RIGIDITY OF FOUR-DIMENSIONAL KÄHLER-RICCI SOLITONS

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Abstract. In this article, we investigate four-dimensional gradient shrinking Ricci solitons close to a Kähler model. The first theorem could be considered as a rigidity result for the Kähler-Ricci soliton structure on $\mathbb{S}^2 \times \mathbb{R}^2$ (in the sense of Remark 1). Moreover, we show that if the quotient of norm of the self-dual Weyl tensor and scalar curvature is close to that on a Kähler metric in a specific sense, then the gradient Ricci soliton must be either half-conformally flat or locally Kähler.

1. Introduction

A complete Riemannian metric g on an n-dimensional smooth manifold M^n is called a *gradient shrinking Ricci soliton* (GSRS) if there exists a smooth potential function f such that the Ricci tensor Ric and metric g satisfy the equation

$$Ric + Hess\, f = \frac{1}{2}g. \tag{1.1}$$

Here, $Hess\, f$ denotes the Hessian of f. Ricci solitons are self-similar solutions to Hamilton's Ricci flow and play a crucial role in the singularity analysis of the Ricci flow [26]. They also provide a natural extension of Einstein manifolds, see [5] for an overview on Ricci solitons.

In [26], Hamilton showed that any two-dimensional GSRS is either isometric to the plane $\mathbb{R}^2$ or a quotient of the sphere $\mathbb{S}^2$ (both are Kähler by the identifications $\mathbb{R}^2 \simeq \mathbb{C}$ and $\mathbb{S}^2 \simeq \mathbb{CP}^1$). Furthermore, it follows by the works of Ivey [27], Perelman [35], Naber [33], Ni-Wallach [34], and H.-D. Cao-Chen-Zhu [7], that any three-dimensional GSRS is a finite quotient of either the round sphere $\mathbb{S}^3$, the Gaussian shrinking soliton $\mathbb{R}^3$, or the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.

In dimension $n = 4$, the first non-Einstein example of compact gradient shrinking Ricci soliton is a rotationally Kähler metric on $\mathbb{CP}^2 \sharp (-\mathbb{CP}^2)$, where $(-\mathbb{CP}^2)$ is the complex projective space with the opposite orientation. This was first discovered by H.-D. Cao [6] and Koiso [29]. Later, Wang and Zhu [41] constructed a new gradient Kähler-Ricci soliton on $\mathbb{CP}^2 \sharp 2(-\mathbb{CP}^2)$. A well-known folklore conjecture asks *whether any compact non-Einstein gradient Ricci soliton in dimension four is necessarily Kähler*. For noncompact examples, we have the gradient Kähler-Ricci soliton $\mathbb{S}^2 \times \mathbb{R}^2$ with the natural orientation induced from the complex structure on $\mathbb{CP}^1 \times \mathbb{C}$ and the family constructed by Feldman, Ilmanen and Knopf in [24], which is $U(n)$-invariant and cone-like at infinity.

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Despite of recent important advancements, the classification of four-dimensional GSRS remains open, even in the case of complete Kähler-Ricci solitons; see, e.g., [2, 8–10, 12, 17, 19, 20, 22, 23, 25, 28, 30–34, 36, 43] for recent progress. By the works [12,23,30,34,36,43], locally conformally flat (i.e., $\mathrm{W} = 0$) four-dimensional complete GSRS are isometric to finite quotients of either $\mathbb{S}^4$, or $\mathbb{R}^4$, or $\mathbb{S}^3 \times \mathbb{R}$. Other relevant classification results have been obtained under various curvature conditions: half-conformally flat ($\mathrm{W}^+ = 0$ by a choice of orientation) by Chen and Wang [18]; Bach-flat by H.-D. Cao and Chen [8]; harmonic Weyl tensor ($\mathrm{div}\mathrm{W} = 0$) by Fernández-López and García-Río [25] together with Munteanu and Sesum [30]; and harmonic self-dual Weyl tensor ($\mathrm{div}\mathrm{W}^+ = 0$) by Wu et al. [42], also see [15].

On the other hand, there are results based on pinching conditions of the Weyl curvature. Catino [14] showed that any complete four-dimensional GSRS with nonnegative Ricci curvature and

$$|\mathrm{W}|R \leq \sqrt{3}\left(|\mathring{Ric}| - \frac{1}{2\sqrt{3}}R\right)^2 \tag{1.2}$$

must be conformally flat, here, $\mathring{Ric}$ and R denote traceless Ricci tensor and scalar curvature, respectively. The nonnegative Ricci curvature assumption was later removed in [42]. Some related results involving integral pinching conditions were established in [11, 16]. By a different method, Zhang [44] shows that any four-dimensional GSRS with $0 \leq Ric \leq C$ and

$$|\mathrm{W}| \leq \gamma\left||\mathring{Ric}| - \frac{1}{2\sqrt{3}}R\right| \tag{1.3}$$

is either flat or has 2-positive Ricci curvature, for some constant $\gamma < 1 + \sqrt{3}$.

It turns out that neither pinching conditions (1.2) nor (1.3) recovers the non-Einstein Kähler-Ricci soliton $\mathbb{S}^2 \times \mathbb{R}^2$, the (normalized) GSRS $\left(\mathbb{S}^2(\sqrt{2}) \times \mathbb{R}^2,\, g,\, f\right)$ with product metric g and potential function

$$f(x, y) = \frac{|y|^2}{4} + 1,$$

for $(x, y) \in \mathbb{S}^2(\sqrt{2}) \times \mathbb{R}^2$. A recent result, due to H.-D. Cao, Ribeiro and Zhou [9, Theorem 1], states that a complete four-dimensional GSRS satisfying

$$|\mathrm{W}^+|^2 - \sqrt{6}|\mathrm{W}^+|^3 \geq \frac{1}{2}\langle(\mathring{Ric} \odot \mathring{Ric})^+, \mathrm{W}^+\rangle \tag{1.4}$$

is either Einstein, conformally flat, or $\mathbb{S}^2 \times \mathbb{R}^2$. Here, $\odot$ denotes the Kulkarni-Nomizu product. One important new feature in the above result is that they only need a condition on the self-dual part of Weyl tensor, but no point-wise Ricci curvature bound is needed. Nonetheless, as mentioned in [9, Remark 2], it is interesting to replace the right hand side of (1.4) by a sharp expression depending on the norm of $\mathring{Ric}$ instead of a Kulkarni-Nomizu product.

In this article, we first prove a classification result, in the spirit of [9, Theorem 1], using the norm of $\mathring{Ric}$. More precisely, we have established the following result.

Theorem 1. *Let $(M^4,\, g,\, f)$ be a complete four-dimensional gradient shrinking Ricci soliton (1.1) satisfying the pinching condition*

$$|\mathrm{W}^+|^2 - \sqrt{6}|\mathrm{W}^+|^3 \geq \frac{\sqrt{6}}{6}|\mathring{Ric}|^2|\mathrm{W}^+|. \tag{1.5}$$

Then $(M^4,\, g,\, f)$ *is either*

(i) *isometric to the standard round sphere* $\mathbb{S}^4$*, the complex projective space* $\mathbb{CP}^2$*, or*
(ii) *a finite quotient of the Gaussian shrinking soliton* $\mathbb{R}^4$*, the round cylinder* $\mathbb{S}^3\times\mathbb{R}$*, or*
(iii) *a compact Einstein manifold with* $\nabla \mathrm{W}^+\equiv 0$ *and* W^+ *has precisely two distinct eigenvalues (including* $\mathbb{S}^2\times\mathbb{S}^2$ *with the product metric), or*
(iv) *a finite quotient of the Kähler-Ricci soliton* $\mathbb{S}^2\times\mathbb{R}^2$.

A key ingredient in the proof of Theorem 1 is the novel sharp estimate

$$\langle (\mathring{Ric}\odot\mathring{Ric})^+, \mathrm{W}^+\rangle \leq \frac{\sqrt{6}}{3}|\mathring{Ric}|^2|\mathrm{W}^+|. \tag{1.6}$$

Interestingly, its proof is based on the notion of the curvature of the second kind and its algebraic consequences outlined by X. Cao, Gursky and Tran [13] in the resolution of a conjecture of Nishikawa.

Remark 1. *We point out that the equality in Theorem 1 holds for* $\mathbb{S}^2\times\mathbb{R}^2$:

$$|\mathrm{W}^+|^2-\sqrt{6}|\mathrm{W}^+|^3=\frac{\sqrt{6}}{6}|\mathring{Ric}|^2|\mathrm{W}^+|=\frac{1}{48}.$$

Thereby, Theorem 1 can be seen as a gap theorem for $\mathbb{S}^2\times\mathbb{R}^2$.

Remark 2. *A relevant observation is that, by using essentially the same arguments as in the proof of Theorem 1, if we replace the assumption (1.5) by the condition*

$$|\mathrm{W}^+|-\sqrt{6}|\mathrm{W}^+|^2\geq\frac{\sqrt{6}}{6}|\mathring{Ric}|^2. \tag{1.7}$$

we derive the same classification of Theorem 1 with the exception of the round cylinder $\mathbb{S}^3\times\mathbb{R}$.

In our next result, we establish a general rigidity theorem for four-dimensional Kähler-Ricci solitons. It is known from [21] that a four-dimensional Kähler metric with a natural of orientation must satisfy

$$|\mathrm{W}^+|^2=\frac{R^2}{24}.$$

We will show that on a four-dimensional GSRS, if the quotient $\frac{|\mathrm{W}^+|^2}{R^2}$ is close enough to $1/24$ from below, then the manifold must be either locally Kähler or one of the standard models. To be precise, we have the following result.

Theorem 2. *Let* $(M^4,\, g,\, f)$ *be a four-dimensional complete (nonflat) gradient shrinking Ricci soliton (1.1) satisfying the pinching condition*

$$\frac{1}{24}\geq\frac{|\mathrm{W}^+|^2}{R^2}\geq\frac{\sqrt{6}}{3}\frac{|\mathring{Ric}|^2}{R^3}|\mathrm{W}^+|. \tag{1.8}$$

Then $(M^4,\, g,\, f)$ *is either*

(i) *isometric to* $\mathbb{S}^4$*, or* $\mathbb{CP}^2$*, or*
(ii) *a finite quotient of the round cylinder* $\mathbb{S}^3\times\mathbb{R}$*, or*
(iii) *a locally Kähler-Ricci soliton.*

Remark 3. *Locally Kähler in Theorem 2 means Kähler after possibly pulling back to a double cover of* M^4.

Remark 4. *We highlight that the condition in Theorem 2 is sharp as both inequalities in (1.8) become equalities for the Kähler-Ricci soliton* $\mathbb{S}^2 \times \mathbb{R}^2$.

In the compact case, the reverse of (1.8) leads to the following characterization.

Corollary 1. *Let* $(M^4,\, g,\, f)$ *be a four-dimensional compact gradient shrinking Ricci soliton (1.1) satisfying*

$$\frac{1}{24} \leq \frac{|\mathrm{W}^+|^2}{R^2} \leq \frac{\sqrt{6}}{3}\frac{|\mathring{Ric}|^2}{R^3}|\mathrm{W}^+|. \tag{1.9}$$

Then $(M^4,\, g,\, f)$ *is a locally Kähler-Ricci soliton.*

Remark 5. *In the compact case, it is known (see* [5, 41]*) that a nontrivial Kähler-Ricci soliton is Fano (i.e., the first Chern class is positive) and the Futaki-invariant is nonzero. Moreover, it follows from the works of Tian and Zhu* [39, 40] *that the soliton vector field is unique up to holomorphic automorphisms of the underlying complex manifold.*

This article is organized as follows. In Section 2, we review some basic facts and useful lemmas on four-dimensional GSRS. Section 3 contains relevant contributions with several novel estimates. Finally, Section 4 collects the proofs of our main results.

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2. Background

In this section, we review some basic facts and present key results that will be used for our main theorems.

Throughout this paper, for an n-dimensional Riemannian manifold $(M^n,\, g)$, we denote Ric, $\mathring{Ric}$, R and K to be the Ricci, traceless Ricci, scalar and sectional curvatures, respectively. Given a point $p \in M$, let $\{e_1,\, \ldots\, , e_n\}$ be a local normal orthogonal coordinate of T_pM, then $\{e^1,\, \ldots\, , e^n\}$ denotes the associated dual basis.

For a finite dimensional vector space V, $S^2(V)$ and $\Lambda^2(V)$ are the space of symmetric and anti-symmetric linear maps on V (called symmetric 2-tensors and 2-forms, respectively). In particular, $S^2_0(V)$ contains only traceless symmetric maps. When $V = T_pM$, we normally subdue the vector space for convenience. The convention for the inner products is as follows. For $u, v \in S^2(V)$ and $\alpha, \beta \in \Lambda^2(V)$:

$$\langle u, v\rangle = \mathrm{Tr}\,(u^T v) \quad \text{and} \quad \langle \alpha, \beta\rangle = \frac{1}{2}\mathrm{Tr}\,(\alpha^T \beta).$$

Let $\mathcal{R}(V)$ be the space of algebraic curvatures, that is, $(4, 0)$-tensors satisfying certain symmetric properties and the first Bianchi identity. More precisely, for $T \in \mathcal{R}(V)$, we have

$$T(e_i, e_j, e_k, e_l) = -T(e_j, e_i, e_k, e_l) = -T(e_i, e_j, e_l, e_k) = T(e_k, e_l, e_i, e_j)$$

and

$$0 = T(e_i, e_j, e_k, e_l) + T(e_i, e_k, e_l, e_j) + T(e_i, e_l, e_j, e_k).$$

Thus, any element $T \in \mathcal{R}(V)$ can be considered as an element in either $\mathrm{End}(\Lambda^2(V))$ or $\mathrm{End}(S^2(V))$. They are so called the *curvature of the first* or *second kind*, respectively (see [13] and references therein). Precisely, for $\omega \in \Lambda^2(V)$ and $A \in S^2(V)$, we define

$$T(\omega)(e_i, e_j) := \sum_{k<l} T(e_i, e_j, e_k, e_l)\omega(e_k, e_l)$$

and

$$(\hat{T}A)(e_i, e_k) := \sum_{j,l} T(e_i, e_j, e_l, e_k)A(e_j, e_l).$$

The inner product in $\mathrm{End}(\Lambda^2(V))$ is given by

$$\langle T_1, T_2 \rangle = \sum_{i<j, k<l} T_1(e_{ij}, e_{kl})T_2(e_{ij}, e_{kl}) = \frac{1}{4}\sum_{i,j,k,l}(T_1)_{ijkl}(T_2)_{ijkl}.$$

Recall that the Kulkarni-Nomizu product $\odot$: $S^2(V) \times S^2(V) \mapsto \mathcal{R}(V)$ is defined by

$$(A \odot B)_{ijkl} = A_{ik}B_{jl} + A_{jl}B_{ik} - A_{il}B_{jk} - A_{jk}B_{il}. \tag{2.1}$$

In terms of the Kulkarni-Nomizu product, we have the following intrinsic relation, its proof follows from a straightforward calculation.

Lemma 1. *Let $A, B \in S^2(V)$, then we have:*

$$(\hat{T}A, B) = -\langle T, A \odot B \rangle .$$

We further recall the following curvature decomposition

$$\begin{aligned} R_{ijkl} \quad &= \quad \mathrm{W}_{ijkl} + \frac{1}{n-2}\big(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\big) \\ &\quad - \frac{R}{(n-1)(n-2)}\big(g_{jl}g_{ik} - g_{il}g_{jk}\big), \end{aligned} \tag{2.2}$$

where R_{ijkl} stands for the Riemann curvature tensor and W_{ijkl} is the Weyl curvature tensor.

2.1. **Four-dimensional Manifolds.** In this subsection, we focus on dimension $n = 4$. The bundle of 2-forms Λ^2 can be invariantly decomposed into a direct sum,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-. \tag{2.3}$$

Moreover, as observed in [1] (see also [21, Lemma 2]), these components have special structures:

Lemma 2. *Let (M^4, g) be an oriented four-dimensional Riemannian manifold and p be a point in M^4. Then the following assertions hold:*

1. *$\Lambda^\pm$ are mutually commuting, each isomorphic to $\mathfrak{so}(3)$.*
2. *Elements of lengths $\sqrt{2}$ in $\Lambda^\pm$ coincides with the almost complex structure compatible with the metric.*

(3) *Each oriented orthogonal basis* $\{\omega_1, \omega_2, \omega_3\}$ *of* $\Lambda^{\pm}$ *with* $|\omega_1| = |\omega_2| = |\omega_3| = \sqrt{2}$ *forms a quaternionic structure in* T_pM. *That is,*

$$\omega_1^2 = \omega_2^2 = \omega_3^2 = -I \text{ and } \omega_1\omega_2 = \omega_3 = -\omega_2\omega_1.$$

(4) *Given* $\omega \in \Lambda^+$, $\alpha \in \Lambda^-$, $\omega\alpha = \alpha\omega$ *is an orientation preserving involution of* T_pM. *Its* (± 1)*-eigenspaces form an orthogonal decomposition of* T_pM *into a direct sum of two planes. In particular, we have* $\omega\alpha \in S_0^2$.

The decomposition (2.3) is in particular conformally invariant and induces a decomposition for the Weyl curvature:

$$\mathrm{W} = \mathrm{W}^+ \oplus \mathrm{W}^-, \tag{2.4}$$

where $\mathrm{W}^{\pm} : \Lambda^{\pm} \longrightarrow \Lambda^{\pm}$ are called the *self-dual* part and *anti-self-dual* part of the Weyl tensor, respectively. Furthemore, at a point $p \in M^4$, one can diagonalize $\mathrm{W}^{\pm}$ with eigenforms $\{\omega_i\}_{i=1}^3 \in \Lambda^+$ and $\{\alpha_i\}_{i=1}^3 \in \Lambda^-$ such that λ_i and μ_i, $1 \leq i \leq 3$, are the respective eigenvalues. In particular, one observes that

$$\begin{cases} \lambda_1 \geq \lambda_2 \geq \lambda_3 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \mu_1 \geq \mu_2 \geq \mu_3 \text{ and } u_1 + \mu_2 + \mu_3 = 0. \end{cases} \tag{2.5}$$

So, it follows that

$$2\mathrm{W} = \sum_{i=1}^{3} \lambda_i \omega_i \otimes \omega_i + \mu_i \alpha_i \otimes \alpha_i. \tag{2.6}$$

In a local neighborhood, we construct a local frame so that (2.6) holds. In particular, $\Lambda^{\pm}$ is invariant under parallel displacement. By using Derdzinski's argument [21, p. 414], we arrive at, for (i, j, k) an orientation-preserving permutation of $(1, 2, 3)$,

$$\begin{aligned} \nabla\omega_i &= a_j \otimes \omega_k - a_k \otimes \omega_j, \\ 2\nabla\mathrm{W}^+ &= \sum_{i=1}^{3} \left(d\lambda_i \otimes \omega_i + (\lambda_i - \lambda_k) a_j \otimes \omega_k - (\lambda_i - \lambda_j) a_k \otimes \omega_j\right) \otimes \omega_i. \end{aligned} \tag{2.7}$$

The following Kato inequality will be also useful.

Lemma 3. *Let* $(M^4,\, g)$ *be an oriented four-dimensional Riemannian manifold. Then we have:*

$$|\nabla|\mathrm{W}^+|| \leq |\nabla\mathrm{W}^+|.$$

Equality holds if and only if there is an one-form ν *such that* $\nabla\mathrm{W}^+ = \nu \otimes \mathrm{W}^+$.

We will also need to use the following algebraic inequality

$$\mathrm{det}\mathrm{W}^+ \leq \frac{\sqrt{6}}{18}|\mathrm{W}^+|^3, \tag{2.8}$$

moreover, equality holds if and only if $\lambda_3 = \lambda_2 = -\frac{1}{2}\lambda_1$.

2.2. **Two-forms and Two-tensors.** In dimension four, there is an interesting connection between 2-tensors and 2-forms. A choice of orthonormal bases of $\Lambda^{\pm}$ is given by

$$
\begin{aligned}
\mathbb{B}^{+} &= \frac{1}{\sqrt{2}}(e_{12}+e_{34}, e_{13}-e_{24}, e_{14}+e_{23}) = \frac{1}{\sqrt{2}}(\omega_1,\omega_2,\omega_3), \\
\mathbb{B}^{-} &= \frac{1}{\sqrt{2}}(e_{12}-e_{34}, e_{13}+e_{24}, e_{14}-e_{23}) = \frac{1}{\sqrt{2}}(\alpha_1,\alpha_2,\alpha_3).
\end{aligned} \tag{2.9}
$$

By Lemma 2, their cross products form a basis for $S_0^2(V)$, i.e.,

$$
\mathbb{B}_2 = \{\omega_i\alpha_j\} = \{h_1, \ldots, h_9\}. \tag{2.10}
$$

More specifically, one has

$$
h_1 = \omega_1\alpha_1 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \qquad h_2 = \omega_1\alpha_2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix};
$$

$$
h_3 = \omega_1\alpha_3 = \begin{pmatrix} & & -1 & \\ & & & -1 \\ -1 & & & \\ & -1 & & \end{pmatrix}; \qquad h_4 = \omega_2\alpha_1 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & -1 & & \\ -1 & & & \end{pmatrix};
$$

$$
h_5 = \omega_2\alpha_2 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}; \qquad h_6 = \omega_2\alpha_3 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix};
$$

$$
h_7 = \omega_3\alpha_1 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{pmatrix}; \qquad h_8 = \omega_3\alpha_2 = \begin{pmatrix} & -1 & & \\ -1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix};
$$

$$
h_9 = \omega_3\alpha_3 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.
$$

At the same time, according to Berger [3] (see also [38]), at a point, there is a normal form for the Weyl tensor: a basis $\{e_i\}_{i=1}^4$ such that the bases (2.9) consist of eigenforms of $\mathrm{W}^{\pm}$. Whence, X. Cao, Gursky and Tran in [13, Proposition 4.3] are able to compute $\widehat{\mathrm{W}}$ with respect to (2.10),

$$
\widehat{\mathrm{W}} = \begin{pmatrix} \mathcal{D}_1 & 0 & 0 \\ 0 & \mathcal{D}_2 & 0 \\ 0 & 0 & \mathcal{D}_3 \end{pmatrix}, \tag{2.11}
$$

here the $\mathcal{D}_i$'s are diagonal matrices given by

$$
\mathcal{D}_i = \begin{pmatrix} -4(\lambda_i+\mu_1) & & \\ & -4(\lambda_i+\mu_2) & \\ & & -4(\lambda_i+\mu_3) \end{pmatrix}.
$$

Before proceeding, we list the curvature operator of the *second kind* $\widehat{R}$ for some basic examples:

1. Let $(\mathbb{S}^4(\sqrt{6}),\, g_0)$ be the 4-dimensional round sphere of radius $\sqrt{6}$ and scalar curvature $R = 2$. In this case, one has

$$\widehat{R} = \sqrt{6}\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix.

2. Let $(\mathbb{CP}^2,\, g_{_{FS}})$ be the complex projective space of complex dimension 2 with the Fubini-Study metric and scalar curvature $R = 8$. Then, up to the ordering of eigenvalues, one has

$$\widehat{R} = 16\begin{pmatrix} -\frac{1}{2}\mathbb{I} & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & \mathbb{I} \end{pmatrix}.$$

3. Let $\left(\mathbb{S}^2(\sqrt{2})\times\mathbb{R}^2,\, g,\, f\right)$ be the GSRS with the product metric g and scalar curvature $R = 1$. In this case, up to the ordering of eigenvalues, one has

$$\widehat{R} = \sqrt{2}\begin{pmatrix} -\frac{1}{2} & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{pmatrix}.$$

2.3. **Four-Dimensional Shrinking Ricci Solitons.** In this subsection, we are going to collect some well-known identities for four-dimensional GSRS satisfying (1.1). First, let us recall the following lemma.

Lemma 4 ([26]). *Let $\left(M^4,\, g,\, f\right)$ be a four-dimensional gradient shrinking Ricci soliton satisfying (1.1). Then we have:*

(1) $R + \Delta f = 2$;
(2) $\frac{1}{2}\nabla R = Ric(\nabla f)$;
(3) $\Delta_f R = R - 2|Ric|^2$;
(4) $R + |\nabla f|^2 = f$ *(after normalizing);*
(5) $\Delta_f R_{ij} = R_{ij} - 2R_{ikjl}R_{kl}$.

Here, $\Delta_f \cdot := \Delta \cdot - \nabla_{\nabla f}\cdot$ denotes the drifted Laplacian.

Chen showed in [17] that any complete ancient solution to the Ricci flow has nonnegative scalar curvature, it follows that $R \geq 0$ for any complete GSRS. Moreover, R is strictly positive unless $(M^4,\, g,\, f)$ is the Gaussian shrinking soliton (see [37]).

Regarding the potential function f, H.-D. Cao and Zhou [10] proved that

$$\frac{1}{4}\Big(r(x) - c\Big)^2 \leq f(x) \leq \frac{1}{4}\Big(r(x) + c\Big)^2, \tag{2.12}$$

for all $r(x) \geq r_0$, where $r = r(x)$ is the distance function to a fixed point in M. Additionally, they showed that every complete noncompact GSRS has at most

Euclidean volume growth (see [10, Theorem 1.2]). These asymptotic estimates are optimal in the sense that they are achieved by the Gaussian shrinking soliton.

Next, we recall a Weitzenböck type formula established by the first and third authors [11] that will play a crucial role in our proofs.

Proposition 1 ([11]). *Let $(M^4,\, g,\, f)$ be a four-dimensional gradient shrinking Ricci soliton satisfying (1.1). Then we have:*

$$\Delta_f|\mathrm{W}^\pm|^2 = 2|\nabla \mathrm{W}^\pm|^2 + 2|\mathrm{W}^\pm|^2 - 36\,\det \mathrm{W}^\pm - \langle(\mathring{Ric}\odot\mathring{Ric})^\pm, \mathrm{W}^\pm\rangle,$$

where $\odot$ stands for the Kulkarni-Nomizu product.

3. Key Estimates

In this section, we will establish some key lemmas that will be used in our proofs.

Lemma 5. *Let $(M^4,\, g)$ be a four-dimensional Riemannian manifold. If $\nabla \mathrm{W}^+ = \nu\otimes \mathrm{W}^+$ for some one-form ν and $\lambda_2=\lambda_3$ at each point, then ω_1 is a locally Kähler form.*

Proof. Since $\lambda_2=\lambda_3$ at each point, from (2.5), it is clear that

$$\lambda_1 = -2\lambda_2 = -2\lambda_3$$

is a non-negative function. Its eigenspace ω_1 is therefore locally defined (for example, see [21]). Hence, it follows from (2.7) that

$$\begin{aligned} 2\nabla \mathrm{W}^+ &= (d\lambda_1\otimes\omega_1 + \frac{3}{2}\lambda_1 a_2\otimes\omega_3 - \frac{3}{2}\lambda_1 a_3\otimes\omega_2)\otimes\omega_1 \\ &\quad +(d\lambda_2\otimes\omega_2 - \frac{3}{2}\lambda_1 a_3\otimes\omega_1)\otimes\omega_2 \\ &\quad +(d\lambda_3\otimes\omega_3 + \frac{3}{2}\lambda_1 a_2\otimes\omega_1)\otimes\omega_3. \end{aligned}$$

At the same time, since $\nabla \mathrm{W}^+ = \nu\otimes \mathrm{W}^+$, we have

$$2\nabla \mathrm{W}^+ = \sum_{i=1}^3 \lambda_i \nu\otimes\omega_i\otimes\omega_i.$$

The equations above then imply that

$$\begin{aligned} \lambda_1\nu &= d\lambda_1, \\ a_2 &= a_3 \equiv 0. \end{aligned}$$

Consequently, $\nabla\omega_1\equiv 0$ and, since $\omega_1\in\Lambda^+$, it is a locally Kähler form. □

Next, we obtain a sharp estimate for $\langle(\mathring{Ric}\odot\mathring{Ric})^+, \mathrm{W}^+\rangle$.

Lemma 6. *Let $(M^4,\, g)$ be an oriented four-dimensional Riemannian manifold. Then we have:*

$$\langle(\mathring{Ric}\odot\mathring{Ric})^+, \mathrm{W}^+\rangle \le \frac{\sqrt{6}}{3}|\mathring{Ric}|^2|\mathrm{W}^+|. \tag{3.1}$$

Moreover, equality holds if and only if W^+ has eigenvalues

$$0\le\lambda_1 = -2\lambda_2 = -2\lambda_3$$

and

$$\mathring{Ric} = a_1h_1 + a_2h_2 + a_3h_3.$$

Proof. A priori, using the orthogonal basis (2.10), one has $\mathring{Ric} = \sum_{i=1}^{9} a_i h_i$. It follows from (2.11) that

$$\begin{aligned}\widehat{\mathrm{W}^+}(\mathring{Ric}, \mathring{Ric}) &= a_i^2 \widehat{\mathrm{W}^+}(h_i, h_i)\\ &= -4\sum_{i=1}^{3}\lambda_i \sum_{j=3i-2}^{3i} a_j^2.\end{aligned}$$

Denote $A_i^2 = \sum_{j=3i-2}^{3i} a_j^2$ and hence, since $|h_i| = 2$, one obtains that

$$|\mathring{Ric}|^2 = 4\sum_i A_i^2.$$

Without loss of generality, we may consider $|\mathring{Ric}|^2 = \rho \geq 0$, $|\mathrm{W}^+| = \sigma \geq 0$ and

$$\begin{aligned}\vec{\lambda} &= (\lambda_1, \lambda_2, \lambda_3),\\ \vec{A} &= (A_1^2, A_2^2, A_3^2).\end{aligned}$$

Therefore, $\vec{\lambda}$ is on the circular intersection of the sphere $x^2+y^2+z^2 = \sigma^2$ and region $x \geq y \geq z$ in the plane $x+y+z = 0$. Besides, $\vec{A}$ is on the triangular intersection of the plane $x+y+z = \rho/4$ and the quadrant $x, y, z \geq 0$. Consequently,

$$-\widehat{\mathrm{W}^+}(\mathring{Ric}, \mathring{Ric}) = 4\sum_{i=1}^{3}\lambda_i A_i^2 = 4\left\langle \vec{\lambda}, \vec{A}\right\rangle = 4\left\langle \vec{\lambda}, \vec{\vec{A}}\right\rangle. \tag{3.2}$$

Here, $\vec{\vec{A}}$ is the projection of $\vec{A}$ on the plane $x+y+z = 0$, that is,

$$\vec{\vec{A}} = \frac{1}{3}\left(2A_1^2 - A_2^2 - A_3^2, -A_1^2 + 2A_2^2 - A_3^2, -A_1^2 - A_2^2 + 2A_3^2\right). \tag{3.3}$$

Observe that if $\vec{\vec{A}}$ is $(0,0,0)$, then it suffices to use (3.2) and (3.3) in order to infer that the asserted inequality is trivially satisfied. Therefore, from the description of $\vec{A}$, one can deduce that $\vec{\vec{A}}$ is within an equilateral triangle centered at the origin and vertices $(\rho/6, -\rho/12, -\rho/12)$, $(-\rho/12, \rho/6, -\rho/12)$ and $(-\rho/12, -\rho/12, \rho/6)$. In particular, the maximum value of $\langle \vec{\lambda}, \vec{\vec{A}}\rangle$ is attained if and only if $\vec{\vec{A}}$ coincides with a vertex of the triangle and $\vec{\lambda}$ is parallel to it. Thereby, $\vec{A}$ coincides with a vertex and $\vec{\lambda}$ is a positive multiple of the projection of $\vec{A}$ on the plane $x+y+z = 0$. By a direct computation at three vertices, one sees that the maximum is achieved at

$$\vec{A} = (\rho/4, 0, 0).$$

Then, $\vec{\lambda}$ is a positive multiple of the projection of $\vec{A}$ on the plane $x+y+z = 0$ and hence, it is not difficult to check that $\vec{\lambda}$ must be a multiple of $(2, -1, -1)$. Of which, we obtain

$$-\widehat{\mathrm{W}^+}(\mathring{Ric}, \mathring{Rc}) \leq \frac{2}{\sqrt{6}}|\mathrm{W}^+||\mathring{Ric}|^2. \tag{3.4}$$

Furthermore, equality holds if and only if W^+ has eigenvalues

$$0 \leq \lambda_1 = -2\lambda_2 = -2\lambda_3$$

and

$$\mathring{Ric} = a_1h_1 + a_2h_2 + a_3h_3.$$

Finally, it suffices to use Lemma 1 and (3.4) to calculate that

$$\left\langle \mathrm{W}^{+}, \mathring{Ric} \odot \mathring{Ric} \right\rangle = -\widehat{\mathrm{W}^{+}}(\mathring{Ric}, \mathring{Rc}) \le \frac{2}{\sqrt{6}}|\mathrm{W}^{+}||\mathring{Ric}|^{2},$$

which gives the desired result. □

To conclude this section, we shall use Proposition 1 to establish a lemma that will be employed in the proof of Theorem 2.

Lemma 7. *Let $(M^4,\, g,\, f)$ be a four-dimensional gradient shrinking Ricci soliton. Then, for $\Psi = f - 2\ln R$, we have*

$$\begin{aligned}\Delta_{\Psi}\left(\frac{|W^{+}|^{2}}{R^{2}}\right) &= 4\frac{|Ric|^{2}}{R^{3}}|W^{+}|^{2} + 2\frac{|\nabla R|^{2}}{R^{4}}|W^{+}|^{2} + \frac{2}{R^{2}}|\nabla W^{+}|^{2}\\ &\quad -36\frac{\det W^{+}}{R^{2}} - \frac{1}{R^{2}}\langle(\mathring{Ric}\odot\mathring{Ric})^{+}, W^{+}\rangle\\ &\quad -\frac{2}{R^{3}}\langle\nabla|W^{+}|^{2}, \nabla R\rangle.\end{aligned} \tag{3.5}$$

Proof. First, one observes that

$$\Delta\left(\frac{|W^{+}|^{2}}{R^{2}}\right) = R^{-2}\Delta|W^{+}|^{2} + |W^{+}|^{2}\Delta\left(R^{-2}\right) + 2\langle\nabla|W^{+}|^{2}, \nabla\left(R^{-2}\right)\rangle. \tag{3.6}$$

Moreover, we have $\nabla(R^{-2}) = -2R^{-3}\nabla R$ so that

$$\Delta(R^{-2}) = -2R^{-3}\Delta R + 6R^{-4}|\nabla R|^{2}.$$

This substituted into (3.6) gives

$$\begin{aligned}\Delta\left(\frac{|W^{+}|^{2}}{R^{2}}\right) &= |W^{+}|^{2}\left(-2\frac{\Delta R}{R^{3}} + 6\frac{|\nabla R|^{2}}{R^{4}}\right) + \frac{1}{R^{2}}\Delta|W^{+}|^{2}\\ &\quad -\frac{4}{R^{3}}\langle\nabla|W^{+}|^{2}, \nabla R\rangle.\end{aligned}$$

Now, we may use Proposition 1 and Lemma 4 to infer

$$\begin{aligned}\Delta\left(\frac{|W^{+}|^{2}}{R^{2}}\right) &= |W^{+}|^{2}\left[-\frac{2}{R^{3}}\left(\langle\nabla R, \nabla f\rangle + R - 2|Ric|^{2}\right) + 6\frac{|\nabla R|^{2}}{R^{4}}\right]\\ &\quad +\frac{1}{R^{2}}\Big[\langle\nabla f, \nabla|W^{+}|^{2}\rangle + 2|\nabla W^{+}|^{2} + 2|W^{+}|^{2}\\ &\quad -36\det W^{+} - \langle(\mathring{Ric}\odot\mathring{Ric})^{+}, W^{+}\rangle\Big]\\ &\quad -\frac{4}{R^{3}}\langle\nabla|W^{+}|^{2}, \nabla R\rangle.\end{aligned}$$

Consequently,

$$\begin{aligned}\Delta\left(\frac{|W^{+}|^{2}}{R^{2}}\right) &= -2\frac{|W^{+}|^{2}}{R^{3}}\langle\nabla R, \nabla f\rangle + 4\frac{|Ric|^{2}}{R^{3}}|W^{+}|^{2} + 6\frac{|\nabla R|^{2}}{R^{4}}|W^{+}|^{2}\\ &\quad +\frac{1}{R^{2}}\langle\nabla f, \nabla|W^{+}|^{2}\rangle + 2\frac{|\nabla W^{+}|^{2}}{R^{2}} - 36\,\frac{\det W^{+}}{R^{2}}\\ &\quad -\frac{1}{R^{2}}\langle(\mathring{Ric}\odot\mathring{Ric})^{+}, W^{+}\rangle - \frac{4}{R^{3}}\langle\nabla|W^{+}|^{2}, \nabla R\rangle.\end{aligned} \tag{3.7}$$

On the other hand, observe that

$$\left\langle \nabla\Big(\frac{|W^+|^2}{R^2}\Big), \nabla f \right\rangle = \frac{1}{R^2}\langle \nabla |W^+|^2, \nabla f\rangle - 2\frac{|W^+|^2}{R^3}\langle \nabla R, \nabla f\rangle.$$

Therefore, (3.7) becomes

$$\begin{aligned}\Delta_f\left(\frac{|W^+|^2}{R^2}\right) &= 4\frac{|Ric|^2}{R^3}|W^+|^2 + 6\frac{|\nabla R|^2}{R^4}|W^+|^2 + 2\frac{|\nabla W^+|^2}{R^2} \\ &\quad -36\frac{\det W^+}{R^2} - \frac{1}{R^2}\langle(\mathring{Ric}\odot\mathring{Ric})^+, W^+\rangle \\ &\quad -\frac{4}{R^3}\langle\nabla|W^+|^2, \nabla R\rangle.\end{aligned} \tag{3.8}$$

Now, choosing $\varphi = 2\ln R$ we have

$$\left\langle \nabla\varphi, \nabla\left(\frac{|W^+|^2}{R^2}\right)\right\rangle = \frac{2}{R^3}\langle\nabla R, \nabla|W^+|^2\rangle - \frac{4}{R^4}|W^+|^2|\nabla R|^2.$$

Plugging this into (3.8) yields

$$\begin{aligned}\Delta_\Psi\left(\frac{|W^+|^2}{R^2}\right) &= 4\frac{|Ric|^2}{R^3}|W^+|^2 + 2\frac{|\nabla R|^2}{R^4}|W^+|^2 + \frac{2}{R^2}|\nabla W^+|^2 \\ &\quad -36\frac{\det W^+}{R^2} - \frac{1}{R^2}\langle(\mathring{Ric}\odot\mathring{Ric})^+, W^+\rangle \\ &\quad -\frac{2}{R^3}\langle\nabla|W^+|^2, \nabla R\rangle,\end{aligned} \tag{3.9}$$

where $\Psi = f - 2\ln R$. This finishes the proof of the lemma.

□

4. Proof of the Main Results

In this section, we will present the proofs of Theorem 1, Theorem 2 and Corollary 1. In the first part, we adapt the arguments from H.-D. Cao-Ribeiro-Zhou [9].

4.1. **Proof of Theorem 1.**

Proof. By Proposition 1, we have

$$\begin{aligned}2|\mathrm{W}^+|\Delta_f|\mathrm{W}^+| &= 2|\nabla \mathrm{W}^+|^2 - 2|\nabla|\mathrm{W}^+||^2 + 2|\mathrm{W}^\pm|^2 - 36\det \mathrm{W}^+ \\ &\quad -\langle(\mathring{Ric}\odot\mathring{Ric})^+, \mathrm{W}^+\rangle,\end{aligned} \tag{4.1}$$

where we have used that

$$\Delta_f|\mathrm{W}^+|^2 = 2|\mathrm{W}^+|\Delta_f|\mathrm{W}^+| + 2|\nabla|\mathrm{W}^+||^2.$$

Using the Kato inequality (2.8) and Lemma 6, one sees that

$$\begin{aligned}|\mathrm{W}^+|\Delta_f|\mathrm{W}^+| &\geq |\mathrm{W}^+|^2 - \sqrt{6}|\mathrm{W}^+|^3 - \frac{1}{2}\langle(\mathring{Ric}\odot\mathring{Ric})^+, \mathrm{W}^+\rangle, \\ &\geq |\mathrm{W}^+|^2 - \sqrt{6}|\mathrm{W}^+|^3 - \frac{\sqrt{6}}{6}|\mathring{Ric}|^2|\mathrm{W}^+|.\end{aligned} \tag{4.2}$$

Hence, our assumption guarantees that $|\mathrm{W}^+|\Delta_f|\mathrm{W}^+|$ does not change sign.

In order to proceed, we need to show that $|\mathrm{W}^+|$ is L^2_f-integrable, i.e., $|\mathrm{W}^+| \in L^2(e^{-f}dV_g)$. To do so, we first observe that the assumption also implies that

$$\begin{aligned}\sqrt{6}|\mathrm{W}^+|^2 &\leq |\mathrm{W}^+| - \frac{\sqrt{6}}{6}|\mathring{Ric}|^2 \\ &\leq \frac{\sqrt{6}}{2}|\mathrm{W}^+|^2 + \frac{1}{2\sqrt{6}}.\end{aligned}$$

Integrating over M^4, we obtain

$$\int_M |\mathrm{W}^+|^2 e^{-f}dV_g \leq \frac{1}{6}\int_M e^{-f}dV_g.$$

Thus, since M^4 has finite weighted volume, i.e., $\int_M e^{-f}dV_g < \infty$ (see, [10, Corollary 1.1]), one concludes that $|W^+|$ is L^2_f-integrable.

Proceeding, we consider a cut-off function $\rho : M \to \mathbb{R}$ such that $\rho = 1$ on a geodesic ball $B_p(r)$ centered at a fixed point $p \in M$ of radius r, $\rho = 0$ outside of $B_p(2r)$ and $|\nabla\rho| \leq \frac{c}{r}$, where c is a constant. By our assumption and (4.2), we have

$$\begin{aligned}0 &\geq -\int_M \rho^2|\mathrm{W}^+|\Delta_f|\mathrm{W}^+|e^{-f}dV_g \\ &= \int_M \left\langle \nabla\left(\rho^2|\mathrm{W}^+|\right), \nabla|\mathrm{W}^+|\right\rangle e^{-f}dV_g \\ &= \int_M \left|\rho\nabla|\mathrm{W}^+| + |\mathrm{W}^+|\nabla\rho\right|^2 e^{-f}dV_g - \int_M |\mathrm{W}^+|^2|\nabla\rho|^2 e^{-f}dV_g.\end{aligned}$$

It follows that

$$\int_M \left|\nabla\left(\rho|W^+|\right)\right|^2 e^{-f}dV_g \leq \int_M |W^+|^2|\nabla\rho|^2 e^{-f}dV_g. \tag{4.3}$$

Hence, we deduce that

$$\begin{aligned}\int_{B(r)} \left|\nabla|W^+|\right|^2 e^{-f}dV_g &\leq \int_M \left|\nabla\left(\rho|W^+|\right)\right|^2 e^{-f}dV_g \\ &\leq \int_M |W^+|^2|\nabla\rho|^2 e^{-f}dV_g \\ &\leq \int_{M\setminus B(2r)} |\mathrm{W}^+|^2|\nabla\rho|^2 dV_g + \int_{B(2r)\setminus B(r)} |\mathrm{W}^+|^2|\nabla\rho|^2 dV_g \\ &\quad + \int_{B(r)} |\mathrm{W}^+|^2|\nabla\rho|^2 dV_g \\ &\leq \int_{B(2r)\setminus B(r)} |W^+|^2|\nabla\rho|^2 e^{-f}dV_g \\ &\leq \frac{c^2}{r^2}\int_M |W^+|^2 e^{-f}dV_g. \end{aligned} \tag{4.4}$$

Since $|W^+|$ is a L^2_f-integrable function on M^4, we conclude that the right hand side tends to zero as $r \to \infty$ and hence, $|W^+|$ must be constant.

Now, if $|W^+| = 0$, then, by Theorem 1.2 of [18], one deduces that the soliton is either Einstein, the Gaussian soliton $\mathbb{R}^4$, the round cylinder $\mathbb{S}^3 \times \mathbb{R}$, or their quotients. In the case that it is a non-flat Einstein manifold, one invokes Myer's theorem and the Hitchin classification to conclude that it is isometric to either the round sphere $\mathbb{S}^4$ or the complex projective space $\mathbb{CP}^2$ (see [4, Theorem 13.30]).

On the other hand, if $|W^+|$ is a nonzero constant, then it suffices to use (4.1), (2.8), Lemma 6 and the assumption, in order to infer that $\nabla W^+ \equiv 0$. Consequently, by Theorem 1.1 in [42], one concludes that (M^4, g, f) is either a finite quotient of $\mathbb{S}^2 \times \mathbb{R}^2$, or compact Einstein with $\nabla W^+ \equiv 0$ and W$^+$ has 2 distinct eigenvalues. This finishes the proof of Theorem 1.

□

4.2. **Proof of Theorem 2.**

Proof. Initially, one verifies that

$$\begin{aligned}\Delta_\Psi\left(\frac{|W^+|^2}{R^2}\right) &= 2\frac{|W^+|}{R}\Delta_\Psi\left(\frac{|W^+|}{R}\right) + 2\frac{|W^+|^2}{R^4}|\nabla R|^2 + \frac{2}{R^2}|\nabla|W^+||^2\\ &\quad -\frac{2}{R^3}\langle \nabla R, \nabla |W^+|^2\rangle.\end{aligned}$$

Now, substituting this data into Lemma 7, one obtains that

$$\begin{aligned}2\frac{|W^+|}{R}\Delta_\Psi\left(\frac{|W^+|}{R}\right) &= 4\frac{|Ric|^2}{R^3}|W^+|^2 + \frac{2}{R^2}\left(|\nabla W^+|^2 - |\nabla|W^+||^2\right)\\ &\quad -36\frac{\det W^+}{R^2} - \frac{1}{R^2}\langle(\mathring{Ric}\odot\mathring{Ric})^+, W^+\rangle.\end{aligned}$$

Thereby, by the Kato inequality, (2.8) and Lemma 6, we compute

$$\begin{aligned}2\frac{|W^+|}{R}\Delta_\Psi\left(\frac{|W^+|}{R}\right) &\geq 4\frac{|Ric|^2}{R^3}|W^+|^2 - \frac{2\sqrt{6}}{R^2}|W^+|^3\\ &\quad -\frac{1}{R^2}\langle(\mathring{Ric}\odot\mathring{Ric})^+, W^+\rangle\\ &\geq 4\frac{|\mathring{Ric}|^2}{R^3}|W^+|^2 + \frac{|W^+|^2}{R} - \frac{2\sqrt{6}}{R^2}|W^+|^3\\ &\quad -\frac{1}{R^2}\frac{\sqrt{6}}{3}|W^+||\mathring{Ric}|^2\\ &= \frac{1}{R}\left(1-2\sqrt{6}\frac{|W^+|}{R}\right)\left(|W^+|^2 - \frac{\sqrt{6}}{3}\frac{|\mathring{Ric}|^2}{R}|W^+|\right).\end{aligned} \tag{4.5}$$

Hence, our assumption guarantees that $\frac{|W^+|}{R}\Delta_\Psi\left(\frac{|W^+|}{R}\right)$ does not change sign.

Now, we need to show that $\frac{|W^+|}{R}$ is a L^2_Ψ-integrable function on M^4. Indeed, we observe that

$$\int_M \frac{|W^+|^2}{R^2}e^{-\Psi}dV_g = \int_M \frac{|W^+|^2}{R^2}e^{-f}e^{2\ln R}dV_g = \int_M |W^+|^2 e^{-f}dV_g. \tag{4.6}$$

At the same time, our assumption gives

$$\int_M |W^+|^2 e^{-f}dV_g \leq \frac{1}{24}\int_M R^2 e^{-f}dV_g \leq \frac{1}{6}\int_M |Ric|^2 e^{-f}dV_g. \tag{4.7}$$

By Theorem 1.1 in [30], we have

$$\int_M |Ric|^2 e^{-f}dV_g < \infty.$$

Thus, it follows from (4.6) and (4.7) that $\frac{|W^+|}{R}$ is a L^2_Ψ-integrable.

Next, we are going to apply a cut-off function argument similar as in the proof of Theorem 1 to conclude that $\frac{|W^+|}{R}$ is constant. Indeed, we set a cut-off function $\rho : M \to \mathbb{R}$ such that $\rho = 1$ on a geodesic ball $B_p(r)$ centered at a fixed point $p \in M$ of radius r, $\rho = 0$ outside of $B_p(2r)$ and $|\nabla\rho| \leq \frac{c}{r}$, where c is a constant. Hence, taking into account that $\frac{|W^+|}{R}$ is a Ψ-subharmonic function on M^4 and $R > 0$ unless $(M^4,\, g,\, f)$ is the Gaussian shrinking soliton, one easily verifies that

$$\begin{aligned} 0 \;&\geq\; -\int_M \rho^2 \frac{|W^+|}{R}\Delta_\Psi\left(\frac{|W^+|}{R}\right) e^{-\Psi} dV_g \\ &=\; \int_M \left|\rho\nabla\left(\frac{|W^+|}{R}\right) + \frac{|W^+|}{R}\nabla\rho\right|^2 e^{-\Psi} dV_g - \int_M \frac{|W^+|^2}{R^2}|\nabla\rho|^2 e^{-\Psi} dV_g, \end{aligned}$$

so that

$$\int_M \left|\nabla\left(\rho\frac{|W^+|}{R}\right)\right|^2 e^{-\Psi} dV_g \leq \int_M \frac{|W^+|^2}{R^2}|\nabla\rho|^2 e^{-\Psi} dV_g. \tag{4.8}$$

It follows that

$$\begin{aligned} \int_{B(r)} \left|\nabla\left(\frac{|\mathrm{W}^+|}{R}\right)\right|^2 e^{-\Psi} dV_g \;&\leq\; \int_M \left|\nabla\left(\rho\frac{|\mathrm{W}^+|}{R}\right)\right|^2 e^{-\Psi} dV_g \\ &\leq\; \int_{B(2r)\setminus B(r)} \frac{|\mathrm{W}^+|^2}{R^2}|\nabla\rho|^2 e^{-\Psi} dV_g \\ &\leq\; \frac{c^2}{r^2}\int_M \frac{|W^+|^2}{R^2} e^{-\Psi} dV_g. \end{aligned} \tag{4.9}$$

Since $\frac{|\mathrm{W}^+|}{R}$ is a L^2_Ψ-integrable function on M^4, we conclude that the right hand side tends to zero as $r \to \infty$ and consequently, $\frac{|W^+|}{R}$ must be constant on M^4, as asserted.

If $\frac{|\mathrm{W}^+|}{R} = 0$, then $\mathrm{W}^+ \equiv 0$. As in the proof of Theorem 1, the soliton must be isometric to either $\mathbb{S}^4$, $\mathbb{CP}^2$, or a finite quotient of the round cylinder $\mathbb{S}^3 \times \mathbb{R}$.

Otherwise, if $\frac{|W^+|}{R}$ is a nonzero constant, each inequality above becomes an equality. In particular, by Lemma 3, one obtains that $\nabla\mathrm{W}^+ = \nu \otimes \mathrm{W}^+$ for some one-form ν. Besides, it follows from Lemma 6 that W^+ has $\lambda_2 = \lambda_3$ at each point. Finally, it suffices to use Lemma 5 to conclude that $(M^4,\, g)$ is a locally Kähler-Ricci soliton.

□

4.3. **Proof of Corollary 1.**

Proof. To begin with, we follow the initial steps in the proof of Theorem 2 up till (4.5) in order to obtain

$$2\frac{|W^+|}{R}\Delta_\Psi\left(\frac{|\mathrm{W}^+|}{R}\right) \geq \frac{1}{R}\left(1 - 2\sqrt{6}\frac{|\mathrm{W}^+|}{R}\right)\left(|\mathrm{W}^+|^2 - \frac{\sqrt{6}}{3}\frac{|\mathring{Ric}|^2}{R}|\mathrm{W}^+|\right). \tag{4.10}$$

Thereby, our assumption guarantees that the right hand side of (4.10) is nonnegative and therefore, by using the Maximum Principle, we conclude that $\frac{|W^+|}{R}$ is a constant. In particular, each previous inequality obtained in the establishment of (4.10) becomes an equality. Now, observe that $W^+ = 0$ leads to a contradiction with the assumption (1.9). Consequently, $W^+ \neq 0$ and we may use again the equality case in Lemma 3, Lemma 6 and Lemma 5 in order to infer that (M^4, g) is a locally Kähler-Ricci soliton, which gives the desired result. □

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