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# Approximations of Quasi-Variational Inequalities and Traffic Networks

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**Abstract** This paper aims at studying global approximations in terms of variational convergence of set-valued quasi-variational inequalities and traffic network problems with arc capacity constraints, set-valued travel costs, and elastic demands depending on equilibrium flows. We consider two types of quasi-variational inequalities, the weak and the usual strong variants, and the above traffic problem. We propose some weak types of epi/hypo- and lopsided convergence, new concepts of approximate solutions, new definitions of saturatedness of arcs and paths, and new notions of equilibrium flows. By defining a suitable bifunction for each case and basing on suitable types of variational convergence of such a bifunction, we obtain the set convergence of approximate global solutions of the approximating problems in question. This is the first attempt to consider global approximations of the aforementioned problems. Hence, the novelty of the results is high and also suggests further developments of the topic.

**Keywords** Set-valued quasi-variational inequalities · traffic networks with arc capacity constraints, set-valued travel costs, and elastic demands depending on equilibrium flows · types of epi/hypo- and lopsided convergence · approximate solutions

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## 1 Introduction

Approximations are met in any mathematical problem. One of the major reasons is that although mathematical models greatly simplify real-world problems, they are often very difficult to deal with so require approximation methods for consideration. Local approximations have been intensively studied with variety applications in many problem classes, see e.g., the books [6,27,29]. Variational or quasi-variational inequalities and traffic network problems are among the most important optimization-related models (variational inequalities are also involved in research of partial differential equations). The former encompasses many significant particular cases and the latter reflects interesting real-world situations. The reader is referred to the book [22] by two of the pioneers in variational inequality studies, the coherent and comprehensive books [12,13], and papers [4,5,9,14,19,20,26] among a huge number of contributions. Traffic network problems constitute one of the most important special cases of quasi-variational inequalities. The history of traffic network problems may be sketched as follows. The classical network problem is of seeking to determine the users' paths of minimal cost from origins to destinations in a congested transportation network. It appeared in 1920 in the work of Pigou and was developed by Knight in 1924. In 1952, Wardrop [35] introduced a notion of equilibrium flows for transportation network problems, which has been the basis for predicting traffic equilibria of urban transportation and various kinds of "transportation" in engineering, economics, management, etc. Smith [32] converted these problems to variational inequalities in 1979 and started a substantial number of contributions to applications of variational inequalities to traffic networks; see, e.g., [7,9,19–21,26,34] among others. The works [15,37] extended the notion of Wardrop equilibrium depending on the pattern equilibrium flow to the vector case. The papers [9,26] proposed to consider the costs as multifunctions of the flows to make the traffic model more elastic and suitable for diverse practical situations. Developing this idea, the contributions [19,20] extended the Wardrop definition of equilibrium to models with multivalued costs. In [24,25], traffic networks with arc constraints (instead of path constraints) were studied. We can observe that the aforementioned works on quasi-variational inequalities and traffic networks mainly consider global issues and global solutions. However, to the best of our knowledge, there have not been contributions to global approximations of these problems. On the other hand, variational convegence is a powerful tool for global approximations, especially for global sotutions. Variational convegence is not a particular type of convergence, but a common name for the types of convergence of a sequence of functions or bifunctions (i.e., bivariate/component functions) which preserve variational properties of these functions or bifunctions. Here, variational properties means those related to extremality/optimality such as being an infimum or supremum, a minimizer of maximizer, a minsup-value or maxinf-value, a minsup-point or maxinf-point, a saddle point, etc. Epi-convergence of functions was introduced in [36]. For bifunctions, epi/hypo-convergence was proposed in [2] and lopsided convergence appeared first in [3]. These three concepts are the basic types of variational convergence. Approximations in terms of variational convergence have been investigated for many classes of optimization-related problems such as quasi-equilibrium problems, generalized Nash equilibria, multiobjective minimization problems, Walras equilibria, consistency of approximate statistical estimators, problems of optimization under stochastic ambiguity, see [10,11,17,18,30,31]. The above observations ensure the importance of a study of such approximations of quasi-variational inequalities and traffic network problems in this paper.

Besides the novelty in the topic, our paper also contains a number of other significant newnesses. Namely, additionally to the basic types being epi/hypo- and lopsided convergence, we propose inside and weak types for both of them to have more tools for approximation research. Since the present work focuses on approximations of chosen problems, we limit the explanation to simply saying that these new concepts are indeed effective tools in establishing new approximation results under assumptions lighter than the known basic types of variational convergence, see Propositions 4.1 - 4.3, Theorem 4.1, and Corollaries 5.1 - 5.3. We also introduce new concepts of approximate solutions to quasi-variational inequalities and of sateratedness of arcs and paths as well as equilibrium notions for traffic problems. We would like to specifically mention that by the end of Sect. 4, we study for the first time so-called second-level approximations of optimization problems in our research context (Propositions 4.4 and 4.5). So, our results are new and no comparisons with the known ones are needed.

The layout of the paper is simple. In Sect. 2, the used standard notation is briefly presented and then the variational convergence concepts are provided including some new notions. They are the main tools for approximation investigations. Section 3 is devoted to the formulation of the quasi-variational inequality and its weak variant together with definitions of approximate solutions and their properties. Approximations of the above two quasi-variational inequalities are studied in Sect. 4. In Sect. 5, the definitions of the considered traffic networks and their properties are presented followed by their approximations in terms of types of variational convergence. The final short Sect. 6 contains concluding remarks about what are the achievements of the paper and what are the perspectives for further developments of the obtained results.

## 2 Preliminaries

Throughout the paper, $\mathbb{N}, \mathbb{R}, \mathbb{R}_+$, $\mathbb{R}^m$, $\mathbb{R}^m_+$, and $(\mathbb{R}^m)^*$ denote the set of the natural numbers, the set of the real numbers, the nonnegative numbers, an $m$-dimensional space, the $m$-positive orthant, and the dual space of $\mathbb{R}^m$, respectively (resp). Set $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. For $M \subset \mathbb{R}^m$, cl$M$ and conv$M$ are its closure and convex hull. If $M$ is convex and $x \in M$, the normal cone of $M$ at $x$ is

$$N(M, x) := \{p \in (\mathbb{R}^m)^* | \langle p, y - x\rangle \leq 0, \forall y \in M\}.$$

For a sequence $r^k \in \mathbb{R}$, its upper and lower limits are

$$\text{limsup}_{k\to+\infty} r^k := \lim_{p\to+\infty} \sup_{k>p} r^k, \;\; \text{liminf}_{k\to+\infty} r^k := \lim_{p\to+\infty} \inf_{k>p} r^k, \text{ resp.}$$

Let $X$ be a metric space, $A \subset X$, and $h : A \to \bar{\mathbb{R}}$. The epigraph and hypograph of $h$ are

$$\text{epi}h := \{(x, r) \in X \times \mathbb{R} \mid h(x) \leq r\}, \quad \text{hypo}h := \{(x, r) \in X \times \mathbb{R} \mid h(x) \geq r\}.$$

For function $\varphi : A \to \mathbb{R}$, define $\zeta\varphi : X \to \mathbb{R} \cup \{+\infty\}$ by

$$\zeta\varphi(x) := \begin{cases} \varphi(x) \text{ if } \; x \in A, \\ +\infty \text{ otherwise} \end{cases}$$

and regard points $(x, +\infty)$ as belonging to epi$\zeta\varphi$ when $x \notin A$. Then, define epi$\varphi :=$ epi$\zeta\varphi$. The hypograph of $\varphi$ can be defined similarly.

Now recall some notions of set-valued analysis. For a metric space $X$ and $M^k \subset X$, the *outer/upper limit (inner/lower limit, resp)* of the sequence $M^k$ is

$$\text{Limsup}_{k\to+\infty} M^k := \{x \in X \mid \exists x^{k_j} \in M^{k_j}, x^{k_j} \to x\}$$

$$(\text{Liminf}_{k\to+\infty} M^k := \{x \in X \mid \exists x^k \in M^k, x^k \to x\}, \text{ resp}).$$

Obviously, an inner limit of a set sequence is a subset of its outer limit and even outer limits may be empty. $M^k$ are said to *Painlevé-Kuratowski converge* to $M$, denoted by $M^k \overset{P-K}{\to} M$ or $M = \text{Lim}_{k\to+\infty} M^k$, iff

$$\text{Limsup}_{k\to+\infty} M^k = \text{Liminf}_{k\to+\infty} M^k = M.$$

For brevity, later on we usually use the abbreviations ls$_k$, li$_k$, Ls$_k$, Li$_k$, and Lim$_k$ for limsup$_{k\to+\infty}$, liminf$_{k\to+\infty}$, Limsup$_{k\to+\infty}$, Liminf$_{k\to+\infty}$, Lim$_{k\to+\infty}$, resp. For metric spaces $X, Y$ and a set-valued map $G : X \rightrightarrows Y$, its domain and graph are defined as

$$\text{dom}G := \{x \in X \mid G(x) \neq \emptyset\}, \;\; \text{gph}G := \{(x, y) \in X \times Y \mid y \in G(x)\}, \text{resp.}$$

Finally, let us speak about variational convergence of functions and bifunctions. Let $X$ be a metric space. For $k \in \mathbb{N}$, let $A^k, A \subset X$ be nonempty, $\varphi^k : A^k \to \mathbb{R}$, and $\varphi : A \to \mathbb{R}$. The following two definitions are taken from [11].

**Definition 2.1** (types of epi-convergence) $\varphi^k$ are said to *epi-converge* to $\varphi$, denoted by $\varphi^k \xrightarrow{e} \varphi$ or $\varphi =$ e-lim$_k\varphi^k$, iff

(a) for all subsequences $x^{k_j} \in A^{k_j} \to x$, $\text{li}_j\varphi^{k_j}(x^{k_j}) \geq \varphi(x)$ if $x \in A$ and $\varphi^{k_j}(x^{k_j}) \to +\infty$ if $x \notin A$;

(b) for all $x \in A$, there exist points $x^k \in A^k \to x$ such that $\text{ls}_k\varphi^k(x^k) \leq \varphi(x)$.

Omitting the case that $x \notin A$ with the infinity condition, one has *inside epi-convergence* which is denoted by $\varphi^k \xrightarrow{i-e} \varphi$ or $\varphi =$ i-e-lim$_k\varphi^k$.

Recall the well-known geometric reformulation of the concept of epi-convergence: condition (a) of $\varphi =$ e-lim$_k\varphi^k$ means that $\text{Ls}_k(\text{epi}\varphi^k) \subset \text{epi}\varphi$ and its condition (b) is equivalent to $\text{epi}\varphi \subset \text{Li}_k(\text{epi}\varphi^k)$.

**Definition 2.2** (types of hypo-convergence) $\varphi^k$ are said to *hypo-converge* (*inside hypo-converge*, resp) to $\varphi$, denoted by by $\varphi^k \xrightarrow{h} \varphi$ or $\varphi =$ h-lim$_k\varphi^k$ ($\varphi^k \xrightarrow{i-h} \varphi$ or $\varphi =$ i-h-lim$_k\varphi^k$, resp), iff $-\varphi^k$ epi-converge (i-epi-converge) to $-\varphi$.

Now let $A, A^k, B, B^k \subset \mathbb{R}^m$, $\mathcal{D}^k : A^k \rightrightarrows B^k$, $\mathcal{D} : A \rightrightarrows B$, $\Psi^k : \text{gph}\mathcal{D}^k \to \mathbb{R}$, and $\Psi : \text{gph}\mathcal{D} \to \mathbb{R}$. Assume, without loss of generality, that $\mathcal{D}^k$ and $\mathcal{D}$ are defined on the entire $A^k$ and $A$, resp, and surjective.

**Definition 2.3** (types of epi/hypo-convergence) Bifunctions $\Psi^k$ are said to *inside epi/hypo converge (i-e/h-converge)* to $\Psi$ (wrt $\text{gph}\mathcal{D}^k$ and $\text{gph}\mathcal{D}$), denoted by $\Psi^k \xrightarrow{i-e/h} \Psi$ or $\Psi =$ i-e-lim$_k\Psi^k$, iff

(a) for any subsequence $x^{k_j} \in A^{k_j} \to x \in A$ and $y \in \mathcal{D}(x)$, there exists a subsequence $y^{k_j} \in \mathcal{D}^{k_j}(x^{k_j}) \to y$ such that $\text{li}_j\Psi^{k_j}(x^{k_j}, y^{k_j}) \geq \Psi(x, y)$;

(b) for any subsequence $y^{k_j} \in B^{k_j} \to y \in B$ and $x \in \mathcal{D}^{-1}(y)$, there exists a subsequence $x^{k_j} \in (\mathcal{D}^{k_j})^{-1}(y^{k_j}) \to x$ such that $\text{ls}_j\Psi^{k_j}(x^{k_j}, y^{k_j}) \leq \Psi(x, y)$.

Bifunctions $\Psi^k$ are called *e/h-convergent* to $\Psi$, denoted by $\Psi^k \xrightarrow{e/h} \Psi$ or $\Psi =$ e-lim$_k\Psi$, iff additionally the following infinity conditions are fulfilled: for any subsequence $x^{k_j} \in A^{k_j} \to x \notin A$, there exists a subsequence $y^{k_j} \in \mathcal{D}^{k_j}(x^{k_j})$ such that $\Psi^{k_j}(x^{k_j}, y^{k_j}) \to +\infty$, and for any subsequence $y^{k_j} \in B^{k_j} \to y \notin B$, there exists a subsequence $x^{k_j} \in (\mathcal{D}^{k_j})^{-1}(y^{k_j})$ such that $\Psi^{k_j}(x^{k_j}, y^{k_j}) \to -\infty$.

Bifunctions $\Psi^k$ are *weak e/h-convergent (w-e/h-convergent)* or *weak inside e/h-convergent (w-i-e/h-convergent)*, resp, iff in the above statements, all the subsequences are replaced by the corresponding (whole) sequences.

For brevity, instead of writing "Bifunctions $\Psi^k$ are ...", "there exist points $y^k$", etc, we simply write "$\Psi^k$ are ...", "there exist $y^k$", etc, since from the indices $k$, the plural form is clear.

*Remark 2.1* (i) In the above definition, the weak convergence concepts are newly proposed, while the others are taken from [11]. To see that the weak e/h-convergence is really weaker than the e/h-convergence, consider a simple example with $\mathcal{D}^k(x) = B^k$ for all $x \in A^k$ and $\mathcal{D}(x) = B$ for all $x \in A$. Let $A^k$ be equal to $\{0,1\}$ for the odd $k$'s and to $\{0\}$ for the even $k$'s, $A = \{0\}$, $B^k = B = \{1\}$ for all $k$, $\Phi^k(x,y) = 0$ for all $k \in \mathbb{N}$, $(x,y) \in A^k \times B^k$, and $\Phi(x,y) = 0$ for all $(x,y) \in A \times B$. Then, direct verification shows that $\Phi^k$ do not e/h-converge to $\Phi$ as the infinity condition in (a) of Definition 2.3 is violated. But, we have all the other three weaker types: inside e/h, weak e/h-, and inside weak e/h-convergence. Sects. 4 and 5 will provide approximation results under these types of convergence, including the types weaker than the e/h-convergence.

(ii) Each type of e/h-convergence is symmetric in the sense that conditions (a) and (b) are symmetric (wrt $x$ and $y$, liminf and limsup, etc). Furthermore, if we insist on $\min_x \max_y$ and only change the order of the two operations, then e/h-convergence is also symmetric in the sense that bifunctions $(x,y) \mapsto \Psi^k(x,y)$ e/h-converge to $(x,y) \mapsto \Psi(x,y)$ if and only if bifunctions $(x,y) \mapsto \Psi^k(y,x)$ h/e-converge to $(x,y) \mapsto \Psi(y,x)$. So, h/e-convergence may be considered to coincide with e/h-convergence (with condition (b) coming before (a)). For the weaker types of e/h-convergence, we have the same situation.

(iii) In Sections 4 and 5, we will see that the considerations in this paper employ largely the types of variational convergence of bifunctions $-\Psi^k$ to $-\Psi$ due to our problem formulations. However, to be consistent with the previous publications in the literature, we keep the terminology as in Definition 2.3. Note that a type of convergence of $-\Psi^k$ to $-\Psi$ corresponds to $\max_x \min_y$ and we do not call it the hypo/epi-convergence of $\Psi^k$ to $\Psi$.

The second basic type of variational convergence of bifunctions is lopsided convergence (lop-convergence). Unlike the e/h-convergence, from the definition below, it is clear that the lop-convergence is not symmetric. Hence, we must have two concepts given in the following two definitions. Moreover, similar to e/h-convergence, in fact, we have two groups of the corresponding types of lop-convergence.

**Definition 2.4** (types of mis-lop convergence) We say that $\Psi^k$ *mis-lop converge* to $\Psi$ ([31]) iff

(a) $\forall x^{k_j} \in A^{k_j} \to x \in A$, $\forall y \in \mathcal{D}(x)$, $\exists y^{k_j} \in \mathcal{D}^{k_j}(x^{k_j}) \to y$, $\text{li}_j \Psi^{k_j}(x^{k_j}, y^{k_j}) \geq \Psi(x,y)$, and $\forall x^{k_j} \in A^{k_j} \to x \notin A$, $\exists y^{k_j} \in \mathcal{D}^{k_j}(x^{k_j})$, $\Psi^{k_j}(x^{k_j}, y^{k_j}) \to +\infty$;

(b) $\forall x \in A$, $\exists x^{k_j} \in A^{k_j} \to x$, $\forall y^{k_j} \in \mathcal{D}^{k_j}(x^{k_j}) \to y$, $\text{ls}_j \Psi^{k_j}(x^{k_j}, y^{k_j}) \leq \Psi(x,y)$ if $y \in \mathcal{D}(x)$, and $\Psi^{k_j}(x^{k_j}, y^{k_j}) \to -\infty$ if $y \notin \mathcal{D}(x)$.

We use the notation $\Psi^k \overset{mis-lop}{\to} \Psi$ or $\Psi = \text{mis-lop-lim}_k \Psi^k$. If we consider only $x \in A$ in (a) and $y \in \mathcal{D}(x)$ in (b) (and omit the infinity conditions), $\Psi^k$ is called *inside mis-lop convergent* and we have "i-" added to the notation.

Replacing the subsequences in the above definitions by the corresponding sequences, we have the concepts of *weak mis-lop convergence* and *weak inside mis-lop convergence*, resp. Similar to the w-e/h-convergence, we use the notation "w-mis-lop convergence".

Note that condition (a) in Definition 2.4 is the same as (a) in Definition 2.3, but in the former definition, condition (b) is different from (b) in Definition 2.3, so it is not symmetric to (a). This leads to various differences in properties and applications of the two types of convergence. These differences also confirm the usefulness of both the convergence types.

In [31] only mis-lop convergence is discussed and called simply lop-convergence (the above three weaker variants are not involved). However, in studies related to duality phenomena, we need to clearly distinguish the following parallelism and its weaker variants.

**Definition 2.5** (types of mai-lop convergence) $\Psi^k$ are called *mai-lop convergent* to $\Psi$ iff

(a) $\forall y^{k_j} \in B^{k_j} \to y \in B$, $\forall x \in \mathcal{D}^{-1}(y)$, $\exists x^{k_j} \in (\mathcal{D}^{k_j})^{-1}(y^{k_j}) \to x$, $\text{ls}_j \Psi^{k_j}(x^{k_j}, y^{k_j}) \leq \Psi(x,y)$; $\forall y^{k_j} \in B^{k_j} \to y \notin B$, $\exists x^{k_j} \in (\mathcal{D}^{k_j})^{-1}(y^{k_j})$, $\Psi^{k_j}(x^{k_j}, y^{k_j}) \to -\infty$;

(b) $\forall y \in B$, $\exists y^{k_j} \in B^{k_j} \to y$, $\forall x^{k_j} \in (\mathcal{D}^{k_j})^{-1}(y^{k_j}) \to x$, $\text{li}_j \Psi^{k_j}(x^{k_j}, y^{k_j}) \geq \Psi(x,y)$ if $x \in \mathcal{D}^{-1}(y)$, and $\Psi^{k_j}(x^{k_j}, y^{k_j}) \to +\infty$ if $x \notin \mathcal{D}^{-1}(y)$.

We denote this convergence by $\Psi^k \overset{mai-lop}{\to} \Psi$ or $\Psi = \text{mai-lop-lim}_k \Psi^k$. The *inside mai-lop* can be defined and denoted similarly, with the infinity conditions being removed.

Also by replacing the subsequences by the corresponding sequences, we have *weak lop-convergence* with a similar notation.

Now, we concentrate on approximation studies for our chosen optimization problems. Note that in fact, the above concepts can be defined for more general underlying spaces, but we provide them just for the context of this paper.

*Example 2.1* (mis-lop convergence holds; w-mai-lop and w-e/h-convergence fail) Let $A^k = A = (0, e-1]$, $B^k = B = (0,1]$, $\mathcal{D}^k(x) = \mathcal{D}(x) = [\ln(\mathrm{x}+1), 1]$, and $\Psi^k(x,y) = \Psi(x,y) = \ln(y/x)$. We verify first condition (a) of the mis-lop convergence. As $\Psi^k$ form a constant sequence, we work with sequences instead of subsequences. For all $x^k \in A^k \to x$ and $y \in \mathcal{D}(x)$, we take in $\mathcal{D}^k(x^k)$

$$y^k = \begin{cases} y & \text{if } 0 < x^k \le x, \\ y + \gamma^k & \text{such that } \gamma^k \to 0^+ \; and \; \ln(x^k+1) \le y + \gamma^k \le 1 \text{ if } x < x^k \le e-1. \end{cases}$$

Then, $\lim_k \Psi^k(x^k, y^k) = \Psi(x,y)$ by the continuity of $\Psi^k \equiv \Psi$. For all $x^k \in A^k \to 0 \notin A$, $y^k \equiv 1 \in \mathcal{D}^k(x^k)$ satisfy $\Psi^k(x^k, y^k) \to +\infty$.

For condition (b), with any $x \in A$, we take $x^k \equiv x$ to get that, for all $y^k \in \mathcal{D}^k(x^k) \to y \in \mathcal{D}(x)$, $\mathrm{ls}_k \Psi^k(x^k, y^k) = \Psi(x,y)$. Furthermore, for all $x \in A = (0, e-1]$, taking arbitrary $x^k \in A^k \to x$, there is no $y^k \in \mathcal{D}^k(x^k) = \mathcal{D}(x)$ tending to a point outside $\mathcal{D}(x)$. Hence, (b) is fulfilled and the mis-lop convergence holds.

We show that condition (a) of the weak mai-lop convergence is not satisfied. The "inside part" of (a) is checked since for any $y^k \in B^k \to y$ and $x \in \mathcal{D}^{-1}(y)$, we pick in $(\mathcal{D}^k)^{-1}(y^k)$ the points

$$x^k = \begin{cases} x + \gamma^k & \text{such that } \gamma^k \to 0^+ \; and \; x + \gamma^k \le e^{y^k} - 1 \text{ if } 0 < y^k \le y, \\ x & \text{if } y < y^k \le 1 \end{cases}$$

to see that $\lim_k \Psi^k(x^k, y^k) = \Psi(x,y)$ (by the continuity). But, the infinity condition is violated since for any $y^k \in B^k \to 0 \notin B$, taking $x^k = e^{y^k} - 1 \in (\mathcal{D}^k)^{-1}(y^k)$, one sees that $\Psi^k(x^k, y^k) \ge \ln(y^k(e^{y^k} - 1)^{-1}) \to 0$, not $-\infty$ as required. So, $\Psi^k$ do not w-mai-lop converge to $\Psi$. Condition (b) of the i-mai-lop convergence is satisfied by the continuity and so the i-mai-lop convergence holds.

Moving on to the weak e/h-convergence, let $y^k \in B^k \to 0 \notin B$ and $x^k$ be in $(\mathcal{D}^k)^{-1}(y^k)$. Then, $x^k \le e^{y^k} - 1$ and $\Psi^k(x^k, y^k) \ge \ln(y^k(e^{y^k} - 1)^{-1}) \to 0$, not $-\infty$ as required. So, $\Psi^k$ do not weakly e/h-converge to $\Psi$.

It should be noted that in the rectangular case (i.e., the domain of $\Psi$ is a constant map: $\mathcal{D}(x) = B$ for all $x \in A$), condition (b) of the mis-lop convergence is properly stronger than (b) of the e/h-convergence.

But, in this nonrectangular case, the mis-lop convergence does not imply the e/h-convergence. So, the need of using both types of convergence is clearer. Note also that in this paper we consider quasi-inequalities and traffic networks with elastic demands. So the types variational convergence of bifunctions on general (nonrectangular) domains are needed since the bifunctions with rectangular domains cannot describe these quasi-variational problems.

## 3 Set-Valued Quasi-Variational Inequalities

### 3.1 Primal Problems

Given two nonempty-set-valued maps $\mathcal{T} : \mathbb{R}^m \rightrightarrows (\mathbb{R}^m)^*$ with compact values and $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, the so-called *weak set-valued quasi-variational inequality* is

$$\text{(wQVI)} \quad \text{find } \bar{x} \in \mathcal{K}(\bar{x}) \text{ such that } \forall y \in \mathcal{K}(\bar{x}),\ \exists \bar{t}_y \in \mathcal{T}(\bar{x}),\ \langle \bar{t}_y, y - \bar{x} \rangle \geq 0.$$

Clearly, $\bar{x} \in \mathcal{K}(\bar{x})$ means that $\bar{x} \in \text{fix}\mathcal{K}$, the fixed-point set of $\mathcal{K}$. We also denote this quasi-variational inequality by (wQVI($\mathcal{T}, \mathcal{K}$)) to make the data clear, and its solution set by Sol(wQVI). This style of notation is always adopted later for various problems.

To consider approximations of (wQVI($\mathcal{T}, \mathcal{K}$)), we use the following bifunction $\Phi : \mathbb{R}^m \times \mathbb{R}^m$

$$\Phi(x, y) := \max_{t \in \mathcal{T}(x)} \langle t, y - x \rangle,$$

which is equivalent to

$$\Phi(x, y) := \max_{t \in \text{conv}\mathcal{T}(x)} \langle t, y - x \rangle. \tag{1}$$

Hence, when considering (wQVI($\mathcal{T}, \mathcal{K}$)) via $\Phi$, we can always assume that $\mathcal{T}$ is convex-valued (but, in the general problem setting, $\mathcal{T}$ may be nonconvex-valued). Moreover, $\Phi(x, \cdot)$ is convex for any $x \in \mathbb{R}^m$. Recall that for a convex function $h : \mathbb{R}^m \to \bar{\mathbb{R}}$, its *subdifferential* at $x$ is defined by $\partial h(x) = \{t \in (\mathbb{R}^m)^* \mid h(y) - h(x) \geq \langle t, y - x \rangle \ \ \forall y \in \mathbb{R}^m\}$. If $x \in \text{dom}h := \{x \in \mathbb{R}^m \mid h(x) \in \mathbb{R}\}$, then $\partial h(x)$ is nonempty, closed, and convex (see [28], Theorem 23.4).

**Lemma 3.1** (the subdifferential of $\Phi$ in terms of $\mathcal{T}$) *Let $\mathcal{T} : \mathbb{R}^m \rightrightarrows (\mathbb{R}^m)^*$ be nonempty-compact-convex-valued and $\Phi$ defined by (3.1). Then, $\partial\Phi(x, \cdot)(x) = \mathcal{T}(x)$ for all $x \in \mathbb{R}^m$.*

*Proof* Since $\Phi(x,x)=0$, by the definition of $\Phi$, $\Phi(x,y)-\Phi(x,x)=\Phi(x,y)\geq\langle t,y-x\rangle$ for all $t\in\mathcal{T}(x)$ and $y\in\mathbb{R}^m$. Hence, $\mathcal{T}(x)\subset\partial\Phi(x,\cdot)(x)$. For the reverse inclusion, suppose $t_0\notin\mathcal{T}(x)$. The separation theorem gives $\bar{y}\in(\mathbb{R}^m)^*$ such that $\max_{t\in\mathcal{T}(x)}\langle t,\bar{y}\rangle<\langle t_0,\bar{y}\rangle$. Setting $\bar{y}=y_0-x$ yields

$$\Phi(x,y_0)-\Phi(x,x)<\langle t_0,y_0-x\rangle,$$

and so $t_0\notin\partial\Phi(x,\cdot)(x)$. It follows that $\partial\Phi(x,\cdot)(x)\subset\mathcal{T}(x)$. Consequently $\partial\Phi(x,\cdot)(x)=\mathcal{T}(x)$. □

We denote the fixed-point set of $\mathcal{K}$ by $A:=\text{fix}\mathcal{K}$ and its image by $B:=\mathcal{K}(A)$, and always assume that $A\neq\emptyset$ (there have been a huge number of results about such a nonemptiness in the literature, see, e.g., [1,23]). We only focus on approximation studies and so we also always assume that the quasi-variational inequalities under consideration have solutions. Furthermore, clearly we can restrict $\mathcal{K}$ as a map $\mathcal{K}:A\rightrightarrows B$, and so $\mathcal{K}$ is surjective. Then, (wQVI($\mathcal{T},\mathcal{K}$)) is equivalent to the following quasi-equilibrium problems

$$(\text{QEP}(\Phi,\mathcal{K}))\quad \text{find } \bar{x}\in A \text{ such that for all } y\in\mathcal{K}(\bar{x}),\ \Phi(\bar{x},y)\geq 0.$$

We also consider the following so-called *strong set-valued quasi-variational inequality*, which is usually considered in the literature,

$$(\text{sQVI})\quad \text{find } \bar{x}\in\mathcal{K}(\bar{x}) \text{ such that } \exists\bar{t}\in\mathcal{T}(\bar{x}),\ \forall y\in\mathcal{K}(\bar{x}),\ \langle\bar{t},y-\bar{x}\rangle\geq 0.$$

**Proposition 3.1** (relationship between (sQVI) and (wQVI)) Sol(sQVI) $\subset$ Sol(wQVI) *always holds. Moreover, under the condition that ($\mathcal{T}$ is compact-convex-valued and) $\mathcal{K}$ is closed-convex-valued,* Sol(wQVI) $\subset$ Sol(sQVI)*, i.e., the two problems are equivalent.*

*Proof* By definition, if $\bar{x}\in$ Sol(wQVI), then $\sup_{y\in\mathcal{K}(\bar{x})}\min_{t\in\mathcal{T}(\bar{x})}\langle t,y-\bar{x}\rangle\geq 0$. By the closed-convex-valuedness of $\mathcal{K}$, due to Sion's minimax theorem ([33], Corollary 3.3), that inequality is equivalent to $\min_{t\in\mathcal{T}(\bar{x})}\sup_{y\in\mathcal{K}(\bar{x})}\langle t,y-\bar{x}\rangle\geq 0$. Therefore, there exists $\bar{t}\in\mathcal{T}(\bar{x})$ such that $\langle\bar{t},y-\bar{x}\rangle\geq 0$, i.e., $\bar{x}\in$ Sol(sQVI). □

The following example shows that the inclusion in Proposition 3.1 may be proper.

*Example 3.1* ((sQVI) is properly stronger than (wQVI)) Let $m=2$, $\mathcal{K}(x)=\{\gamma x\mid\gamma\geq 0\}$, and $\mathcal{T}(x)=\{(-1,-1),(1,1)\}$ for all $x\in\mathbb{R}^2$. Then, fix$\mathcal{K}=\mathbb{R}^2$. We first consider Sol(sQVI). Let $x=(x_1,x_2)\in$ fix$\mathcal{K}=\mathbb{R}^2$. For the case $x_1>x_2$, $t=(-1,-1)$ is not appropriate since $\inf_{y\in\{\gamma x|\gamma\geq 0\}}(1-\gamma)(x_1+x_2)=-\infty$. The situation with $t=(1,1)$ is similar since the corresponding infimum is less than 0. Hence, $x$ with

$x_1 > x_2$ do not belong to Sol(sQVI). Now it suffices to show that such points $x$ belong to Sol(wQVI). We have to check two cases for $y \in \{\gamma x \mid \gamma \geq 0\}$. If $\gamma \in [0,1]$, one takes $t = (-1,-1)$ to see that the above infimum is nonnegative. If $\gamma > 1$, one takes $t = (1,1)$ to have that this infimum is positive. Hence, $x$ with $x_1 > x_2$ do belong to Sol(wQVI).

*Example 3.2* ((sQVI) is properly stronger than (wQVI) even with convex-valued $\mathcal{T}$) Assume that $m = 2$, $\mathcal{T}(x) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, x_2 \in [x_1, 1] \cap [0.1, 1]\}$, and

$$\mathcal{K}(x) = \begin{cases} \{(0,0), (-1, 0.5), (1,-1)\} & \text{if } x \in \mathbb{R}^2_+, \\ \mathbb{R}^2 \setminus \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

Then, fix$\mathcal{K} = \{(0,0)\} \cup (\mathbb{R}^2 \setminus \mathbb{R}^2_+)$. We first consider $x = (0,0) \in$ fix$\mathcal{K}$ for (sQVI). Clearly, there does not exist $t \in \mathcal{T}(0,0)$ such that $\inf_{y \in \mathcal{K}(0,0)} \langle t, y - (0,0)\rangle \geq 0$, so $x = (0,0) \notin$ Sol(sQVI). However, this point is a solution of (wQVI) since for each $y \in \mathcal{K}(0,0)$, there obviously exists $t \in \mathcal{T}(0,0)$ such that $\langle t, y - (0,0)\rangle \geq 0$. Therefore, Sol(sQVI) is properly smaller than Sol(wQVI).

*Example 3.3* (Proposition 3.1 provides a sufficient condition, not a necessary one) Assume that $m = 1$, $\mathcal{T}(x) = \{-1, 1\}$ for all $x \in \mathbb{R}$, and

$$\mathcal{K}(x) = \begin{cases} \{0\} & \text{if } x \in [0,1), \\ \{1\} & \text{if } x \in [1,+\infty), \\ \{-1\} & \text{if } x \in (-\infty, 0]. \end{cases}$$

Then, fix$\mathcal{K} = \{-1\} \cup \{0\} \cup \{1\}$. To have Sol(wQVI) = Sol(sQVI)), we show that Sol(sQVI) = fix$\mathcal{K}$. For $x = -1$, we take $t = -1$ to obtain $\inf_{y \in \mathcal{K}(-1)}[-(-1-(-1))] = 0$, and so $x = -1$ is a strong solution. For $x = 0$, we take $t = 1$ to obtain the same infimum value and so $x = 0$ is a strong solution. For $x = 1$, since $\mathcal{K}(1) = \{1\}$, the corresponding minimum is equal to 0 for any $t \in \{-1, 1\}$. Therefore, Sol(sQVI) = fix$\mathcal{K}$ and is evidently equal to Sol(wQVI).

In many cases, exact solutions of a problem do not exist or are difficult to compute. Hence, approximate solutions play a crucial role.

**Definition 3.1** (approximate solutions) Let $\varepsilon \geq 0$.

(i) A point $\bar{x} \in$ fix$\mathcal{K}$ is called an *$\varepsilon$-solution* of (QEP($\Phi, \mathcal{K}$)), denoted by $\bar{x} \in \varepsilon$-Sol(QEP($\Phi, \mathcal{K}$)), iff $\forall y \in \mathcal{K}(\bar{x})$, $\Phi(\bar{x}, y) \geq -\varepsilon$. Later on, we adopt this writing style for any $\varepsilon$-solution set.

(ii) A point $\bar{x} \in \text{fix}\mathcal{K}$ is called an *$\varepsilon$-solution* of (wQVI($\mathcal{T},\mathcal{K}$)) iff $\forall y \in \mathcal{K}(\bar{x})$, $\exists \bar{t}_y \in \mathcal{T}(\bar{x})$, $\langle \bar{t}_y, y-\bar{x}\rangle \geq -\varepsilon$.

(iii) A point $\bar{x} \in \text{fix}\mathcal{K}$ such that $\exists \bar{t} \in \mathcal{T}(\bar{x})$, $\forall y \in \mathcal{K}(\bar{x})$, $\langle \bar{t}, y-\bar{x}\rangle \geq -\varepsilon$ is said to be an *$\varepsilon$-solution* of (sQVI).

**Proposition 3.2** (relationships between approximate solutions)

(i) *If* $\bar{x} \in \varepsilon$-Sol(QEP($\Phi,\mathcal{K}$)), *then for all* $\varepsilon_1 > \varepsilon$, $\bar{x} \in \varepsilon_1$-Sol(wQVI($\mathcal{T},\mathcal{K}$)). *Conversely, if* $\bar{x} \in \varepsilon$-Sol(wQVI($\mathcal{T},\mathcal{K}$)), *then* $\bar{x} \in \varepsilon$-Sol(QEP($\Phi,\mathcal{K}$)).

(ii) $\bar{x} \in \varepsilon$-Sol(QEP($\Phi,\mathcal{K}$)) *if and only if* $-\varepsilon \leq \inf_{y\in\mathcal{K}(\bar{x})}\Phi(\bar{x},y) \leq 0$. *This implies that* $\bar{x}$ *is in* $\varepsilon$-$\text{argmin}_{\mathcal{K}(\bar{x})}\Phi(\bar{x},\cdot) := \{\bar{y} \in \mathcal{K}(\bar{x}) \mid \Phi(\bar{x},\bar{y}) \leq \inf_{y\in\mathcal{K}(\bar{x})}\Phi(\bar{x},y)+\varepsilon\}$.

(iii) *Assume that* $\mathcal{K}$ *is closed-convex-valued. Then,* $\bar{x} \in \varepsilon$-Sol(QEP($\Phi,\mathcal{K}$)) *if and only if* $\bar{x} \in \varepsilon$-Sol(sQVI($\mathcal{T},\mathcal{K}$)) *if and only if* $\bar{x} \in \varepsilon$-Sol(wQVI($\mathcal{T},\mathcal{K}$)).

*Proof* (i) By definition, $\bar{x} \in \varepsilon$-Sol(QEP) means that $\max_{t\in\mathcal{T}(\bar{x})}\langle t, y-\bar{x}\rangle \geq -\varepsilon$ for all $y \in \mathcal{K}(\bar{x})$. For any $\varepsilon_1 > \varepsilon$, set $\varepsilon_2 := \varepsilon_1 - \varepsilon$. Then, we have that $\forall y \in \mathcal{K}(\bar{x})$, $\exists \bar{t}_y \in \mathcal{T}(\bar{x})$,

$$\langle \bar{t}_y, y-\bar{x}\rangle \geq \max_{t\in\mathcal{T}(\bar{x})}\langle t, y-\bar{x}\rangle - \varepsilon_2 \geq -\varepsilon-\varepsilon_2 = -\varepsilon_1.$$

Hence, $\bar{x} \in \varepsilon_1$-Sol(wQVI).

Conversely, $\bar{x} \in \varepsilon$-Sol(wQVI) implies that $\forall y \in \mathcal{K}(\bar{x})$, $\max_{t\in\mathcal{T}(\bar{x})}\langle t, y-\bar{x}\rangle \geq -\varepsilon$, i.e., $\bar{x} \in \varepsilon$-Sol(QEP).

(ii) $\bar{x} \in \varepsilon$-Sol(QEP) means that $\bar{x} \in \mathcal{K}(\bar{x})$ and $\inf_{y\in\mathcal{K}(\bar{x})}\Phi(\bar{x},y) \geq -\varepsilon$. Since $\Phi(\bar{x},\bar{x}) = 0$, the assertion follows. Then, the required inclusion is evident.

(iii) $\bar{x} \in \varepsilon$-Sol(QEP($\Phi,\mathcal{K}$)) means that $\inf_{y\in\mathcal{K}(\bar{x})}\max_{t\in\mathcal{T}(\bar{x})}\langle t, y-\bar{x}\rangle \geq -\varepsilon$. In view of Sion's minimax theorem, that is equivalent to $\max_{t\in\mathcal{T}(\bar{x})}\inf_{y\in\mathcal{K}(\bar{x})}\langle t, y-\bar{x}\rangle \geq -\varepsilon$, i.e., there exists $\bar{t} \in \mathcal{T}(\bar{x})$ such that $\inf_{y\in\mathcal{K}(\bar{x})}\langle t, y-\bar{x}\rangle \geq -\varepsilon$. This means that $\bar{x} \in \varepsilon$-Sol(sQVI($\mathcal{T},\mathcal{K}$)). □

Observe that in Proposition 1(iii) of [8], Sion's minimax theorem was applied for a similar result in the special case of variational inequalities (not "quasi-variational", i.e., with $\mathcal{K}(x) = K$ for all $x \in \mathbb{R}^m$).

## 3.2 Dual Problems

The *dual problem* (DwQVI($\mathcal{T},\mathcal{K}$)) of (wQVI($\mathcal{T},\mathcal{K}$)) is defined in a natural way:

$$\text{(DwQVI}(\mathcal{T},\mathcal{K})) \quad \text{find } \bar{y} \in \mathcal{K}(\text{fix}\mathcal{K}) \text{ such that } \forall x \in \mathcal{K}^{-1}(\bar{y}), \exists \bar{t} \in \mathcal{T}(\bar{y}), \langle \bar{t}, \bar{y}-x\rangle \leq 0.$$

This problem can be reformulated as

$$(\mathrm{DwQVI}(\mathcal{T},\mathcal{K})) \quad \text{find } \bar{y} \in \mathcal{K}(\mathrm{fix}\mathcal{K}) \text{ such that } \forall x \in \mathcal{K}^{-1}(\bar{y}),\ \min_{t\in\mathcal{T}(\bar{y})}\langle t, \bar{y}-x\rangle \leq 0.$$

This reformulation means that (DwQVI($\mathcal{T},\mathcal{K}$)) is just (DQEP($\hat{\Phi},\mathcal{K}$)) for

$$\hat{\Phi}(x,y) := \min_{t\in\mathcal{T}(y)}\langle t, y-x\rangle,$$

which is the dual of (QEP($\hat{\Phi},\mathcal{K}$)), not of (QEP($\Phi,\mathcal{K}$)) as in the Stampacchia-Minty duality scheme for quasi-equilibrium problem (QEP($\Phi,\mathcal{K}$)). It should be noted that if $\mathcal{T}=t$, a single-valued map, then (DwQVI($t,\mathcal{K}$)) collapses to the classical Minty quasi-variational inequality. The mentioned mismath when expressing our dual couple in terms of quasi-equilibrium problems (intermediate problems) is due to the set-valuedness of the quasi-variational inequalities. This note also holds for the dual strong problems defined as follows.

The *dual problem* (DsQVI) of (sQVI) is

$$(\mathrm{DsQVI}) \quad \text{find } \bar{y} \in \mathcal{K}(\mathrm{fix}\mathcal{K}) \text{ such that } \exists \bar{t} \in \mathcal{T}(\bar{y}), \forall x \in \mathcal{K}^{-1}(\bar{y}), \langle \bar{t}, \bar{y}-x\rangle \leq 0.$$

**Proposition 3.3** (relationships between convex problems (DsQVI) and (DwQVI)) *It always holds that* Sol(DsQVI) $\subset$ Sol(DwQVI). *Moreover, under the condition that ($\mathcal{T}$ is compact-convex-valued and) $\mathcal{K}^{-1}$ is closed-convex-valued, one has that $\bar{y} \in \mathrm{fix}\mathcal{K}^{-1}$ and $\bar{y} \in$ Sol(DwQVI) imply that $\bar{y} \in$ Sol(DsQVI).*

*Proof* By the definitions of the solutions, clearly Sol(DsQVI) $\subset$ Sol(DwQVI). Now impose the value condition and that $\bar{y} \in \mathrm{fix}\mathcal{K}^{-1}$. By definition, $\bar{y} \in$ Sol(DwQVI) means that for all $x \in \mathcal{K}^{-1}(\bar{y})$, $\sup_{\mathcal{K}^{-1}(\bar{y})}\hat{\Phi}(\cdot,\bar{y}) \leq 0$. As $\bar{y} \in \mathrm{fix}\mathcal{K}^{-1}$ and $\Phi(\bar{y},\bar{y})=0$, this means that $\bar{y} \in \mathrm{argmin}_{\mathcal{K}^{-1}(\bar{y})}(-\hat{\Phi}(\cdot,\bar{y}))$. By Lemma 3.1, this implies that $0 \in \mathcal{T}(\bar{y}) + N(\mathcal{K}^{-1}(\bar{y}),\bar{y})$, i.e., $\bar{y} \in$ Sol(DsQVI). □

**Definition 3.2** (approximate dual solutions) Let $\varepsilon \geq 0$.

(i) A point $\bar{y} \in B \equiv \mathcal{K}(\mathrm{fix}\mathcal{K})$ is called an *$\varepsilon$-solution* of (DQEP($\hat{\Phi},\mathcal{K}$)), denoted by $\bar{y} \in \varepsilon$-Sol(DQEP($\hat{\Phi},\mathcal{K}$)), iff $\forall x \in \mathcal{K}^{-1}(\bar{y})$, $\hat{\Phi}(x,\bar{y}) \leq \varepsilon$.

(ii) A point $\bar{y} \in B$ is called an *$\varepsilon$-solution* of (DwQVI($\mathcal{T},\mathcal{K}$)) iff $\forall x \in \mathcal{K}^{-1}(\bar{y})$, $\exists \bar{t} \in \mathcal{T}(\bar{y})$, $\langle \bar{t}, \bar{y}-x\rangle \leq \varepsilon$.

(iii) A point $\bar{y} \in \mathcal{K}(\mathrm{fix}\mathcal{K})$ such that $\exists \bar{t} \in \mathcal{T}(\bar{y})$, $\forall x \in \mathcal{K}^{-1}(\bar{y})$, $\langle \bar{t}, \bar{y}-x\rangle \leq \varepsilon$ is said to be an *$\varepsilon$-solution* of (DsQVI).

**Proposition 3.4** (relationships between approximate dual solutions)

(i) $\bar{y} \in \varepsilon$-Sol(DQEP($\hat{\Phi}, \mathcal{K}$)) *implies that* $\bar{y} \in \varepsilon_1$-Sol(DwQVI($\mathcal{T}, \mathcal{K}$)) *for all* $\varepsilon_1 > \varepsilon$. *Conversely, if* $\bar{y} \in \varepsilon$-Sol(DwQVI($\mathcal{T}, \mathcal{K}$)), *then* $\bar{y} \in \varepsilon$-Sol(DQEP($\hat{\Phi}, \mathcal{K}$))

(ii) *Assume that* $\bar{y} \in \mathcal{K}^{-1}(\bar{y})$. *Then,* $\bar{y} \in \varepsilon$-Sol(DQEP($\hat{\Phi}, \mathcal{K}$)) *if and only if* $0 \le \sup_{x \in \mathcal{K}^{-1}(\bar{y})} \hat{\Phi}(x, \bar{y}) \le \varepsilon$. *The latter implies that* $\bar{y}$ *belongs to* $\varepsilon$-argmax$_{\mathcal{K}^{-1}(\bar{y})} \hat{\Phi}(\cdot, \bar{y})$.

(iii) *Assume that* $\mathcal{K}^{-1}$ *is closed-convex-valued and* $\bar{y} \in \mathcal{K}^{-1}(\bar{y})$. *Then,* $\bar{y} \in \varepsilon$-Sol(DQEP($\hat{\Phi}, \mathcal{K}$)) *if and only if* $\bar{y} \in \varepsilon$-Sol(DsQVI).

*Proof* The proofs of (i) and (ii) are similar/symmetric to those of Proposition 3.2.

(iii) Impose the additional assumptions. $\bar{y} \in \varepsilon$-Sol(DQEP($\hat{\Phi}, \mathcal{K}$)) means that $\sup_{x \in \mathcal{K}^{-1}(\bar{y})} \min_{t \in \mathcal{T}(\bar{y})} \langle t, \bar{y} - x \rangle \le \varepsilon$. By Sion's minimax theorem, this is equivalent to $\min_{t \in \mathcal{T}(\bar{y})} \sup_{x \in \mathcal{K}^{-1}(\bar{y})} \langle t, \bar{y} - x \rangle \le \varepsilon$, i.e., $\bar{y} \in \varepsilon$-Sol(DsQVI). □

**Definition 3.3** (second-kind approximate solutions) Assume that $\theta : \mathbb{R}^m \to \mathbb{R}$ satisfies $\theta(x) > 0$ for $x \neq 0$ and $\varepsilon \ge 0$.

(i) A point $\bar{x} \in \mathrm{fix}\mathcal{K}$ is called an *$\varepsilon\theta$-solution* of (QEP) iff $\Phi(\bar{x}, y) \ge -\varepsilon\theta(\bar{x} - y)$ for all $y \in \mathcal{D}(\bar{x})$.

(ii) A point $\bar{x} \in \mathrm{fix}\mathcal{K}$ is called an *$\varepsilon\theta$-solution* of (wQVI) iff $\forall y \in \mathcal{K}(\bar{x})$, $\exists \bar{t}_y \in \mathcal{T}(\bar{x})$, $\langle \bar{t}_y, y - \bar{x} \rangle \ge -\varepsilon\theta(\bar{x} - y)$.

**Proposition 3.5** (relationship between second-kind approximate solutions) *If* $\bar{x} \in \varepsilon_1\theta$-Sol(QEP($\Phi, \mathcal{K}$)), *then for all* $\varepsilon > \varepsilon_1$, $\bar{x} \in \varepsilon\theta$-Sol(wQVI($\mathcal{T}, \mathcal{K}$)). *Conversely, if* $\bar{x} \in \varepsilon\theta$-Sol(wQVI($\mathcal{T}, \mathcal{K}$)), *then* $\bar{x} \in \varepsilon\theta$-Sol(QEP($\Phi, \mathcal{K}$)).

*Proof* The proposition can be verified analogously to Proposition 3.2(i). □

With a reference to Definition 3.3 and Propositions 3.4 and 3.5, we easily have the statement about the corresponding relationship between second-kind dual approximate solutions.

## 4 Approximations of Set-Valued Quasi-Variational Inequalities

In this section we consider the variants of set-valued quasi-variational inequalities presented in Sect. 3. Assume that we have sequences of problems approximating our problems defined by $\mathcal{T}^k : \mathbb{R}^m \rightrightarrows (\mathbb{R}^m)^*$ with compact values, $\mathcal{K}^k : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, and $\Phi^k : \mathrm{gph}\mathcal{K}^k \to \mathbb{R}$ defined by $\Phi^k(x, y) := \max_{t \in \mathcal{T}^k(x)} \langle t, y - x \rangle$. (When considering quasi-variational inequalities and without connections with subdifferentials, view $\Phi$

and $\Phi^k$ as bifunctions on gph$\mathcal{K}$ and gph$\mathcal{K}^k$, resp, not on $\mathbb{R}^m \times \mathbb{R}^m$.) Similar to $\mathcal{K}$, we regard $\mathcal{K}^k$ as a set-valued map from the whole $A^k := \text{fix}\mathcal{K}^k$ and onto $B^k := \mathcal{K}^k(\text{fix}\mathcal{K}^k)$. So, (QEP($\Phi^k, \mathcal{K}^k$)) is the problem of finding $\bar{x} \in A^k$ such that for all $y \in \mathcal{K}^k(\bar{x})$, $\Phi^k(\bar{x}, y) \geq 0$. The other approximating problems under our consideration are stated in the same way.

We propose the following notion. For $F^k : A^k \subset X \rightrightarrows Y$ and $F : A \subset X \rightrightarrows Y$, we say that gph$\mathcal{F}$ is said to be *strongly included* in Li$_k$(gph$F$), denoted by gph$F \sqsubset$ Li$_k$ gph$F^k$, if for $(x, y) \in$ gph$F$ and subsequence $\{k_j\}_j$, each type of the convergence $x^{k_j} \in A^{k_j} \to x$ and $y^{k_j} \in F^{k_j}(x^{k_j}) \to y$ implies the other one.

**Proposition 4.1** (convergence of approximate solutions of (wQVI$^k$)) *Let* $\varepsilon^k, \varepsilon \geq 0$, $\varepsilon^k \to \varepsilon$, *and* $\varepsilon_1 > \varepsilon$.

(i) *If* $-\Phi^k$ *i-e/h- or i-mis-lop converge to* $-\Phi$, *then* $A \cap \text{Ls}_k(\varepsilon^k\text{-Sol(wQVI}^k)) \subset \varepsilon_1\text{-Sol(wQVI)}$. *If* $-\Phi^k$ *e/h- or mis-lop converge to* $-\Phi$, *then* $\text{Ls}_k(\varepsilon^k\text{-Sol(wQVI}^k)) \subset \varepsilon_1\text{-Sol(wQVI)}$.

(ii) *If* gph$\mathcal{K} \sqsubset$ Li$_k$(gph$\mathcal{K}^k$) *and the i-e- or e-convergence of* $-\Phi^k$ *to* $-\Phi$ *(as unifunctions on* $\mathbb{R}^m \times \mathbb{R}^m$*) replaces the convergence of* $-\Phi^k$ *assumed in* (i), *then the conclusion of* (i) *is still valid.*

(iii) *If* $-\Phi^k$ *w-e/h- or w-mis-lop converge to* $-\Phi$ *and* $x^k \in \varepsilon^k\text{-Sol(wQVI}^k) \to x$, *then* $x \in \varepsilon_1\text{-Sol(wQVI)}$.

*Proof* (i) Because of similarity, we only consider in detail the case when $-\Phi^k \overset{e/h}{\to} -\Phi$. Let $\bar{x} \in \text{Ls}_k(\varepsilon^k$-Sol(wQVI$^k$)) $\subset \text{Ls}_k(\varepsilon^k$-Sol(QEP$^k$)) (by Proposition 3.2(i)), i.e., there exists a sequence $\{x^{k_j}\}_j$ in $\varepsilon^{k_j}$-Sol(QEP$^{k_j}$) converging to $\bar{x}$. If $\bar{x} \in A$, then for all $y \in \mathcal{K}(\bar{x})$, condition (a) of the e/h-convergence yields $y^{k_j} \in \mathcal{K}^{k_j}(x^{k_j}) \to y$ such that $\text{li}_j(-\Phi^{k_j}(x^{k_j}, y^{k_j})) \geq -\Phi(\bar{x}, y)$, i.e., $\text{ls}_j \Phi^{k_j}(x^{k_j}, y^{k_j}) \leq \Phi(\bar{x}, y)$. As $\Phi^{k_j}(x^{k_j}, y^{k_j}) \geq -\varepsilon^{k_j}$, for all $y^{k_j} \in \mathcal{K}^{k_j}(x^{k_j})$, it holds $\Phi(\bar{x}, y) \geq -\varepsilon$. Suppose that $\bar{x} \notin A$. Again condition (a) gives $y^{k_j} \in \mathcal{K}^{k_j}(x^{k_j})$ such that $\Phi^{k_j}(x^{k_j}, y^{k_j}) \to -\infty$, contradicting the inequality $\Phi^{k_j}(x^{k_j}, y^{k_j}) \geq -\varepsilon^{k_j}$ for all $y^{k_j} \in \mathcal{K}^{k_j}(x^{k_j})$. Hence, $\bar{x} \in \varepsilon$-Sol(QEP) and so $\bar{x} \in \varepsilon_1$-Sol(wQVI) also by Proposition 3.2(i). Thus, $\text{Ls}_k(\varepsilon^k$-Sol(wQVI$^k$) $\subset \varepsilon_1$-Sol(wQVI). The assertion for the mis-lop convergence is automatic since the above reasoning only utilizes the common condition (a) of the e/h- and mis-lop convergence.

(ii) As argued in (i), $\bar{x} \in \text{Ls}_k(\varepsilon^k$-Sol(wQVI$^k$)) implies that there exist $\{x^{k_j}\}_j$ in $\varepsilon^{k_j}$-Sol(QEP$^{k_j}$) tending to $\bar{x}$. Suppose that $\bar{x} \notin A$. Then, for all $y^{k_j} \in \mathcal{K}^{k_j}(x^{k_j}) \to y$, $(\bar{x}, y) \notin$ gph$\mathcal{K}$. Hence, condition (a) of the epi-convergence ensures that $\Phi^{k_j}(x^{k_j}, y^{k_j}) \to -\infty$, which contradicts the fact that $\Phi^{k_j}(x^{k_j}, y^{k_j}) \geq -\varepsilon^{k_j}$ for all $y^{k_j} \in \mathcal{K}^{k_j}(x^{k_j})$. So, $\bar{x} \in A$. The proposition mentioned in (i) shows that it suffices to verify that $\Phi(\bar{x}, y) \geq -\varepsilon$ for each $y \in \mathcal{K}(\bar{x})$. Since $(\bar{x}, y) \in$ gph$\mathcal{K}$, by the assumed strong inclusion of gph$\mathcal{K}$ in Li$_k$(gph$\mathcal{K}^k$), there exist $y^{k_j} \in \mathcal{K}^{k_j}(x^{k_j}) \to y$. Then, condition (a) of the epi-convergence implies that

$\mathrm{ls}_j \Phi^{k_j}(x^{k_j}, y^{k_j}) \le \Phi(\bar{x}, y)$. As $\Phi^{k_j}(x^{k_j}, y^{k_j}) \ge -\varepsilon^{k_j}$, $\Phi(\bar{x}, y) \ge -\varepsilon$. The assertion for the remaining case $-\Phi^k \overset{i-e}{\to} -\Phi$ is clear.

(iii) $x^k \in \varepsilon^k$-Sol(wQVI$^k$) implies that $x^k \in \varepsilon^k$-Sol(QEP$^k$) by Proposition 3.2(i), i.e., $\Phi^k(x^k, y^k) \ge -\varepsilon^k$ for all $y^k \in \mathcal{K}^k(x^k)$. If $x \in A$, then for all $y \in \mathcal{K}(x)$, condition (a) of the w-e/h-convergence yields $y^k \in \mathcal{K}^k(x^k) \to y$ such that $\mathrm{ls}_k \Phi^k(x^k, y^k) \le \Phi(x, y)$. The rest of the argument is similar to that in (i) with sequences replacing subsequences. The case of w-mis-lop convergence is automatic as only a common condition of the two types of convergence is used. □

**Proposition 4.2** (convergence of approximate solutions of (DwQVI$^k$)) *Let* $\varepsilon^k, \varepsilon \ge 0$, $\varepsilon^k \to \varepsilon$, *and* $\varepsilon_1 > \varepsilon$.

(i) *If* $-\hat{\Phi}^k$ *i-e/h- or i-mai-lop converge to* $-\hat{\Phi}$, *then* $B \cap \mathrm{Ls}_k(\varepsilon^k$-Sol(DwQVI$^k$))$\subset \varepsilon_1$-Sol(DwQVI). *If* $-\hat{\Phi}^k$ *e/h- or mai-lop converge to* $-\hat{\Phi}$, *then* $\mathrm{Ls}_k(\varepsilon^k$-Sol(DwQVI$^k$))$\subset \varepsilon_1$-Sol(DwQVI).

(ii) *If* gph$\mathcal{K} \sqsubset \mathrm{Li}_k$(gph$\mathcal{K}^k$) *and the types of convergence of* $-\hat{\Phi}^k$ *to* $-\hat{\Phi}$ *in* (i) *is replaced by the inside hypo- or hypo-convergence, then one also has the conclusion in* (i).

(iii) *If* $-\hat{\Phi}^k$ *w-e/h- or w-mai-lop converge to* $-\hat{\Phi}$ *and* $y^k \in \varepsilon^k$-Sol(DwQVI$^k$) $\to y$, *then* $y \in \varepsilon_1$-Sol(DwQVI).

*Proof* (i) We only discuss the case $-\hat{\Phi}^k \overset{e/h}{\to} -\hat{\Phi}$. In view of Proposition 3.4(i), $\bar{y} \in \mathrm{Ls}_k(\varepsilon^k$-Sol(DwQVI$^k$)) implies that $\bar{y} \in \mathrm{Ls}_k(\varepsilon^k$-Sol(DQEP$^k(\hat{\Phi}, \mathcal{K})$)), i.e., there exist $y^{k_j} \in \varepsilon^{k_j}$-Sol(DQEP$^{k_j}$) $\to \bar{y}$. If $\bar{y} \in B$, then condition (b) of the e/h-convergence gives that, for all $x \in \mathcal{K}^{-1}(\bar{y})$, there exist $x^{k_j} \in (\mathcal{K}^{k_j})^{-1}(y^{k_j}) \to x$ such that $\mathrm{li}_j \hat{\Phi}^{k_j}(x^{k_j}, y^{k_j}) \ge \hat{\Phi}(x, \bar{y})$. Since $\hat{\Phi}^{k_j}(x^{k_j}, y^{k_j}) \le \varepsilon^{k_j}$ for all $x^{k_j} \in (\mathcal{K}^{k_j})^{-1}(y^{k_j})$, $\hat{\Phi}(x, \bar{y}) \le \varepsilon$. Suppose $\bar{y} \notin B$. Then, for all $x \in \mathcal{K}^{-1}(\bar{y})$, the above condition (b) yields $x^{k_j} \in (\mathcal{K}^{k_j})^{-1}(y^{k_j})$ such that $\hat{\Phi}^{k_j}(x^{k_j}, y^{k_j}) \to +\infty$, which is a contradiction as $\hat{\Phi}^{k_j}(x^{k_j}, y^{k_j}) \le \varepsilon^{k_j}$ for all $x^{k_j} \in (\mathcal{K}^{k_j})^{-1}(y^{k_j})$. Hence, $\bar{y} \in \varepsilon$-Sol(DQEP$(\hat{\Phi}, \mathcal{K})$), so $\bar{y} \in \varepsilon_1$-Sol(DwQVI) by Proposition 3.4(i).

(ii) The proof is similar to (ii) of the preceding proposition with condition (a) of the epi-convergence replaced by condition (a) of the hypo-convergence.

(iii) The proof is similar to (i) (cf. (iii) in the proof of the preceding proposition). □

**Theorem 4.1** (convergence of couples of approximate solutions of (wQVI$^k$)-(DwQVI$^k$)). *Assume that* $\varepsilon^k, \varepsilon \ge 0$, $\varepsilon^k \to \varepsilon$, *and* $\varepsilon_1 > \varepsilon$.

(i) *If* $-\Phi^k$ *e/h- or mis-lop converge to* $-\Phi$ *and* $\hat{\Phi}^k$ *e/h- or mai-lop converge to* $-\hat{\Phi}$, *then*

$$\mathrm{Ls}_k[(\varepsilon^k\text{-Sol(wQVI}^k)) \times (\varepsilon^k\text{-Sol(DwQVI}^k))] \subset (\varepsilon_1\text{-Sol(wQVI)}) \times (\varepsilon_1\text{-Sol(DwQVI)}).$$

(ii) *If* gph$\mathcal{K} \sqsubset \mathrm{Li}_k$gph$\mathcal{K}^k$, $-\Phi^k$ *epi-converge to* $-\Phi$, *and* $-\hat{\Phi}^k$ *hypo-converge to* $-\hat{\Phi}$, *then the same conclusion as in* (i) *holds.*

(iii) *If* $-\Phi^k$ *w-e/h- or w-mis-lop converge to* $-\Phi$, $-\hat{\Phi}^k$ *w-e/h- or w-mai-lop converge to* $-\hat{\Phi}$, $x^k \in \varepsilon^k$-Sol(wQVI$^k$) $\to x$, *and* $y^k \in \varepsilon^k$-Sol(DwQVI$^k$) $\to y$, *then* $(x, y) \in (\varepsilon_1$-Sol(wQVI))$\times(\varepsilon_1$-Sol(DwQVI)).

*Proof* (i) Since the Limsup of a product of sets is equal to the product of the Limsups of these sets, applying Propositions 4.1(i) and 4.2(i) leads to the conclusion.

(ii) This assertion follows from Propositions 4.1(ii) and 4.2(ii).

(iii) This assertion follows from Propositions 4.1(iii) and 4.2(iii). □

For second-kind approximate solutions, we have the following corresponding results.

**Proposition 4.3** (convergence of second-kind approximate solutions of (wQVI$^k$)) *Let* $\varepsilon^k, \varepsilon \geq 0$, $\varepsilon^k \to \varepsilon$, $\varepsilon_1 > \varepsilon$, *and* $\theta$ *be as defined in Definition 3.3.*

(i) *If* $-\Phi^k$ *i-e/h- or i-mis-lop converge to* $-\Phi$, *then* $A \cap \mathrm{Ls}_k(\varepsilon^k\theta$-Sol(wQVI$^k$)) $\subset \varepsilon_1\theta$-Sol(wQVI). *If* $-\Phi^k$ *e/h- or mis-lop converge to* $-\Phi$, *then* $\mathrm{Ls}_k(\varepsilon^k\theta$-Sol(wQVI$^k$)) $\subset \varepsilon_1\theta$-Sol(wQVI).

(ii) *If* gph$\mathcal{K} \sqsubset \mathrm{Li}_k$(gph$\mathcal{K}^k$) *and the i-e- or e-convergence of* $-\Phi^k$ *to* $-\Phi$ *in* (i) *replaces the types of convergence of* $-\Phi^k$ *in* (i), *then the conclusion in* (i) *is still valid.*

(iii) *If* $-\Phi^k$ *w-e/h- or w-mis-lop converge to* $-\Phi$ *and* $x^k \in \varepsilon^k\theta$-Sol(wQVI$^k$)$\to x$, *then* $x \in \varepsilon_1\theta$-Sol(wQVI).

The proof is similar to that of Proposition 4.1. Furthermore, we also have the convergence of second-kind approximate solutions of (DwQVI$^k$) similar to Proposition 4.2.

Now move on to discuss approximations for strong set-valued quasi-variational inequalities. Clearly, we are only interested in the case with nonconvex-valued $\mathcal{K}$, since the approximate solution set of a strong inequality with $\mathcal{K}$ closed-convex-valued coincides with that of the corresponding weak inequality as we saw above. (Hence, Propositions 4.1, 4.2, and Theorem 4.1 are also about approximations of strong set-valued quasi-variational inequalities when $\mathcal{K}$ is closed-convex-valued.) For such a nonconvex case, an inconvenience is that bifunctions (like $\Phi$ in (3.1)) cannot be used and so variational convergence of bifunctions is still inapplicable. However, we can go deeper into concepts of approximations as follows. Solving our strong problem (sQVI) consists of finding two elements: a solution $\bar{x} \in$ fix$\mathcal{K}$ together with a point $\bar{t} \in \mathcal{T}(\bar{x})$. Hence, one can be interested in the following "second-level" approximations of (sQVI) associated with a sequence $\{x^k\}_k$ of its $\varepsilon^k$-solutions and an $\varepsilon$-solution $\bar{x}$, where $\varepsilon^k, \varepsilon \geq 0$ and $\varepsilon^k \to \varepsilon$.

Assume that $x^k \in \varepsilon^k$-Sol(sQVI), i.e., we have a problem of finding $t^k$ solving $\max_{t\in\mathcal{T}(x^k)}\inf_{y\in\mathcal{K}(x^k)} \langle t, y-x^k\rangle \geq -\varepsilon^k$, and that $\bar{x} \in \varepsilon$-Sol(sQVI), i.e., we have a problem of finding $\bar{t}$ solving $\max_{t\in\mathcal{T}(\bar{x})}\inf_{y\in\mathcal{K}(\bar{x})} \langle t, y-\bar{x}\rangle \geq -\varepsilon$. Setting $A^k := \mathcal{T}(x^k)$, $B^k := \mathcal{K}(x^k)$, and $F^k(t,y) := \langle t, y-x^k\rangle$ for $t \in (\mathbb{R}^m)^*$ and $y \in \mathbb{R}^m$, we have a set-valued strong equilibrium problem (not "quasi-"!) (sEP($F^k, A^k, B^k$)) (the following definition of strong equilibrium problems was given in [11]) of finding $t^k \in A^k$ to solve $\max_{t\in A^k}\inf_{y\in B^k}\langle t, y-x^k\rangle \geq -\varepsilon^k$ (hence $t^k$ is necessarily a maxinf-point of $F^k$ on $A^k \times B^k$). Similarly, with $A := \mathcal{T}(\bar{x})$, $B := \mathcal{K}(\bar{x})$, and $F(t,y) := \langle t, y-\bar{x}\rangle$, we have that $\bar{t}$ is a maxinf-point of $F$ on $A\times B$ and it solves problem (sEP($F, A, B$)): $\max_{t\in\mathcal{T}(\bar{x})}\inf_{y\in\mathcal{K}(\bar{x})}\langle t, y-\bar{x}\rangle \geq -\varepsilon$. Now we have new bifunctions $F^k$ and $F$, so we can employ variational convergence to investigate global approximations of (sEP($F^k, A^k, B^k$)) and (sEP($F, A, B$)), which can be viewed as second-level approximations for (sQVI). It should be emphasized that in the following study of second-level approximations of strong quasi-variational inequalities, we do not impose neither compactness nor convexity assumptions. We only need approximate solutions of our approximating problems exist. We recall some facts needed in this investigation, for a metric space $X$, $A^k, A, B^k, B \subset X$, $F^k : A^k \times B^k \to \mathbb{R}$, and $F : A\times B \to \mathbb{R}$.

A function $\eta : A \to \mathbb{R}$ defined by $\eta(t) := \inf_{y\in B} F(t,y)$ for $t \in A$ is called the *inf-projection* of $F$.

The following notion of tightness modifies the corresponding one in [11] for our current research context.

**Definition 4.1** (ancillary tightness for convergence of $-F^k$) Assume that $X$ is a metric space, $A^k, A, B^k, B \subset X$, $F^k : A^k \times B^k \to \mathbb{R}$, and $F : A \times B \to \mathbb{R}$. Iff $-F^k$ e/h-converge (mis-lop converge or mai-lop converge, resp) to $-F$ and for all $t \in A$, there exist points $t^k \in A^k \to t$ such that $\mathrm{li}_k(\eta^k(t^k)) \geq \eta(t)$, then we say that $-F^k$ e/h-converge (mis-lop converge or mai-lop converge, resp) $t$-ancillary tightly to $-F$.

**Proposition 4.4** (second-level approximations of (sQVI)) *Assume that* $\varepsilon^k, \varepsilon \geq 0$, $\varepsilon^k \to \varepsilon$, $x^k \in \varepsilon^k$-Sol(sQVI), *and* $\bar{x} \in \varepsilon$-Sol(sQVI). *If* $-F^k$ *e/h- or mis-lop or mai-lop converge* $t$*-ancillary tightly to* $-F$, *then* $\mathrm{Ls}_k(\varepsilon^k$-Sol(sEP($F^k, A^k, B^k$))) $\subset \varepsilon$-Sol(sEP($F, A, B$)).

*Proof* (i) Let $\bar{t} \in \mathrm{Ls}_k(\varepsilon^k$-Sol(sEP($F^k, A^k, B^k$))). Then, there exist $t^{k_j} \in \varepsilon^{k_j}$-Sol(sEP($F^{k_j}, A^{k_j}, B^{k_j}$))$\to \bar{t}$ as $j \to +\infty$. Assume the $t$-ancillary tight e/h-convergence of $-F^k$ to $-F$. If $\bar{t} \in A$, then for all $y \in B$, by condition (a) of the assumed e/h-convergence, there exist $y^{k_j} \in B^{k_j} \to y$ such that $\mathrm{li}_j(-F^{k_j}(t^{k_j}, y^{k_j})) \geq -F(\bar{t}, y)$. As $t^{k_j}$ is an $\varepsilon^{k_j}$-solution of (sEP($F^{k_j}, A^{k_j}, B^{k_j}$)), $F^{k_j}(t^{k_j}, y^{k_j})) \geq -\varepsilon^{k_j}$. So, $F(\bar{t}, y) \geq -\varepsilon$, i.e.,

$\bar{t} \in \varepsilon$-Sol(EP$(F, A, B)$). Suppose $\bar{t} \notin A$. Then, the above condition (a) says that there exist $y^{k_j} \in B^{k_j}$ such that $-F^{k_j}(t^{k_j}, y^{k_j}) \to +\infty$, contradicting the fact that $t^{k_j}$ is an $\varepsilon^{k_j}$-solution as aforementioned.

It suffices now to show that $\bar{t}$ is a maxinf-point of $F$ on $A \times B$. The first step is to ensure that $\mathrm{Ls}_k(\mathrm{hypo}\eta^k) \subset \mathrm{hypo}\eta$. Assume that $(t, \alpha) \in \mathrm{Ls}_k(\mathrm{hypo}\eta^k)$, i.e., there exist $t^{k_j} \in \mathrm{dom}\eta^{k_j}$ and $\alpha^{k_j} \leq \eta^{k_j}(t^{k_j})$ such that $(t^{k_j}, \alpha^{k_j}) \to (t, \alpha)$. If $t \notin A$, then by condition (a) of the assumed e/h-convergence, there exist $y^{k_j} \in B^{k_j}$ such that $-F^{k_j}(t^{k_j}, y^{k_j}) \to +\infty$. Since $\alpha^{k_j} \leq \eta^{k_j}(t^{k_j}) \leq F^{k_j}(t^{k_j}, y^{k_j})$, it holds that $\alpha^{k_j} \to -\infty$, which is a contradiction as $\alpha^{k_j} \to \alpha$. So, $t \in A$. If $t \notin \mathrm{dom}\eta$, then one has $y \in B$ with $F(t, y) \leq \alpha - 1$. By the above (a), there are $y^{k_j} \in B^{k_j} \to y$ such that $\mathrm{li}_j(-F^{k_j}(t^{k_j}, y^{k_j})) \geq -F(t, y)$. Hence,

$$\alpha = \mathrm{lim}_j \alpha^{k_j} \leq \mathrm{ls}_j \eta^{k_j}(t^{k_j}) \leq \mathrm{ls}_j(F^{k_j}(t^{k_j}, y^{k_j})) \leq F(t, y) \leq \alpha - 1,$$

which is impossible. Consequently, $t$ must be in $\mathrm{dom}\eta$. For any positive $\varepsilon$, take $y_\varepsilon \in B$ such that $F(t, y_\varepsilon) \leq \eta(t) + \varepsilon$. Again by condition (a) of the e/h-convergence, one has $y^{k_j} \in B^{k_j} \to y_\varepsilon \in B$ such that

$$\mathrm{ls}_j \eta^{k_j}(t^{k_j}) \leq \mathrm{ls}_j F^{k_j}(t^{k_j}, y^{k_j}) \leq F(t, y_\varepsilon) \leq \eta(t) + \varepsilon.$$

Therefore,

$$\alpha = \mathrm{lim} \alpha^{k_j} \leq \mathrm{ls}_j \eta^{k_j}(t^{k_j}) \leq \eta(t) + \varepsilon$$

and so $(t, \alpha) \in \mathrm{hypo}\eta$ as $\varepsilon$ is arbitrary.

The second step is to verify that $\mathrm{hypo}\eta \subset \mathrm{Li}_k(\mathrm{hypo}\eta^k)$. By the assumed ancillary tightness, for all $t \in A$, there exist points $t^k \in A^k \to t$ such that $\mathrm{li}_k(\eta^k(t^k)) \geq \eta(t)$, i.e., $\mathrm{ls}_k(-\eta^k(t^k)) \leq -\eta(t)$. This is equivalent to condition (b) of the epi-convergence of $-\eta^k$ to $-\eta$, whose geometric formulation is $\mathrm{epi}(-\eta) \subset \mathrm{Li}_k(\mathrm{epi}(-\eta^k))$. The last inclusion is equivalent to $\mathrm{hypo}\eta \subset \mathrm{Li}_k \mathrm{hypo}\eta^k$.

The conclusion for the $t$-ancillary case of these types of convergence is clearly also verified because both the mis-lop and mai-lop convergence are stronger than the e/h-convergence of bifunctions defined on rectangles (i.e., the domains of the form $A \times B$). (Although these types of convergence are incomparable for (general) bifunctions defined on domains of the form $\mathrm{gph}\mathcal{D}$ as in Definitions 2.3-2.5.) □

Now consider dual strong problem (DsQVI). Assume that $y^k \in \varepsilon^k$-Sol(DsQVI), i.e., we have the problem of finding $t^k$ solving problem $\mathrm{min}_{t \in \mathcal{T}(y^k)} \mathrm{sup}_{x \in \mathcal{K}^{-1}(y^k)} \langle t, y^k - x \rangle \leq \varepsilon^k$, and $\bar{y} \in \varepsilon$-Sol(DsQVI), i.e., we have the problem of finding $\bar{t}$ solving $\mathrm{min}_{t \in \mathcal{T}(\bar{y})} \mathrm{sup}_{x \in \mathcal{K}^{-1}(\bar{y})} \langle t, \bar{y} - x \rangle \leq \varepsilon$. Setting $C^k := \mathcal{T}(y^k)$, $D^k := \mathcal{K}^{-1}(y^k)$, $G^k := \langle t, y^k - x \rangle$, $C := \mathcal{T}(\bar{y})$, $D := \mathcal{K}^{-1}(\bar{y})$, $G := \langle t, \bar{y} - x \rangle$, the above two problems become dual strong set-valued equilibrium problems (DsEP$(G^k, C^k, D^k)$) and (DsEP$(G, C, D)$), resp.

Global approximations of them in terms of variational convergence of $G^k$ and $G$ can be viewed as second-level global approximations of (DsQVI). For these problems, we need the following notions.

A function $\xi : C \to \mathbb{R}$ defined by $\xi(t) := \sup_{x\in D} G(t,x)$ for $t \in A$ is called the *sup-projection* of $G$.

**Definition 4.2** (ancillary tightness for convergence of $G^k$) Assume that $X$ is a metric space, $C^k, C, D^k, D \subset X$, $G^k : C^k \times D^k \to \mathbb{R}$, and $G : C \times D \to \mathbb{R}$. If $G^k$ e/h-converge (mis-lop converge or mai-lop converge, resp) to $G$ and for all $t \in C$, there exist points $t^k \in C^k \to t$ such that $\mathrm{ls}_k(\xi^k(t^k)) \le \xi(t)$, then we say that $G^k$ e/h-converge (mis-lop converge or mai-lop converge, resp) $t$-ancillary tightly to $G$.

**Proposition 4.5** (second-level approximations of (DsQVI)) *Assume that* $\varepsilon^k, \varepsilon \ge 0$*,* $\varepsilon^k \to \varepsilon$*,* $y^k \in \varepsilon^k$-Sol(DsQVI)*, and* $\bar{y} \in \varepsilon$-Sol(DsQVI)*. If* $G^k$ *e/h- or mis-lop or mai-lop converge t-ancillary tightly to G, then* $\mathrm{Ls}_k(\varepsilon^k\text{-Sol}(\mathrm{DsEP}(G^k, C^k, D^k))) \subset \varepsilon\text{-Sol}(\mathrm{DsEP}(G, C, D))$*.*

The proof is similar to that of Proposition 4.4 and omitted.

## 5 Approximations of Traffic Networks with Set-Valued Costs and Elastic Demands Depending on Equilibrium Flows

### 5.1 Traffic Networks with Set-Valued Costs and Elastic Demands Depending on Equilibrium Flows

Assume for traffic network $\mathcal{N}$ that $N$ is the set of *nodes*, $L$ is that of directed *arcs* (also called *links*), $W$ is the set of *origin-destination pairs* (*O/D pairs*) of nodes. Assume further that O/D pair $w \in W$ is connected by set $P_w$ of paths, $P = \cup_{w\in W} P_w$, and $m = |P|$ (the number of paths in the network). Let $F := (F_1, \dots, F_m) = (F_p)_{p=1,\dots,m}$ denote a *vector flow* (also briefly called *flow*). Let the *capacity restriction* on arc $\ell \in L$ be $c_\ell > 0$. So, any arc flow $v_\ell$ on $\ell$ needs to satisfy $0 \le v_\ell \le c_\ell$ for $\ell \in L$. Expressing in terms of vector flow $F$, $v_\ell$ takes the form

$$v_\ell = \sum_{w\in W}\sum_{p\in P_w} \delta_{\ell p} F_p, \quad \text{where } \delta_{\ell p} = \begin{cases} 1 \text{ if } \ \text{arc } \ell \text{ is contained in path } p, \\ 0 \text{ otherwise.} \end{cases}$$

Then, the set of the vector flows satisfying the arc capacity constraints is

$$\{F \in \mathbb{R}^m_+ \mid \sum_{w\in W}\sum_{p\in P_w} \delta_{\ell p} F_p \le c_\ell \text{ for all } \ell \in L\}.$$

If the (fixed) demand of O/D pair $w$ is $d_w$, then the set of feasible vector flows is

$$K = \{F \in \mathbb{R}^m_+ \mid \sum_{w\in W}\sum_{p\in P_w} \delta_{\ell p} F_p \le c_\ell \text{ for all } \ell \in L \text{ and } \sum_{p\in P_w} F_p = d_w \text{ for all } w \in W\}.$$

Assume that the cost of vector flow $F$ is $t(F) := (t_1(F), ..., t_m(F))$, i.e, we have the cost map $t : \mathbb{R}^m_+ \to (\mathbb{R}^m)^*$. When a given $F$ is clear, we also write simply $t := (t_1, ..., t_m)$.

**Definition 5.1** (types of approximate saturated arcs and approximate saturated paths) Let $F \in K$.

(i) For an $\ell \in L$ and a pair $k, j \in \{1, ..., m\}$, assume that $0 < \frac{\varepsilon}{t_j - t_k} \leq c_\ell$. Iff $v_\ell \geq c_\ell - \frac{\varepsilon}{t_j - t_k}$, then $\ell$ is said to be a $(k, j)$-$\varepsilon$-*saturated arc* of $F$; iff $0 \leq v_\ell < c_\ell - \frac{\varepsilon}{t_j - t_k}$, then $\ell$ is a $(k, j)$-$\varepsilon$-*nonsaturated arc* of $F$.

When $\varepsilon = 0$, $(k, j)$-0-nonsaturatedness is called simply $(k, j)$-*nonsaturatedness* and iff $v_\ell = c_\ell$, then $\ell$ is called a $(k, j)$-*saturated arc*.

(ii) For $\varepsilon \geq 0$, iff there exists a $(k, j)$-$\varepsilon$-saturated arc $\ell$ in the vector flow $F$ such that $\ell$ belongs to path $p$, then $p$ is called a $(k, j)$-$\varepsilon$-*saturated path* of $F$, otherwise it is a $(k, j)$-$\varepsilon$-*nonsaturated path* of $F$.

(iii) Assume that $\theta : \mathbb{R}^m \to \mathbb{R}$ satisfies $\theta(F) > 0$ for all vector flows $F \neq 0$. Iff, for a pair $k, j \in \{1, ..., m\}$, it holds $v_\ell \geq c_\ell - \frac{\varepsilon\theta(F_k - F_j)}{t_j - t_k}$, then arc $\ell$ is termed $(k, j)$-$\varepsilon\theta$-*saturated*; iff $0 \leq v_\ell < c_\ell - \frac{\varepsilon\theta(F_k - F_j)}{t_j - t_k}$, then $\ell$ is a $(k, j)$-$\varepsilon\theta$-*nonsaturated arc*.

(iv) For $\theta$, assume the same as in (iii). Iff there exists a $(k, j)$-$\varepsilon\theta$-saturated arc in path $p$, then $p$ is said to be a $(k, j)$-$\varepsilon\theta$-*saturated path*.

Iff $\varepsilon = 0$ in the above three statements, we remove "$\varepsilon$" and "$\varepsilon\theta$" in the names.

**Definition 5.2** (Wardrop approximate equilibrium flows for (TP$(t, K)$)) Consider the above traffic network with fixed demands. Let $\varepsilon \geq 0$.

(i) Vector flow $H \in K$ is called an $\varepsilon$-*equilibrium flow* iff for all $w \in W$ and $k, j \in P_w$, $t_k < t_j$ implies that $0 \leq H_j \leq \frac{\varepsilon}{t_j - t_k}$ or path $k$ is a $(k, j)$-$\varepsilon$-saturated path of vector flow $H$.

(ii) Assume that $\theta : \mathbb{R}^m \to \mathbb{R}$ satisfies $\theta(F) > 0$ for all vector flows $F \neq 0$. Vector flow $H \in K$ is called an $\varepsilon\theta$-*equilibrium flow* iff for all $w \in W$ and $k, j \in P_w$, $t_k < t_j$ implies that $0 \leq H_j \leq \frac{\varepsilon\theta(F_k - F_j)}{t_j - t_k}$ or path $k$ is a $(k, j)$-$\varepsilon\theta$-saturated path of vector flow $H$.

Iff $\varepsilon = 0$, we remove "$\varepsilon$" and "$\varepsilon\theta$" in the names.

The *traffic equilibrium problem* (briefly, *traffic problem*), denoted by (TP$(t, K)$), is to find equilibrium flows or approximate equilibrium flows in the above sense.

It is well-known that $H$ is a vector equilibrium flow of (TP(t,K)) if $H$ is a solution of the variational inequality: find $H \in K$ such that for all $F \in K$, $\langle t(H), F - H\rangle \geq 0$. But, to the best of our knowledge, there are no contributions to approximate equilibrium flows in the sense of Definition 5.2.

In this paper, our traffic problem is more general than (TP$(t, K)$) (to suit diverse practical situations) as follows. Assume that the cost $\mathcal{T}(F)$ of vector flow $F$ is a compact subset of $(\mathbb{R}^m_+)^*$, i.e., one has a compact-set-valued map $\mathcal{T} : \mathbb{R}^m_+ \rightrightarrows (\mathbb{R}^m_+)^*$. Assume further that each demand $d_w$ depends on equilibrium costs, or more directly, on equilibrium pattern flows $H$, i.e., we have a map $d_w : \mathbb{R}^m_+ \to \mathbb{R}_+$, which is assumed to be continuous. Moreover, we are interested in the following case with a tolerance of the demands. Let $\beta : \mathbb{R}^m_+ \to \mathbb{R}$ be a continuous function. The feasible path follows are defined as

$$\mathcal{K}(H) := \{F \in \mathbb{R}^m_+ \mid \sum_{w \in W} \sum_{p \in P_w} \delta_{\ell p} F_p \leq c_\ell \text{ for all } \ell \in L$$

$$\text{and} \sum_{p \in P_w} F_p \in [d_w(H) - \beta(H), d_w(H) + \beta(H)] \text{ for all } w \in W\}.$$

Then, $\mathcal{K}(H)$ is clearly a closed, compact, and convex set for each $H$. Definition 5.1 of approximate saturateness applies also to our traffic problem with the only replacement of $K$ by $\mathcal{K}$. The Wardrop approximate equilibrium concepts are extended as follows.

**Definition 5.3** (Wardrop concepts for the case of elastic demands and costs depending on equilibrium flows) Let $\varepsilon \geq 0$.

(i) Iff vector flow $H \in \mathcal{K}(H)$ such that $\forall w \in W$, $\forall k, j \in P_w$, $\exists t \in \mathcal{T}(H)$ such that

$$t_k < t_j \ \Rightarrow 0 \leq H_j \leq \frac{\varepsilon}{t_j - t_k} \quad \text{or} \quad k \text{ is a (k,j)} - \varepsilon - \text{saturated path},$$

then $H$ is called a *weak $\varepsilon$-equilibrium flow*

(ii) Iff for $H \in \mathcal{K}(H)$, $\exists t \in \mathcal{T}(H)$ such that $\forall w \in W$, $\forall k, j \in P_w$,

$$t_k < t_j \ \Rightarrow 0 \leq H_j \leq \frac{\varepsilon}{t_j - t_k} \quad \text{or} \quad k \text{ is a (k,j)} - \varepsilon - \text{saturated path},$$

then $H$ is called an *$\varepsilon$-equilibrium flow.*

(iii) Assume that $\theta : \mathbb{R}^m \to \mathbb{R}$ satisfies $\theta(F) > 0$ for all vector flows $F \neq 0$. Iff vector flow $H \in \mathcal{K}(H)$ satisfies that $\forall w \in W$, $\forall k, j \in P_w$, $\exists t \in \mathcal{T}(H)$ such that

$$t_k < t_j \ \Rightarrow 0 \leq H_j \leq \frac{\varepsilon\theta}{t_j - t_k} \quad \text{or} \quad k \text{ is a (k,j)} - \varepsilon\theta - \text{saturated path},$$

then $H$ is called a *weak $\varepsilon\theta$-equilibrium flow*

Iff $\varepsilon = 0$ in the above statements, $H$ is said to be a weak equilibrium flow or an equilibrium flow, resp.

The traffic problem of finding equilibrium or approximate equilibrium flows in the sense given in Definition 5.3 is denoted by (TP$(\mathcal{T}, \mathcal{K})$). We also use the simple notation (TP) for this problem since hereafter we always deal with this general model.

Using a modification of traditional techniques, we demonstrate the following relationship between our traffic problem with elastic demands and a set-valued quasi-variational inequality.

**Theorem 5.1** (approximate solutions of (wQVI($\mathcal{T},\mathcal{K}$)) or (sQVI($\mathcal{T},\mathcal{K}$)) are approximate weak equilibrium or approximate equilibrium, resp, flows of (TP($\mathcal{T},\mathcal{K}$)))

(i) *Vector flow $H$ is a weak $\varepsilon$-equilibrium flow of* (TP) *if $H$ is an $\varepsilon$-solution of the weak set-valued quasi-variational inequality*

$$(\text{wQVI}(\mathcal{T},\mathcal{K}))\quad \text{find } H\in\mathcal{K}(H) \text{ such that } \forall F\in\mathcal{K}(H), \exists t_F\in\mathcal{T}(H), \langle t_F, F-H\rangle \geq -\varepsilon.$$

(ii) *$H$ is an $\varepsilon$-equilibrium flow of* (TP) *if $H$ is an $\varepsilon$-solution of the set-valued quasi-variational inequality* (sQVI).

(iii) *Vector flow $H$ is a weak $\varepsilon\theta$-equilibrium flow of* (TP) *if $H$ is an $\varepsilon\theta$-solution of the weak set-valued quasi-variational inequality*

$$\text{find } H\in\mathcal{K}(H) \text{ such that } \forall F\in\mathcal{K}(H), \exists t_F\in\mathcal{T}(H), \langle t_F, F-H\rangle \geq -\varepsilon\theta(H-F).$$

*If $\varepsilon = 0$ in the above statements, then they are about a weak equilibrium flow or an equilibrium flow, resp.*

*Proof* (i) Assume that $H\in\varepsilon$-Sol(wQVI($\mathcal{T},\mathcal{K}$)) and that for all $w\in W$ and $k,j\in P_w$, there exists $t\in\mathcal{T}(H)$ such that $t_k<t_j$ and $k$ is a $(k,j)$-$\varepsilon$-nonsaturated path of flow $H$. We need to verify that $0\leq H_j\leq\frac{\varepsilon}{t_j-t_k}$. Arguing by contradiction, suppose that $H_j>\frac{\varepsilon}{t_j-t_k}$. Let

$$\delta := \min\{H_j - \frac{\varepsilon}{t_j-t_k}, \min_{\ell\in k}(c_\ell - H_k)(t_j-t_k)-\varepsilon\}.$$

Since $k$ is $(k,j)$-$\varepsilon$-nonsaturated, $(c_\ell - H_k)(t_j-t_k)-\varepsilon>0$ for all $\ell\in k$. Hence, $\delta>0$.

Consider the following vector flow $F=(F_i)_{i=1,\dots,m}$

$$F_i=\begin{cases} H_i & \text{if } i\neq k, i\neq j,\\ H_j-\frac{\delta+\varepsilon}{t_j-t_k} & \text{if } i=j,\\ H_k+\frac{\delta+\varepsilon}{t_j-t_k} & \text{if } i=k.\end{cases}$$

It is not hard to check that

$$0\leq \sum_{i\neq j, i\neq k} H_i + \delta_{\ell k}(H_k+\frac{\delta+\varepsilon}{t_j-t_k}) + \delta_{\ell j}(H_j-\frac{\delta+\varepsilon}{t_j-t_k}) \leq c_\ell,$$

for all $\ell \in L$, i.e., $F$ is a feasible vector flow: $F \in \mathcal{K}(H)$. We have

$$\langle t, F - H\rangle = \sum_{i=1}^{m} t_i(F_i - H_i) = t_k(F_k - H_k) + t_j(F_j - H_j) = -\delta - \varepsilon < -\varepsilon,$$

which is a contradiction.

(ii) The proof for $\varepsilon$-solutions of the two problems under consideration is almost the same as that of (i) (with some clear modifications).

(iii) Assume that $H \in \varepsilon\theta$-Sol(wQVI$(\mathcal{T}, \mathcal{K})$) and for all $w \in W$ and $k, j \in P_w$, there exists $t \in \mathcal{T}(H)$ with $t_k < t_j$ and $k$ being a $(k, j)$-$\varepsilon\theta$-nonsaturated path of flow $H$. We need to verify that $0 \le H_j \le \frac{\varepsilon\theta(F_k - F_j)}{t_j - t_k}$. Suppose to the contrary that $H_j > \frac{\varepsilon\theta(F_k - F_j)}{t_j - t_k}$. Set

$$\delta := \min\{H_j - \frac{\varepsilon\theta(F_k - F_j)}{t_j - t_k}, \min_{\ell\in k}(c_\ell - H_k)(t_j - t_k) - \varepsilon\theta(F_k - F_j)\},$$

$$F_i = \begin{cases} H_i & \text{if } i \neq k, i \neq j, \\ H_j - \frac{\delta + \varepsilon\theta(F_k - F_j)}{t_j - t_k} & \text{if } i = j, \\ H_k + \frac{\delta + \varepsilon\theta(F_k - F_j)}{t_j - t_k} & \text{if } i = k. \end{cases}$$

Similar to part (i), we have that $F \in \mathcal{K}(H)$ and

$$\langle t, F - H\rangle = \sum_{i=1}^{m} t_i(F_i - H_i) = t_k(F_k - H_k) + t_j(F_j - H_j) = -\delta - \varepsilon\theta(H_k - H_j) < -\varepsilon\theta(H_k - H_j).$$

This contradiction with the assumption that $H \in \varepsilon\theta$-Sol(wQVI$(\mathcal{T}, \mathcal{K})$) ends the proof. □

### 5.2 Approximations of Traffic Networks

In this subsection, we will apply the results of Sects. 3 and 4 for approximations of quasi-variational inequalities to approximations of traffic networks. Hence, the following bifunction on $\mathbb{R}^m_+ \times \mathbb{R}^m_+$

$$\Phi(F_1, F_2) := \max_{t\in\mathcal{T}(F_1)}\langle t, F_2 - F_1\rangle \tag{2}$$

will be in use.

Assume that we have maps $\mathcal{T}^k : \mathbb{R}^m_+ \rightrightarrows (\mathbb{R}^m_+)^*$ with compact values, which approximate $\mathcal{T}$, and $\Phi^k :$ gph$\mathcal{K}^k \to \mathbb{R}$ defined by $\Phi^k(F_1, F_2) := \max_{t\in\mathcal{T}^k(F_1)}\langle t, F_2 - F_1\rangle$, which approximate $\Phi$. Here, the map $\mathcal{K}^k$ is produced by approximations of $d^k_w(\cdot)$ as follows

$$\mathcal{K}^k(H) := \{F \in \mathbb{R}^m_+ \mid 0 \le v_\ell \le \sum_{w\in W}\sum_{p\in P_w} \delta_{\ell p}F_p \text{ for all } \ell \in L$$

$$\text{and} \sum_{p\in P_w} F_p \in [d_w^k(H) - \beta(H), d_w^k(H) + \beta(H)] \text{ for all } w \in W\}.$$

We now investigate the set convergence of the weak approximate equilibrium flows of approximating problems ($\mathrm{TP}^k(\mathcal{T}^k, \mathcal{K}^k)$), which are usually briefly written as ($\mathrm{TP}^k$), in terms of variational convergence of $\Phi^k$.

We also denote the set of weak $\varepsilon$-equilibrium and $\varepsilon$-equilibrium flows, resp, of problem (TP), by $\varepsilon$-weq(TP) and $\varepsilon$-eq(TP).

**Corollary 5.1** (convergence of approximate weak equilibria) *Assume that* $\varepsilon^k, \varepsilon \geq 0$, $\varepsilon^k \to \varepsilon$, *and* $\varepsilon_1 > \varepsilon$.

(i) *If* $-\Phi^k$ *i-e/h- or i-mis-lop converge to* $-\Phi$, *then* $\mathrm{fix}\mathcal{K} \cap \mathrm{Ls}_k(\varepsilon^k\text{-Sol(wQVI}^k)) \subset \varepsilon_1\text{-weq(TP)}$. *If* $-\Phi^k$ *e/h- or mis-lop converge to* $-\Phi$, *then* $\mathrm{Ls}_k(\varepsilon^k\text{-Sol(wQVI}^k)) \subset \varepsilon_1\text{-weq(TP)}$.

(ii) *If* $-\Phi^k \stackrel{e}{\to} -\Phi$ *and* $\mathrm{gph}\mathcal{K} \sqsubset \mathrm{Li}_k(\mathrm{gph}\mathcal{K}^k)$, *then*

$$\mathrm{Ls}_k(\varepsilon^k\text{-Sol(wQVI}^k)) \subset \varepsilon_1\text{-weq(TP)}.$$

*If* $-\Phi^k \stackrel{i-e}{\to} -\Phi$ *replaces the above epi-convergence, then*

$$\mathrm{fix}\mathcal{K} \cap \mathrm{Ls}_k(\varepsilon^k\text{-Sol(wQVI}^k)) \subset \varepsilon_1\text{-weq(TP)}.$$

(iii) *If* $-\Phi^k$ *w-e/h- or w-mis-lop converge to* $-\Phi$ *and* $H^k \in \varepsilon^k\text{-Sol(wQVI}^k) \to H$, *then* $H \in \varepsilon_1\text{-weq(TP)}$.

*Proof* (i) Consider the case of i-e/h-convergence. In virtue of Proposition 4.1(i), $\mathrm{fix}\mathcal{K} \cap \mathrm{Ls}_k(\varepsilon^k\text{-Sol(wQVI}^k)) \subset \varepsilon_1\text{-Sol(wQVI)}$. By Theorem 5.1(i), $\varepsilon_1$-Sol(wQVI)$\subset \varepsilon_1$-weq(TP) as what we need to verify. The other cases of assertion (i) can be proved similarly.

(ii) The verification of this assertion is similar to that of (i), with the use of (ii) in Proposition 4.1 instead of (i).

(iii) The proof is also analogous, using (iii) of Proposition 4.1 instead of (ii). □

**Corollary 5.2** (convergence of approximate equilibria) *Assume that* $\varepsilon^k, \varepsilon \geq 0$, $\varepsilon^k \to \varepsilon$.

(i) *If* $-\Phi^k$ *e/h- or mis-lop converge to* $-\Phi$, *then* $\mathrm{Ls}_k(\varepsilon^k\text{-Sol(QVI}^k)) \subset \varepsilon\text{-eq(TP)}$. *If the above types of convergence reduce to the inside ones, then* $\mathrm{fix}\mathcal{K} \cap \mathrm{Ls}_k(\varepsilon^k\text{-Sol(QVI}^k)) \subset \varepsilon\text{-eq(TP)}$

(ii) *If* $-\Phi^k$ *epi- or i-epi-converge to* $-\Phi$ *and* $\mathrm{gph}\mathcal{K} \sqsubset \mathrm{Li}_k\mathrm{gph}\mathcal{K}^k$, *then one has the same inclusion as in* (i).

(iii) *If* $-\Phi^k$ *w-e/h- or w-mis-lop converge to* $-\Phi$ *and* $H^k \in \varepsilon^k\text{-Sol(QVI}^k) \to H$, *then* $H \in \varepsilon\text{-eq(TP)}$.

*Proof* Apply Propositions 3.2, 4.1, and Theorems 5.1(ii). □

**Corollary 5.3** (convergence of second-kind approximate weak equilibria) *Assume that* $\varepsilon^k, \varepsilon \geq 0$, $\varepsilon^k \to \varepsilon$, *and* $\varepsilon_1 > \varepsilon$.

(i) *If* $-\Phi^k$ *i-e/h- or i-mis-lop converge to* $-\Phi$, *then* $\mathrm{fix}\mathcal{K} \cap \mathrm{Ls}_k(\varepsilon^k\theta\text{-Sol(wQVI}^k)) \subset \varepsilon_1\theta\text{-weq(TP)}$. *If* $-\Phi^k$ *e/h- or mis-lop converge to* $-\Phi$, *then* $\mathrm{Ls}_k(\varepsilon^k\theta\text{-Sol(wQVI}^k)) \subset \varepsilon_1\theta\text{-weq(TP)}$.

(ii) *If* $\mathrm{gph}\mathcal{K} \sqsubset \mathrm{Li}_k(\mathrm{gph}\mathcal{K}^k)$ *and the i-e- or e-convergence of* $-\Phi^k$ *to* $-\Phi$ *replaces the assumed types of convergence of* $-\Phi^k$ *in* (i), *then the conclusion in* (i) *holds as well.*

(iii) *If* $-\Phi^k$ *w-e/h- or w-mis-lop converge to* $-\Phi$ *and* $H^k \in \varepsilon^k\theta\text{-Sol(wQVI}^k) \to H$, *then* $H \in \varepsilon_1\theta\text{-weq(TP)}$.

*Proof* The argument is similar to that for Corollary 5.1, but now we employ Proposition 4.3 and Theorem 5.1(iii).

## 6 Concluding Remarks

This paper is the first attempt to study global approximations of quasi-variational inequalities and traffic networks in terms of variational convergence. Weak and usual strong variants of quasi-variational inequalities and traffic problems with arc capacity constraints, set-valued costs, and elastic demands depending on the pattern equilibrium flow are the research objects. The inside and weak types of epi/hypo- and lopsided convergence are introduced and used as tools for approximation studies. New concepts of approximate solutions of quasi-variational inequalities and saturatedness of arcs and paths together with new notions of equilibrium flows of traffic network problems are also proposed. The obtained approximation results are novel.

Since this is only the first step of the research topic, we can expect a significant possibility for a continuation of this paper. For instance, approximations of the above problems but direct in terms of the problem data, not via the associated bifunction $\Phi$, and a definition and study of dual traffic problems following the research in the paper for quasi-variational inequalities are perspectives.

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