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A UNIFIED APPROACH TO EXPONENTIAL STABILITY ANALYSIS FOR A GENERAL CLASS OF SWITCHED TIME-DELAY LINEAR SYSTEMS

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Abstract. This paper proposes a unified approach to study global exponential stability for a class of switched time-delay linear systems described by general linear functional differential equations. Several new delay-dependent criteria of global exponential stability are established for these systems over the sets of switchings satisfying the assumption on the minimum dwell time or having the average dwell time. As particular cases, the obtained results are shown to include and improve many previously known results. An example is given to illustrate the proposed method.

Keywords. Exponential stability; Switched systems; Functional differential equations; Positive systems; Average dwell time.

1. INTRODUCTION

A switched system is a type of hybrid dynamic system which consists of a family of subsystems and a rule called a switching signal that chooses an active subsystem from the family at every instant of time. Switched systems have attracted a lot of attention from researchers in control and systems theory due to their abilities in modeling various physical systems in engineering practice. Among a large number of interesting topics on switched systems the study on systems dynamic behavior, in particular, stability problems have been always the focus issues. The reader is referred to the monograph [1] and the survey paper [2, 3] and the references therein for more details. It has been indicated, for instance, that the switched linear system is exponentially stable under arbitrary switching signals if all constituent subsystems have a common quadratic Lyapunov function (QLF). Recently, similar problems have been considered intensively also for time-delay switched systems, where different kinds of the so-called Lyapunov - Krasovskii functionals are playing a similar role.

Dedicated to Professor Phan Dinh Dieu on the occasion of his 85th birth anniversary.

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In the meantime, for the class of positive or compartmental switched linear systems, besides traditional quadratic Lyapunov functions, a more restrictive notion of linear copositive Lyapunov functions (LCLF), combining with the comparison principle for solutions, is exploited effectively in the study of stability problems (see e.g. [12, 14, 15] and also [16, 17, 18] for time-delay systems and the comparison method).

The main purpose of this paper is to develop a unified approach to study exponential stability for a general class of time-delay switched linear systems, described by linear functional differential equations (LFDEs), based on the comparison principle, with the use of LCLFs and the average dwell time (ADT) switching concept. Namely, we will first over-bound each constituent subsystem by an appropriate positive subsystem and prove that the original switched time-delay system is uniformly exponentially stable over the set of all switchings which have positive minimum dwell time, if all over-bounding subsystems have a common LCLF. As a second main result, we will remove the above restrictive assumption on common LCLF and show that exponential stability of all over-bounding subsystems is enough to guarantee uniform exponential stability of the original switched time-delay system over the set of all switchings satisfying some ADT assumption. While stability of LFDEs has been widely investigated in the literature, the stability problems for switched systems of this general type is considered for the first time in this paper, to the best of our knowledge. As particular cases, the obtained stability criteria include and, what is more, improve many known results in the literature.

The following notation will be used throughout the paper. $\mathbb{R}$ and $\mathbb{N}$ will stand for the sets of real numbers and non-negative intergers, respectively. For $r \in \mathbb{N}$, $\underline{r}$ will stand for the set of numbers $\{1, 2, \ldots, r\}$. For matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{n\times m}$, we write $A \geq B$ and $A \gg B$ iff $a_{ij} \geq b_{ij}$ and $a_{ij} > b_{ij}$ for $i \in \underline{n}$, $j \in \underline{m}$, respectively. $|A|$ stands for the matrix $(|a_{ij}|)$ and $A^\top$ is the transpose of A. Similar notation is applied for vectors $x \in \mathbb{R}^n$. Without loss of generality, the norm of vectors $x \in \mathbb{R}^n$ is assumed to be the ∞-norm $\|x\| = \|x\|_\infty = \max_{1\leq i\leq n} |x_i|$. For $h > 0, C([-h, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions $\varphi : [-h, 0] \to \mathbb{R}^n$ with the norm $\|\varphi\| = \max_{\theta\in[-h,0]} \|\varphi(\theta)\|$ and $NBV([-h, 0], \mathbb{R})$ is the linear space of all normalized functions $\psi : [-h, 0] \to \mathbb{R}$ with bounded variation $Var([-h, 0], \psi)$ (so that ψ is left-side continuous on the interval $(-h, 0)$ and $\psi(-h) = 0$). It is well-known that, for any $\psi \in NBV([-h, 0], \mathbb{R})$ and any continuous function $\beta \in C([-h, 0], \mathbb{R})$, we have

$$\int_{-h}^{0} d[\psi(\theta)]\beta(\theta) \leq Var([-h, 0], \psi) \max_{\theta\in[-h,0]} |\beta(\theta)|, \tag{1}$$

where the integral is understood in the sense of Riemann-Stieltjes. Similarly, $NBV([-h, 0], \mathbb{R}^{n\times n})$ will stand for the linear space of all matrix functions $\eta : [-h, 0] \to \mathbb{R}^{n\times n}$ such that $\eta_{ij}(\cdot) \in NBV([-h, 0], \mathbb{R}), \forall i, j \in \underline{n}$. Thus, to each $\eta \in NBV([-h, 0], \mathbb{R}^{n\times n})$ we can associate a non-negative $(n \times n)$-matrix of variations

$$V(\eta) := \big(Var([-h, 0], \eta_{ij})\big) \geq 0. \tag{2}$$

Recall that $A \in \mathbb{R}^{n\times n}$ is said to be a *Metzler matrix* if all off-diagonal elements of A are nonnegative: $a_{ij} \geq 0, \forall i \neq j$. Finally, for any matrix $A \in \mathbb{R}^{n\times n}$ we associate the Metzler matrix $\mathcal{M}(A) = (\hat{a}_{ij})$, by setting $\hat{a}_{ij} = |a_{ij}|$ if $i \neq j$ and $\hat{a}_{ii} = a_{ii},\ \forall i \in \underline{n}$.

2. MAIN RESULTS

Consider a switched time-delay linear system of the general form

$$\dot{x}(t) = A^0_{\sigma(t)}x(t) + \int_{-h}^{0} d[\eta_{\sigma(t)}(\theta)]x(t+\theta), t \geq 0, \tag{3}$$

where $\sigma(\cdot)$ is a switching signal such that for each $t \geq 0, A^0_{\sigma(t)} \in \{A^0_k, k \in \underline{N}\} \subset \mathbb{R}^{n\times n}$ - a given family of N real matrices and $\eta_{\sigma(t)} \in \{\eta_k, k \in \underline{N}\} \subset NBV([-h, 0], \mathbb{R}^{n\times n})$ - a given family of N matrix functions with normalized bounded variation elements $\eta_{k,ij}$. Denote by Σ_+ the set of all switching signals $\sigma : [0, +\infty) \to \underline{N}$ which are piece-wise constant, right-side continuous functions, with points of discontinuity $\tau_k, k = 1, 2, \ldots$ (known as the *switching instances*) satisfying the following assumption on the *minimum dwell time*

$$\tau_{min}(\sigma) := \inf_{k\in\mathbb{N}}(\tau_{k+1} - \tau_k) > 0. \tag{4}$$

It is clear that Σ_+ does not contain any switching signal whose discontinuities have a finite accumulation point. Also, any signal σ having switching instances $\tau_{2k} = k, \tau_{2k+1} = k + \frac{1}{2k+1}, k = 0, 1, 2, \ldots$ does not satisfy (4) because in this case $\tau_{min}(\sigma) = 0$.

Thus, each $\sigma \in \Sigma_+$ performs switchings between the following N time-delay linear subsystems (A^0_k, η_k) of the form

$$\dot{x}(t) = A^0_k x(t) + \int_{-h}^{0} d[\eta_k(\theta)]x(t+\theta),\ t \geq 0,\ k \in \underline{N}, \tag{5}$$

where the i-th component of the second term in (5), for each $k \in \underline{N}$ and $i = 1, \ldots, n$, is defined as

$$\left(\int_{-h}^{0} d[\eta_k(\theta)]x(t+\theta)\right)_i = \sum_{j=1}^{n} \int_{-h}^{0} d[\eta_{k,ij}(\theta)]x_j(t+\theta). \tag{6}$$

For any $\varphi \in C([-h, 0], \mathbb{R}^n)$ and any switching signal $\sigma \in \Sigma_+$, the system (3) admits a unique solution $x(t) = x(t, \varphi, \sigma),\ t \geq -h$, satisfying the initial condition $x(\theta) = \varphi(\theta),\ \theta \in [-h, 0]$. Note that the solution $x(t)$ is absolutely continuous function on $[0, +\infty)$ and differentiable everywhere, except for the set of switching instances $\{\tau_k\}$ of σ where $x(t)$ has only Dini right- and left-derivatives $D^+x(\tau_k), D^-x(\tau_k)$ which are generally different.

Definition 1. The switched system (3) is said to be globally exponentialy stable (shortly, GES) over the set of switching signals Σ_+ if there exist positive numbers M, α such that for any $\varphi \in C([-h, 0], \mathbb{R}^n)$ and any $\sigma \in \Sigma_+$, the solutions $x(t, \varphi, \sigma)$ of (3) satisfies

$$\|x(t, \varphi, \sigma)\| \leq Me^{-\alpha t}\|\varphi\|, \quad \forall t \geq 0. \tag{7}$$

Obviously, for each $k \in \underline{N}$, the switching law $\sigma(t) \equiv k, t \geq 0$, belongs to Σ_+. Therefore, if the switched system (3) is GES over Σ_+ then all of the constituent subsystems (5) are uniformly exponentially stable or, equivalently, all zeros of their characteristic quasi-polynomials $P_k(s) = sI - A_k^0 - \int_{-h}^0 e^{s\theta} d[\eta_k(\theta)], k \in \underline{N}$, have negative real parts (see e.g. [20]). The last condition is, however, not sufficient for exponential stability of (3) under arbitrary switching (see, e.g. [1] for the case when $\eta_k = 0, \forall k \in \underline{N}$).

Finally, we recall that the system (3) is said to be *positive* if $x(t) \geq 0, \forall t \geq 0$ whenever $\varphi(\theta) \geq 0,\ \forall \theta \in [-h, 0]$. It is trivial to show that (3) is positive if and only if all subsystems (5) are positive. The latter is equivalent to the condition that, for each $k \in \underline{N}, A_k^0$ is a Metzler matrix and η_k is increasing on $[-h, 0] : \eta_k(\theta_1) \leq \eta_k(\theta_2)$, if $-h \leq \theta_1 \leq \theta_2 \leq 0$ (see, e.g. [21, 23]). Clearly, in this case, we have

$$\mathcal{M}(A_k^0) = A_k^0, \quad V(\eta_k) = \eta_k(0),\ \forall k \in \underline{N}.$$

We are now in position to prove the first main result of this paper which gives a verifiable criterion for exponential stability of the class of switched time-delay linear systems of the form (3) over the set of switching signals Σ_+. The main idea of the proof is essentially based on the comparison principle of solutions (see and compare with [16, 18]). The case of non-delay linear systems (i.e. when $\eta_k = 0, \forall k \in \underline{N}$) has been considered in our recent work [22]. We give a detailed proof in Appendix for the convenience of the readers.

Theorem 1. *Consider the switched time-delay linear system* (3). *Assume that there exist a strictly positive vector $\xi \gg 0$ (i.e. all elements of vector ξ are positive) and a number $\alpha > 0$ such that*

$$\big(\mathcal{M}(A_k^0) + e^{\alpha h} V(\eta_k)\big)\xi \ll -\alpha\xi,\ \forall k \in \underline{N}, \tag{8}$$

where the nonnegative matrices $V(\eta_k)$ are defined by (2). *Then the switched time-delay linear system* (3) *is GES over the set of switching signals $\sigma \in \Sigma_+$.*

Remark 1. If (8) holds for some $\alpha > 0$ then it obviously holds also for $\alpha = 0$. The latter in turn implies straightforwardly (see e.g. [13]) that all positive linear subsystems $\dot{x}(t) = D_k x(t), k \in \underline{N}$, are exponentially stable, with $D_k := \mathcal{M}(A_k^0) + V(\eta_k)$ being obviously Metzler matrices. These positive linear subsystems can be considered as over-bounding systems for the original time-delay subsystems $(A_k^0, \eta_k), k \in \underline{N}$. Moreover, (8) implies that the dual systems $\dot{x}(t) = D_k^\top x(t), k \in \underline{N}$, share a common linear co-positive Lyapunov function (shortly LCLF) $v(x) = \xi^\top x$ (see, e.g.[14]). Note additionally that in order to check whether or not Metzler matrices $D_k^\top, k \in \underline{N}$, share a common LCLF one can use the procedure given by Theorem 4 in [15].

As the most important particular case of Theorem 1, let us consider the class of switched linear systems with multiple discrete-time delays and distributed time delays of the form

$$\dot{x}(t) = A_{\sigma(t)}^0 x(t) + \sum_{i=1}^{m_{\sigma(t)}} A_{\sigma(t)}^i x(t - h_{\sigma(t)}^i) + \int_{-h_{\sigma(t)}}^0 B_{\sigma(t)}(\theta) x(t+\theta) d\theta,\ t \geq 0, \tag{9}$$

where, for each $k \in \underline{N}$, $0 = h_k^0 < h_k^1 < \ldots < h_k^{m_k}$, matrices $A_k^i \in \mathbb{R}^{n\times n}$ and matrix functions $B_k(\cdot) \in C([-h_k,0],\mathbb{R}^{n\times n})$ are given. Defining $h \coloneqq \max_{k\in\underline{N}, i\in\underline{m_k}}\{h_k^i, h_k\}, m := \max\{m_k, k \in \underline{N}\}$ and setting, for each $k \in \underline{N}, B_k(\theta) \equiv 0, \theta \in [-h,-h_k)$ (if $h_k < h$) and $A_k^i = 0$ for $i = m_{k+1},\ldots,m$ (if $m_k < m$), it is easy to see that (9) is just a particular case of (3) with $\eta_k(\theta) = \sum_{i=1}^{m} A_k^i \chi_{(-h_k^i,0]}(\theta) + \int_{-h}^{\theta} B_k(s)ds, k \in \underline{N}$, where χ_M denotes the characteristic function of a set $M \subset \mathbb{R}$. Obviously, $V(\eta_k) \le \sum_{i=1}^{m} |A_k^i| + \int_{-h}^{0} |B_k(s)|ds$, but if the system (9) is positive (i.e. when $\mathcal{M}(A_k^0) = A_k^0,\ A_k^i \ge 0, B_k(\theta) \ge 0$ for all $k \in \underline{N}, i \in \underline{m}, \theta \in [-h,0]$) then $V(\eta_k) = \sum_{i=1}^{m} A_k^i + \int_{-h}^{0} B_k(s)ds$. Therefore, by Theorem 1, we get

Corollary 1. *If there exist $\xi \in \mathbb{R}^n, \xi \gg 0$ and a non-negative number $\alpha > 0$ satisfying*

$$\big(\mathcal{M}(A_k^0) + e^{\alpha h}\sum_{i=1}^{m}|A_k^i| + e^{\alpha h}\int_{-h}^{0}|B_k(s)|ds\big)\xi \ll -\alpha\xi,\ \forall k \in \underline{N}, \tag{10}$$

then the system (9) *is exponentially stable for any switching $\sigma \in \Sigma_+$. Moreover, if the system* (9) *is positive then the modulus symbol in* (10) *can be removed.*

The above corollary gives a new and improved delay-dependent criterion of exponential stability for the switched time-delay linear system(9), which gets back obviously to results proved in [16, 18] by letting $\alpha \downarrow 0$ in (10).

It is worth noticing that in Theorem 1 the switching signals are assumed to be taken arbitrarily from the class Σ_+ defined by the assumption (4) which is a rather mild property. The cost to pay is that we have to impose the conservative condition on existence of a common vector ξ in (8) to ensure GES of (3) for each switching $\sigma \in \Sigma_+$. In the next theorem, by using the *average dwell time* (or ADT, for short) concept, we will relax this condition by assuming only the existence of a set of vectors $\xi_k, k \in \underline{N}$ which maybe different but satisfying (8). However, GES of (3) is then guaranteed only over *a subset* of Σ_+, namely the set of all switching signals having ADT $\tau_a > \tau_*$ where the lower bound τ_* is calculated via $\xi_k, k \in \underline{N}$.

We recall (see, e.g. [5]) that, for a given positive number τ_a, a switching signal σ is said to have an ADT τ_a if for any $t > 0$ the number $N_\sigma(0,t)$ of discontinuities of σ on the interval $(0,t)$ satisfies

$$N_\sigma(0,t) \le \frac{t}{\tau_a}. \tag{11}$$

The set of all switching signals having ADT τ_a is denoted by Σ_{τ_a}. It follows that for any $\sigma \in \Sigma_{\tau_a}$, the average dwell time between any two consecutive switching instances is at least τ_a. We have obviuosly that, for any $\tau_1 > \tau_2 > 0$,

$$\Sigma_{\tau_1} \subset \Sigma_{\tau_2} \subset \Sigma_+.$$

Therefore, if the system (3) is GES over Σ_+ it is also GES over Σ_{τ_a} for any $\tau_a > 0$. On the other side, in case of non-delay linear systems (i.e. when $\eta_k = 0, \forall k \in \underline{N}$), it is well known that if every constituent subsystem is GES then the corresponding switched system is also GES for each switching σ having ADT τ_a sufficiently large (see, e.g. Lemma 1 in [6]) and it is an important problem to find such a lower bound $\tau_* > 0$ that the associate switched system is exponentially stable for any $\sigma \in \Sigma_{\tau_a}$ with ADT $\tau_a > \tau_*$ (see, e.g. [1, 5, 8, 17, 18]). In the following theorem we establish a similar result for the general class of FDE of the form (3).

Theorem 2. *Consider the switched linear time-delay system* (3). *Assume that there exist vectors $\xi_k \in \mathbb{R}^n, \xi_k \gg 0, k \in \underline{N}$ and a positive number $\alpha > 0$ satisfying*

$$\big(\mathcal{M}(A_k^0) + e^{\alpha h} V(\eta_k)\big)\xi_k \ll -\alpha \xi_k,\ \forall k \in \underline{N}. \tag{12}$$

Then the switched system (3) *is GES over the set Σ_{τ_a} of switching signals with ADT*

$$\tau_a > \tau_* := \frac{\ln\,\gamma}{\alpha}, \tag{13}$$

where

$$\gamma := \max\{\frac{\xi_{k,i}}{\xi_{l,i}} : k, l \in \underline{N}, i \in \underline{n}\},\ \xi_k := (\xi_{k,1}\ \xi_{k,2}\ \cdots\ \xi_{k,n})^\top. \tag{14}$$

Proof.

The proof is partly similar to that of Theorem 1. Without loss of generality we can assume that $\|\xi_k\| = 1, k \in \underline{N}$. Let τ_a satisfy (13) and $\sigma \in \Sigma_{\tau_a}$ be an arbitrary switching signal, with switching instances $0 = \tau_0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} < \ldots$ Let $x(t) = x(t, \varphi, \sigma)$ be the corresponding solution of (3) satisfying the initial condition $x(\theta) = \varphi(\theta),\ \theta \in [-h, 0], \|\varphi\| = 1$. Assume that $\sigma(\tau_k) = l_k \in \underline{N}$, i.e. the subsystem $(A_{l_k}^0, \eta_{l_k})$ is active on $[\tau_k, \tau_{k+1}), k = 0, 1, \ldots$. For any $\delta > 1$, define the functions $y_i(t), t \geq -h,\ i \in \underline{n}$, by setting

$$y_i(t) = \begin{cases} M_\delta e^{-\alpha t}\xi_{l_0,i} & \text{if } \ t \in [-h, \tau_0), \\ M_\delta e^{-\alpha t}\xi_{l_k,i} & \text{if } \ t \in [\tau_k, \tau_{k+1}),\ k = 0, 1, 2, \ldots \end{cases} \tag{15}$$

where $M_\delta = \delta.\gamma$. Then we verify readily that $M_\delta > 1$ and

$$|x_i(t)| = |\varphi_i(t)| \leq 1 < M_\delta e^{-\alpha t}\xi_{l_0,i} = y_i(t),\ \forall t \in [-h, 0],\ \forall i \in \underline{n}. \tag{16}$$

Therefore, noticing that the subsystem $(A_{l_0}^0, \eta_{l_0})$ is active on $[0, \tau_1) = [\tau_0, \tau_1)$ we can proceed similarly as in the proof of Theorem 1, by using (12) (with $k = l_0$) and (16), to show that (16) keeps hold on the interval $[\tau_0, \tau_1)$

$$|x_i(t)| \leq y_i(t) = M_\delta e^{-\alpha t}\xi_{l_0,i},\ \forall t \in [\tau_0, \tau_1),\ \forall i \in \underline{n}. \tag{17}$$

Letting t tend to τ_1 and δ tend to 1 we get from (17), (14) and (15) that

$$|x_i(\tau_1))| \leq M_1 e^{-\alpha \tau_1}\xi_{l_0,i} = \frac{\xi_{l_0,i}}{\xi_{l_1,i}} M_1 e^{-\alpha\tau_1}\xi_{l_1,i} < \gamma\, M_\delta e^{-\alpha\tau_1}\xi_{l_1,i} = \gamma\, y_i(\tau_1),\ \forall i \in \underline{n}. \tag{18}$$

Further, using the strict inequality above, we now prove that

$$|x_i(t)| \le \gamma y_i(t) = \gamma M_\delta e^{-\alpha t}\xi_{l_1,i}\ ,\ \forall t \in [\tau_1,\tau_2),\ \forall i \in \underline{n}. \tag{19}$$

Assume again that (19) does not hold then, by continuity, there exist $i_1 \in \underline{n}$, $\bar{t}_1 \in (\tau_1,\tau_2)$ and $\varepsilon > 0$ such that

$$|x_i(t)| < \gamma y_i(t), \forall t \in [\tau_1,\bar{t}_1),\ \forall i \in \underline{n}, \tag{20}$$

and

$$|x_{i_1}(\bar{t}_1)| = \gamma y_{i_1}(\bar{t}_1), |x_{i_1}(t)| > \gamma y_{i_1}(t),\ \forall t \in (\bar{t}_1,\bar{t}_1+\epsilon). \tag{21}$$

Noticing that the subsystem $(A^0_{l_1},\eta_{l_1})$ is active on $[\tau_1,\tau_2)$, by using (1), we can estimate the Dini right-derivative $D_+|x(\bar{t}_1)|$ as

$$D^+|x_{i_1}(\bar{t}_1)| \le a^0_{l_1,i_1\,i_1}|x_{i_1}(\bar{t}_1)| + \sum_{j=1,j\ne i_1}^{n} |a^0_{l_1,i_1j}|\ |x_j(\bar{t}_1)| + \sum_{j=1}^{n} V(\eta_{l_1,i_1j}) \max_{\theta\in[-h,0]} |x_j(\bar{t}_1+\theta)|. \tag{22}$$

Here, we have to consider two cases: $\tau_1 \le \bar{t}_1 - h$ and $\bar{t}_1 - h < \tau_1$. In the first case, we have, by (20), that for any $\theta \in [-h,0]$, $|x_j(\bar{t}_1+\theta)| \le \gamma y_j(\bar{t}_1+\theta) \le \gamma y_j(\bar{t}_1-h) = \gamma M_\delta e^{-\alpha(\bar{t}_1-h)}\xi_{l_1,j}$, $\forall j \in \underline{n}$. In the second case, by using (17) and (14), we deduce that, for any $\theta \in [-h,0]$,

$$|x_j(\bar{t}_1+\theta)| \le y_j(\bar{t}_1+\theta) = M_\delta e^{-\alpha(\bar{t}_1+\theta)}\xi_{l_0,j} \le M_\delta e^{-\alpha(\bar{t}_1-h)}\xi_{l_0,j} \le \gamma M_\delta e^{-\alpha(\bar{t}_1-h)}\xi_{l_1,j},\ \forall j \in \underline{n}.$$

Thus in both cases, using the equality in (21) and (22), (12) we get the following estimate

$$D^+|x_{i_1}(\bar{t}_1)| \le \gamma M_\delta e^{-\alpha\bar{t}_1}\big((\mathcal{M}(A^0_{l_1}) + e^{\alpha h}V(\eta_{l_1}))\xi_{l_1}\big)_{i_1} < \gamma M_\delta e^{-\alpha\bar{t}_1}(-\alpha\xi_{i_1}) = \gamma\frac{d}{dt}y_{i_1}(\bar{t}_1). \tag{23}$$

On the other hand, by the inequality in (21) it follows readily that $D^+|x_{i_1}(\bar{t}_1)| \ge \gamma\frac{d}{dt}y_{i_1}(\bar{t}_1)$, a contradiction. Thus (19) is proved. By letting $t \to \tau_2, \delta \to 1$ in (19) we get $|x_i(\tau_2))| < \gamma^2\ M_\delta e^{-\alpha(\tau_2-\tau_1)}\xi_{l_1,i} = \gamma^2\ y_i(\tau_2)$, $\forall i \in \underline{n}$, which implies, from the same reasoning as above, that $|x_i(t)| \le \gamma^2 y_i(t), \forall t \in [\tau_2,\tau_3)$, $\forall i \in \underline{n}$. Proceeding as above steps we conclude that for each $k = 0,1,2,\dots$ and each $i \in \underline{n}$, we have $|x_i(t)| \le \gamma^k y_i(t) = \gamma^k M_\delta e^{-\alpha t}\xi_{l_k,i} \le \gamma^k M_\delta e^{-\alpha t}$, $t \in [\tau_k,\tau_{k+1})$, taking into account that $\|\xi_k\| = \max_{i\in\underline{n}}|\xi_{k,i}| = 1, \forall k \in \underline{N}$. Therefore, by the assumption that $\sigma \in \Sigma_{\tau_a}$ with ADT τ_a satisfying (13) it follows that, for each $t > 0$,

$$|x_i(t)| \le \gamma^{N_\sigma(t,0)}M_\delta e^{-\alpha t} = M_\delta e^{N_\sigma(t,0)\ln\gamma}e^{-\alpha t} \le M_\delta e^{(\frac{\ln\gamma}{\tau_a}-\alpha)t}, \tag{24}$$

where $\frac{\ln\gamma}{\tau_a} - \alpha < 0$. This completes the proof. ∎

Similarly as Corollary 1 we have

Corollary 2. *Consider the switched time-delay linear system* (9). *If there exist* $\xi_k \in \mathbb{R}^n$, $\xi_k \gg 0$, $k \in \underline{N}$ *and a positive number* $\alpha > 0$ *satisfying*

$$\big(\mathcal{M}(A^0_k) + e^{\alpha h}\sum_{i=1}^{m}|A^i_k| + e^{\alpha h}\int_{-h}^{0}|B_k(s)|ds\big)\xi_k \ll -\alpha\xi_k,\ \forall k \in \underline{N}, \tag{25}$$

then the system (9) *is GES over the set of switching signals* Σ_{τ_a} *with ADT* τ_a *satisfying* (13).

It is obvious that the main result in [17] is just a particular case of the above Corollary 2 (by letting $m = 1, B_k(s) \equiv C_k, \forall s, k$, and $\alpha \downarrow 0$ in (25)).

Remark 2. The above proof keeps valid also for a more general definition of switching signals σ having ADT τ_a (see, e.g. [5]) where, instead of (11), the number of discontinuities of σ on each interval $(0, t)$ satisfies

$$N(0,t) \leq N_0 + \frac{t}{\tau_a}, \tag{26}$$

with a given number $N_0 \geq 0$ (called a 'chatter bound' of the signal). Actually, in this case, instead of (24), we can easily deduce

$$|x_i(t)| \leq \gamma^{N_\sigma(t,0)} M_\delta e^{-\alpha t} = M_\delta e^{N_0 \ln \gamma} e^{(\frac{\ln \gamma}{\tau_a} - \alpha)t}, \tag{27}$$

implying again that the system is GES.

Remark 3. Theorem 2 and its corollary above resemble the main results in [24]. However, the analysis in [24] can not be applied to deal with the problem considered in Theorem 2 since conditions (12) and (25) do not necessarily imply that such a common vector ξ exists. Instead, an average dwell time (ADT) approach (that is different from the method in the mentioned work) has been used to establish our above results. Notice, moreover, that Theorem 1 can be obtained from Theorem 2. Indeed, if there exist $\xi_k = \xi \ll 0, \forall k \in \underline{N}$ satisfying (8) with some $\alpha > 0$ then $\gamma = 1$, by (14), and so $\tau_* = 0$, in view of (13). Then, by Theorem 2, the switched system (3) is GES over the set Σ_{τ_a} for any ADT $\tau_a > 0$ and hence over Σ_+.

In the following example, a numerical simulation in $\mathbb{R}^2$ is given to illustrate Theorem 2.

Example 1. Consider the switched time-delay linear system (3) in $\mathbb{R}^2$ with $h = 0.5, N = 2$,

$$A_1^0 = \begin{bmatrix} -1 & 0.15 \\ -0.15 & -0.55 \end{bmatrix},\ A_2^0 = \begin{bmatrix} -0.5 & -0.1 \\ 0.15 & -1 \end{bmatrix},\ \eta_k(\theta) = \begin{cases} 0 \text{ if } \theta = -0.5 \\ B_k \text{ if } \theta \in (-0.5, 0], \end{cases} \quad \text{for } k = 1, 2,$$

where $B_1 = \begin{bmatrix} 0.15 & 0.25 \\ 0.25 & 0.15 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0.25 & 0.15 \\ 0.15 & 0.15 \end{bmatrix}$. Thus, we have

$$\mathcal{M}(A_1^0) = \begin{bmatrix} -1 & 0.15 \\ 0.15 & -0.55 \end{bmatrix},\ \mathcal{M}(A_2^0) = \begin{bmatrix} -0.5 & 0.1 \\ 0.15 & -1 \end{bmatrix},\ V(\eta_1) = \eta_1(0) = B_1,\ V(\eta_2) = \eta_2(0) = B_2.$$

Note that there does not exist a vector $\xi = [c_1, c_2]^\top > 0$ that satisfies (8) with $\alpha = 0$, because otherwise, we would get $c_1 < c_2$ and $c_2 < c_1$, a contradiction. On the other hand, there exist two vectors $\xi_1 = [0.6, 1]^\top$ and $\xi_2 = [1, 0.5]^\top$ which satisfy (12) with $\alpha = 0.1$. Therefore, by Theorem 2, we conclude that the switched time-delay system of the form (3) is exponentially stable GES for any switching signal with ADT $\tau_a > \frac{\ln \gamma}{\alpha} = \frac{\ln 2}{0.1} \approx 6.93$. For instance, choose $\tau_a = 7 > \tau_* = 6.93$, a switching signal $\sigma \in \Sigma_{\tau_a}$ as shown in Figure 1 and the initial condition given by the function $\varphi(\theta) \equiv (1\ \ -1)^\top,\ \theta \in [-0.5, 0]$, then the solution trajectory of the above switched system is shown in Figure 2, where the simulation has been performed with the MATLAB code **dde23**.

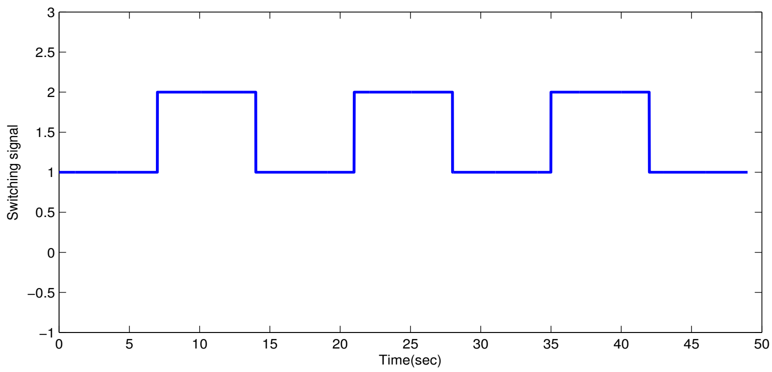

Figure 1: The switching with ADT $\tau_a = 7$

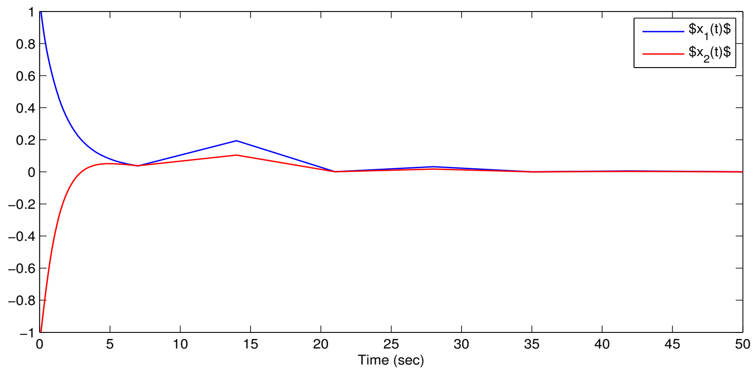

Figure 2: The sulution trajectory of Example 1 under switching with ADT $\tau_a = 7 > 6.93$

3. CONCLUSIONS

We have studied exponential stability problem for a general class of time-delay switched systems described by linear functional differential equations. Unlike the previous results, which were based on the comparison principle and common Lyapunov functions, stability criteria in this paper are derived under ADT switching. Finally, a numerical example is included to illustrate the main results. The results of this paper can be extended, with appropriate modifications, to the case of switched linear systems with time-varying delays of the form

$$\dot{x}(t) = A^0_{\sigma(t)} x(t) + \int_{-h_{\sigma(t)}(t)}^{0} d[\eta_{\sigma(t)}(\theta)] x(t+\theta), t \geq 0, \tag{28}$$

where, for $k \in \underline{N}$, $h_k : [0, +\infty) \to \mathbb{R}_+$ are continuous functions, such that $\sup_{k\in\underline{N}, t\geq 0} h_k(t) \leq h$. The case of unbounded time-varying delays can also be treated by our approach. This will be reported in our future work.

APPENDIX

Proof of Theorem 1. Let $\sigma \in \Sigma_+$ be an arbitrary switching law satisfying (4) and $x(t) = x(t,\varphi,\sigma)$ be the solution of (3) satisfying the initial condition $x(\theta) = \varphi(\theta),\ \theta \in [-h,0]$ with an arbitrary $\varphi \in C([-h,0].\mathbb{R}^n), \|\varphi\| \leq 1$. Assume (8) holds for some $\alpha > 0$ and $\xi := (\xi_1\ \xi_2\ ...\ \xi_n)^\top \in \mathbb{R}^n$ with $\xi_i > 0,\ \forall i \in \underline{n}$. Without loss of generality we can assume $\|\xi\| = 1$. For an arbitrary $\delta > 1$, define $M_\delta = \dfrac{\delta}{\min_{j\in\underline{n}} \xi_j} > 1$ and

$$y(t) = M_\delta e^{-\alpha t}\xi,\ t \in [-h,\infty). \tag{29}$$

Then, since $\mathbb{R}^n$ is equipped with the ∞-norm, in order to prove (7) it suffices show that

$$|x_i(t)| \leq y_i(t),\ \forall t \in [0,\infty),\ \forall i \in \underline{n}, \tag{30}$$

To this end, observe first that (30) clearly holds also for $t \in [-h,0]$, that is

$$|x_i(t)| = |\varphi_i(t)| \leq 1 < y_i(t) = M_\delta e^{-\alpha t}\xi_i,\ \forall t \in [-h,0],\ \forall i \in \underline{n}. \tag{31}$$

Assume to the contrary that (30) does not hold, then, by the continuity, there exist $i_0 \in \underline{n},\ \bar{t}_0 > 0$ and $\varepsilon > 0$ such that

$$|x_i(t)| < y_i(t), \forall t \in [-h,\bar{t}_0),\ \forall i \in \underline{n}, \tag{32}$$

and

$$|x_{i_0}(\bar{t}_0)| = y_{i_0}(\bar{t}_0), |x_{i_0}(t)| > y_{i_0}(t),\ \forall t \in (\bar{t}_0,\bar{t}_0+\epsilon). \tag{33}$$

Let $\{\tau_k\}_{k=1}^{+\infty}$ be the sequence of switching instances of σ and $\bar{t}_0 \in [\tau_{m_0},\ \tau_{m_0+1})$, then, by the assumption (4), $\tau_{m_0+1} - \tau_{m_0} \geq \tau_{min}(\sigma) > 0$. Therefore, one can choose a sufficiently small $\epsilon_1 > 0, \epsilon_1 < \epsilon$ such that $[\bar{t}_0,\ \bar{t}_0 + \epsilon_1) \subset [\tau_{m_0},\ \tau_{m_0+1})$. Letting the subsystem $(A^0_{k_0}, \eta_{k_0})$ be active on the interval $[\tau_{m_0},\ \tau_{m_0+1})$ (or equivalently the solution $x(t)$ satisfies (5) with $k = k_0$) and denoting elements of $A^0_{k_0}$ and $\eta_{k_0}(\theta)$, respectively, by $a_{k_0,ij}$ and $\eta_{k_0,ij}(\theta), i,j \in \underline{n}$ then, from (6),(32) and (5) with $k = k_0$, we can deduce, for any $t \in [\tau_{m_0},\ \tau_{m_0+1})$ and every $i \in \underline{n}$, that

$$\begin{aligned} D^+|x_i(t)| &\leq a_{k_0,ii}|x_i(t)| + \sum_{j=1,j\neq i}^{n} |a_{k_0,ij}|\ |x_j(t)| + \sum_{j=1}^{n} V(\eta_{k_0,ij}) \max_{\theta\in[-h,0]} |x_j(t+\theta)| \\ &\leq a_{k_0,ii}y_i(t) + \sum_{j=1,j\neq i}^{n} |a_{k_0,ij}|\ y_j(t) + \sum_{j=1}^{n} V(\eta_{k_0,ij})y_j(t-h). \end{aligned} \tag{34}$$

Therefore, by (29), (31) and (34) (with $i = i_0,\ t = \bar{t}_0$), using the equality in (33) and (8) we get

$$D^+|x_{i_0}(\bar{t}_0)| \leq M_\delta e^{-\alpha\bar{t}_0}\big((\mathcal{M}(A^0_{k_0}) + e^{\alpha h}V(\eta_{k_0}))\xi\big)_{i_0} < M_\delta e^{-\alpha\bar{t}_0}(-\alpha\xi_{i_0}) = \frac{d}{dt}y_{i_0}(\bar{t}_0).$$

On the other hand, by definition of the Dini right-derivative and the inequality in (33), we have

$$D^+|x_{i_0}(\bar{t}_0)| = \lim_{\epsilon\to 0^+} \frac{|x_{i_0}(\bar{t}_0+\epsilon)| - |x_{i_0}(\bar{t}_0)|}{\epsilon} \geq \lim_{\epsilon\to 0^+} \frac{y_{i_0}(\bar{t}_0+\epsilon) - y_{i_0}(\bar{t}_0)}{\epsilon} = \frac{d}{dt}y_{i_0}(\bar{t}_0),$$

a contradiction, completing the proof. ■

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