \in \mathbb{N}$, and Gaussian-weighted space $L_q(\mathbb{R}^d;\gamma)$, it was proven in [16] the following.

(i)

$$\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) \asymp n^{-\frac{r}{2}}(\log n)^{\frac{(d-1)r}{2}} \quad (r \geq 2), \tag{3.6}$$

and

$$\lambda_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) = d_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) \asymp n^{-\frac{r}{2}}(\log n)^{\frac{(d-1)r}{2}}, \tag{3.7}$$

(ii) For $2 < p < \infty$,

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) \asymp n^{-r}(\log n)^{(d-1)r}, \tag{3.8}$$

and for $1 \leq q < p < \infty$,

$$\lambda_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma), L_q(\mathbb{R}^d;\gamma)) \asymp d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma), L_q(\mathbb{R}^d;\gamma)) \asymp n^{-r}(\log n)^{(d-1)r}. \tag{3.9}$$

The proofs of the asymptotic orders in (3.6)–(3.9) require quite different techniques.

The proof of (3.9) for the case $1 \leq q < p < \infty$ is based on a linear approximation in the space $L_q(\mathbb{I}^d)$ of functions from the classical unweighted Sobolev spaces $W^r_p(\mathbb{I}^d)$. By assembling of integer shifts of this approximation we construct a linear approximation in the space $L_q(\mathbb{I}^d;\gamma)$ of functions from Gaussian-weighted Sobolev spaces $W^r_p(\mathbb{R}^d;\gamma)$ which preserves the convergence rate.

The proof of (3.7) is completely different from the proof of the case $1 \leq q < p < \infty$. It is similar to the hyperbolic cross trigonometric approximation in the Hilbert space $\tilde{L}_2(\mathbb{I}^d)$ of periodic functions from the Sobolev space $\tilde{W}^\alpha_2(\mathbb{I}^d)$ (see, e.g., [17] for detail). Here, the approximation is based on finite truncations of the Hermite polynomial expansion of functions to be approximated.

The results (3.6) and (3.8) are deduced from (3.7) and (3.9) and the inequalities beween sampling and Kolmogorov n-widths in Corollaries 2.13 and 2.16, respectively.

It is interesting to compare the results (3.6)–(3.9) with the known results for the corresponding unweighted cases. For the unweighted class $\boldsymbol{W}^r_p(\mathbb{I}^d)$, $r \in \mathbb{N}$, and unweighted space $L_q(\mathbb{I}^d)$, we have the following.

(iii)
$$\varrho_n(\boldsymbol{W}^r_2(\mathbb{I}^d), L_2(\mathbb{I}^d)) \asymp n^{-r}(\log n)^{(d-1)r}, \tag{3.10}$$

and
$$\lambda_n(\boldsymbol{W}^r_2(\mathbb{I}^d), L_2(\mathbb{I}^d)) = d_n(\boldsymbol{W}^r_2(\mathbb{I}^d), L_2(\mathbb{I}^d) \asymp n^{-r}(\log n)^{(d-1)r}, \tag{3.11}$$

(iv) For $2 < p < \infty$,
$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{I}^d), L_2(\mathbb{I}^d)) \asymp n^{-r}(\log n)^{(d-1)r}. \tag{3.12}$$

and for $1 \le q < p < \infty$,
$$\lambda_n(\boldsymbol{W}^r_p(\mathbb{I}^d), L_q(\mathbb{I}^d)) \asymp d_n(\boldsymbol{W}^r_p(\mathbb{I}^d), L_q(\mathbb{I}^d)) \asymp n^{-r}(\log n)^{(d-1)r}. \tag{3.13}$$

Inspecting (3.6)–(3.9) and (3.10)–(3.13), we can see that there is a significant difference between the cases $p = q = 2$ and the cases $p > q$. If $p > q$, the asymptotic orders of the n-widths ϱ_n, λ_n and d_n are the same for the weighted and unweighted cases. If $p = q = 2$, the asymptotic orders of these n-widths in the weighted case are twice worse than in the unweighted case.

In this section, we extend the results (3.8) and (3.9) to the more general case – the weighted class $W^r_p(\mathbb{R}^d; \mu)$ and weighted space $L_q(\mathbb{R}^d; \mu)$ for the measure μ associated with Freud-type weights as the density function given by (3.3).

For technical convenience we use the conventions for $n \in \mathbb{R}_1$:
$$\varrho_n(F, X) := \varrho_{\lfloor n \rfloor}(F, X),$$
$$d_n(F, X) := d_{\lfloor n \rfloor}(F, X), \quad \lambda_n(F, X) := \lambda_{\lfloor n \rfloor}(F, X),$$
and
$$A_n := A_{\lfloor n \rfloor}, \quad R_n := R_{\lfloor n \rfloor}.$$

3.2 Linear sampling recovery and approximation

In this subsection, we give shorten proofs of the results (3.6) and (3.7) by using of the Hilbert space structure of the Gaussian-weighted spaces $\boldsymbol{W}^r_2(\mathbb{R}^d; \gamma)$ and $L_2(\mathbb{R}^d; \gamma)$ and Corollary 2.13, emphasizing possibility of application of the results in [26] to different situations. We also extend the results (3.8) and (3.9) to the weighted function class $\boldsymbol{W}^r_p(\mathbb{R}^d; \mu)$ and weighted space $L_q(\mathbb{R}^d; \mu)$ by using a modification of the technique from [16].

We first give shorten proofs of the results (3.6) and (3.7). Let $(p_m)_{m \in \mathbb{N}_0}$ be the sequence of orthonormal polynomials with respect to the univariate Gaussian weight $g(x) = (2\pi)^{-1/2} \exp(-|x|^2/2)$. For every multi-degree $\boldsymbol{k} \in \mathbb{N}^d_0$, the d-variate polynomial $H_{\boldsymbol{k}}$ is defined by
$$p_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^d p_{k_j}(x_j), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

It is well-known that the polynomials $\{p_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ constitute an orthonormal basis of the Hilbert space $L_2(\mathbb{R}^d;\gamma)$. In particular, every $f \in L_2(\mathbb{R}^d;\gamma)$ can be represented by the series

$$f = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \hat{f}(\boldsymbol{k}) p_{\boldsymbol{k}} \quad \text{with} \quad \hat{f}(\boldsymbol{k}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, p_{\boldsymbol{k}}(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \tag{3.14}$$

converging in the norm of $L_2(\mathbb{R}^d;\gamma)$, and in addition, there holds Parseval's identity

$$\|f\|^2_{L_2(\mathbb{R}^d;\mu)} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} |\hat{f}(\boldsymbol{k})|^2. \tag{3.15}$$

For $r \in \mathbb{N}_0$ and $\boldsymbol{k} \in \mathbb{N}_0^d$, we define

$$\rho_{r,\boldsymbol{k}} := \prod_{j=1}^d (k_j + 1)^r .$$

For any $r > 0$. Denote by $\mathcal{H}^r_w$ the space of all functions $f \in L_2(\mathbb{R}^d;\gamma)$ represented by the series (3.14) for which the norm

$$\|f\|_{\mathcal{H}^r_g} := \left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \rho_{r,\boldsymbol{k}} |\hat{f}(\boldsymbol{k})|^2 \right)^{1/2} \tag{3.16}$$

is finite.

For the proof of the following norm equivalence see [16, Lemma 3.4] (cf. also [23, pages 687–689]).

Lemma 3.1 *Let $r \in \mathbb{N}_0$. Then we have the norm equivalence*

$$\|f\|_{W^r_2(\mathbb{R}^d,\gamma)} \asymp \|f\|_{\mathcal{H}^r_g} . \tag{3.17}$$

Due to this norm equivalence, we identify the space $W^r_2(\mathbb{R}^d,\gamma)$ with the space $\mathcal{H}^r_g$ for $r \in \mathbb{N}$.

Theorem 3.2 *Let $r > 0$. Then we have the right asymptotic orders*

$$\lambda_n(\boldsymbol{\mathcal{H}}^r_g, L_2(\mathbb{R}^d;\gamma)) = d_n(\boldsymbol{\mathcal{H}}^r_g, L_2(\mathbb{R}^d;\gamma)) \asymp n^{-\frac{r}{2}} (\log n)^{\frac{(d-1)r}{2}}. \tag{3.18}$$

The proof of this theorem which is given in [16, Theorem 3.5], is similar the proof of the right asymptotic order of the Kolmogorov n-widths $d_n(\boldsymbol{W}^r_2(\mathbb{I}^d), L_2(\mathbb{I}^d)$ in the unweighted case.

From Theorem 3.2 and Corollary 2.13 we prove

Theorem 3.3 *Let $r > 1$. Then there holds the right asymptotic order*

$$\varrho_n(\boldsymbol{\mathcal{H}}^r_g, L_2(\mathbb{R}^d;\gamma)) \asymp n^{-\frac{r}{2}} (\log n)^{\frac{(d-1)r}{2}}. \tag{3.19}$$

Proof. The lower bound of (3.19) follows from (3.18) and the inequality

$$\varrho_n(\boldsymbol{\mathcal{H}}^r_g, L_2(\mathbb{R}^d;\gamma)) \ge \lambda_n(\boldsymbol{\mathcal{H}}^r_g, L_2(\mathbb{R}^d;\gamma)).$$

We verify the upper one. By (3.18),

$$d_n(\boldsymbol{\mathcal{H}}^r_g, L_2(\mathbb{R}^d;\gamma)) \ll n^{-\frac{r}{2}}(\log n)^{\frac{(d-1)r}{2}}. \tag{3.20}$$

Notice that for $r > 1$, $\mathcal{H}^r_g$ is a separable reproducing kernel Hilbert space with the reproducing kernel

$$K(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \rho_{r,\boldsymbol{k}}^{-1}\, p_{\boldsymbol{k}}(\boldsymbol{x}) p_{\boldsymbol{k}}(\boldsymbol{y}). \tag{3.21}$$

From the orthonormality of the system $\{p_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ it is easily seen that $K(\boldsymbol{x},\boldsymbol{y})$ satisfies the finite trace assumption

$$\int_{\mathbb{R}^d} K(\boldsymbol{x},\boldsymbol{x}) g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \;<\; \infty. \tag{3.22}$$

Hence by Corollary 2.13 and (3.20) we obtain

$$\varrho_n(\boldsymbol{\mathcal{H}}^r_g, L_2(\mathbb{R}^d;\gamma)) \ll d_n(\boldsymbol{\mathcal{H}}^r_g, L_2(\mathbb{R}^d;\gamma)).$$

This and (3.20) prove the upper bound of (3.19). ∎

For the Sobolev function class $\boldsymbol{W}^r_2(\mathbb{R}^d;\mu)$, Theorems 3.2 and 3.3 are the following results on sampling, linear and Kolmogorov n-widths.

Theorem 3.4 *Let $r \in \mathbb{N}$. Then there hold the right asymptotic orders*

$$\lambda_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) = d_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) \asymp n^{-\frac{r}{2}}(\log n)^{\frac{(d-1)r}{2}}. \tag{3.23}$$

Theorem 3.5 *Let $r \in \mathbb{N}$ and $r \ge 2$. Then there holds the right asymptotic order*

$$\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) \asymp n^{-\frac{r}{2}}(\log n)^{\frac{(d-1)r}{2}}, \tag{3.24}$$

We stress that the assumptions $r > 1$ in Theorem 3.3 for (3.19) is vital since it is a necessary and sufficient condition for $\mathcal{H}^r_g$ to be a separable reproducing kernel Hilbert space with the finite trace condition (3.22) and therefore, Corollary 2.13 can be applied. We conjecture that the consequent right asymptotic order (3.24) still holds true for $r = 1$. Here it may require a different technique.

We now extend the results (3.8) and (3.9) to the weighted function class $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ and weighted space $L_q(\mathbb{R}^d;\mu)$ for the measure μ associated with Freud-type weights as the density function given by (3.3).

For given r and p, q, we make use of the abbreviations:

$$\lambda_n := \lambda_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)), \quad d_n := d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)),$$

$$\varrho_n := \varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)).$$

We prove the right asymptotic orders

$$\varrho_n \asymp n^{-r}(\log n)^{(d-1)r} \tag{3.25}$$

for $1 \le q \le 2 < p < \infty$, and

$$\lambda_n \asymp d_n \asymp n^{-r}(\log n)^{(d-1)r}. \tag{3.26}$$

for $1 \le q < p < \infty$. Moreover, we explicitly construct asymptotically optimal approximation methods for λ_n and d_n in the case $1 \le q < p < \infty$.

Let us describe our strategy to prove the upper and lower bounds of (3.26) for linear and Kolmogorov n-widths, and construct asymptotically optimal linear approximation methods when $1 \le q < p < \infty$. For the sampling n-widths, the result (3.25) can be deduced from (3.26) and the results from [26] presented in Subsection 2.4.

Let $r \in \mathbb{N}$, $1 \le q < p < \infty$ and $\alpha > 0$, $\beta \ge 0$. Denote by $\tilde{L}^q(\mathbb{I}^d)$ and $\tilde{W}^r_p(\mathbb{I}^d)$ the subspaces of $L_q(\mathbb{I}^d)$ and $W^r_p(\mathbb{I}^d)$, respectively, of all functions f which can be extended to the whole $\mathbb{R}^d$ as 1-periodic functions in each variable (denoted again by f). Let A_m be a linear operator in $\tilde{L}^q(\mathbb{I}^d)$ of rank $\le m$. Assume it holds that

$$\|f - A_m(f)\|_{\tilde{L}^q(\mathbb{I}^d)} \le Cm^{-\alpha}(\log m)^\beta \|f\|_{\tilde{W}^r_p(\mathbb{I}^d)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d). \tag{3.27}$$

Then based on A_m, we will construct a linear operator A^μ_m in $L_q(\mathbb{R}^d;\mu)$ which approximates $f \in W^r_p(\mathbb{R}^d;\mu)$ with the same bound as in (3.27). Therefore, with $\alpha = r$ and $\beta = (d-1)r$ we prove the upper bound of (3.9).

Fix a number θ with $1 < \theta < 2$. Denote by $\tilde{L}_q(\mathbb{I}^d_\theta)$ and $\tilde{W}^r_p(\mathbb{I}^d_\theta)$ the subspaces of $L_q(\mathbb{I}^d_\theta)$ and $W^r_p(\mathbb{I}^d_\theta)$, respectively, of all functions f which can be extended to the whole $\mathbb{R}^d$ as θ-periodic functions in each variable (denoted again by f). A linear operator A_m induces the linear operator $A_{\theta,m}$ in $\tilde{L}_q(\mathbb{I}^d_\theta)$, defined for $f \in \tilde{L}_q(\mathbb{I}^d_\theta)$ by

$$A_{\theta,m}(f) := A_m(f(\cdot/\theta)).$$

From (3.27) it follows that

$$\|f - A_{\theta,m}(f)\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \le Cm^{-\alpha}(\log m)^\beta \|f\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d_\theta).$$

Since $q < p$, we can choose a fixed $\delta > 0$ such that

$$e^{\frac{a|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^\lambda}{p} - \frac{a|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^\lambda}{q}} \le Ce^{-\delta|\boldsymbol{k}|}, \quad \boldsymbol{k} \in \mathbb{Z}^d. \tag{3.28}$$

(Here, a and λ are the parameters in the definition of the weigh w in (3.3)–(3.4).)

We define for $n \in \mathbb{R}_1$,

$$\xi_n = \sqrt{\delta^{-1} 2\alpha(\log n)}\,, \tag{3.29}$$

and for $\boldsymbol{k} \in \mathbb{Z}^d$,

$$n_{\boldsymbol{k}} = \begin{cases} \varrho n e^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|^\lambda} & \text{if } |\boldsymbol{k}| < \xi_n, \\ 0 & \text{if } |\boldsymbol{k}| \geq \xi_n, \end{cases} \tag{3.30}$$

where $\varrho := 2^{-d}\left(1 - e^{-\frac{a\delta}{\alpha}}\right)^d$. We write $\mathbb{I}^d_{\theta,\boldsymbol{k}} := \boldsymbol{k} + \mathbb{I}^d_\theta$ for $\boldsymbol{k} \in \mathbb{Z}^d$, and denote by $f_{\theta,\boldsymbol{k}}$ the restriction of f on $\mathbb{I}^d_{\theta,\boldsymbol{k}}$ for a function f on $\mathbb{R}^d$.

It is well-known that one can constructively define a unit partition $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ such that

(i) $\varphi_{\boldsymbol{k}} \in C_0^\infty(\mathbb{R}^d)$ and $0 \leq \varphi_{\boldsymbol{k}}(\boldsymbol{x}) \leq 1, \quad \boldsymbol{x} \in \mathbb{R}^d, \quad \boldsymbol{k} \in \mathbb{Z}^d$;

(ii) $\operatorname{supp}\varphi_{\boldsymbol{k}}$ are contained in the interior of $\mathbb{I}^d_{\theta,\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{Z}^d$;

(iii) $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_{\boldsymbol{k}}(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \mathbb{R}^d$;

(iv) $\|\varphi_{\boldsymbol{k}}\|_{W^r_p(\mathbb{I}^d_{\theta,\boldsymbol{k}})} \leq C_{r,d,\theta}, \quad \boldsymbol{k} \in \mathbb{Z}^d$,

(see, e.g., [41, Chapter VI, 1.3]).

By using the items (ii) and (iv) we have that if $f \in W^r_p(\mathbb{R}^d;\mu)$, then

$$f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in \tilde{W}^r_p(\mathbb{I}^d_\theta),$$

and it holds that

$$\|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} \ll e^{\frac{a|\boldsymbol{k}+(\theta \operatorname{sign}\boldsymbol{k})/2|^\lambda}{p}} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \tag{3.31}$$

We define the linear operator $A^w_{\theta,n}$ in $L_q(\mathbb{R}^d;\mu)$ by

$$\left(A^w_{\theta,n} f\right)(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|<\xi_n} \left(A_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}}\right)(\boldsymbol{x} - \boldsymbol{k}), \tag{3.32}$$

where $\tilde{f}_{\theta,\boldsymbol{k}}(\boldsymbol{x}) = f_{\theta,\boldsymbol{k}}(\boldsymbol{x} + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\boldsymbol{x} + \boldsymbol{k})$. The rank of this operator is not greater than n. Indeed,

$$\operatorname{rank} A^w_{\theta,n} \leq \sum_{|\boldsymbol{k}|<\xi_n} \operatorname{rank} A_{\theta,n_{\boldsymbol{k}}} \leq \sum_{|\boldsymbol{k}|<\xi_n} n_{\boldsymbol{k}},$$

and

$$\begin{aligned} \sum_{|\boldsymbol{k}|<\xi_n} n_{\boldsymbol{k}} = \sum_{|\boldsymbol{k}|<\xi_n} \varrho n e^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|^\lambda} &\leq 2^d \varrho n \sum_{s=0}^{\lfloor \xi_n \rfloor} \binom{s+d-1}{d-1} e^{-\frac{a\delta}{\alpha}s^2} \\ &\leq 2^d \varrho n \sum_{s=0}^{\infty} \binom{s+d-1}{d-1} e^{-\frac{a\delta}{\alpha}s} \leq n, \end{aligned} \tag{3.33}$$

where in the last estimate we used the well-known formula

$$\sum_{j=0}^{\infty} x^j \binom{k+j}{k} = (1-x)^{-k-1}, \; k \in \mathbb{N}_0, \; x \in (0,1). \tag{3.34}$$

Theorem 3.6 *Let $r \in \mathbb{N}$, $1 \le q < p < \infty$ and $\alpha > 0$, $\beta \ge 0$, $1 < \theta < 2$. Assume that for any $m \in \mathbb{R}_1$, there is a linear operator A_m in $\tilde{L}_q(\mathbb{I}^d)$ of rank $\le m$ such that the convergence rate (3.27) holds. Then for any $n \in \mathbb{R}_1$, based on this linear operator one can construct the linear operator $A^{\mu}_{\theta,n}$ in $L_q(\mathbb{R}^d;\gamma)$ of rank $\le n$ as in (3.32) so that*

$$\|f - A^{\mu}_{\theta,n}(f)\|_{L_q(\mathbb{R}^d;\mu)} \le Cn^{-\alpha}(\log n)^{\beta}\|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \quad f \in W^r_p(\mathbb{R}^d;\mu). \tag{3.35}$$

Proof. We preliminarily decompose a function in $W^r_p(\mathbb{R}^d;\mu)$ into a sum of functions on $\mathbb{R}^d$ having support contained in integer translations of the d-cube $\mathbb{I}^d_\theta := \left[-\frac{\theta}{2}, \frac{\theta}{2}\right]$. Then a desired linear operator for $W^r_p(\mathbb{R}^d,\gamma)$ will be the sum of integer-translated dilations of A_m. Details of this construction are presented below.

First observe that

$$\mathbb{R}^d = \bigcup_{\boldsymbol{k}\in\mathbb{Z}^d} \mathbb{I}^d_{\theta,\boldsymbol{k}},$$

where $\mathbb{I}^d_{\theta,\boldsymbol{k}} := \mathbb{I}^d_\theta + \boldsymbol{k}$. From the items (ii) and (iii) it is implied that

$$f = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}, \tag{3.36}$$

where $f_{\theta,\boldsymbol{k}}$ denotes the restriction to $\mathbb{I}^d_{\theta,\boldsymbol{k}}$. Hence we have

$$\begin{aligned}\|f - A^{w}_{\theta,n}(f)\|^q_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} = &\sum_{|\boldsymbol{k}|<\xi_n} \left\|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left(A_{\theta,n_{\boldsymbol{k}}}\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|^q_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \\ &+ \sum_{|\boldsymbol{k}|\ge\xi_n} \|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\|^q_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)}.\end{aligned} \tag{3.37}$$

By (3.30), (3.27) and (3.31) we derive the estimates

$$\begin{aligned}&\left\|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left(A_{\theta,n_{\boldsymbol{k}}}\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \\ &\ll e^{-\frac{a|\boldsymbol{k}-(\theta\operatorname{sign}\boldsymbol{k})/2|^\lambda}{q}} \left\|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - A_{\theta,n_{\boldsymbol{k}}}\tilde{f}_{\theta,\boldsymbol{k}}\right\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \\ &\ll e^{-\frac{a|\boldsymbol{k}-(\theta\operatorname{sign}\boldsymbol{k})/2|^\lambda}{q}} n_{\boldsymbol{k}}^{-\alpha}(\log n_{\boldsymbol{k}})^{\beta}\|f(\cdot + \boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)}. \\ &\ll e^{\frac{a|\boldsymbol{k}+(\theta\operatorname{sign}\boldsymbol{k})/2|^\lambda}{p} - \frac{a|\boldsymbol{k}-(\theta\operatorname{sign}\boldsymbol{k})/2|^\lambda}{q}} \left(ne^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|^\lambda}\right)^{-\alpha}(\log n)^{\beta}\|f\|_{W^r_p(\mathbb{R}^d;\mu)}.\end{aligned}$$

Using (3.28) we get

$$\left\|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left(A_{\theta,n_{\boldsymbol{k}}}\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \ll e^{-a\delta|\boldsymbol{k}|^\lambda} n^{-\alpha}(\log n)^{\beta}\|f\|_{W^r_p(\mathbb{R}^d;\mu)},$$

which implies

$$\begin{aligned}\sum_{|\boldsymbol{k}|<\xi_n} \left\|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left(A_{\theta,n_{\boldsymbol{k}}}\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|^q_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} &\ll \sum_{|\boldsymbol{k}|<\xi_n} e^{-qa\delta|\boldsymbol{k}|^\lambda}\left(n^{-\alpha}(\log n)^{\beta}\right)^q\|f\|^q_{W^r_p(\mathbb{R}^d;\mu)} \\ &\ll \left(n^{-\alpha}(\log n)^{\beta}\right)^q\|f\|^q_{W^r_p(\mathbb{R}^d;\mu)}.\end{aligned}$$

We have for a fixed $\varepsilon \in (0, 1/2)$,

$$\begin{aligned}\sum_{|\boldsymbol{k}|\ge\xi_n} \|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\|^q_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} &\ll \sum_{|\boldsymbol{k}|\ge\xi_n} e^{-q\left(\frac{a|\boldsymbol{k}-(\theta\operatorname{sign}\boldsymbol{k})/2|^\lambda}{q} - \frac{a|\boldsymbol{k}+(\theta\operatorname{sign}\boldsymbol{k})/2|^\lambda}{p}\right)} \|f\|^q_{W^r_p(\mathbb{R}^d;\mu)} \\ &\ll \sum_{|\boldsymbol{k}|\ge\xi_n} e^{-q\delta|\boldsymbol{k}|^\lambda} \|f\|^q_{W^r_p(\mathbb{R}^d;\mu)} \ll e^{-q\delta(1-\varepsilon)\xi_n^2} \|f\|^q_{W^r_p(\mathbb{R}^d;\mu)} \\ &\ll e^{-2qa(1-\varepsilon)\log n} \|f\|^q_{W^r_p(\mathbb{R}^d;\mu)} \ll \left(n^{-\alpha}(\log n)^\beta\right)^q \|f\|^q_{W^r_p(\mathbb{R}^d;\mu)}.\end{aligned}$$

From the last two estimates and (3.37) we obtain (3.35). $\square$

The lower bound of (3.9) relies on the following known results on Kolmogorov n-widths in the space $\tilde{L}_q(\mathbb{I}^d)$ the unweighted Sobolev class $\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d)$.

Lemma 3.7 *Let $r \in \mathbb{N}$ and $1 \le q < p < \infty$. Then we have the right asymptotic order*

$$d_m(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) \asymp m^{-r}(\log m)^{(d-1)r}.$$

Moreover, truncations on certain hyperbolic crosses of the Fourier series form an asymptotically optimal linear operator A_n in $\tilde{L}_q(\mathbb{I}^d)$ of rank $\le n$ such that

$$\|f - A_m f\|_{\tilde{L}_q(\mathbb{I}^d)} \ll m^{-r}(\log m)^{(d-1)r} \|f\|_{\tilde{W}^r_p(\mathbb{I}^d)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d). \tag{3.38}$$

For details on this lemma see, e.g., in [17, Theorems 4.2.5, 4.3.1 & 4.3.7] and related comments on the asymptotic optimality of the hyperbolic cross approximation.

We are now in the position to prove the main result in this section.

Theorem 3.8 *Let $r \in \mathbb{N}$ and $1 \le q < p < \infty$. Then for any $n \in \mathbb{R}_1$, based on the linear operator A_m in Lemma 3.7 one can construct the linear operator A^μ_n in $L_q(\mathbb{R}^d, \gamma)$ of rank $\le n$ as in* (3.32) *so that there hold the right asymptotic orders*

$$\sup_{f\in\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)} \|f - A^\mu_n(f)\|_{L_q(\mathbb{R}^d;\mu)} \asymp \lambda_n \asymp d_n \asymp n^{-r}(\log n)^{(d-1)r}. \tag{3.39}$$

Proof. For a fixed $1 < \theta < 2$, we define $A^\mu_n := A^\mu_{\theta,n}$ as the linear operator described in Theorem 3.6. The upper bounds in (3.39) follow from (3.38) and Theorem 3.6 with $\alpha = r$, $\beta = (d-1)r$.

If f is a 1-periodic function on $\mathbb{R}^d$ and $f \in \tilde{W}^r_p(\mathbb{I}^d)$, then

$$\begin{aligned}\|f\|_{W^r_p(\mathbb{R}^d;\mu)} &= \left(e^{db} \sum_{|\boldsymbol{r}|_\infty\le r} \int_{\mathbb{R}^d} |D^{\boldsymbol{r}} f|^p e^{-a|\boldsymbol{x}|^\lambda} \mathrm{d}\boldsymbol{x}\right)^{1/p} \\ &\ll \left(\sum_{|\boldsymbol{r}|_\infty\le r} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \int_{\mathbb{I}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x}+\boldsymbol{k})|^p e^{-a|\boldsymbol{x}+\boldsymbol{k}|^\lambda} \mathrm{d}\boldsymbol{x}\right)^{1/p} \\ &\ll \left(\sum_{|\boldsymbol{r}|_\infty\le r} \int_{\mathbb{I}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} e^{-a|\boldsymbol{k}-(\operatorname{sign}\boldsymbol{k})/2|^\lambda}\right)^{1/p} \\ &\ll \|f\|_{\tilde{W}^r_p(\mathbb{I}^d)},\end{aligned}$$

and

$$\|f\|_{\tilde{L}_q(\mathbb{I}^d)} \le e^{\frac{db}{q}} e^{\frac{d}{8q}} \|f\|_{L_q(\mathbb{R}^d;\mu)}.$$

Hence we get

$$\lambda_n \ge d_n \gg d_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)).$$

Now Lemma 3.7 implies the lower bounds in (3.39). $\square$

Theorem 3.9 *Let $r \in \mathbb{N}$ and $1 \le q \le 2 < p < \infty$. Then there holds the right asymptotic order*

$$\varrho_n \asymp n^{-r} (\log n)^{(d-1)r}. \tag{3.40}$$

Proof. The lower bound of (3.40) follows from (2.1) and (3.39). By the norm inequality $\|\cdot\|_{L_q(\mathbb{R}^d;\mu)} \ll \|\cdot\|_{L_2(\mathbb{R}^d;\mu)}$ for $1 \le q \le 2$, it is sufficient to prove the upper bound of (3.40) for $q = 2$. By (3.39) we have that

$$d_n \ll n^{-r} (\log n)^{(d-1)r}. \tag{3.41}$$

Notice that the separable normed space $W^r_p(\mathbb{R}^d;\mu)$ is continuously embedded into $L_2(\mathbb{R}^d;\mu)$, and the evaluation functional $f \mapsto f(\boldsymbol{x})$ is continuous on the space $W^r_p(\mathbb{R}^d;\mu)$ for each $\boldsymbol{x} \in \mathbb{R}^d$. This means that $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ satisfies Assumption A in Subsection 2.4. By Corollary 2.16 and (3.41) we prove the upper bound for $q = 2$:

$$\varrho_n \ll d_n \ll n^{-r} (\log n)^{(d-1)r}.$$

$\square$

4 Numerical weighted integration in the space $W^r_p(\mathbb{R}^d;\mu)$

4.1 Introducing remarks

The aim of this section is to present and extend some recent results of [16] on numerical weighted integration over $\mathbb{R}^d$ for functions from weighted Sobolev spaces $W^r_p(\mathbb{R}^d;\mu)$ of mixed smoothness $r \in \mathbb{N}$ for $1 < p < \infty$ and the measure μ with the density function given by (3.3). The next section is devoted to this problem in the case $p = 1$ which requires a different approach. Extending the results from [16] for the Gaussian-weighted case, we prove the right asymptotic order of the quantity of optimal quadrature for the Sobolev function class $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ and construct asymptotically optimal quadratures.

There is a large number of works on high-dimensional unweighted integration over the unit d-cube $\mathbb{I}^d := [0,1]^d$ for functions having a mixed smoothness (see [17, 24, 47] for results and bibliography). However, there are only a few works on high-dimensional weighted integration for functions having a mixed smoothness. The problem of optimal weighted integration (1.2)–(1.4) has been studied in [29, 30, 23] for functions in certain

Hermite spaces, in particular, the space $\mathcal{H}_{d,r}$ which coincides with $W_2^r(\mathbb{R}^d;\gamma)$ in terms of norm equivalence. It has been proven in [23] that

$$n^{-r}(\log n)^{(d-1)/2} \ll \mathrm{Int}_n\big(\boldsymbol{W}_2^r(\mathbb{R}^d;\gamma)\big) \ll n^{-r}(\log n)^{d(2r+3)/4-1/2}.$$

Moreover, the upper bound is achieved by a translated and scaled quasi-Monte Carlo (QMC) quadrature based on Dick's high order digital nets.

Recently, in [16, Theorem 2.3] for the space $W_p^r(\mathbb{R}^d,\gamma)$ with $r \in \mathbb{N}$ and $1 < p < \infty$, we have constructed an asymptotically optimal quadrature Q_n^γ of the form (1.3) which gives the asymptotic order

$$\sup_{f\in \boldsymbol{W}_p^r(\mathbb{R}^d;\gamma)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\gamma(\mathrm{d}\boldsymbol{x}) - Q_n^\gamma f \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{R}^d;\gamma)\big) \asymp n^{-r}(\log n)^{(d-1)/2}. \tag{4.1}$$

In constructing Q_n^γ, we proposed a novel method assembling an asymptotically optimal quadrature for the related Sobolev spaces of the same mixed smoothness r on the unit d-cube to the integer-shifted d-cubes which cover $\mathbb{R}^d$. The asymptotically optimal quadrature Q_n^γ is based on very sparse integration nodes contained in a d-ball of radius $\sqrt{\log n}$.

In this section, we show that the result (4.1) can be extended by a modification of this technique to the weighted Sobolev spaces $W_p^r(\mathbb{R}^d;\mu)$ for $1 < p < \infty$.

4.2 Assembling quadratures

In this subsection, based on a quadrature on the d-cube $\mathbb{I}^d := \big[-\frac{1}{2},\frac{1}{2}\big]^d$ for numerical integration of functions from classical Sobolev spaces of mixed smoothness r on $\mathbb{I}^d$, by assembling we construct a quadrature on $\mathbb{R}^d$ for numerical integration of functions from weighted Sobolev spaces $W_p^r(\mathbb{R}^d;\mu)$ which preserves the convergence rate. In the next subsection, as a consequence, we prove the right asymptotic order of $\mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{R}^d;\mu)\big)$.

Let $r \in \mathbb{N}$, $1 < p < \infty$ and $\alpha > 0$, $\beta \geq 0$. Assume that for the quadrature

$$Q_m(f) := \sum_{i=1}^m \lambda_i f(\boldsymbol{x}_i), \quad \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m\} \subset \mathbb{I}^d, \tag{4.2}$$

holds the convergence rate

$$\left| \int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_m(f) \right| \leq C m^{-\alpha}(\log m)^\beta \|f\|_{W_p^r(\mathbb{I}^d)}, \quad f \in W_p^r(\mathbb{I}^d). \tag{4.3}$$

Then by using Q_m, we will construct a quadrature on $\mathbb{R}^d$ which approximates the integral (1.2) with the same bound of the error approximation as in (4.3) for $f \in W_p^r(\mathbb{R}^d;\mu)$.

Our strategy is as follows. The weighted integral

$$\int_{\mathbb{R}^d} f(\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{4.4}$$

can be represented as the sum of component integrals over the integer-shifted d-cubes $\mathbb{I}^d_{\boldsymbol{k}}$ as

$$\int_{\mathbb{R}^d} f(\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x})w_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{4.5}$$

where for $\boldsymbol{k} \in \mathbb{Z}^d$, $\mathbb{I}^d_{\boldsymbol{k}} := \boldsymbol{k} + \mathbb{I}^d$ and for a function h on $\mathbb{R}^d$, $h_{\boldsymbol{k}}$ denotes the restriction of h to $\mathbb{I}^d_{\boldsymbol{k}}$. For a given $n \in \mathbb{R}_1$, we take "shifted" quadratures $Q_{n_{\boldsymbol{k}}}$ of the form (4.2) for approximating the component integrals in the sum in (4.5). The integration nodes in $Q_{n_{\boldsymbol{k}}}$, $\boldsymbol{k} \in \mathbb{Z}^d$, are taken so that

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \lfloor n_{\boldsymbol{k}} \rfloor \le n.$$

In the next step, we "assemble" these shifted integration nodes to form a quadrature $Q_n(f)$ for approximating the integral (4.4). Let us describe this construction in detail.

It is clear that if $f \in W^r_p(\mathbb{R}^d;\mu)$, then $f_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in W^r_p(\mathbb{I}^d)$, and

$$\|f_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d)} \le e^{\frac{a|\boldsymbol{k}+(\mathrm{sign}\boldsymbol{k})/2|^\lambda}{p}} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \tag{4.6}$$

where $\mathrm{sign}\boldsymbol{k} := (\mathrm{signk_1}, \ldots, \mathrm{signk_d})$ and $\mathrm{signx} := 1$ if $x \ge 0$, and $\mathrm{signx} := -1$ otherwise for $x \in \mathbb{R}$. We have

$$\|w_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d)} = \left(\sum_{|\boldsymbol{r}|_\infty \le r} \|D^{\boldsymbol{r}} w\|^p_{L_p(\mathbb{I}^d_{\boldsymbol{k}})} \right)^{1/p}.$$

A direct computation shows that for any $\boldsymbol{s} \in \mathbb{N}^d_0$ with $|\boldsymbol{s}|_\infty \le r$, we have

$$D^{\boldsymbol{s}} w(\boldsymbol{x}) = F_{\boldsymbol{s}}(\boldsymbol{x}) w(\boldsymbol{x}),$$

where

$$F_{\boldsymbol{s}}(\boldsymbol{x}) := \prod_{i=1}^d F_{s_i}(x_i),$$

and for $s \in \mathbb{N}_0$, the univariate function F_s is defined by

$$F_s(x) :== (\mathrm{sign(x)})^{\mathrm{s}} \sum_{\mathrm{j=1}}^{\mathrm{s}} \mathrm{c_{s,j}}(\lambda, \mathrm{a}) |\mathrm{x}|^{\lambda_{\mathrm{s,j}}},$$

where $\mathrm{sign(x)} := 1$ if $x \ge 0$, and $\mathrm{sign(x)} := -1$ if $x < 0$,

$$\lambda_{s,s} = s(\lambda - 1) > \lambda_{s,s-1} > \cdots > \lambda_{s,1} = \lambda - s, \tag{4.7}$$

and $c_{s,j}(\lambda, a)$ are polynomials in the variables λ and a of degree at most s with respect to each variable. Therefore, for $\boldsymbol{x} \in \mathbb{I}^d_{\boldsymbol{k}}$ we get

$$|D^{\boldsymbol{s}} w(\boldsymbol{x})| \le C e^{-\frac{a|\boldsymbol{k}-(\mathrm{sign}\boldsymbol{k})/2|^\lambda}{\tau'}} \le C e^{-\frac{a|\boldsymbol{k}|^\lambda}{\tau}}$$

for some τ' and τ such that $1 < \tau < \tau' < p < \infty$. This implies

$$\|w_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d)} \le C e^{-\frac{a|\boldsymbol{k}|^\lambda}{\tau}} \tag{4.8}$$

with C independent of $\boldsymbol{k} \in \mathbb{Z}^d$. Since $W^r_p(\mathbb{I}^d)$ is a multiplication algebra (see [37, Theorem 3.16]), from (4.6) and (4.8) we have that

$$f_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) w_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in W^r_p(\mathbb{I}^d), \tag{4.9}$$

and

$$\begin{aligned} \|f_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) w_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d)} &\le C \|f_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d)} \cdot \|w_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d)} \\ &\le C e^{\frac{a|\boldsymbol{k}+(\operatorname{sign}\boldsymbol{k})/2|^\lambda}{p} - \frac{a|\boldsymbol{k}|^\lambda}{\tau}} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \end{aligned} \tag{4.10}$$

For $1 < \tau < p < \infty$, we choose $\delta > 0$ such that

$$\max\left\{ e^{-a|\boldsymbol{k}-(\operatorname{sign}\boldsymbol{k})/2|^\lambda \left(1-\frac{1}{\mathrm{p}}\right)}, e^{\frac{a|\boldsymbol{k}+(\operatorname{sign}\boldsymbol{k})/2|^\lambda}{p} - \frac{a|\boldsymbol{k}|^\lambda}{\tau}} \right\} \le C e^{-\delta|\boldsymbol{k}|^\lambda} \tag{4.11}$$

for $\boldsymbol{k} \in \mathbb{Z}^d$, and therefore,

$$\|f_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) w_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d)} \le C e^{-\delta|\boldsymbol{k}|^\lambda} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \quad \boldsymbol{k} \in \mathbb{Z}^d. \tag{4.12}$$

For $n \in \mathbb{R}_1$, let ξ_n and $n_{\boldsymbol{k}}$ be given as in (3.29) and (3.30). We have by (3.33)

$$\sum_{|\boldsymbol{k}|<\xi_n} n_{\boldsymbol{k}} \le n. \tag{4.13}$$

We define

$$Q_n(f) := \sum_{|\boldsymbol{k}|<\xi_n} Q_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) w_{\boldsymbol{k}}(\cdot + \boldsymbol{k})) = \sum_{|\boldsymbol{k}|<\xi_n} \sum_{j=1}^{\lfloor n_{\boldsymbol{k}} \rfloor} \lambda_j f_{\boldsymbol{k}}(\boldsymbol{x}_j + \boldsymbol{k}) w_{\boldsymbol{k}}(\boldsymbol{x}_j + \boldsymbol{k}), \tag{4.14}$$

or equivalently,

$$Q_n(f) := \sum_{|\boldsymbol{k}|<\xi_n} \sum_{j=1}^{\lfloor n_{\boldsymbol{k}} \rfloor} \lambda_{\boldsymbol{k},j} f(\boldsymbol{x}_{\boldsymbol{k},j}) \tag{4.15}$$

as a quadrature for the approximate weighted integration of functions f on $\mathbb{R}^d$, where

$$\boldsymbol{x}_{\boldsymbol{k},j} := \boldsymbol{x}_j + \boldsymbol{k}, \quad \lambda_{\boldsymbol{k},j} := \lambda_j w_{\boldsymbol{k}}(\boldsymbol{x}_j + \boldsymbol{k})$$

(here for simplicity, with an abuse the dependence of integration nodes and weights on the quadratures $Q_{n_{\boldsymbol{k}}}$ is omitted). The integration nodes of the quadrature Q_n are

$$\{\boldsymbol{x}_{\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\, j = 1, \dots, \lfloor n_{\boldsymbol{k}} \rfloor\} \subset \mathbb{R}^d, \tag{4.16}$$

and the integration weights

$$(\lambda_{\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\, j = 1, \dots, \lfloor n_{\boldsymbol{k}} \rfloor).$$

Due to (4.13), the number of integration nodes is not greater than n. From the definition we can see that the integration nodes are contained in the ball of radius $\xi_n^* := \sqrt{d}/2 + \xi_n$, i.e.,

$$\{\boldsymbol{x}_{\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\ j = 1,\dots,\lfloor n_{\boldsymbol{k}} \rfloor\} \subset B(\xi_n^*) := \left\{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| \le \xi_n^*\right\}.$$

The density of the integration nodes is exponentially decreasing in $|\boldsymbol{k}|$ to zero from the origin of $\mathbb{R}^d$ to the boundary of the ball $B(\xi_n^*)$, and the set of integration nodes is very sparse because of the choice of $n_{\boldsymbol{k}}$ as in (3.30).

Theorem 4.1 *Let $r \in \mathbb{N}$, $1 < p < \infty$ and $\alpha > 0$, $\beta \ge 0$. Assume that for any $m \in \mathbb{R}_1$, there is a quadrature Q_m of the form* (4.2) *satisfying* (4.3). *Then for the quadrature Q_n defined as in* (4.15), *we have*

$$\left|\int_{\mathbb{R}^d} f(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}) - Q_n(f)\right| \ll n^{-\alpha}(\log n)^{\beta}\|f\|_{W_p^r(\mathbb{R}^d;\mu)}, \quad f \in W_p^r(\mathbb{R}^d;\mu). \tag{4.17}$$

Proof. Let $f \in W_p^r(\mathbb{R}^d;\mu)$ and $m \in \mathbb{R}_1$. For the quadrature I_m for functions on $\mathbb{I}^d$ in the assumption, by (4.3) and (4.12), we have

$$\begin{aligned}
&\left|\int_{\mathbb{I}^d} f_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k})w_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k})\mathrm{d}\boldsymbol{x} - I_m(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})w_{\boldsymbol{k}}(\cdot+\boldsymbol{k}))\right| \\
&\ll m^{-r}(\log m)^{\frac{d-1}{2}}\|f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})w_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{W_p^r(\mathbb{I}^d)} \\
&\ll m^{-\alpha}(\log m)^{\beta}e^{-\delta|\boldsymbol{k}|^{\lambda}}\|f\|_{W_p^r(\mathbb{R}^d;\mu)}.
\end{aligned} \tag{4.18}$$

From (4.5) and (4.14) it follows that

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} f(\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_n(f)\right| \le & \sum_{|\boldsymbol{k}|<\xi_n}\left|\int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x})w_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})w_{\boldsymbol{k}}(\cdot+\boldsymbol{k}))\right| \\
&+ \sum_{|\boldsymbol{k}|\ge\xi_n}\left|\int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x})w_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|.
\end{aligned}$$

For each term in the first sum, by (4.18) we derive the estimates

$$\begin{aligned}
&\left|\int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x})w_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})w_{\boldsymbol{k}}(\cdot+\boldsymbol{k}))\right| \\
&\qquad = \left|\int_{\mathbb{I}^d} f_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k})w_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k})\mathrm{d}\boldsymbol{x} - Q_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k})w_{\boldsymbol{k}}(\cdot+\boldsymbol{k}))\right| \\
&\qquad \ll n_{\boldsymbol{k}}^{-\alpha}(\log n_{\boldsymbol{k}})^{b}e^{-\delta|\boldsymbol{k}|^{\lambda}}\|f\|_{W_p^r(\mathbb{R}^d;\mu)} \\
&\qquad \ll (ne^{-\frac{\delta}{2\alpha}|\boldsymbol{k}|^{\lambda}})^{-\alpha}(\log n)^{b}e^{-\delta|\boldsymbol{k}|^{\lambda}}\|f\|_{W_p^r(\mathbb{R}^d;\mu)} \\
&\qquad \ll n^{-\alpha}(\log n)^{b}e^{-\frac{a|\boldsymbol{k}|^{\lambda}\delta}{2}}\|f\|_{W_p^r(\mathbb{R}^d;\mu)}.
\end{aligned}$$

Hence,

$$\sum_{|\boldsymbol{k}|<\xi_n} \left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) w_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_{n_{\boldsymbol{k}}}(f_{\boldsymbol{k}}(\cdot+\boldsymbol{k}) w_{\boldsymbol{k}}(\cdot+\boldsymbol{k})) \right|$$
$$\ll \sum_{|\boldsymbol{k}|<\xi_n} n^{-\alpha}(\log n)^b e^{-\frac{a|\boldsymbol{k}|^\lambda \delta}{2}} \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \ll n^{-\alpha}(\log n)^b \|f\|_{W^r_p(\mathbb{R}^d;\mu)}.$$

For each term in the second sum we get by Hölder's inequality and (4.11),

$$\begin{aligned} \left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) w_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right| &\le \left(\int_{\mathbb{I}^d_{\boldsymbol{k}}} |f_{\boldsymbol{k}}(\boldsymbol{x})|^p w_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}} \left(\int_{\mathbb{I}^d_{\boldsymbol{k}}} w_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right)^{1-\frac{1}{p}} \\ &\ll e^{-a|\boldsymbol{k}-(\operatorname{sign}\boldsymbol{k})/2|^\lambda(1-\frac{1}{p})} \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \\ &\ll e^{-\delta|\boldsymbol{k}|^\lambda} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \end{aligned}$$

which implies

$$\begin{aligned} \sum_{|\boldsymbol{k}|\ge\xi_n} \left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) w_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right| &\le \sum_{|\boldsymbol{k}|\ge\xi_n} e^{-\delta|\boldsymbol{k}|^\lambda} \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \\ &\le 2^d \sum_{s=\lceil\xi_n\rceil}^{\infty} e^{-s^2\delta} \binom{s+d-1}{d-1} \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \\ &\le 2^d e^{-\xi_n^2\delta(1-\varepsilon)} \sum_{s=\lceil\xi_n\rceil}^{\infty} e^{-s^2\varepsilon\delta} \binom{s+d-1}{d-1} \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \\ &\ll e^{-\xi_n^2\delta(1-\varepsilon)} \sum_{s=0}^{\infty} e^{-s\varepsilon\delta} \binom{s+d-1}{d-1} \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \end{aligned} \tag{4.19}$$

with $\varepsilon \in (0, 1/2)$. Using (3.34) we get

$$\sum_{|\boldsymbol{k}|\ge\xi_n} \left| \int_{\mathbb{I}^d_{\boldsymbol{k}}} f_{\boldsymbol{k}}(\boldsymbol{x}) w_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right| \ll e^{-2\alpha(1-\varepsilon)\log n} \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \ll n^{-\alpha}(\log n)^b \|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \tag{4.20}$$

Summing up, we prove (4.17). ∎

Some important quadratures such as the Frolov quadratue and the QMC quadrature based on Fibonacci lattice rules ($d = 2$) are constructively designed for functions on $\mathbb{R}^d$ with support contained in the unit d-cube or for 1-periodic functions. To employ them for constructing a quadrature for functions on $\mathbb{R}^d$ we need to modify those constructions.

Assume that there is a quadrature Q_m of the form (4.2) with the integration nodes $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$ and weights $(\lambda_1, \dots, \lambda_m)$ such that the convergence rate

$$\left| \int_{\mathbb{I}^d} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_m(f) \right| \le C m^{-\alpha}(\log m)^\beta \|f\|_{W^r_p(\mathbb{I}^d)}, \quad f \in \mathring{W}^r_p(\mathbb{I}^d) \tag{4.21}$$

holds where $\mathring{W}^r_p(\mathbb{I}^d)$ denotes the space of functions in $W^r_p(\mathbb{R}^d)$ with support contained in $\mathbb{I}^d$. Then based on the quadrature Q_m, we propose two constructions of quadratures which approximate the integral (4.4) with the same convergence rate for $f \in W^r_p(\mathbb{R}^d;\mu)$.

The first method is a preliminary change of variables to transform the quadrature Q_m into a quadrature for functions in $W^r_p(\mathbb{I}^d)$ which gives the same convergence rate, and then apply the construction as in (4.15). Let us describe it. Let ψ_k be the function defined by

$$\psi_k(t) = \begin{cases} C_k \int_0^t \xi^k (1-\xi)^k \,\mathrm{d}\xi, & t \in [0,1]; \\ 1, & t > 1; \\ 0, & t < 0\,, \end{cases} \tag{4.22}$$

where

$$C_k = \Big(\int_0^1 \xi^k (1-\xi)^k \,\mathrm{d}\xi \Big)^{-1}.$$

If $f \in W^r_p(\mathbb{I}^d)$, a change of variable yields that

$$\int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbb{I}^d} \big(T_{\psi_k} f\big)(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where the function $T_{\psi_k} f$ is defined by

$$\big(T_{\psi_k} f\big)(\boldsymbol{x}) := \left(\prod_{i=1}^d \psi_k'(x_i) \right) f\big(\psi_k(x_1), \dots, \psi_k(x_d)\big). \tag{4.23}$$

Observe that the support of $T_{\psi_k} f$ is contained in $\mathbb{I}^d$. If $T_{\psi_k} f$ belongs to $\mathring{W}^\alpha_p(\mathbb{I}^d)$, then a quadrature with the integration nodes $\{\tilde{\boldsymbol{x}}_1, \dots, \tilde{\boldsymbol{x}}_m\} \subset \mathbb{I}^d$ and weights $(\tilde{\lambda}_1, \dots, \tilde{\lambda}_m)$ for the function f can be defined as

$$\tilde{Q}_m(f) := Q_m(T_{\psi_k} f) = \sum_{j=1}^m \tilde{\lambda}_j f(\tilde{\boldsymbol{x}}_j),$$

where

$$\tilde{\boldsymbol{x}}_j = (\psi_k(x_{j,1}), \dots, \psi_k(x_{j,d})) \ \text{ and } \ \tilde{\lambda}_j = \lambda_j \psi_k'(x_{j,1}) \cdot \ldots \cdot \psi_k'(x_{j,d}).$$

Hence, our task is finding a condition on k so that the mapping

$$T_{\psi_k} : W^r_p(\mathbb{I}^d) \to \mathring{W}^r_p(\mathbb{I}^d)$$

defined by (4.23), is a bounded operator from $W^r_p(\mathbb{I}^d)$ to $\mathring{W}^r_p(\mathbb{I}^d)$, see, e.g., [44, Theorem IV.4.1].

The second method is based on the decomposition (3.36) of functions in $W^r_p(\mathbb{R}^d;\mu)$. Let $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ be the unit partition satisfying items (i)–(iv), which is defined in Subsection 3.2. By the items (ii) and (iii), the integral (4.4) can be represented as

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \int_{\mathbb{I}^d_{\theta,\boldsymbol{k}}} f_{\theta,\boldsymbol{k}}(\boldsymbol{x}) w_{\theta,\boldsymbol{k}}(\boldsymbol{x}) \varphi_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{4.24}$$

where $f_{\theta,\boldsymbol{k}}$ and $w_{\theta,\boldsymbol{k}}$ denote the restrictions of f and w on $\mathbb{I}^d_{\theta,\boldsymbol{k}}$, respectively. The quadrature (4.2) induces the quadrature

$$Q_{\theta,m}(f) := \sum_{i=1}^{m} \lambda_{\theta,i} f(\boldsymbol{x}_{\theta,i}), \tag{4.25}$$

for functions f on $\mathbb{I}^d_\theta$, where $\boldsymbol{x}_{\theta,i} := \theta \boldsymbol{x}_i$ and $\lambda_{\theta,i} := \theta\lambda_i$.

Denote by $\mathring{W}^r_p(\mathbb{I}^d_\theta)$ the subspace of function in $W^r_p(\mathbb{R}^d)$ with support contained in $\mathbb{I}^d_\theta$. From (4.21) the error bound

$$\left| \int_{\mathbb{I}^d_\theta} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_{\theta,m}(f) \right| \ll m^{-\alpha} (\log m)^\beta \|f\|_{W^r_p(\mathbb{I}^d_\theta)}$$

holds for every $f \in \mathring{W}^r_p(\mathbb{I}^d_\theta)$. Let $f \in W^r_p(\mathbb{R}^d;\mu)$. It is clear that $f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in W^r_p(\mathbb{I}^d_\theta)$ and

$$\|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d_\theta)} \ll e^{\frac{a|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^\lambda}{p}} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \quad f \in W^r_p(\mathbb{R}^d;\mu), \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

Similarly to (4.9) and (4.10), by additionally using the items (ii) and (iv) we have that

$$f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) w_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) \varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in \mathring{W}^r_p(\mathbb{I}^d_\theta),$$

and

$$\|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) w_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) \varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{W^r_p(\mathbb{I}^d_\theta)} \ll e^{\frac{a|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^\lambda}{p} - \frac{a|\boldsymbol{k}|^\lambda}{\tau}} \|f\|_{W^r_p(\mathbb{R}^d;\mu)},$$

where τ is a fixed number satisfying the inequalies $1 < \tau < p < \infty$. We choose $\delta > 0$ so that

$$\max\left\{ e^{-a|\boldsymbol{k}-(\theta \operatorname{sign} \boldsymbol{k})/2|^\lambda \left(1-\frac{1}{\mathrm{p}}\right)}, e^{\frac{a|\boldsymbol{k}+(\theta \operatorname{sign} \boldsymbol{k})/2|^\lambda}{p} - \frac{a|\boldsymbol{k}|^\lambda}{\tau}} \right\} \le C e^{-\delta |\boldsymbol{k}|^\lambda}, \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

For $n \in \mathbb{R}_1$, let ξ_n and $n_{\boldsymbol{k}}$ be given as in (3.29) and (3.30), respectively. Noting (4.24) and (4.25), we define

$$Q_{\theta,n}(f) := \sum_{|\boldsymbol{k}| < \xi_n} Q_{\theta,n_{\boldsymbol{k}}} \big(f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) w_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k}) \varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \big),$$

or equivalently,

$$Q_{\theta,n}(f) := \sum_{|\boldsymbol{k}| < \xi_n} \sum_{j=1}^{\lfloor n_{\boldsymbol{k}} \rfloor} \lambda_{\theta,\boldsymbol{k},j} f(\boldsymbol{x}_{\theta,\boldsymbol{k},j}) \tag{4.26}$$

as a linear quadrature for the approximate integration of weighted functions f on $\mathbb{R}^d$, where

$$\boldsymbol{x}_{\theta,\boldsymbol{k},j} := \boldsymbol{x}_{\theta,i} + \boldsymbol{k}, \quad \lambda_{\theta,\boldsymbol{k},j} := \lambda_{\theta,i} w_{\boldsymbol{k}}(\boldsymbol{x}_{\theta,\boldsymbol{k},j}) \varphi_{\boldsymbol{k}}(\boldsymbol{x}_{\theta,\boldsymbol{k},j}).$$

The integration nodes and the weights of the quadrature $Q_{\theta,n}$ are

$$\{\boldsymbol{x}_{\theta,\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\, j = 1, \ldots, \lfloor n_{\boldsymbol{k}} \rfloor\} \subset \mathbb{R}^d, \tag{4.27}$$

and

$$(\lambda_{\theta,\boldsymbol{k},j} : |\boldsymbol{k}| < \xi_n,\ j = 1, \ldots, \lfloor n_{\boldsymbol{k}} \rfloor).$$

Due to (4.13), the number of integration nodes is not greater than n. Moreover, from the definition we can see that the integration nodes are contained in the ball of radius $\xi^*_{\theta,n} := \theta\sqrt{d}/2 + \xi_n$, i.e.,

$$\{\boldsymbol{x}_{\theta,\boldsymbol{k},j} : |\boldsymbol{k}| < \xi^*_{\theta,n},\ j = 1, \ldots, \lfloor n_{\boldsymbol{k}} \rfloor\} \subset B(\xi^*_{\theta,n}) := \left\{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| \le \xi^*_{\theta,n}\right\}.$$

Notice that the set of integration nodes (4.27) possesses similar sparsity properties as the set (4.16).

In a way similar to the proof of Theorem 4.1 we derive

Theorem 4.2 *Let $r \in \mathbb{N}$, $1 < p < \infty$ and $\alpha > 0$, $\beta \ge 0$, $1 < \theta < 2$. Assume that for any $m \in \mathbb{R}_1$, there is a quadrature Q_m of the form* (4.2) *with $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$ satisfying* (4.21)*. Then for the quadrature $Q_{\theta,n}$ defined as in* (4.26) *we have*

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}) - Q_{\theta,n}(f) \right| \ll n^{-\alpha}(\log n)^{\beta} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \quad f \in W^r_p(\mathbb{R}^d;\mu). \tag{4.28}$$

4.3 Optimal quadrature

In this subsection, we prove the right asymptotic order of the quantity of optimal quadrature as formulated in (4.1) based on Theorem 4.2 and known results on numerical integration for functions from $W^r_p(\mathbb{I}^d)$.

Theorem 4.3 *Let $r \in \mathbb{N}$ and $1 < p < \infty$. Then one can construct an asymptotically optimal family of quadratures of the form* (4.26) $\left(Q^\mu_n(f)\right)_{n\in\mathbb{R}_1}$ *so that there hold the right asymptotic orders*

$$\sup_{f \in \boldsymbol{W}^r_p(\mathbb{R}^d;\mu)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}) - Q^\mu_n f)(f) \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)\big) \asymp n^{-r}(\log n)^{\frac{d-1}{2}}. \tag{4.29}$$

Proof. Let $Q_{\mathrm{F},m}$ be the Frolov quadrature for functions in $\mathring{W}^r_p(\mathbb{I}^d)$ (see, e.g., [17, Chapter 8] for the definition) in the form (4.2) with $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$. It was proven in [27] for $p = 2$, and in [39] for $1 < p < \infty$ that

$$\left| \int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - I_{\mathrm{F},m} \right| \le C n^{-r}(\log n)^{\frac{d-1}{2}} \|f\|_{\mathring{W}^r_p(\mathbb{I}^d)}, \quad f \in \mathring{W}^r_p(\mathbb{I}^d). \tag{4.30}$$

For a fixed $1 < \theta < 2$, we define $Q^w_n(f) := Q_{\theta,n}$ as the quadrature described in Theorem 4.2 for $a = r$ and $b = \frac{d-1}{2}$, based on $Q_m = Q_{\mathrm{F},m}$. By Theorem 4.2 and (4.30) we prove the upper bound in (4.29).

Since for $f \in \mathring{W}^r_p(\mathbb{I}^d)$

$$\|f\|_{W^r_p(\mathbb{R}^d;\mu)} \le e^{\frac{db}{p}} \|f\|_{\mathring{W}^r_p(\mathbb{I}^d)},$$

we get

$$\operatorname{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)\big) \gg \operatorname{Int}_n(\mathring{\boldsymbol{W}}{}^r_p(\mathbb{I}^d)).$$

Hence the lower bound in (4.29) follows from the lower bound

$$\operatorname{Int}_n(\mathring{\boldsymbol{W}}{}^r_p(\mathbb{I}^d)) \gg n^{-r}(\log n)^{\frac{d-1}{2}}$$

proven in [42]. ∎

Besides Frolov quadratures, there are many quadratures for efficient numerical integration for functions on $\mathbb{I}^d$ to list. We refer the reader to [17, Chapter 8] for bibliography and historical comments as well as related results, in particular, the asymptotic order

$$\operatorname{Int}_m\big(\boldsymbol{W}^r_p(\mathbb{I}^d)\big) \asymp m^{-r}(\log m)^{\frac{d-1}{2}}.$$

We recall only some of them, especially those which give asymptotic order of optimal integration.

The QMC quadrature based on a set of integration nodes $\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m\} \subset \mathbb{I}^d$ is defined as

$$Q_m(f) = \frac{1}{m}\sum_{i=1}^m f(\boldsymbol{x}_i).$$

In [21, 22] for a prime number q the author introduced higher order digital nets over the finite field $\mathbb{F}_q := \{0,1,\ldots,q-1\}$ equipped with the arithmetic operations modulo q. Such digital nets can achieve the convergence rate $m^{-r}(\log m)^{dr}$ with $m = q^s$ [25] for functions from $W^r_2(\mathbb{I}^d)$. In the recent paper [28], the authors have shown that the asymptotic order of $\operatorname{Int}_m\big(\boldsymbol{W}^r_2(\mathbb{I}^d)\big)$ can be achieved by Dick's digital nets $\{\boldsymbol{x}^*_1,\ldots,\boldsymbol{x}^*_{q^s}\}$ of order $(2r+1)$. Namely, they proved that

$$\left|\int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \frac{1}{m}\sum_{i=1}^m f(\boldsymbol{x}^*_i)\right| \le Cm^{-r}(\log m)^{\frac{d-1}{2}}\|f\|_{W^r_2(\mathbb{I}^d)}, \quad f \in W^r_2(\mathbb{I}^d), \quad m = q^s. \tag{4.31}$$

In the case $d = 2$ the QMC quadrature $Q_m = Q_{\Phi,m}$ based on Fibonacci lattice rules ($d = 2$) is also asymptotically optimal for numerical integration of periodic functions in $\tilde{W}^r_p(\mathbb{I}^2)$, that is,

$$\left|\int_{\mathbb{I}^2} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_{\Phi,m}(f)\right| \le Cm^{-r}(\log m)^{\frac{1}{2}}\|f\|_{W^r_p(\mathbb{T}^2)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^2), \tag{4.32}$$

where $\tilde{W}^r_p(\mathbb{I}^2)$ denotes the subspace of $W^r_p(\mathbb{I}^2)$ of all functions which can be extended to the whole $\mathbb{R}^2$ as 1-periodic functions in each variable. The estimate (4.32) was proven in [1] for $p = 2$ and in [43] for $1 < p < \infty$. The QMC quadrature $Q_m = Q_{\Phi,m}$ based on Fibonacci lattice rules ($d = 2$) is defined by

$$Q_{\Phi,m}(f) := \frac{1}{b_m}\sum_{i=1}^{b_m} f\bigg(\Big\{\frac{i}{b_m}\Big\} - \frac{1}{2}, \Big\{\frac{ib_{m-1}}{b_m}\Big\} - \frac{1}{2}\bigg),$$

where $b_0 = b_1 = 1$, $b_m := b_{m-1} + b_{m-2}$ are the Fibonacci numbers and $\{x\}$ denotes the fractional part of the number x.

Therefore, from Theorems 4.1–4.3 and (4.31), (4.32) it follows that the QMC quadratures based on Dick's digital nets of order $(2r+1)$ and Fibonacci lattice rules ($d = 2$) can be used for assembling asymptotically optimal quadratures $Q_n^w(f)$ and $Q_{\theta,n}^\gamma$ of the forms (4.15) and (4.26) for $\mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{R}^d;\mu)\big)$, in the particular cases $p = 2$, $d \ge 2$, and $1 < p < \infty$, $d = 2$, respectively.

The sparse Smolyak grid $SG(\xi)$ in $\mathbb{I}^d$ is defined as the set of points:

$$SG(\xi) := \left\{\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}} := 2^{-\boldsymbol{k}}\boldsymbol{s} \in \mathbb{Z}^d : |\boldsymbol{k}|_1 \le \xi, \quad |s_i| \le 2^{k_i - 1}, \ i = 1, \dots, d\right\}, \quad \xi \in \mathbb{R}_1.$$

For a given $m \in \mathbb{R}_1$, let ξ_m be the maximal number satisfying $|SG(\xi_m)| \le m$. Then we can constructively define a quadrature $Q_m = Q_{\mathrm{S},m}$ based on the integration nodes in $SG(\xi_m)$ so that

$$\left|\int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_{\mathrm{S},m}(f)\right| \le Cm^{-r}(\log m)^{(d-1)(r+1/2)}\|f\|_{W_p^r(\mathbb{I}^d)}, \quad f \in W_p^r(\mathbb{I}^d). \tag{4.33}$$

To understand this quadrature let us recall a detailed construction from [18, p. 760]. Indeed, the well-known embedding of $W_p^r(\mathbb{I}^d)$ into the Besov space of mixed smoothness $B_{p,\max(p,2)}^r(\mathbb{I}^d)$ (see, e.g., [17, Lemma 3.4.1(iv)]), and the result on B-spline sampling recovery of functions from the last space it follows that one can constructively define a sampling recovery algorithm of the form

$$R_m(f) := \sum_{\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}} \in SG(\xi_m)} f(\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}})\phi_{\boldsymbol{k},\boldsymbol{s}}$$

with certain B-splines $\phi_{\boldsymbol{k},\boldsymbol{s}}$, such that

$$\|f - R_m(f)\|_{L_1(\mathbb{I}^d)} \le Cm^{-r}(\log m)^{(d-1)(r+1/2)}\|f\|_{W_p^r(\mathbb{I}^d)}, \quad f \in W_p^r(\mathbb{I}^d).$$

Then the quadrature $Q_{\mathrm{S},m}$ can be defined as

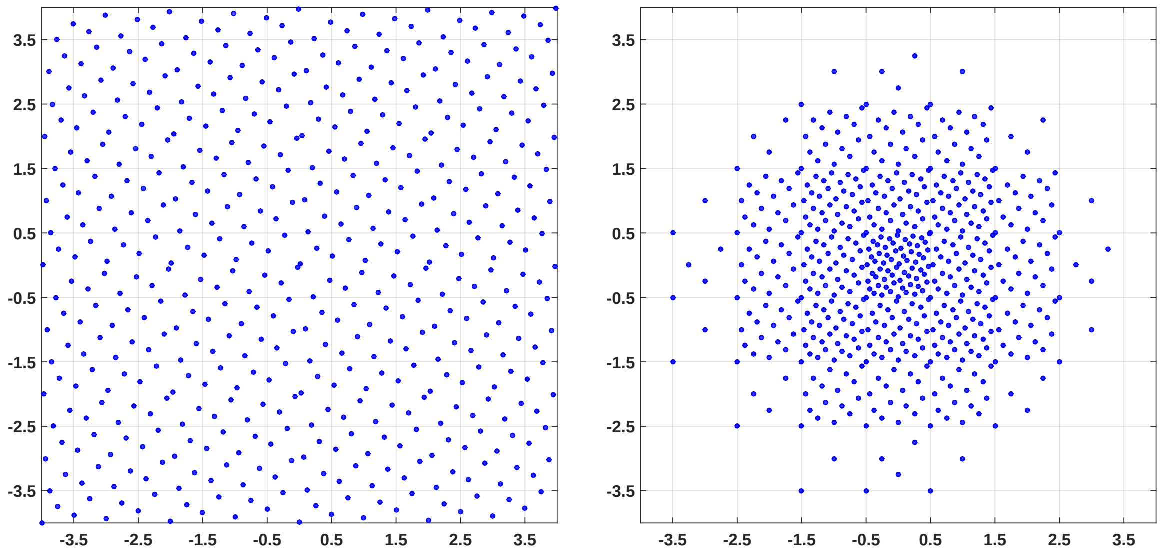
$$Q_{\mathrm{S},m}(f) := \sum_{\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}} \in SG(\xi_m)} \lambda_{\boldsymbol{k},\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}}), \quad \lambda_{\boldsymbol{k},\boldsymbol{s}} := \int_{\mathbb{I}^d} \phi_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

and (4.33) is implied by the obvious inequality

$$\big|\int_{\mathbb{I}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - S_m(f)\big| \le \|f - R_m(f)\|_{L_1(\mathbb{I}^d)}.$$

Therefore, from Theorem 4.1 and (4.33) we can see that the Smolyak quadrature $Q_{\mathrm{S},m}$ can be used for assembling a quadrature $Q_{\mathrm{S},n}^\mu$ of the form (4.15) with "double" sparse integration nodes which gives the convergence rate

$$\left|\int_{\mathbb{R}^d} f(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}) - Q_{\mathrm{S},n}^\mu(f)\right| \ll n^{-r}(\log n)^{(d-1)(r+1/2)}, \quad f \in \boldsymbol{W}_p^r(\mathbb{R}^d;\mu).$$



Point set of Dick et al. [23] (512 points) Point set of our construction (560 points)

Figure 1: Pictures of integration nodes from [16]

We illustrate the integration nodes of the quadratures constructed in this subsection, in comparison with the integration nodes used in [23]. Assume that $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ are the integration nodes for an optimal quadrature Q_n for functions in $W^r_p(\mathbb{I}^2)$. Then the integration nodes in [23] are just a dilation of these nodes to the cube $[-C\sqrt{\log n}, C\sqrt{\log n}]^2$. Hence these nodes are distributed similarly on this cube. Differently, the integration nodes in our construction are formed from certain integer-shifted dilations of $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}$ and contained in the ball of radius $C\sqrt{\log n}$. These nodes are dense when they are near the origin and getting sparser as they are farther from the origin. The illustration is given in Figure 1.

5 Numerical weighted integration in the space $W^r_1(\mathbb{R}^d;\mu)$

5.1 Introducing remarks

In this section, we present some recent results of [15] on numerical weighted integration over $\mathbb{R}^d$ for functions from weighted Sobolev spaces $W^r_1(\mathbb{R}^d;\mu)$ of mixed smoothness $r \in \mathbb{N}$, in particular, upper and lower bounds of the quantity of optimal quadrature of the functions class $W^r_1(\mathbb{R}^d;\mu)$. Here the measure μ is defined by the Freud-type weight as the density function given by (3.3). We first briefly describe these results and then give comments on related works.

Throughout this section, for $\lambda > 1$, we make use of the notation

$$r_\lambda := (1 - 1/\lambda)r.$$

For the Sobolev class $\boldsymbol{W}_1^r(\mathbb{R}^d;\mu)$, there hold the upper and lower bounds

$$n^{-r_\lambda}(\log n)^{r_\lambda(d-1)} \ll \mathrm{Int}_n(\boldsymbol{W}_1^r(\mathbb{R}^d;\mu)) \ll n^{-r_\lambda}(\log n)^{(r_\lambda+1)(d-1)}, \qquad (5.1)$$

in particular, in the case of Gaussian measure

$$n^{-r/2}(\log n)^{r(d-1)/2} \ll \mathrm{Int}_n(\boldsymbol{W}_1^r(\mathbb{R}^d;\gamma)) \ll n^{-r/2}(\log n)^{(r/2+1)(d-1)}. \qquad (5.2)$$

In the one-dimensional case, we have the right asymptotic order

$$\mathrm{Int}_n(\boldsymbol{W}_1^r(\mathbb{R};\mu)) \asymp n^{-r_\lambda}.$$

The difference between the upper and lower bounds in (5.1) is the logarithmic factor $(\log n)^{d-1}$.

In Theorem 4.3, for $1 < p < \infty$, we have constructed an asymptotically optimal quadrature Q_n^μ of the form (1.3) which gives the right asymptotic order

$$\sup_{f \in \boldsymbol{W}_p^r(\mathbb{R}^d;\mu)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}) - Q_n^\mu f \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{R}^d;\mu)\big) \asymp n^{-r}(\log n)^{(d-1)/2}. \qquad (5.3)$$

The results (5.2) and (5.3) show a substantial difference of the convergence rates between the cases $p = 1$ and $1 < p < \infty$. In constructing the asymptotically optimal quadrature Q_n^μ in (5.3), we used a technique assembling a quadrature for the Sobolev spaces on the unit d-cube to the integer-shifted d-cubes. Unfortunately, this technique is not suitable to constructing a quadrature realizing the upper bound in (5.1) for the space $W_1^r(\mathbb{R}^d;\mu)$ which is the largest among the spaces $W_p^r(\mathbb{R}^d;\mu)$ with $1 \le p < \infty$. It requires a different technique based on Smolyak algorithms [40]. Such a quadrature relies on sparse grids of integration nodes which are step hyperbolic crosses in the function domain $\mathbb{R}^d$, and some generalization of the results on univariate numerical integration by truncated Gaussian quadratures from [20]. To prove the lower bound in (5.1) we adopt a traditional technique to construct for arbitrary n integration nodes a fooling function vanishing at these nodes.

It is interesting to compare the results (5.2) and (5.3) on $\mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{R}^d;\mu)\big)$ with known results on $\mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{I}^d)\big)$ for the unweighted Sobolev space $W_p^r(\mathbb{I}^d)$ of mixed smoothness r. For $1 < p < \infty$, there holds the asymptotic order

$$\mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{I}^d)\big) \asymp n^{-r}(\log n)^{(d-1)/2},$$

and for $p = 1$ and $r > 1$, there hold the bounds

$$n^{-r}(\log n)^{(d-1)/2} \ll \mathrm{Int}_n(\boldsymbol{W}_1^r(\mathbb{I}^d)) \ll n^{-r}(\log n)^{d-1}$$

which are so far the best known (see, e.g., [17, Chapter 8], for detail). Hence we can see that $\mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{R}^d;\mu)\big)$ and $\mathrm{Int}_n\big(\boldsymbol{W}_p^r(\mathbb{I}^d)\big)$ have the same asymptotic order in the case $1 < p < \infty$, and very different lower and upper bounds in both power and logarithmic terms in the case $p = 1$. The right asymptotic orders of the both $\mathrm{Int}_n\big(\boldsymbol{W}_1^r(\mathbb{I}^d)\big)$ and $\mathrm{Int}_n\big(\boldsymbol{W}_1^r(\mathbb{R}^d;\mu)\big)$ are still open problems (cf. [17, Open Problem 1.9]).

The problem of numerical integration considered in this section is related to the research direction of optimal approximation and integration for functions having mixed smoothness on one hand, and the other research direction of univariate weighted polynomial approximation and integration on $\mathbb{R}$, on the other hand. For survey and bibliography, we refer the reader to the books [17, 47] on the first direction, and [36, 34, 32] on the other hand.

5.2 Univariate numerical integration

In this subsection, for one-dimensional numerical integration, we present the right asymptotic order of the quantity of optimal quadrature $\mathrm{Int}_n\big(\boldsymbol{W}^r_1(\mathbb{R};\mu)\big)$ and its shortened proof from [15].

The following lemma which is implied directly from the definition (1.4) is quite useful for lower estimation of $\mathrm{Int}_n(\boldsymbol{W})$.

Lemma 5.1 *Let $\boldsymbol{W}$ be a set of continuous functions on $\mathbb{R}^d$. Then we have*

$$\mathrm{Int}_n(\boldsymbol{W}) \ge \inf_{\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\}\subset\mathbb{R}^d} \ \sup_{f\in\boldsymbol{W}:\ f(\boldsymbol{x}_i)=0,\ i=1,\dots,n} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x})\,\mu(\mathrm{d}\boldsymbol{x})\right|. \tag{5.4}$$

We now consider the problem of approximation of integral (1.2) for univariate functions from $W^r_1(\mathbb{R};\mu)$. Let $(p_m(w))_{m\in\mathbb{N}_0}$ be the sequence of orthonormal polynomials with respect to the weight w. In the classical quadrature theory, a possible choice of integration nodes is to take the zeros of the polynomials $p_m(w)$. Denote by $x_{m,k}$, $1\le k\le \lfloor m/2\rfloor$ the positive zeros of $p_m(w)$, and by $x_{m,-k}=-x_{m,k}$ the negative ones (if m is odd, then $x_{m,0}=0$ is also a zero of $p_m(w)$). These zeros are located as

$$-a_m+\frac{Ca_m}{m^{2/3}} < x_{m,-\lfloor m/2\rfloor} < \cdots < x_{m,-1} < x_{m,1} < \cdots < x_{m,\lfloor m/2\rfloor} \le a_m - \frac{Ca_m}{m^{2/3}}, \tag{5.5}$$

with a positive constant C independent of m (see, e. g., [32, (4.1.32)]). Here a_m is the Mhaskar-Rakhmanov-Saff number which is

$$a_m = a_m(w) = (\gamma_\lambda m)^{1/\lambda}, \quad \gamma_\lambda := \frac{2\Gamma((1+\lambda)/2)}{\sqrt{\pi}\Gamma(\lambda/2)}, \tag{5.6}$$

and Γ is the gamma function. Notice that the formula (5.6) is given in [32, (4.1.4)] for the particular case $a=1,\ b=0$ of the weight w, but inspecting the definition of Mhaskar-Rakhmanov-Saff number (see, e.g., [32, Page 116]), one easily verify that it still holds true for the general weight w for any $a>0$ and $b\in\mathbb{R}$.

For a continuous function on $\mathbb{R}$, the classical Gaussian quadrature is defined as

$$Q^{\mathrm{G}}_m f := \sum_{|k|\le m/2} \lambda_{m,k}(w) f(x_{m,k}), \tag{5.7}$$

where $\lambda_{m,k}(w)$ are the corresponding Cotes numbers. This quadrature is based on Lagrange interpolation (for details, see, e.g., [36, 1.2. Interpolation and quadrature]). Unfortunately, it does not give the optimal convergence rate for functions from $\boldsymbol{W}_1^r(\mathbb{R};\mu)$, see comments in Remark 5.3 below.

In [20], for the weight $w(x)$ with $a=1,\ b=0$, the authors proposed truncated Gaussian quadratures which not only improve the convergence rate but also give the asymptotic order of $\mathrm{Int}_n\big(\boldsymbol{W}_1^r(\mathbb{R};\mu)\big)$ as shown in Theorem 5.2 below. Let us introduce in the same manner truncated Gaussian quadratures for the weight $w(x)$ with any $a>0$ and $b\in\mathbb{R}$.

In what follows, we fix a number θ with $0<\theta<1$, and denote by $j(m)$ the smallest integer satisfying $x_{m,j(m)}\ge\theta a_m$. It is useful to remark that

$$d_{m,k}\asymp\frac{a_m}{m}\asymp m^{1/\lambda-1},\quad |k|\le j(m);\quad x_{m,j(m)}\asymp m^{1/\lambda},$$

where $d_{m,k}:=x_{m,k}-x_{m,k-1}$ is the distance between consecutive zeros of the polynomial $p_m(w)$. These relations were proven in [20, (13)] for the case particular $w(x)=e^{-|x|^\lambda}$. From their proofs there, one can easily see that they are still hold true for the general case of the weight w. Hence by the definitions and (5.5), for m sufficiently large we have that

$$Cm\le j(m)\le m/2 \tag{5.8}$$

with a positive constant C depending on λ,a,b and θ only.

For a continuous function on $\mathbb{R}$, consider the truncated Gaussian quadrature

$$Q_{2j(m)}^{\mathrm{TG}}f:=\sum_{|k|\le j(m)}\lambda_{m,k}(w)f(x_{m,k}). \tag{5.9}$$

Notice that the number $2j(m)$ of samples in the quadrature $Q_{2j(m)}^{\mathrm{TG}}f$ is strictly smalller than m – the number of samples in the quadrature $Q_m^{\mathrm{G}}f$. However, due to (5.8) it has the same asymptotic order as $2j(m)\asymp m$ when m going to infinity.

Theorem 5.2 *For any $n\in\mathbb{N}$, let m_n be the largest integer such that $2j(m_n)\le n$. Then the family of quadratures $\big(Q_{2j(m_n)}^{\mathrm{TG}}\big)_{n\in\mathbb{R}_1}$ is asymptotically optimal for $\boldsymbol{W}_1^r(\mathbb{R};\mu)$, and*

$$\sup_{f\in\boldsymbol{W}_1^r(\mathbb{R};\mu)}\left|\int_{\mathbb{R}}f(x)\mu(\mathrm{d}x)-Q_{2j(m_n)}^{\mathrm{TG}}f\right|\asymp\mathrm{Int}_n\big(\boldsymbol{W}_1^r(\mathbb{R};\mu)\big)\asymp n^{-r_\lambda}. \tag{5.10}$$

Proof. For $f\in W_1^r(\mathbb{R};\mu)$, there holds the inequality

$$\left|\int_{\mathbb{R}}f(x)\mu(\mathrm{d}x)-Q_{2j(m)}^{\mathrm{TG}}f\right|\le C\left(m^{-(1-1/\lambda)r}\left\|f^{(r)}\right\|_{L_1(\mathbb{R};\mu)}+e^{-Km}\left\|f\right\|_{L_1(\mathbb{R};\mu)}\right) \tag{5.11}$$

with some constants C and K independent of m and f. The inequality (5.11) implies the upper bound in (5.10):

$$\mathrm{Int}_n\big(\boldsymbol{W}_1^r(\mathbb{R};\mu)\big)\le\sup_{f\in\boldsymbol{W}_1^r(\mathbb{R};\mu)}\left|\int_{\mathbb{R}}f(x)\mu(\mathrm{d}x)-Q_{2j(m_n)}^{\mathrm{TG}}f\right|\ll n^{-r_\lambda}.$$

In order to prove the lower bound in (5.10) we apply Lemma 5.1. Let $\{\xi_1, ..., \xi_n\} \subset \mathbb{R}$ be arbitrary n points. For a given $n \in \mathbb{N}$, we put $\delta = n^{1/\lambda - 1}$ and $t_j = \delta j$, $j \in \mathbb{N}_0$. Then there is $i \in \mathbb{N}$ with $n+1 \le i \le 2n+2$ such that the interval (t_{i-1}, t_i) does not contain any point from the set $\{\xi_1, ..., \xi_n\}$. Take a nonnegative function $\varphi \in C_0^\infty([0,1])$, $\varphi \neq 0$, and put

$$b_0 := \int_0^1 \varphi(y)\mathrm{d}y > 0, \quad b_s := \int_0^1 |\varphi^{(s)}(y)|\mathrm{d}y, \ s = 1, ..., r.$$

Define the functions g and h on $\mathbb{R}$ by

$$g(x) := \begin{cases} \varphi(\delta^{-1}(x - t_{i-1})), & x \in (t_{i-1}, t_i), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h(x) := (gw^{-1})(x).$$

By a direct computation we find that

$$\|h\|_{W_1^r(\mathbb{R};\mu)} \le C n^{(1-1/\lambda)(k-1)} \le C n^{(1-1/\lambda)(r-1)}.$$

If we define

$$\bar{h} := C^{-1} n^{-(1-1/\lambda)(r-1)} h,$$

then $\bar{h}$ is nonnegative, $\bar{h} \in \boldsymbol{W}_1^r(\mathbb{R};\mu)$, $\sup \bar{h} \subset (t_{i-1}, t_i)$ and

$$\int_{\mathbb{R}} (\bar{h}w)(x)\mathrm{d}x \gg n^{-(1-1/\lambda)r}$$

Since the interval (t_{i-1}, t_i) does not contain any point from the set $\{\xi_1, ..., \xi_n\}$, we have $\bar{h}(\xi_k) = 0$, $k = 1, ..., n$. Hence, by Lemma 5.1,

$$\mathrm{Int}_n\big(\boldsymbol{W}_1^r(\mathbb{R};\mu)\big) \ge \int_{\mathbb{R}} \bar{h}(x)w(x)\mathrm{d}x \gg n^{-r\lambda}.$$

$\square$

Remark 5.3 In the case of the Gaussian measure γ, the truncated Gaussian quadratures $Q_{2j(m)}^{\mathrm{TG}}$ in Theorem 5.2 give

$$\sup_{f \in \boldsymbol{W}_1^1(\mathbb{R};\mu)} \left| \int_{\mathbb{R}} f(x)\gamma(\mathrm{d}x) - Q_{2j(m_n)}^{\mathrm{TG}} f \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}_1^1(\mathbb{R};\gamma)\big) \asymp n^{-r/2}. \tag{5.12}$$

On the other hand, for the full Gaussian quadratures Q_n^{G}, it has been proven in [20, Proposition 1] the right asymptotic order

$$\sup_{f \in \boldsymbol{W}_1^1(\mathbb{R};\gamma)} \left| \int_{\mathbb{R}} f(x)\gamma(\mathrm{d}x) - Q_n^{\mathrm{G}} f \right| \asymp n^{-1/6}$$

which is much worse than the right asymptotic order of $\mathrm{Int}_n\big(\boldsymbol{W}_1^1(\mathbb{R};\gamma)\big) \asymp n^{-1/2}$ as in (5.12) for $r = 1$.

5.3 Multivariate numerical integration

In this section, for multivariate numerical integration ($d \geq 2$), we present some results and their shortened proofs from [15] on upper and lower bounds of the quantity of optimal quadrature $\mathrm{Int}_n\big(\boldsymbol{W}^r_1(\mathbb{R}^d;\mu)\big)$ and a construction of quadratures based on step-hyperbolic-cross grids of integration nodes which give the upper bounds.

We need some auxiliary lemmata.

For $\boldsymbol{x} \in \mathbb{R}^d$ and $e \subset \{1,...,d\}$, let $\boldsymbol{x}^e \in \mathbb{R}^d$ be defined by $x^e_i := x_i$, $i \in e$, and $x^e_i := 0$ otherwise, and put $\bar{\boldsymbol{x}}^e := \boldsymbol{x} - \boldsymbol{x}^e$. With an abuse we write $(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e) = \boldsymbol{x}$.

Lemma 5.4 *Let $1 \leq p \leq \infty$, $e \subset \{1,...,d\}$ and $\boldsymbol{r} \in \mathbb{N}^d_0$. Assume that f is a function on $\mathbb{R}^d$ such that for every $\boldsymbol{k} \leq \boldsymbol{r}$, $D^{\boldsymbol{k}} f \in L_p(\mathbb{R}^d;\mu)$. Put for $\boldsymbol{k} \leq \boldsymbol{r}$ and $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$,*

$$g(\boldsymbol{x}^e) := D^{\bar{\boldsymbol{k}}^e} f(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e).$$

Then $D^{\boldsymbol{s}} g \in L_p(\mathbb{R}^{|e|};\mu)$ for every $\boldsymbol{s} \leq \boldsymbol{k}^e$ and almost every $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$.

Assume that there exists a sequence of quadratures $(Q_{2^k})_{k\in\mathbb{N}_0}$ with

$$Q_{2^k} f := \sum_{s=1}^{2^k} \lambda_{k,s} f(x_{k,s}), \quad \{x_{k,1}, \dots, x_{k,2^k}\} \subset \mathbb{R}, \tag{5.13}$$

such that

$$\left| \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x) - Q_{2^k} f \right| \leq C 2^{-\alpha k} \|f\|_{W^r_1(\mathbb{R};\mu)}, \quad k \in \mathbb{N}_0, \quad f \in W^r_1(\mathbb{R};\mu), \tag{5.14}$$

for some number $a > 0$ and constant $C > 0$. Based on a sequence $(Q_{2^k})_{k\in\mathbb{N}_0}$ of the form (5.13) satisfying (5.14), we construct quadratures on $\mathbb{R}^d$ by using the well-known Smolyak algorithm. We define for $k \in \mathbb{N}_0$, the one-dimensional operators

$$\Delta^Q_k := Q_{2^k} - Q_{2^{k-1}}, \ k > 0, \quad \Delta^Q_0 := Q_1,$$

and

$$E^Q_k f := \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x) - Q_{2^k} f.$$

For $\boldsymbol{k} \in \mathbb{N}^d$, the d-dimensional operators $Q_{2^{\boldsymbol{k}}}$, $\Delta^Q_{\boldsymbol{k}}$ and $E^Q_{\boldsymbol{k}}$ are defined as the tensor product of one-dimensional operators:

$$Q_{2^{\boldsymbol{k}}} := \bigotimes_{i=1}^d Q_{2^{k_i}}, \quad \Delta^Q_{\boldsymbol{k}} := \bigotimes_{i=1}^d \Delta^Q_{k_i}, \quad E^Q_{\boldsymbol{k}} := \bigotimes_{i=1}^d E^Q_{k_i}, \tag{5.15}$$

where $2^{\boldsymbol{k}} := (2^{k_1}, \cdots, 2^{k_d})$ and the univariate operators $Q_{2^{k_j}}$, $\Delta^Q_{k_j}$ and $E^Q_{k_j}$ are successively applied to the univariate functions $\bigotimes_{i<j} Q_{2^{k_i}}(f)$, $\bigotimes_{i<j} \Delta^Q_{k_i}(f)$ and $\bigotimes_{i<j} E^Q_{k_i}$, respectively, by considering them as functions of variable x_j with the other variables held fixed. The

operators $Q_{2^{\boldsymbol{k}}}$, $\Delta_{\boldsymbol{k}}^Q$ and $E_{\boldsymbol{k}}^Q$ are well-defined for continuous functions on $\mathbb{R}^d$, in particular for ones from $W_1^r(\mathbb{R}^d;\mu)$.

If f is a continuous function on $\mathbb{R}^d$, then $Q_{2^{\boldsymbol{k}}}(f)$ is a quadrature on $\mathbb{R}^d$ which is given by

$$Q_{2^{\boldsymbol{k}}} f = \sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}}} \lambda_{\boldsymbol{k},\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}}), \quad \{\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}}\}_{\boldsymbol{1}\le \boldsymbol{s}\le 2^{\boldsymbol{k}}} \subset \mathbb{R}^d, \tag{5.16}$$

where

$$\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{s}} := (x_{k_1,s_1}, ..., x_{k_d,s_d}), \quad \lambda_{\boldsymbol{k},\boldsymbol{s}} := \prod_{i=1}^d \lambda_{k_i,s_i},$$

and the summation $\sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}}}$ means that the sum is taken over all $\boldsymbol{s}$ such that $\boldsymbol{1} \le \boldsymbol{s} \le 2^{\boldsymbol{k}}$. Hence we derive that

$$\Delta_{\boldsymbol{k}}^Q f = \sum_{e\subset\{1,...,d\}} (-1)^{d-|e|} Q_{2^{\boldsymbol{k}(e)}} f = \sum_{e\subset\{1,...,d\}} (-1)^{d-|e|} \sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}(e)}} \lambda_{\boldsymbol{k}(e),\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}), \tag{5.17}$$

where $\boldsymbol{k}(e) \in \mathbb{N}_0^d$ is defined by $k(e)_i = k_i$, $i \in e$, and $k(e)_i = \max(k_i - 1, 0)$, $i \notin e$. We also have

$$E_{\boldsymbol{k}}^Q f = \sum_{e\subset\{1,...,d\}} (-1)^{|e|} \int_{\mathbb{R}^{d-|e|}} Q_{2^{\boldsymbol{k}^e}} f(\cdot, \bar{\boldsymbol{x}}^e) w(\bar{\boldsymbol{x}}^e) \mathrm{d}\bar{\boldsymbol{x}}^e, \tag{5.18}$$

where $w(\bar{\boldsymbol{x}}^e) := \prod_{j\notin e} w(x_j)$.

Notice that as mappings from $C(\mathbb{R}^d)$ to $\mathbb{R}$, the operators $Q_{2^{\boldsymbol{k}}}$, $\Delta_{\boldsymbol{k}}^Q$ and $E_{\boldsymbol{k}}^Q$ possess commutative and associative properties with respect to applying the component operators $Q_{2^{k_j}}$, $\Delta_{k_j}^Q$ and $E_{k_j}^Q$. In particular, we have for any $e \subset \{1, ..., d\}$,

$$Q_{2^{\boldsymbol{k}}} f = Q_{2^{\boldsymbol{k}^e}} \left(Q_{2^{\bar{\boldsymbol{k}}^e}} f\right), \quad \Delta_{\boldsymbol{k}}^Q f = \Delta_{\boldsymbol{k}^e}^Q \left(\Delta_{\bar{\boldsymbol{k}}^e}^Q f\right), \quad E_{\boldsymbol{k}}^Q f = E_{\boldsymbol{k}^e}^Q \left(E_{\bar{\boldsymbol{k}}^e}^Q f\right),$$

and for any reordered sequence $\{i(1), ..., i(d)\}$ of $\{1, ..., d\}$,

$$Q_{2^{\boldsymbol{k}}} = \bigotimes_{j=1}^d Q_{2^{k_{i(j)}}}, \quad \Delta_{\boldsymbol{k}}^Q = \bigotimes_{j=1}^d \Delta_{k_{i(j)}}^Q, \quad E_{\boldsymbol{k}}^Q = \bigotimes_{j=1}^d E_{k_{i(j)}}^Q. \tag{5.19}$$

These properties directly follow from (5.16)–(5.18).

Lemma 5.5 *Under the assumption* (5.13)–(5.14)*, we have*

$$\left|E_{\boldsymbol{k}}^Q f\right| \le C 2^{-\alpha|\boldsymbol{k}|_1} \|f\|_{W_1^r(\mathbb{R}^d;\mu)}, \quad \boldsymbol{k} \in \mathbb{N}_0^d, \quad f \in W_1^r(\mathbb{R}^d;\mu),$$

We say that $\boldsymbol{k} \to \infty$, $\boldsymbol{k} \in \mathbb{N}_0^d$, if and only if $k_i \to \infty$ for every $i = 1, ..., d$.

Lemma 5.6 *Under the assumption* (5.13)–(5.14)*, we have that for every* $f \in W_1^r(\mathbb{R}^d;\mu)$,

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) \mu(\mathrm{d}\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \Delta_{\boldsymbol{k}}^Q f \tag{5.20}$$

with absolute convergence of the series, and

$$\left|\Delta_{\boldsymbol{k}}^Q f\right| \le C 2^{-\alpha|\boldsymbol{k}|_1} \|f\|_{W_1^r(\mathbb{R}^d;\mu)}, \quad \boldsymbol{k} \in \mathbb{N}_0^d. \tag{5.21}$$

We now define an algorithm for quadrature on sparse grids adopted from the algorithm for sampling recovery initiated by Smolyak (for detail see [17, Sections 4.2 and 5.3]). For $\xi > 0$, we define the operator

$$Q_\xi := \sum_{|\boldsymbol{k}|_1 \le \xi} \Delta_{\boldsymbol{k}}^Q.$$

From (5.17) we can see that Q_ξ is a quadrature on $\mathbb{R}^d$ of the form (1.3):

$$Q_\xi f = \sum_{|\boldsymbol{k}|_1 \le \xi} \sum_{e \subset \{1,\dots,d\}} (-1)^{d-|e|} \sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}(e)}} \lambda_{\boldsymbol{k}(e),\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}) = \sum_{(\boldsymbol{k},e,\boldsymbol{s}) \in G(\xi)} \lambda_{\boldsymbol{k},e,\boldsymbol{s}} f(\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}}), \tag{5.22}$$

where

$$\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}} := \boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}, \quad \lambda_{\boldsymbol{k},e,\boldsymbol{s}} := (-1)^{d-|e|} \lambda_{\boldsymbol{k}(e),\boldsymbol{s}}$$

and

$$G(\xi) := \left\{(\boldsymbol{k}, e, \boldsymbol{s}) : \ |\boldsymbol{k}|_1 \le \xi, \, e \subset \{1, \dots, d\}, \, \boldsymbol{1} \le \boldsymbol{s} \le \boldsymbol{k}(e)\right\}$$

is a finite set. The set of integration nodes in this quadrature

$$H(\xi) := \{\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}}\}_{(\boldsymbol{k},e,\boldsymbol{s}) \in G(\xi)}$$

is a step hyperbolic cross in the function domain $\mathbb{R}^d$. The number of integration nodes in the quadrature Q_ξ is

$$|G(\xi)| = \sum_{|\boldsymbol{k}|_1 \le \xi} \sum_{e \subset \{1,\dots,d\}} 2^{|\boldsymbol{k}(e)|_1}$$

which can be estimated as

$$|G(\xi)| \asymp \sum_{|\boldsymbol{k}|_1 \le \xi} 2^{|\boldsymbol{k}|_1} \ \asymp \ 2^\xi \xi^{d-1}, \quad \xi \ge 1. \tag{5.23}$$

This quadrature plays a crucial role in the proof of the upper bound in (5.1).

From Lemmata 5.4–5.6, by using a modification of a technique for establishing upper bounds of the error of unweighted sampling recovery by Smolyak algorithms of functions having mixed smoothness on a bounded domain (see, e.g., [17, Section 5.3] and [50, Section 6.9] for detail of related techniques), we prove the following upper bound.

Lemma 5.7 *Under the assumption* (5.13)–(5.14), *we have that*

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \mu(\mathrm{d}\boldsymbol{x}) - Q_\xi f \right| \le C 2^{-\alpha\xi} \xi^{d-1} \|f\|_{W_1^r(\mathbb{R}^d;\mu)}, \quad \xi \ge 1, \quad f \in W_1^r(\mathbb{R}^d;\mu). \tag{5.24}$$

Proof. From Lemma 5.6 we derive that for $\xi \geq 1$ and $f \in W^r_1(\mathbb{R}^d;\mu)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^d} f(\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_\xi f\right| &\leq \sum_{|\boldsymbol{k}|_1>\xi} \left|\Delta^Q_{\boldsymbol{k}} f\right| \leq C \sum_{|\boldsymbol{k}|_1>\xi} 2^{-\alpha|\boldsymbol{k}|_1}\|f\|_{W^r_1(\mathbb{R}^d;\mu)} \\ &\leq C\|f\|_{W^r_1(\mathbb{R}^d;\mu)} \sum_{|\boldsymbol{k}|_1>\xi} 2^{-\alpha|\boldsymbol{k}|_1} \leq C2^{-\alpha\xi}\xi^{d-1}\|f\|_{W^r_1(\mathbb{R}^d;\mu)}.\end{aligned}$$

□

Remark 5.8 From Theorem 5.2 we can see that the truncated Gaussian quadratures $Q^{\mathrm{TG}}_{2j(m)}$ form a sequence $(Q_{2^k})_{k\in\mathbb{N}_0}$ of the form (5.13) satisfying (5.14) with $a = r_\lambda$.

Theorem 5.9 *We have that*

$$n^{-r_\lambda}(\log n)^{r_\lambda(d-1)} \ll \mathrm{Int}_n(\boldsymbol{W}^r_1(\mathbb{R}^d;\mu)) \ll n^{-r_\lambda}(\log n)^{(r_\lambda+1)(d-1)}. \tag{5.25}$$

Proof. Let us first prove the upper bound in (5.12). We will construct a quadrature of the form (5.22) which realizes it. In order to do this, we take the truncated Gaussian quadrature $Q^{\mathrm{TG}}_{2j(m)}f$ defined in (5.7). For every $k \in \mathbb{N}_0$, let m_k be the largest number such that $2j(m_k) \leq 2^k$. Then we have $2j(m_k) \asymp 2^k$. For the sequence of quadratures $(Q_{2^k})_{k\in\mathbb{N}_0}$ with

$$Q_{2^k} := Q^{\mathrm{TG}}_{2j(m_k)} \in \mathcal{Q}_{2^k},$$

from Theorem 5.2 it follows that

$$\left|\int_{\mathbb{R}} f(x)w(x)\mathrm{d}x - Q_{2^k}f\right| \leq C2^{-r_\lambda k}\|f\|_{W^r_1(\mathbb{R};\mu)}, \quad k \in \mathbb{N}_0, \quad f \in W^r_1(\mathbb{R};\mu),$$

This means that the assumption (5.13)–(5.14) holds for $a = r_\lambda$. To prove the upper bound in (5.12) we approximate the integral (4.4) by the quadrature Q_ξ which is formed from the sequence $(Q_{2^k})_{k\in\mathbb{N}_0}$. For every $n \in \mathbb{N}$, let ξ_n be the largest number such that $|G(\xi_n)| \leq n$. Then the corresponding operator Q_{ξ_n} defines a quadrature belonging to $\mathcal{Q}_n$. From (5.23) it follows

$$2^{\xi_n}\xi_n^{d-1} \asymp |G(\xi_n)| \asymp n.$$

Hence we deduce the asymptotic equivalences

$$2^{-\xi_n} \asymp n^{-1}(\log n)^{d-1}, \quad \xi_n \asymp \log n,$$

which together with Lemma 5.7 yield that

$$\begin{aligned}\mathrm{Int}_n(\boldsymbol{W}^r_1(\mathbb{R}^d;\mu)) &\leq \sup_{f\in\boldsymbol{W}^r_1(\mathbb{R}^d;\mu)} \left|\int_{\mathbb{R}^d} f(\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_{\xi_n}f\right| \\ &\leq C2^{-r_\lambda\xi_n}\xi_n^{d-1} \asymp n^{-r_\lambda}(\log n)^{(r_\lambda+1)(d-1)}.\end{aligned}$$

The upper bound in (5.12) is proven.

We now prove the lower bound in (5.12) by using the inequality (5.4) in Lemma 5.1. For $M \ge 1$, we define the set

$$\Gamma_d(M) := \left\{ \boldsymbol{s} \in \mathbb{N}^d : \prod_{i=1}^d s_i \le 2M,\ s_i \ge M^{1/d},\ i = 1, ..., d \right\}.$$

Then we have

$$|\Gamma_d(M)| \asymp M(\log M)^{d-1}, \quad M > 1. \tag{5.26}$$

For a given $n \in \mathbb{N}$, let $\{\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n\} \subset \mathbb{R}^d$ be arbitrary n points. Denote by M_n the smallest number such that $|\Gamma_d(M_n)| \ge n + 1$. We define the d-cube $K_{\boldsymbol{s}}$ for $\boldsymbol{s} \in \mathbb{N}_0^d$ of size

$$\delta := M_n^{\frac{1/\lambda - 1}{d}}$$

by

$$K_{\boldsymbol{s}} := \prod_{i=1}^d K_{s_i}, \quad K_{s_i} := (\delta s_i, \delta s_{i-1}).$$

Since $|\Gamma_d(M_n)| > n$, there exists a multi-index $\boldsymbol{s} \in \Gamma_d(M_n)$ such that $K_{\boldsymbol{s}}$ does not contain any point from $\{\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n\}$.

We take a nonnegative function $\varphi \in C_0^\infty([0, 1])$, $\varphi \ne 0$, and put

$$b_0 := \int_0^1 \varphi(y) \mathrm{d}y > 0, \quad b_s := \int_0^1 |\varphi^{(s)}(y)| \mathrm{d}y,\ s = 1, ..., r. \tag{5.27}$$

For $i = 1, ..., d$, we define the univariate functions g_i in variable x_i by

$$g_i(x_i) := \begin{cases} \varphi(\delta^{-1}(x_i - \delta s_{i-1})), & x_i \in K_{s_i}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.28}$$

Then the mulitivariate functions g and h on $\mathbb{R}^d$ are defined by

$$g(\boldsymbol{x}) := \prod_{i=1}^d g_i(x_i),$$

and

$$h(\boldsymbol{x}) := (gw^{-1})(\boldsymbol{x}) = \prod_{i=1}^d g_i(x_i) w^{-1}(x_i) =: \prod_{i=1}^d h_i(x_i). \tag{5.29}$$

Let us estimate the norm $\|h\|_{W_1^r(\mathbb{R}^d;\mu)}$. For every $\boldsymbol{k} \in \mathbb{N}_0^d$ with $0 \le |\boldsymbol{k}|_\infty \le r$, we prove the inequality

$$\int_{\mathbb{R}^d} \left|(D^{\boldsymbol{k}} h) w\right|(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \le C M_n^{(1-1/\lambda)(r-1)}. \tag{5.30}$$

This inequality means that $h \in W_1^r(\mathbb{R}^d; \mu)$ and

$$\|h\|_{W_1^r(\mathbb{R}^d;\mu)} \le C M_n^{(1-1/\lambda)(r-1)}.$$

If we define

$$\bar{h} := C^{-1} M_n^{-(1-1/\lambda)(r-1)} h,$$

then $\bar{h}$ is nonnegative, $\bar{h} \in \boldsymbol{W}_1^r(\mathbb{R};\mu)$, $\sup \bar{h} \subset K_{\boldsymbol{s}}$ and by (5.27)–(5.29),

$$\int_{\mathbb{R}^d} (\bar{h}w)(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = C^{-1} M_n^{-(1-1/\lambda)(r-1)} \int_{\mathbb{R}^d} (hw)(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \prod_{i=1}^d \int_{K_{s_i}} g_i(x_i)\mathrm{d}x_i$$
$$= C^{-1} M_n^{-(1-1/\lambda)(r-1)} (b_0\delta)^d = C' M_n^{-r_\lambda}.$$

From the definition of M_n and (5.26) it follows that

$$M_n (\log M_n)^{d-1} \asymp |\Gamma(M_n)| \asymp n,$$

which implies that $M_n^{-1} \asymp n^{-1}(\log n)^{d-1}$. This allows to receive the estimate

$$\int_{\mathbb{R}^d} (\bar{h}w)(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = C' M_n^{-r_\lambda} \gg n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{5.31}$$

Since the interval $K_{\boldsymbol{s}}$ does not contain any point from the set $\{\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n\}$ which has been arbitrarily choosen, we have

$$\bar{h}(\boldsymbol{\xi}_k) = 0, \quad k = 1, ..., n.$$

Hence, by Lemma 5.1 and (5.31) we have that

$$\mathrm{Int}_n(\boldsymbol{W}_1^r(\mathbb{R}^d;\mu)) \ge \int_{\mathbb{R}^d} \bar{h}(\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \gg n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}.$$

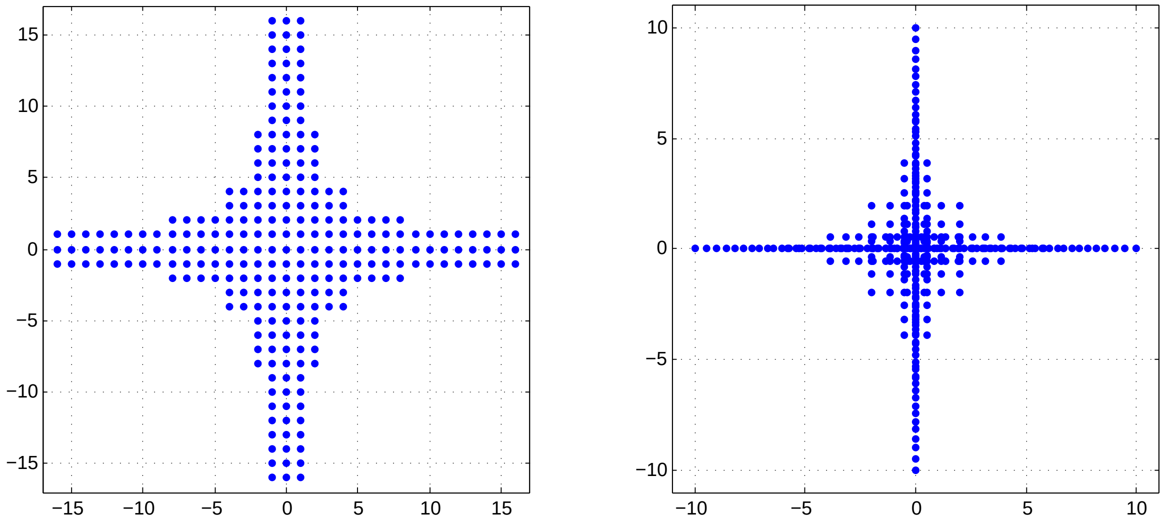
The lower bound in (5.12) is proven. □

Remark 5.10 Let us analyse some properties of the quadratures Q_ξ and their integration nodes $H(\xi)$ which give the upper bound in (5.12).

1. The set of integration nodes $H(\xi)$ in the quadratures Q_ξ which are formed from the non-equidistant zeros of the orthonornal polynomials $p_m(w)$, is a step hyperbolic cross *on the function domain* $\mathbb{R}^d$. This is a contrast to the classical theory of approximation of multivariate periodic functions having mixed smoothness for which the classical step hyperbolic crosses of integer points are *on the frequency domain* $\mathbb{Z}^d$ (see, e.g., [17, Section 2.3] for detail). The terminology 'step hyperbolic cross' of integration nodes is borrowed from there. In Figure 2, in particular, the step hyperbolic cross in the right picture is designed for the Hermite weight $w(\boldsymbol{x}) = \exp(-x_1^2 - x_2^2)$ $(d = 2)$. The set $H(\xi)$ also completely differs from the classical Smolyak grids of fractional dyadic points *on the function domain* $[-1, 1]^d$ (see Figure 3 for $d = 2$) which are used in sparse-grid sampling recovery and numerical integration for functions having a mixed smoothness (see, e.g., [17] and [4] for detail).

2. The set $H(\xi)$ is very sparsely distributed inside the d-cube

$$K(\xi) := \left\{\boldsymbol{x} \in \mathbb{R}^d : |x_i| \le C 2^{\xi/\lambda}, \ i = 1, ..., d\right\},$$



A classical step hyperbolic cross A Hermite step hyperbolic cross

Figure 2: Pictures of step hyperbolic crosses in $\mathbb{R}^2$ from [15]

for some constant $C > 0$. Its diameter which is the length of its symmetry axes is $2C2^{\xi/\lambda}$, i.e., the size of $K(\xi)$. The number of integration nodes in $H(\xi)$ is $|G(\xi)| \asymp 2^\xi \xi^{d-1}$. For the integration nodes $H(\xi) = \{\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}}\}_{(\boldsymbol{k},e,\boldsymbol{s})\in G(\xi)}$, we have that

$$\min_{\substack{(\boldsymbol{k},e,\boldsymbol{s}),(\boldsymbol{k}',e',\boldsymbol{s}')\in G(\xi)\\ (\boldsymbol{k},e,\boldsymbol{s})\neq(\boldsymbol{k}',e',\boldsymbol{s}')}} \min_{1\le i\le d} \left|(x_{\boldsymbol{k},e,\boldsymbol{s}})_i - (x_{\boldsymbol{k}',e',\boldsymbol{s}'})_i\right| \asymp 2^{-(1-1/\lambda)\xi} \to 0, \text{ when } \xi \to \infty.$$

On the other hand, the diameter of $H(\xi)$ is going to ∞ when $\xi \to \infty$.

5.4 Extension to Markov-Sonin weights

In this subsection, we extend the results of the previous subsection to Markov-Sonin weights. A univariate Markov-Sonin weight is a function of the form

$$w_\beta(x) := |x|^\beta \exp(-a|x|^2 + b), \quad \beta > 0, \quad a > 0, \quad b \in \mathbb{R}$$

(here β is indicated in the notation to distinguish Markov-Sonin weights w_β and Freud-type weight w). A d-dimensional Markov-Sonin weight is defined as

$$w_\beta(\boldsymbol{x}) := \prod_{i=1}^d w_\beta(x_i).$$

Markov-Sonin weights are not of the form (3.4) and have a singularity at 0. We will keep all the notations and definitions in Sections 1–5.3 with replacing w by w_β and μ by μ_β, pointing some modifications.

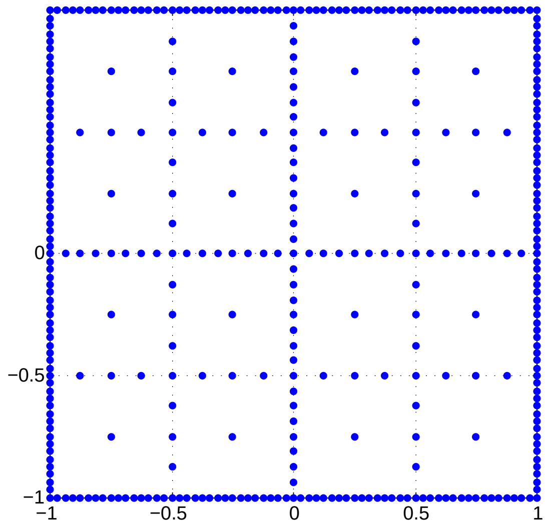


Figure 3: A picture of Smolyak grid in $[-1,1]^2$ from [15]

Denote $\mathring{\mathbb{R}}^d := (\mathbb{R} \setminus \{0\})^d$ and $\mathring{\Omega} := \Omega \cap \mathring{\mathbb{R}}^d$. Besides the spaces $L_p(\Omega;\mu_\beta)$ and $W_p^r(\Omega;\mu_\beta)$ we consider also the spaces $L_p\big(\mathring{\Omega};\mu_\beta\big)$ and $W_p^r\big(\mathring{\Omega};\mu_\beta\big)$ which are defined in a similar manner. For the space $W_p^r\big(\mathring{\Omega};\mu_\beta\big)$, we require one of the following restrictions on r and β to be satisfied:

(i) $\beta > r-1$;

(ii) $0 < \beta < r-1$ and β is not an integer, for $f \in W_p^r\big(\mathring{\Omega};\mu_\beta\big)$, the derivative $D^{\boldsymbol{k}} f$ can be extended to a continuous function on Ω for all $\boldsymbol{k} \in \mathbb{N}_0^d$ such that $|\boldsymbol{k}|_\infty \le r-1-\lceil\beta\rceil$.

Let $(p_m(w_\beta))_{m\in\mathbb{N}}$ be the sequence of orthonormal polynomials with respest to the weight w_β. Denote again by $x_{m,k}$, $1 \le k \le \lfloor m/2 \rfloor$ the positive zeros of $p_m(w_\beta)$, and by $x_{m,-k} = -x_{m,k}$ the negative ones (if m is odd, then $x_{m,0} = 0$ is also a zero of $p_m(w_\beta)$). If m is even, we add $x_{m,0} := 0$. These nodes are located as

$$-\sqrt{m}+Cm^{-1/6} < x_{m,-\lfloor m/2\rfloor} < \cdots < x_{m,-1} < x_{m,0} < x_{m,1} < \cdots < x_{m,\lfloor m/2\rfloor} \le \sqrt{m}-Cm^{-1/6},$$

with a positive constant C independent of m (the Mhaskar-Rakhmanov-Saff number is $a_m(w_\beta) = \sqrt{m}$).

In the case (i), the truncated Gaussian quadrature is defined by

$$Q_{2j(m)}^{\mathrm{TG}} f := \sum_{1\le|k|\le j(m)} \lambda_{m,k}(w_\beta) f(x_{m,k}),$$

and in the case (ii) by

$$Q_{2j(m)}^{\mathrm{TG}} f := \sum_{0\le|k|\le j(m)} \lambda_{m,k}(w_\beta) f(x_{m,k}),$$

where $\lambda_{m,k}(w_\beta)$ are the corresponding Cotes numbers.

In the same ways, by using related results in [35] we can prove the following counterparts of Theorems 5.2 and 5.9 for the unit ball $\boldsymbol{W}^r_{1,w_\beta}\big(\mathring{\mathbb{R}}^d\big)$ of the Markov-Sonin weighted Sobolev space $W^r_{1,w_\beta}\big(\mathring{\mathbb{R}}^d\big)$ of mixed smoothness $r \in \mathbb{N}$.

Theorem 5.11 *For any $n \in \mathbb{N}$, let m_n be the largest integer such that $2j(m_n) \le n$. Then the family of quadratures $\big(Q^{\rm TG}_{2j(m_n)}\big)_{n\in\mathbb{R}_1}$ is asymptotically optimal for $\boldsymbol{W}^r_1\big(\mathring{\mathbb{R}};\mu_\beta\big)$ and*

$$\sup_{f\in\boldsymbol{W}^r_1\left(\mathring{\mathbb{R}};\mu_\beta\right)} \left| \int_{\mathbb{R}} f(x)\mu_\beta(\mathrm{d}x) - Q^{\rm TG}_{2j(m_n)}f\right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}^r_1\big(\mathring{\mathbb{R}};\mu_\beta\big)\big) \asymp n^{-r/2}.$$

Theorem 5.12 *We have that*

$$n^{-r/2}(\log n)^{(d-1)r/2} \ll \mathrm{Int}_n\boldsymbol{W}^r_1\big(\mathring{\mathbb{R}}^d;\mu_\beta\big)\big) \ll n^{-r/2}(\log n)^{(d-1)(r/2+1)}.$$

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