DIFFERENCE ANALOGUE OF THE SECOND MAIN THEOREM FOR HOLOMORPHIC CURVES AND ARBITRARY FAMILIES OF HYPERSURFACES IN PROJECTIVE VARIETIES

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Abstract. Our goal in this paper is to establish some difference analogue of second main theorems for holomorphic curves into projective varieties intersecting arbitrary families of c-periodical hypersurfaces (fixed or moving) with truncated counting functions in various cases. Our results generalize and improve the previous results in this topic.

1. Introduction

The values distribution theory for meromorphic functions on the complex plane was initiated by R. Nevanlinna [14] in 1926. Later on, in 1933, H. Cartan [2] established the second main theorem (SMT) in this theory for linearly non-degenerate holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$ intersecting hyperplanes in general position. In 1983, by introducing the notion of Nochka's weight for the family of hyperplanes, E. Nochka [15] generalized the result of Cartan to the case of hyperplanes in subgeneral position. In some recent years, this theory has been developed to the case of hypersurfaces (fixed or moving) in general or subgeneral position by many mathematicians, such as M. Ru [22, 23], T. T. H. An-H. T. Phuong [1], G. Dethloff-T. V. Tan [7], Q. Yan-G. Yu [25], L. B. Xie-T.B. Cao [24], S. D. Quang-D. P. An [16], S. D. Quang [17, 18] and many others therein. Very recently, by introducing the notion of distributive constant of families of hypersurfaces, the present third author [19] further researched the case of holomorphic curves into projective varieties with arbitrary families of hypersurfaces and proved the following second main theorem.

Theorem A (see [19, Theorem 1.1]) *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective subvariety of dimension $n \geq 1$. Let $\{Q_1, \ldots, Q_q\}$ be a family of hypersurfaces of $\mathbb{P}^N(\mathbb{C})$ with the distributive constant Δ with respect to V, $\deg Q_i = d_i$ $(1 \leq i \leq q)$, and d be the least common multiple of $d_1, \ldots, d_q$. Let f be an algebraically non-degenerate holomorphic*

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map from $\mathbb{C}$ *into* V*. Then, for every* $\epsilon > 0$,

$$\| \ (q - \Delta(n+1) - \epsilon)\, T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[M_0]}(r, f^*Q_i) + o(T_f(r)),$$

where $M_0 = \left[d^{n^2+n} \deg(V)^{n+1} e^n \Delta^n (2n+4)^n (n+1)^n (q!)^n \epsilon^{-n}\right]$.

Here, the notation "$\|\ P$" means the assertion P holds for all $r \in [1, +\infty)$ outside a set of finite measure, the notation $[x]$ stands for the largest integer not exceeding the real number x.

On the other side, motivated by the development of the value distribution of complex difference polynomials and solutions of complex difference equations (refer to see [3]), the difference analogues of second main theorems (DSMT) were established. In 2006, R.G. Halburd and R.J. Korhonen [10] obtained the c-difference analogue of the second main theorem (c-DSMT) for meromorphic functions in the complex plane. Later, P. M. Wong-H.F. Law-P. P. W. Wong [26] and R. G. Halburd-R. J. Korhonen and Tohge [11] have independently obtained the c-DSMT for holomorphic curves into complex projective spaces intersecting hyperplanes in general position. In 2017, the present first author and J. Nie [4] proved an c-DSMT for the case of c-periodical slowly moving hypersurfaces located in subgeneral position of $\mathbb{P}^N(\mathbb{C})$ with the truncation level N. Recently, P. C. Hu and the present second author [12] have considered the case of holomorphic curves with hypersurfaces in subgeneral position with respect to a projective varieties with an explicit truncation level for the counting function. Our goal in this paper is to establish some difference analogues of the SMT for such curves with arbitrary families of hypersurfaces (fixed or c-periodical slowly moving), which are analogues to Theorem A, generalize and extend the above mentioned results for the case of hypersurfaces. In order to state the results, we recall the following.

Throughout this paper, we denote by $\mathcal{M}$ the set of all meromorphic functions on $\mathbb{C}$, by $\mathcal{P}_c$ the subfield of $\mathcal{M}$ consisting of all c-periodical meromorphic functions, by $\mathcal{P}_c^\lambda$ the subfield of $\mathcal{P}_c$ consisting of all functions in $\mathcal{P}_c$ with the hyperorders strictly less than λ, and by $\mathcal{K}_{f,c}^\lambda$ the subfield of $\mathcal{P}_c^\lambda$ consisting of all functions in $\mathcal{P}_c^\lambda$ which are small with respect to f. Obviously, we have the relationship $\mathcal{M} \supset \mathcal{P}_c \supset \mathcal{P}_c^\lambda \supset \mathcal{K}_{f,c}^\lambda$. Here a meromorphic function φ is said to be small (with respect to f) if $\|\ T_\varphi(r) = o(T_f(r))$.

Firstly, motivated by the recent works of the present third author on Nevanlinna theory for entire curves [19] and on Diophantine approximation [20], we prove the following c-DSMT for algebraically nondegenerate holomorphic curves from $\mathbb{C}$ into projective varieties with an explicit truncation level for counting functions.

Theorem 1.1. *Let* $V \subset \mathbb{P}^N(\mathbb{C})$ *be an irreducible algebraic subvariety of dimension* n $(n \leq N)$*. Let* $f : \mathbb{C} \to V$ *be a holomorphic map with hyperorder* $\varsigma(f) < 1$*. Let* $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ *be a set* q *hypersurfaces with the distributive constant* $\Delta_{\mathcal{Q},V}$ *with respect to* V *and* $d = lcm(\deg Q_1, \dots, \deg Q_q)$*. Assume that* f *is algebraically non-degenerate over* $\mathcal{P}_c^1$*. Then for any* $\epsilon > 0$,

$$\| \ (q - \Delta_{\mathcal{Q},V}(n+1) - \epsilon) T_f(r) \leq \sum_{j=1}^{q} \frac{1}{\deg Q_j} \tilde{N}^{[L_0-1,c]}_{Q_j(f)}(r) + o(T_f(r)),$$

where $L_0 = \left[d^{n^2+n}(\deg V)^{n+1}e^n\Delta^n_{\mathcal{Q},V}(2n+5)^n((n+1)\Delta_{\mathcal{Q},V}\epsilon^{-1}+1)^n\right]$.

Here, the distributive constant $\Delta_{\mathcal{Q},V}$ of the family $\mathcal{Q}$ with respect to V, the characteristic function $T_f(r)$ and the truncated counting function $\tilde{N}^{[M_0,c]}_{Q_j(f)}(r)$ are defined in Section 2.

Let d be a positive integer. We denote by $I(V)$ the ideal of homogeneous polynomials in $\mathbb{C}[x_0,\dots,x_n]$ defining V and by $\mathbb{C}[x_0,\dots,x_n]_d$ the $\mathbb{C}$-vector space of all homogeneous polynomials in $\mathbb{C}[x_0,\dots,x_n]$ of degree d (including the zero polynomial) and let $I(V)_d = I(V)\cap\mathbb{C}[x_0,\dots,x_n]_d$. Define

$$I_d(V) := \frac{\mathbb{C}[x_0,\dots,x_n]_d}{I(V)_d} \text{ and } H_V(d) := \dim I_d(V).$$

Then $H_V(d)$ is called the Hilbert function of V. For the second aim, we will establish an c-DSMT with better truncation level as follows.

Theorem 1.2. *Let V be a complex projective subvariety of $\mathbb{P}^N(\mathbb{C})$ of dimension n ($n \le N$). Let $\{Q_i\}_{i=1}^q$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in ℓ-subgeneral position with respect to V. Let d be the least common multiple of* $\deg Q_1,\dots,\deg Q_q$, *i.e.,* $d = lcm(\deg Q_1,\dots,\deg Q_q)$. *Let f be a holomorphic map of $\mathbb{C}$ into V which is $\mathcal{P}^1_c$-algebraically non-degenerate over $I_d(V)$ with hyperorder $\varsigma(f) < 1$. Then, we have*

$$\Big\| \left(q - \frac{(2\ell-n+1)H_V(d)}{n+1}\right)T_f(r) \le \sum_{i=1}^q \frac{1}{\deg Q_i}\tilde{N}^{[H_V(d)-1,c]}_{Q_i(f)}(r) + o(T_f(r)).$$

For the last purpose, we will consider the case of arbitrary families of c-periodical slowly moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$. We will prove the following.

Theorem 1.3. *Let f be a $\mathcal{P}^1_c$-algebraic nondegenerate holomorphic curve of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ with hyperorder $\varsigma(f) < 1$. Let $\mathcal{Q} = \{Q_i\}_{i=1}^q$ be a family of c-periodical slowly (with respect to f) moving hypersurfaces with the distributive constant $\Delta_{\mathcal{Q}}$. Then for any $\epsilon > 0$, we have*

$$\| \ (q - \Delta_{\mathcal{Q}}(n+1) - \epsilon)T_f(r) \le \sum_{i=1}^q \frac{1}{\deg Q_i}\tilde{N}^{[L_i,c]}_{Q_i(f)}(r) + o(T_f(r)),$$

where $L_i = \frac{\deg Q_i}{d}L_0 - \left[\frac{\deg Q_i}{d}\right]$ *and d, L_0 are positive numbers defined by:*

$$d := lcm(\deg Q_1,\dots,\deg Q_q), L_0 := \binom{L+n}{n}p_0^{\binom{L+n}{n}(\binom{L+n}{n}-1)\binom{q}{n}-2}$$

$$\text{with } \ L := \lceil n+1+2\Delta_{\mathcal{Q}}(n+1)^3\epsilon^{-1}\rceil d \text{ and } \ p_0 := \left[\frac{\binom{L+n}{n}(\binom{L+n}{n}-1)\binom{q}{n}-1}{\log(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}})}\right]^2.$$

Here, by $\lceil x\rceil$ stands for the smallest integer not less than x.

2. Preliminaries and lemmas

(A) *Some notation and definitions.* For a divisor ν on $\mathbb{C}$ and a positive integer M or $M = +\infty$, as usual we denote by $N^{[M]}(r,\nu)$ the counting function of ν with multiplicities truncated to level M.

For a meromorphic function φ on $\mathbb{C}$, denote by ν_φ its divisor of zeros and set

$$N_\varphi(r) = N(r, \nu_\varphi),\ N_\varphi^{[M]}(r) = N^{[M]}(r, \nu_\varphi)\ (r_0 < r < R_0).$$

For brevity, we will omit the character $^{[M]}$ if $M = +\infty$. In this paper we fix a nonzero constant $c \in \mathbb{C}$. Define

$$\varphi(z) \equiv \varphi := \bar{\varphi}^{[0]}, \varphi(z+c) \equiv \bar{\varphi} =: \bar{\varphi}^{[1]}, f(z+2c) \equiv \overline{\overline{\varphi}} =: \bar{\varphi}^{[2]}, \cdots, f(z+Mc) \equiv \bar{\varphi}^{[M]}$$

and set $\nu_\varphi^{[M,c]} = \min_{0\le k\le M} \nu_{\bar{\varphi}^{[k]}}, \tilde{\nu}_\varphi^{[M,c]} = \nu_\varphi - \nu_\varphi^{[M,c]}$. We define the following valence functions (cf. [6, inq. (2.5)] and see also [13, Definition 4.1] for a modification):

$$N_\varphi^{[M,c]}(r) = N(r, \nu_\varphi^{[M,c]}) \text{ and } \tilde{N}_\varphi^{[M,c]}(r) = N(r, \tilde{\nu}_\varphi^{[M,c]}).$$

Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic map with a reduced representation $\mathbf{f} = (f_0, \ldots, f_N)$. Set $\|\mathbf{f}\| = \big(|f_0|^2 + \cdots + |f_N|^2\big)^{1/2}$. The characteristic function of f is defined by

$$T_f(r) = \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} - \int_0^{2\pi} \log \|\mathbf{f}(e^{i\theta})\| \frac{d\theta}{2\pi}.$$

The hyper-order and order of f are defined respectively by

$$\varsigma(f) = \limsup_{r\to\infty} \frac{\log\log T_f(r)}{\log r}, \quad \sigma(f) = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

The Casorati determinant of f is defined by

$$C(f) = \det\left(\overline{f_0}^{[w]}, \ldots, \overline{f_N}^{[w]}\right)_{0\le w\le N}.$$

The definition of the function $C(f)$ depends on the choice of the reduced representation of f, but its divisor $\nu_{C(f)}$ does not depend on this choice.

For a positive integer d, we set

$$\mathcal{T}_d := \{(i_0, \ldots, i_N) \in \mathbb{N}_0^{N+1} : i_0 + \cdots + i_N = d\}.$$

In this paper, we call each hypersurface in $\mathbb{P}^N(\mathbb{C})$ a nonzero homogeneous polynomial Q in $\mathbb{C}[x_0, \ldots, x_N]$ and denote by Q^* its support, i.e.,

$$Q^* = \{(\omega_0 : \cdots : \omega_N) | Q(\omega_0, \ldots, \omega_N) = 0\}.$$

We call each moving hypersurface of degree d a homogeneous polynomial $P(z)$ of the form

$$P = \sum_{I\in\mathcal{T}_d} a_I(z) x^I,$$

where $x^I = x_0^{i_0} \cdots x_N^{i_N}$ for $I = (i_0, \ldots, i_N) \in \mathcal{T}_d$ and a_I $(I \in \mathcal{T}_d)$ are holomorphic functions on $\mathbb{C}$ without common zero. Then $P(z)$ is a hypersurface in $\mathbb{P}^N(\mathbb{C})$ for every $z \in \mathbb{C}$. We may also consider P as a holomorphic mapping from $\mathbb{C}$ into $\mathbb{P}^M(\mathbb{C})$ where $M = \binom{N+d}{N} - 1$. If all the meromorphic functions a_I are c-periodic, then we say that P is c-periodical moving hypersurface.

Since the number of moving hypersurfaces occurring in this paper is finite, by changing the homogeneous coordinates of $\mathbb{P}^N(\mathbb{C})$, we may assume that $a_{I_0} \not\equiv 0$, where $I_0 = (d, 0, \ldots, 0)$, for each given moving hypersurface $P(z) = \sum_{I \in \mathcal{T}_d} a_I(z) x^I$, and set

$$\tilde{P}(z) = \sum_{I \in \mathcal{T}_d} \frac{a_I(z)}{a_{I_0}(z)} x^I.$$

The proximity function of f with respect to P, denoted by $m_f(r, P)$, is defined by

$$m_f(r, P) = \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\|^d}{|P(\mathbf{f}(re^{i\theta}))|} \frac{d\theta}{2\pi} - \int_0^{2\pi} \log \frac{\|\mathbf{f}(e^{i\theta})\|^d}{|P(\mathbf{f}(e^{i\theta}))|} \frac{d\theta}{2\pi}.$$

This definition is independent of the choice of the reduced representation of f.

If Q is a slowly moving hypersurface with respect to f, i.e., $\| \ T_Q(r) = o(T_f(r))$, then the first main theorem in Nevanlinna theory for holomorphic maps and moving hypersurfaces is stated as follows:

$$dT_f(r) = m_f(r, Q) + N_{Q(\mathbf{f})}(r) + o(T_f(r)).$$

Definition 2.1. *Let f be a holomorphic curve from $\mathbb{C}$ into a subvariety $V \subset \mathbb{P}^N(\mathbb{C})$ with a presentation $(f_0, \ldots, f_N)$. The map f is said to be $\mathcal{P}^1_c$-algebraic nondegenerate over $I_d(V)$ for a positive integer d if there is no non-zero homogeneous polynomial $Q \in \mathcal{P}^1_c[x_0, \ldots, x_N]$ of degree d with $V \not\subset Q(z)^*$ for some $z \in \mathbb{C}$ such that $Q(f_0, \ldots, f_N) \equiv 0$. If f is $\mathcal{P}^1_c$-algebraic nondegenerate over $I_d(V)$ for every $d \geq 1$ then f is said to be $\mathcal{P}^1_c$-algebraic nondegenerate.*

Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be a set of q moving hypersurfaces, $\deg Q_j = d_j \geq 1$, of the forms $Q_j = \sum_{I \in \mathcal{T}_{d_j}} a_{jI} x^I (j = 1, \ldots, q)$. Denote by $\mathcal{K}_\mathcal{Q}$ the smallest subfield of $\mathcal{M}$ containing all functions $\frac{a_{jI}}{a_{jJ}}$ (for $a_{jJ} \not\equiv 0$), and by $\mathcal{C}_\mathcal{Q}$ the set of all non-negative function $h : \mathbb{C} \to [0, +\infty]$, which are of the form $\frac{|u_1|^{c_1} + \cdots + |u_k|^{c_k}}{|v_1|^{b_1} + \cdots + |v_l|^{b_l}}$, where $k, l \in \mathbb{N}, u_i, v_j \in \mathcal{K}_\mathcal{Q} \backslash \{0\}, c_j, b_j \in \mathbb{R}_+$. Then, for every moving hypersurface Q in $\mathcal{K}_\mathcal{Q}[x_0, \ldots, x_N]$ of degree d, we have $|Q(z)(\mathbf{x})| \leq c(z)\|\mathbf{x}\|^d$ for some $c \in \mathcal{C}_\mathcal{Q}$.

Definition 2.2 (see [21, Definition 3.4]). *Let V be a subvariety of $\mathbb{P}^N(\mathbb{C})$. Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be a family of moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ such that $V \not\subset Q_j(z)^*$ for generic points z and for all $j = 1, \ldots, q$. The distributive constant of $\mathcal{Q}$ with respect to V is defined by*

$$\Delta_{\mathcal{Q},V} := \max_{\Gamma \subset \{1, \ldots, q\}} \frac{\sharp \Gamma}{\dim V - \dim \bigcap_{j \in \Gamma} Q_j(z)^* \cap V}$$

for generic points $z \in \mathbb{C}$.

Remark 2.3. If $Q_1, \ldots, Q_q$ $(q \geq m+1)$ are in weakly $l-$subgeneral position with respect to V then $\Delta_{\mathcal{Q},V} \leq l - \dim V + 1$ (see [21, Remark 3.7]). If $V = \mathbb{P}^N(\mathbb{C})$, then we write $\Delta_\mathcal{Q}$ for $\Delta_{\mathcal{Q},V}$ and call it the distributive constant of the family $\mathcal{Q}$.

For $\mathbf{a} = (a_0, \ldots, a_N) \in \mathbb{Z}^{N+1}$ we write $\mathbf{x}^\mathbf{a}$ for the monomial $x_0^{a_0} \cdots x_N^{a_N}$ and set

$$\mathbf{a}_i \cdot \mathbf{c} = \sum_{i=0}^N a_i c_i$$

for $\mathbf{c} = (c_0, \ldots, c_N) \in \mathbb{C}^{N+1}$. Let V be a projective subvariety of $\mathbb{P}^N(\mathbb{C})$ of dimension n and of degree δ. By the usual theory of Hilbert polynomials,

$$H_V(u) = \delta \cdot \frac{u^n}{n!} + O(u^{n-1}).$$

The u-th Hilbert weight $S_V(u, \mathbf{c})$ of V with respect to the tuple $\mathbf{c} = (c_0, \ldots, c_N) \in \mathbb{R}^{N+1}$ is defined by

$$S_V(u, \mathbf{c}) := \max \left(\sum_{i=1}^{H_V(u)} \mathbf{a}_i \cdot \mathbf{c} \right),$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{H_V(u)}}$ whose residue classes modulo $I(V)$ form a basis of $\mathbb{C}[x_0, \ldots, x_N]_u / I_u(V)$.

(B) *Some auxiliary results.*

Lemma 2.4 (General Nochka's weight [16, Lemma 1])**.** *Let V be a complex projective subvariety of $\mathbb{P}^N(\mathbb{C})$ of dimension n $(n \le N)$. Let $Q_1, \ldots, Q_q$ be q $(q > 2m - n + 1)$ hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ in m-subgeneral position with respect to V of the common degree d. Then there are positive rational constants ω_i $(1 \le i \le q)$ satisfying the following:*

i) $0 < \omega_i \le 1,\ \forall i \in \{1, \ldots, q\}$.

ii) Setting $\tilde{\omega} = \max_{j \in Q} \omega_j$, *one gets*

$$\sum_{j=1}^{q} \omega_j = \tilde{\omega}(q - 2m + n - 1) + n + 1.$$

iii) $\dfrac{k+1}{2m - n + 1} \le \tilde{\omega} \le \dfrac{n}{m}$.

iv) For $R \subset \{1, \ldots, q\}$ *with* $\sharp R = m + 1$, *then* $\sum_{i \in R} \omega_i \le n + 1$.

v) Let $E_i \ge 1$ $(1 \le i \le q)$ *be arbitrarily given numbers. For* $R \subset \{1, \ldots, q\}$ *with* $\sharp R = m + 1$, *there is a subset* $R^o \subset R$ *such that* $\sharp R^o = \operatorname{rank}\{Q_i\}_{i \in R^o} = n + 1$ *and*

$$\prod_{i \in R} E_i^{\omega_i} \le \prod_{i \in R^o} E_i.$$

Lemma 2.5 (see [21, Lemma 3.8])**.** *Let V be as in Lemma 3.3. Let $Q_1, \ldots, Q_l$ be l hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of the same degree $d \ge 1$, such that $\bigcap_{i=1}^{l} Q_i^* \cap V = \varnothing$ and*

$$\dim \left(\bigcap_{i=1}^{s} Q_i^* \right) \cap V = n - u,\ \forall t_{u-1} \le s < t_u, 1 \le u \le n,$$

where $t_0, t_1, \ldots, t_n$ integers with $1 = t_0 < t_1 < \cdots < t_n = l$. Then there exist $n + 1$ hypersurfaces $P_1, \ldots, P_{n+1}$ in $\mathbb{P}^N(\mathbb{C})$ of the forms

$$P_u = \sum_{j=1}^{t_u} c_{uj} Q_j,\ c_{uj} \in \mathbb{C},\ u = 0, \ldots, n,$$

such that $\left(\bigcap_{u=1}^{n+1} P_u^* \right) \cap V = \varnothing$.

Lemma 2.6 (see [19, Lemma 3.9]). *Let $t_0, t_1, \dots, t_n$ be $n+1$ integers such that $1 = t_0 < t_1 < \cdots < t_n$, and let $\Delta = \max_{1\le s\le n} \frac{t_s - t_0}{s}$. Then for every n real numbers $a_0, a_1, \dots, a_{n-1}$ with $a_0 \ge a_1 \ge \cdots \ge a_{n-1} \ge 1$, we have*

$$a_0^{t_1-t_0} a_1^{t_2-t_1} \cdots a_{n-1}^{t_n-t_{n-1}} \le (a_0 a_1 \cdots a_{n-1})^{\Delta}.$$

Theorem 2.7 (see [8, Theorem 4.1]). *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be an algebraic subvariety of dimension n and degree δ. Let $u > \delta$ be an integer and let $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}_{\geqslant 0}$. Then*

$$\frac{1}{uH_V(u)} S_V(u, \mathbf{c}) \ge \frac{1}{(n+1)\delta} e_V(\mathbf{c}) - \frac{(2n+1)\delta}{u} \cdot \left(\max_{i=0,\dots,N} c_i \right).$$

Here, $e_V(\mathbf{c})$ is the Chow weight of V with respect to $\mathbf{c}$ (see [8, 9] for detail definition).

Theorem 2.8 (see [20, Lemma 3.2]). *Let Y be a projective subvariety of $\mathbb{P}^R(\mathbb{C})$ of dimension $n \ge 1$ and degree δ_Y. Let m $(m \ge n)$ be an integer and let $\mathbf{c} = (c_0, \dots, c_R)$ be a tuple of non-negative reals. Let $\mathcal{H} = \{H_0, \dots, H_R\}$ be a set of hyperplanes in $\mathbb{P}^R(\mathbb{C})$ defined by $H_i = \{y_i = 0\}$ $(0 \le i \le R)$. Let $\{i_0, \dots, i_m\}$ be a subset of $\{0, \dots, R\}$ such that:*

(1) $c_{i_m} = \min\{c_{i_0}, \dots, c_{i_m}\}$,
(2) $Y \cap \bigcap_{j=0}^{m-1} H_{i_j} \ne \varnothing$,
(3) *and* $Y \not\subset H_{i_j}$ *for all* $j = 0, \dots, m$.

Let $\Delta_{\mathcal{H},Y}$ be the distributive constant of the family $\mathcal{H} = \{H_{i_j}\}_{j=0}^m$ with respect to Y. Then

$$e_Y(\mathbf{c}) \ge \frac{\delta_Y}{\Delta_{\mathcal{H},Y}} (c_{i_0} + \cdots + c_{i_m}).$$

Note that [20, Lemma 3.2] is stated for the case of number field, but its proof automatically works for the case of $\mathbb{C}$.

Lemma 2.9 (see [12, Lemma 7]). *Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a $\mathcal{P}_c^1$-linearly non-degenerate holomorphic map with $\varsigma(f) < 1$. Let $H_1, \dots, H_q$ be q arbitrary hyperplanes in $\mathbb{P}^N(\mathbb{C})$. Then we have*

$$\Big\| \int_0^{2\pi} \max_{\mathcal{J}} \log \prod_{j\in\mathcal{J}} \frac{\|\mathbf{f}\|}{|H_j(\mathbf{f})|}(re^{i\theta}) \frac{d\theta}{2\pi} \le (N+1)T_f(r) - N_{C(f)}(r) + o(T_f(r)),$$

where $\mathbf{f} = (f_0, f_1, \dots, f_n)$ is a reduced representation of f, the maximal $\max_{j\in\mathcal{J}}$ is taken over all subsets $\mathcal{J} \subset \{1, \dots, q\}$ such that $\{H_j | j \in \mathcal{J}\}$ is linearly independent.

3. Proof of main results

Proof of Theorem 1.1. Without loss of generality, we may assume that $\|Q_j\| = 1$ for all $j = 1, \dots, q$. Also, replacing Q_j by $Q^{\frac{d}{d_j}}$ if necessary, we may assume further that $Q_1, \dots, Q_q$ have the same degree d.

It suffices for us to consider the case where $\Delta_{\mathcal{Q},V} < \frac{q}{n+1}$. Note that $\Delta_{\mathcal{Q},V} \ge 1$, and hence $q > n+1$. If there exists $i \in \{1, \dots, q\}$ such that $\bigcap_{\substack{j=1\\ j\ne i}}^q Q_j^* \cap V \ne \varnothing$ then

$$\Delta_{\mathcal{Q},V} \ge \frac{q-1}{n} > \frac{q}{n+1}.$$

This is a contradiction. Therefore, $\bigcap_{\substack{j=1\\ j\neq i}}^{q} Q_j^* \cap V = \varnothing$ for all $i \in \{1,2,\dots,q\}$.

Let $\mathcal{I} = \{\zeta_1,\dots,\zeta_\lambda\}$ be the set of all bijections from $\{0,\dots,q-1\}$ into $\{1,\dots,q\}$, where $\lambda = q!$. For each ζ_i, since $\bigcap_{j=0}^{q-2} \tilde{Q}^*_{\zeta_i(j)} \cap V = \varnothing$, there exists the smallest index l_i such that $\bigcap_{j=0}^{l_i} Q^*_{\zeta_i(j)} \cap V = \varnothing$.

Let $\mathbf{f} = (f_0,\dots,f_N)$ be a reduced representation of f. By [21, Lemma 3.2], there is a positive constant A, chosen common for all ζ_i, such that

$$\|\mathbf{f}(z)\|^d \le A \max_{0\le j\le l_i} |Q_{\zeta_i(j)}(\mathbf{f}(z))| \ (\forall \zeta_i \in \mathcal{I}).$$

Denote by $S(i)$ the set of all z such that $Q_j(\mathbf{f}(z)) \neq 0$ for all $j = 1,\dots,q$ and

$$|Q_{\zeta_i(0)}(\mathbf{f}(z))| \le |Q_{\zeta_i(1)}(\mathbf{f}(z))| \le \cdots \le |Q_{\zeta_i(q-1)}(\mathbf{f}(z))|.$$

Therefore, for $z \in S(i)$, we have

$$\prod_{j=1}^{q} \frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|} \le A^{q-l_i-1} \prod_{j=0}^{l_i} \frac{\|\mathbf{f}(z)\|^d}{|Q_{\zeta_i(j)}(\mathbf{f}(z))|} \le C \prod_{j=0}^{l_i} \frac{\|\mathbf{f}(z)\|^d}{|Q_{\zeta_i(j)}(\mathbf{f}(z))|}, \tag{3.1}$$

where $C = \sum_{i=1}^{\lambda} A^{q-l_i-1} \prod_{j=l_i+1}^{q-1}$.

Consider the mapping Φ from V into $\mathbb{P}^{q-1}(\mathbb{C})$, which maps a point $x = (x_0 : \cdots : x_N) \in V$ into the point $\Phi(x) \in \mathbb{P}^{q-1}(\mathbb{C})$ given by

$$\Phi(x) = (Q_1(\mathbf{x}) : \cdots : Q_q(\mathbf{x})),$$

where $\mathbf{x} = (x_0,\dots,x_N)$. Set $\tilde{\Phi}(\mathbf{x}) = (Q_1(\mathbf{x}),\dots,Q_q(\mathbf{x}))$.

Let $Y = \Phi(V)$. Since $V \cap \bigcap_{j=1}^{q} Q_j^* = \varnothing$, Φ is a finite morphism on V and Y is a complex projective subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$ with $\dim Y = n$ and of degree

$$\delta := \deg Y \le d^n. \deg V.$$

For every $\mathbf{a} = (a_1,\dots,a_q) \in \mathbb{Z}^q_{\ge 0}$ and $\mathbf{y} = (y_1,\dots,y_q)$ we set

$$\mathbf{y}^{\mathbf{a}} = y_1^{a_1} \dots y_q^{a_q}.$$

Let u be a positive integer. Define

$$Y_u := I_u(Y), \ n_u := H_Y(u) - 1. \tag{3.2}$$

We fix a basis $\{v_0,\dots,v_{n_u}\}$ of Y_u and consider the holomorphic map $F : \mathbb{C} \to \mathbb{P}^{n_u}(\mathbb{C})$ which has a reduced representation

$$\mathbf{F} = (v_0(\tilde{\Phi}\circ\mathbf{f}),\dots,v_{n_u}(\tilde{\Phi}\circ\mathbf{f})) : \mathbb{C} \to \mathbb{C}^{n_u+1}.$$

Hence F is $\mathcal{P}^1_c$-linearly nondegenerate, since f is $\mathcal{P}^1_c$-algebraically nondegenerate.

Now, we fix a point $z \notin \bigcup_{j=1}^{q} (Q_j(\mathbf{f}))^{-1}(0)$. Suppose that $z \in S(i_0)$. We define

$$\mathbf{c}_z = (c_{1,z},\dots,c_{q,z}) \in \mathbb{R}^q,$$

where

$$c_{j,z} := \log \frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|} \text{ for } j = 1,\dots,q. \tag{3.3}$$

We see that $c_{j,z} \geq 0$ for all j. By the definition of the Hilbert weight, there are $\mathbf{a}_{0,z}, \dots, \mathbf{a}_{n_u,z} \in \mathbb{Z}^q_{\geq 0}$ with

$$\mathbf{a}_{i,z} = (a_{i,1,z}, \dots, a_{i,q,z}) \text{ with } a_{i,s,z} \in \{0, \dots, u\},$$

such that the residue classes modulo $I(Y)_u$ of $\mathbf{y}^{\mathbf{a}_{0,z}}, \dots, \mathbf{y}^{\mathbf{a}_{n_u,z}}$ form a basis of $I_u(Y)$ and

$$S_Y(u, \mathbf{c}_z) = \sum_{i=0}^{n_u} \mathbf{a}_{i,z} \cdot \mathbf{c}_z. \tag{3.4}$$

We see that $\mathbf{y}^{\mathbf{a}_{i,z}} \in Y_u$ (modulo $I(Y)_u$). Then we may write

$$\mathbf{y}^{\mathbf{a}_{i,z}} = L_{i,z}(v_0, \dots, v_{n_u}),$$

where $L_{i,z}$ $(0 \leq i \leq n_u)$ are linearly independent linear forms with coefficients in $\mathbb{C}$. We have

$$\begin{aligned}\log \prod_{i=0}^{n_u} |L_{i,z}(\mathbf{F}(z))| &= \log \prod_{i=0}^{n_u} \prod_{1 \leq j \leq q} |Q_j(\mathbf{f}(z))|^{a_{i,j,z}} \\ &= -S_Y(m, \mathbf{c}_z) + \Delta u(n_u + 1) \log \|\mathbf{f}(z)\| + O(u(n_u + 1)).\end{aligned}$$

This implies that

$$\begin{aligned}\log \prod_{i=0}^{n_u} \frac{\|\mathbf{F}(z)\| \cdot \|L_{i,z}\|}{|L_{i,z}(\mathbf{F}(z))|} =& S_Y(u, \mathbf{c}_z) - du(n_u + 1) \log \|\mathbf{f}(z)\| \\ &+ (n_u + 1) \log \|\mathbf{F}(z)\| + O(u(n_u + 1)).\end{aligned}$$

Here we note that $L_{i,z}$ depends on i, z and u, but the number of these linear forms is finite. We denote by $\mathcal{L}$ the set of all $L_{i,z}$ occurring in the above inequalities. Then,

$$\begin{aligned}S_Y(u, \mathbf{c}_z) \leq & \max_{\mathcal{J} \subset \mathcal{L}} \log \prod_{L \in \mathcal{J}} \frac{\|\mathbf{F}(z)\| \cdot \|L\|}{|L(\mathbf{F}(z))|} + \Delta u(n_u + 1) \log \|\mathbf{f}(z)\| \\ & - (n_u + 1) \log \|\mathbf{F}(z)\| + O(u(n_u + 1)),\end{aligned} \tag{3.5}$$

where the maximum is taken over all subsets $\mathcal{J} \subset \mathcal{L}$ with $\sharp \mathcal{J} = n_u + 1$ and $\{L; L \in \mathcal{J}\}$ is linearly independent. From Theorem 2.7 we have

$$\frac{1}{u(n_u + 1)} S_Y(u, \mathbf{c}_z) \geq \frac{1}{(n+1)\delta} e_Y(\mathbf{c}_z) - \frac{(2n+1)\delta}{u} \max_{1 \leq j \leq q} c_{j,z}. \tag{3.6}$$

It is clear that

$$\max_{1 \leq j \leq q} c_{j,z} \leq \sum_{1 \leq j \leq q} \log \frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|}.$$

Combining (3.5), (3.6) and the above remark, we get

$$\begin{aligned}\frac{1}{(n+1)\delta}e_Y(\mathbf{c}_z) \leq& \frac{1}{u(n_u+1)}\left(\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\mathbf{F}(z)\|\cdot\|L\|}{|L(\mathbf{F}(z))|}-(n_u+1)\log\|\mathbf{F}(z)\|\right)\\ &+d\log\|\mathbf{f}(z)\|+\frac{(2n+1)\delta}{u}\max_{1\leq j\leq q}c_{j,z}+O(1)\\ \leq&\frac{1}{u(n_u+1)}\left(\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\mathbf{F}(z)\|\cdot\|L\|}{|L(\mathbf{F}(z))|}-(n_u+1)\log\|\mathbf{F}(z)\|\right)\\ &+d\log\|\mathbf{f}(z)\|+\frac{(2n+1)\delta}{u}\sum_{1\leq j\leq q}\log\frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|}+O(1).\end{aligned}\tag{3.7}$$

We have $V\cap\bigcap_{j=0}^{l_{i_0}}Q_{\zeta_{i_0}(j)}=\emptyset$. Then by Theorem 2.8 and (3.1), we have

$$\begin{aligned}e_Y(\mathbf{c}_z)&\geq\frac{\delta}{\Delta_{\mathcal{Q},V}}\cdot(c_{\zeta_{i_0}(0),z}+\cdots+c_{\zeta_{i_0}(l_{i_0}),z})=\frac{\delta}{\Delta_{\mathcal{Q},V}}\cdot\log\prod_{j=0}^{l_{i_0}}\frac{\|\mathbf{f}(z)\|^d}{|Q_{\zeta_{i_0}(j)}(\mathbf{f}(z))|}\\ &\geq\frac{\delta}{C\Delta_{\mathcal{Q},V}}\cdot\log\prod_{j=1}^{q}\frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|}.\end{aligned}\tag{3.8}$$

Then, from (3.7) and (3.8) we have

$$\begin{aligned}&\frac{1}{\Delta_{\mathcal{Q},V}(n+1)}\cdot\log\prod_{j=1}^{q}\frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|}\\ &\qquad\leq\frac{1}{u(n_u+1)}\left(\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\mathbf{F}(z)\|\cdot\|L\|}{|L(\mathbf{F}(z))|}-(n_u+1)\log\|\mathbf{F}(z)\|\right)\\ &\qquad\quad+d\log\|\mathbf{f}(z)\|+\frac{(2n+1)\delta}{u}\sum_{1\leq j\leq q}\log\frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|}+O(1),\end{aligned}\tag{3.9}$$

where the term $O(1)$ does not depend on z.

By applying Lemma 2.9 to the $\mathcal{P}_c^1$-linear nondegenerate holomorphic map F and the system of linear forms $\mathcal{L}$, we get:

$$\Big\|\int_0^{2\pi}\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\left(\frac{\|\mathbf{f}\|}{|L(\mathbf{f})|}(te^{i\theta})\right)\frac{d\theta}{2\pi}\leq(n_u+1)T_F(r)-N_{C(F)}(r)+o(T_f(r)),\tag{3.10}$$

where the maximum is taken over all subsets $\mathcal{J}$ of $\mathcal{L}$, such that $\sharp J=n_u+1$ and $\{L|L\in\mathcal{J}\}$ is linearly independent.

Integrating both sides of (3.9), in the view of (3.10), we obtain

$$\begin{aligned}\Big\|\ \frac{1}{\Delta_{\mathcal{Q},V}}\left(dqT_f(r)-\sum_{i=1}^{q}N_{Q_i(f)}(r)\right)&\leq d(n+1)T_f(r)-\frac{(n+1)}{u(n_u+1)}N_{C(F)}(r)\\ &+\frac{(2n+1)(n+1)\delta}{u}\sum_{i=1}^{q}m_f(r,Q_i)+o(T_f(r)).\end{aligned}\tag{3.11}$$

We now estimate the quantity $N_{C(F)}(r)$. Fix a point $z \in \mathbb{C}$. We set $c_i = \nu^{[n_u]}_{Q_i(f)}(z)$ for $i = 1, \dots, q$, and

$$\mathbf{c} = (c_1, \dots, c_q) \in \mathbb{Z}^q_{\geq 0}.$$

Then, for $i = 0, \dots, n_u$, there are

$$\mathbf{a}_i = (a_{i,1}, \dots, a_{i,q}), a_{i,s} \in \{1, \dots, u\}$$

such that $\mathbf{y}^{\mathbf{a}_0}, \dots, \mathbf{y}^{\mathbf{a}_{n_u}}$ is a basic of $I_u(Y)$ and

$$S_{Y_z}(u, \mathbf{c}) = \sum_{i=0}^{n_u} \mathbf{a}_i \cdot \mathbf{c}.$$

Similarly as above, we write $\mathbf{y}^{\mathbf{a}_i} = L_i(v_0, \dots, v_{n_u})$, where $L_0, \dots, L_{n_u}$ are independent linear forms with coefficients in $\mathbb{C}$. By the property of the Casorati determinant, we see that

$$C(F) = c \det\big(\overline{L_0(\tilde{\Phi}(\mathbf{f}))}^{[w]}), \dots, \overline{L_{n_u}(\tilde{\Phi}(\mathbf{f}))}^{[w]}\big)_{0\leq w\leq n_u},$$

where c is a non zero constant. This yields that

$$\nu_{C(F)}(z) \geq \sum_{i=0}^{n_u} \min_{0\leq w\leq n_u} \nu_{\overline{L_i(\mathbf{f})}^{[w]}}(z).$$

We easily see that

$$\nu_{\overline{L_i(\mathbf{f})}^{[w]}}(z) = \sum_{j=1}^{q} a_{i,j}\nu_{\overline{Q_j(\mathbf{f})}^{[w]}}(z) \geq \sum_{j=1}^{q} a_{i,j}\nu^{[n_u,c]}_{Q_j(\mathbf{f})}(z).$$

Thus, we have

$$\nu_{C(F)}(z) \geq \sum_{i=0}^{n_u} \mathbf{a}_i \cdot \mathbf{c} = S_Y(u, \mathbf{c}). \tag{3.12}$$

Take an index i_0 such that $\nu^{[n_u,c]}_{Q_{\zeta_{i_0}(0)}(\mathbf{f})}(z) \geq \nu^{[n_u,c]}_{Q_{\zeta_{i_0}(1)}(\mathbf{f})}(z) \geq \cdots \geq \nu^{[n_u,c]}_{Q_{\zeta_{i_0}(q-1)}(\mathbf{f})}(z)$. Hence, $\nu^{[n_u,c]}_{Q_{\zeta_{i_0}(j)}(\mathbf{f})}(z) = 0$ for all $j \geq l_{i_0}$. Then by Lemma 2.8 we have

$$\Delta_{\mathcal{Q},V} e_Y(\mathbf{c}) \geq \delta(c_{\zeta_{i_0}(0)} + \cdots + c_{\zeta_{i_0}(l_{i_0})}) = \delta \sum_{j=1}^{q} \nu^{[n_u,c]}_{Q_j(\mathbf{f})}(z).$$

On the other hand, by Theorem 2.7 we have that

$$\begin{aligned}\frac{1}{u(n_u+1)} S_Y(u, \mathbf{c}) &\geq \frac{1}{(n+1)\delta} e_Y(\mathbf{c}) - \frac{(2n+1)\delta}{u} \max_{1\leq i\leq q} c_i \\ &\geq \left(\frac{1}{\Delta_{\mathcal{Q},V}(n+1)} - \frac{(2n+1)\delta}{u}\right) \sum_{j=1}^{q} \nu^{[n_u,c]}_{Q_{\zeta_{i_0}(j)}(\mathbf{f})}(z).\end{aligned}$$

Combining this inequality and (3.12), we have

$$\frac{(n+1)}{u(n_u+1)} \nu^0_{C(F)}(z) \geq \frac{(n+1)}{u(n_u+1)} S_Y(u, \mathbf{c}) \geq \left(\frac{1}{\Delta_{\mathcal{Q},V}} - \frac{(2n+1)(n+1)\delta}{u}\right) \sum_{j=1}^{q} \nu^{[n_u,c]}_{Q_{\zeta_{i_0}(j)}(\mathbf{f})}(z).$$

Integrating both sides of this inequality, we obtain

$$\frac{(n+1)}{u(n_u+1)}N_{C(F)}(r) \geq \left(\frac{1}{\Delta_{\mathcal{Q},V}} - \frac{(2n+1)(n+1)\delta}{u}\right)\sum_{j=1}^{q} N^{[n_u,c]}_{Q_j(f)}(r). \tag{3.13}$$

Combining inequalities (3.11) and (3.13) and the first main theorem, we get

$$\begin{aligned}\Big\| \ \frac{1}{\Delta_{\mathcal{Q},V}}&\left(dqT_f(r) - \sum_{i=1}^{q} N_{Q_i(f)}(r)\right)\\ &\leq d(n+1)T_f(r) - \left(\frac{1}{\Delta_{\mathcal{Q},V}} - \frac{(2n+1)(n+1)\delta}{u}\right) N^{[n_u,c]}_{Q_j(f)}(r)\\ &+ \frac{(2n+1)(n+1)\delta}{u}\sum_{i=1}^{q}(dT_f(r) - N_{Q_i(f)}(r)) + o(T_f(r)).\end{aligned}$$

By setting $m_0 = \frac{1}{\Delta_{\mathcal{Q},V}} - \frac{(2n+1)(n+1)\delta}{u}$, the above inequality implies that

$$\Big\| \ \left(q - \frac{n+1}{m_0}\right)T_f(r) \leq \sum_{j=1}^{q}\frac{1}{d}\tilde{N}^{[n_u,c]}_{Q_j(f)}(r) + o(T_f(r)).$$

We choose $u = \lceil \Delta_{\mathcal{Q},V}(2n+1)(n+1)d^n \deg V((n+1)\Delta_{\mathcal{Q},V} + \epsilon)\epsilon^{-1}\rceil$. Then we have

$$u \geq \Delta_{\mathcal{Q},V}(2n+1)(n+1)\delta((n+1)\Delta_{\mathcal{Q},V}\epsilon^{-1} + 1)$$

and hence

$$\begin{aligned}\frac{n+1}{m_0} &\leq (n+1)\Delta_{\mathcal{Q},V} + \epsilon,\\ n_u &\leq H_Y(u) - 1 \leq d^n \deg V\binom{u+n}{n} - 1.\end{aligned}$$

Note that, we may suppose that $\epsilon < q - \Delta_{\mathcal{Q},V}(n+1)$. Hence, if $n = 1$ then

$$\begin{aligned}n_u + 1 &< d^n \deg V(1+u)\\ &< d^n \deg V(\Delta_{\mathcal{Q},V}(2n+1)(n+1)d^n \deg V((n+1)\Delta_{\mathcal{Q},V}\epsilon^{-1} + 1) + 2)\\ &\leq d^{n^2+n}(\deg V)^{n+1}e^n\Delta^n_{\mathcal{Q},V}(2n+5)^n((n+1)\Delta_{\mathcal{Q},V}\epsilon^{-1} + 1)^n.\end{aligned}$$

Otherwise, if $n \geq 2$, we have

$$\begin{aligned}n_u + 1 &< d^n \deg V e^n\left(1 + \frac{u}{n}\right)^n\\ &\leq d^n \deg V e^n\left(1 + \frac{\Delta_{\mathcal{Q},V}(2n+1)(n+1)d^n \deg V((n+1)\Delta_{\mathcal{Q},V}\epsilon^{-1} + 1) + 1}{n}\right)^n\\ &\leq d^{n^2+n}(\deg V)^{n+1}e^n\Delta^n_{\mathcal{Q},V}(2n+5)^n((n+1)\Delta_{\mathcal{Q},V}\epsilon^{-1} + 1)^n.\end{aligned}$$

Therefore, we always have

$$n_u \leq \left[d^{n^2+n}(\deg V)^{n+1}e^n\Delta^n_{\mathcal{Q},V}(2n+5)^n((n+1)\Delta_{\mathcal{Q},V}\epsilon^{-1} + 1)^n\right] - 1 = L_0 - 1.$$

Then, we get

$$\| \ (q - \Delta_{\mathcal{Q},V}(n+1) - \epsilon)T_f(r) \le \sum_{j=1}^{q} \frac{1}{d}\tilde{N}^{[L_0-1,c]}_{Q_i(f)}(r) + o(T_f(r)).$$

The proof of the theorem is completed. □

Proof of Theorem 1.2. Take $\mathbf{f} = (f_0, \dots, f_N)$ be a reduced representation of f. Similarly as the proof of Theorem 1.1, we may assume that all Q_i $(i = 1, \dots, q)$ have the same degree d and $\|Q_i\| = 1$.

Take an $\mathbb{C}$-basis $\{[A_i]\}_{i=0}^{n_d}$ of $I_d(V)$, where $A_i \in \mathbb{C}[x_0, \dots, x_N]_d$. Consider a holomorphic map $F : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ with a reduced representation $\mathbf{F} = (A_0(\mathbf{f}), \dots, A_{n_d}(\mathbf{f}))$. Since f is $\mathcal{P}^1_c$-nondegenerate over $I_d(V)$, F is $\mathcal{P}^1_c$-linearly nondegenerate. Then $C(F) \not\equiv 0$. Let $\{\omega_i\}_{i=1}^q$ be as in Lemma 2.4 for the family $\{Q_i\}_{i=1}^q$. For each $i \in \{1, \dots, q\}$, there is a linear form L_i with coefficients in $\mathbb{C}$ such that $[Q_i] = L_i([A_0], \dots, [A_{n_d}])$. This implies that $Q_i(\mathbf{f}) = L_i(\mathbf{F})$.

Let z be a fixed point such that $Q_i(\mathbf{f}(z)) \ne 0$. There is a permutation $(i_1, \dots, i_q)$ of $\{1, \dots, q\}$ such that

$$|Q_{i_1}(\mathbf{f}(z))| \le |Q_{i_2}(\mathbf{f}(z))| \le \dots \le |Q_{i_q}(\mathbf{f}(z))|.$$

Set $R = \{i_1, \dots, i_{l+1}\}$. Since $\bigcap_{j\in R} Q_j^* \cap V = \emptyset$, there exists a positive constant $c > 1$ (chosen not depending on z) such that $\|\mathbf{f}(z)\| \le c \max_{j\in R} |Q_{i_j}(\mathbf{f}(z))| = c|Q_{i_{l+1}}(\mathbf{f}(z))|$. We choose $R^o \subset R$ such that $R^o \in R$ and R^o satisfies Lemma 2.4 v) with respect to numbers $\left\{ \frac{\|\mathbf{f}(z)\|^d}{|Q_i(\mathbf{f}(z))|} \right\}_{i=1}^q$. Then, we get

$$\frac{\|\mathbf{f}(z)\|^{d(\sum_{i=1}^q \omega_i)}}{|Q_1^{\omega_1}(\mathbf{f}(z)) \cdots Q_q^{\omega_q}(\mathbf{f}(z))|} \le c^q \prod_{j\in R} \left(\frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|} \right)^{\omega_j} \le c^q \prod_{j\in R^o} \frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f}(z))|}$$
$$= A \prod_{j\in R^o} \frac{\|\mathbf{F}(z)\|}{|L_j(\mathbf{F}(z))|},$$

where A is a positive constant (chosen not depending on z). This implies that

$$\log \frac{\|\mathbf{f}(z)\|^{d(\sum_{i=1}^q \omega_i)}}{|Q_1^{\omega_1}(\mathbf{f}(z)) \cdots Q_q^{\omega_q}(\mathbf{f}(z))|} \le \max_{\mathcal{J}} \sum_{L\in\mathcal{J}} \log \frac{\|\mathbf{F}(z)\|}{|L(\mathbf{F}(z))|} + O(1),$$

where the maximum is taken over all subsets $\mathcal{J} \subset \{L_1, \dots, L_q\}$ such that the family $\{L | L \in \mathcal{J}\}$ is linearly independent.

Integrating both sides of the above inequality and applying Lemma 2.8 for the map F and the system of linear forms $\mathcal{L}$, we get

$$d(\sum_{i=1}^q \omega_i)T_f(r) - \sum_{i=1}^q \omega_i N_{Q_i(f)}(r) \le (n_d+1)T_F(r) - N_{C(F)}(r) + o(T_f(r)). \tag{3.14}$$

Claim. $\sum_{i=1}^q \omega_i N_{Q_i(f)}(r) - N_{C(F)}(r) \le \sum_{i=1}^q \omega_i \tilde{N}^{[n_d,c]}_{Q_i(f)}(r)$.

Indeed, let z be a fixed point. We suppose that $\nu^{[n_d,c]}_{Q_1(\mathbf{f})}(z) \geq \nu^{[n_d,c]}_{Q_2(\mathbf{f})}(z) \geq \cdots \geq \nu^{[n_d,c]}_{Q_q(\mathbf{f})}(z)$. Then, we have $\nu^{[n_d,c]}_{Q_j(\mathbf{f})}(z) = 0$ for all $j \geq l+1$. Put $R = \{1, \ldots, l+1\}$. Choose $R^1 \subset R$ such that $\sharp R^1 = \operatorname{rank}\{Q_i\}_{i \in R^1} = n_d + 1$ and R^1 satisfies Lemma 3.3 v) with respect to numbers $\{e^{\nu^{[n_d,c]}_{Q_i(\mathbf{f})}(z)}\}_{i=1}^q$. Then we have

$$\sum_{i \in R} \omega_i \nu^{[n_d,c]}_{Q_i(\mathbf{f})}(z) \leq \sum_{i \in R^1} \nu^{[n_d,c]}_{Q_i(\mathbf{f})}(z).$$

This yields that

$$\nu_{C(F)}(z) = \nu_{\deg\left(\overline{L_i(\mathbf{f})}^{[w]}; 0 \leq w \leq n_d, i \in R^1\right)} \geq \sum_{i \in R^1} \nu^{[n_d,c]}_{Q_i(\mathbf{f})}(z) \geq \sum_{i \in R} \omega_i \nu^{[n_d,c]}_{Q_i(\mathbf{f})}(z).$$

Hence

$$\begin{aligned}
\sum_{i=1}^q \omega_i \nu_{Q_i(f)}(z) - \nu_{C(F)}(z) &= \sum_{i \in R} \omega_i \nu_{Q_i(f)}(z) - \nu_{C(F)}(z) \\
&= \sum_{i \in R} \omega_i \tilde{\nu}^{[n_d,c]}_{Q_i(\mathbf{f})}(z)) + \sum_{i \in R} \omega_i \nu^{[n_d,c]}_{Q_i(\mathbf{f})}(z) - \nu_{C(F)}(z) \\
&\leq \sum_{i \in R} \omega_i \tilde{\nu}^{[n_d,c]}_{Q_i(\mathbf{f})}(z).
\end{aligned}$$

Integrating both sides of this inequality, we get

$$\sum_{i=1}^q \omega_i N_{Q_i(f)}(r) - N_{C(F)}(r) \leq \sum_{i=1}^q \omega_i \tilde{N}^{[n_d,c]}_{Q_i(f)}(r).$$

This proves the claim.

Combining the claim and (3.14), we obtain

$$\begin{aligned}
\| \ d(q - 2l + n - 1 - \frac{n_d - n}{\tilde{\omega}}) T_f(r) &\leq \sum_{i=1}^q \frac{\omega_i}{\tilde{\omega}} \tilde{N}^{[n_d,c]}_{Q_i(f)}(r) + o(T_f(r)) \\
&\leq \sum_{i=1}^q \tilde{N}^{[H_V(d)-1,c]}_{Q_i(f)}(r) + o(T_f(r)).
\end{aligned}$$

Since $\tilde{\omega} \geq \dfrac{n+1}{2l-n+1}$, the above inequality implies that

$$\Big\| \quad d\left(q - \frac{(2l-n+1)H_V(d)}{n+1}\right) T_f(r) \leq \sum_{i=1}^q \tilde{N}^{[H_V(d)-1,c]}_{Q_i(f)}(r) + o(T_f(r)).$$

Hence, the theorem is proved. □

Proof of Theorem 1.3. As usual argument, we assume that $Q_1, \ldots, Q_q$ are of the same degree d. Let $\mathcal{I} = \{\zeta_1, \ldots, \zeta_\lambda\}$ be the set of all bijections from $\{1, \ldots, q\}$ into itself, where $\lambda = q!$. Take a point z_0 such that all coefficients of $\tilde{Q}_i$ $(1 \leq i \leq q)$ are holomorphic at z_0 and

$$\Delta_{\mathcal{Q},f} = \max_{\Gamma \subset \{1,\ldots,q\}} \frac{\sharp \Gamma}{n - \dim \bigcap_{j \in \Gamma} \tilde{Q}_i(z_0)^*}.$$

Similar as the proof of Theorem 1.1, we have $\bigcap_{\substack{j=1\\ j\neq i}}^{q}\tilde{Q}_j(z_0)^*=\varnothing$ and $\bigcap_{\substack{j=1\\ j\neq i}}^{q}\tilde{Q}_j(z)^*=\varnothing$ generically, for all $i\in\{1,2,\dots,q\}$. Let l_i be the smallest index such that $\bigcap_{j=1}^{l_i}Q_{\zeta_v(j)}(z_0)^*\cap V=\varnothing$ for every $j=1,\dots,q$. Let $\mathcal{S}$ be the set of all pole and zero of all nonzero coefficients of $\tilde{Q}_j$ $(1\le j\le q)$. Then $\mathcal{S}$ is a discrete subset of $\mathbb{C}$. Also, we may choose a function $\alpha\in\mathcal{C}_{\mathcal{Q}}$ such that for each given moving hypersurface $Q\in\mathcal{P}_c^1[x_0,\dots,x_n]$ occurring in this proof, we have

$$\tilde{Q}(z)(\mathbf{x})\le\alpha(z)\|\mathbf{x}\|^{\deg\tilde{Q}}$$

for all $\mathbf{x}=(x_0,\dots,x_n)\in\mathbb{C}^{n+1}$, $z\in\mathbb{C}$.

For each $\zeta_v\in\mathcal{I}$, since $\bigcap_{j=0}^{q-2}\tilde{Q}_{\zeta_v(j)}(z_0)^*=\varnothing$, there exist integers $t_{v,0}=1<t_{v,1}<\ldots<t_{v,n}=l_v$ such that $\bigcap_{j=0}^{l_v}\tilde{Q}_{\zeta_v(j)}(z_0)^*=\varnothing$ and

$$\dim\bigcap_{j=1}^{s}\tilde{Q}_{\zeta_v(j)}(z_0)^*=n-u\ \forall t_{v,u-1}\le s<t_{v,u},1\le u\le n.$$

Then, $\Delta_{\mathcal{Q},f}\ge\dfrac{t_{v,u}-t_{v,0}}{u}$ for all $1\le u\le n$. Denote by $P'_{v,1},\dots,P'_{v,n+1}$ the hypersurfaces obtained in Lemma 2.5 with respect to the hypersurfaces $\tilde{Q}_{\zeta_v(1)}(z_0),\dots,\tilde{Q}_{\zeta_v(l_v)}(z_0)$. Now, for each $P'_{v,j}$ constructed by $P'_{v,j}=\sum_{s=1}^{t_{v,j}}a_{v,j,s}\tilde{Q}_{\zeta_v(s)}(z_0)$ $(a_{v,j,s}\in\mathbb{C})$ we define

$$P_{v,j}(z)=\sum_{s=1}^{t_{v,j}}a_{v,j,s}\tilde{Q}_{\zeta_v(s)}(z).$$

Hence $\{P_{v,j}\}_{j=1}^{n+1}\subset\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_N]$ with $P_{v,j}(z_0)=P'_{i,j}$. Then $\bigcap_{j=1}^{n+1}P_{v,j}(z_0)^*=\varnothing$, and hence $\{P_{v,j}(z)\}_{j=1}^{n+1}$ is in weakly general position. We may choose a positive constant $B\ge1$, commonly for all $\zeta_v\in\mathcal{I}$, such that

$$|P_{v,j}(\mathbf{x})|\le B\max_{1\le s\le t_{v,j}}|\tilde{Q}_{\zeta_v(j)}(\mathbf{x})|,$$

for all $1\le j\le n+1$ and for all $\mathbf{x}=(x_0,\dots,x_N)\in\mathbb{C}^{N+1}$. Enlarging the set $\mathcal{S}$ by adding to $\mathcal{S}$ all points $z\in\mathbb{C}$ such that $\bigcap_{j=1}^{n+1}P_{v,j}(z)^*\neq\varnothing$ for some index v. Then $\mathcal{S}$ is still a discrete subset of $\mathbb{C}$.

Fix an element $\zeta_v\in\mathcal{I}$. Denote by $S(v)$ the set of all points

$$z\in\mathbb{C}\setminus\left\{\bigcup_{i=1}^{q}\tilde{Q}_i(z)(\mathbf{f}(z))^{-1}(\{0\})\cup\bigcup_{\substack{0\le j\le n\\ \zeta_i\in\mathcal{I}}}P_{i,j}(z)(\mathbf{f}(z))^{-1}(\{0\})\right\}$$

such that $|\tilde{Q}_{\zeta_v(1)}(z)(\mathbf{f}(z))|\le|\tilde{Q}_{\zeta_v(2)}(z)(\mathbf{f}(z))|\le\cdots\le|\tilde{Q}_{\zeta_v(q)}(z)(\mathbf{f}(z))|$.

Therefore, for generic points $z \in S(v)$, By Lemma 2.6 we have

$$\begin{aligned}
\prod_{i=1}^{q} \frac{\|\mathbf{f}(z)\|^d}{|\tilde{Q}_i(z)(\mathbf{f}(z))|} &\le \frac{A(z)^{q-l_v}}{c(z)^{l_v}} \prod_{j=1}^{l_v-1} \frac{c(z)\|\mathbf{f}(z)\|^d}{|\tilde{Q}_{\zeta_v(j)}(z)(\mathbf{f}(z))|} \\
&\le \frac{A(z)^{q-l_v}}{c(z)^{l_v}} \prod_{j=1}^{n} \left(\frac{c(z)\|\mathbf{f}(z)\|^d}{|\tilde{Q}_{\zeta_v(t_j)}(z)(\mathbf{f}(z))|}\right)^{t_{v,j+1}-t_{v,j}} \\
&\le \frac{A(z)^{q-l_v}}{c(z)^{l_v}} \prod_{j=1}^{n} \left(\frac{c(z)\|\mathbf{f}(z)\|^d}{|\tilde{Q}_{\zeta_v(t_j)}(z)(\mathbf{f}(z))|}\right)^{\Delta_{\mathcal{Q},f}} \\
&\le C(z) \prod_{j=1}^{n} \left(\frac{\|\mathbf{f}(z)\|^d}{|P_{v,j}(z)(\mathbf{f}(z))|}\right)^{\Delta_{\mathcal{Q},f}},
\end{aligned} \tag{3.15}$$

where $C \in \mathcal{C}_{\mathcal{Q}}$, chosen commonly for all $\zeta_v \in \mathcal{I}$.

Now, for each non negative integer L, we denote by V_L the vector space (over $\mathcal{K}_{\mathcal{Q}}$) consisting of all homogeneous polynomials of degree L in $\mathcal{K}_{\mathcal{Q}}[x_0, \ldots, x_n]$ and the zero polynomial. Denote by $(P_{v,1}, \ldots, P_{v,n})$ the ideal in $\mathcal{K}_{\mathcal{Q}}[x_0, \ldots, x_n]$ generated by $P_{v,1}, \ldots, P_{v,n}$.

Lemma 3.16 (see [7, Proposition 3.3]). *Let $\{P_i\}_{i=1}^q$ $(q \ge n+1)$ be a set of homogeneous polynomials of common degree $d \ge 1$ in $\mathcal{K}_{\mathcal{Q}}[x_0, \ldots, x_n]$ in weakly general position. Then for any nonnegative integer N and for any $J := \{j_1, \ldots, j_n\} \subset \{1, \ldots, q\}$, the dimension of the vector space $\frac{V_L}{(P_{j_1}, \ldots, P_{j_n}) \cap V_L}$ is equal to the number of n-tuples $(s_1, \ldots, s_n) \in \mathbf{N}_0^n$ such that $s_1 + \cdots + s_n \le L$ and $0 \le s_1, \ldots, s_n \le d-1$. In particular, for all $L \ge n(d-1)$, we have*

$$\dim \frac{V_L}{(P_{j_1}, \ldots, P_{j_n}) \cap V_L} = d^n.$$

Choose a positive integer L (large enough) divisible by d and for each $(\mathbf{i}) = (i_1, \ldots, i_n) \in \mathbf{N}_0^n$ with $\sigma(\mathbf{i}) = \sum_{s=1}^n i_s \le \frac{L}{d}$, we set

$$W_{(\mathbf{i})}^v = \sum_{(\mathbf{j})=(j_1,\ldots,j_n)\ge(\mathbf{i})} P_{v,1}^{j_1} \cdots P_{v,n}^{j_n} \cdot V_{L-d\sigma(\mathbf{j})}.$$

It is clear that $W_{(0,\ldots,0)}^v = V_L$ and $W_{(\mathbf{i})}^v \supset W_{(\mathbf{j})}^v$ if $(\mathbf{i}) < (\mathbf{j})$ in the lexicographic ordering. Hence, $W_{(\mathbf{i})}^v$ is a filtration of V_L.

Let $(\mathbf{i}) = (i_1, \ldots, i_n), (\mathbf{i}') = (i'_1, \ldots, i'_n) \in \mathbf{N}_0^n$. Suppose that $(\mathbf{i}')$ follows $(\mathbf{i})$ in the lexicographic ordering. Similar as (3.4) in [16], we have

$$\dim \frac{W_{(\mathbf{i})}^v}{W_{(\mathbf{i}')}^v} = \dim \frac{V_{L-d\sigma(\mathbf{i})}}{(P_{v,1}, \ldots, P_{v,n}) \cap V_{L-d\sigma(\mathbf{i})}}. \tag{3.17}$$

Set $u = u_L := \dim V_L = \binom{L+n}{n}$. We assume that

$$V_L = W_{(\mathbf{i}_1)}^v \supset W_{(\mathbf{i}_2)}^v \supset \cdots \supset W_{(\mathbf{i}_K)}^v,$$

where $W^v_{(\mathbf{i}_{s+1})}$ follows $W^v_{(\mathbf{i}_s)}$ in the ordering and $(\mathbf{i}_K) = (\frac{L}{d}, 0, \ldots, 0)$. It is easy to see that K is the number of n-tuples $(i_1, \ldots, i_n)$ with $i_j \geq 0$ and $i_1 + \cdots + i_n \leq \frac{L}{d}$. Then we have

$$K = \binom{\frac{L}{d} + n}{n}.$$

For each $k \in \{1, \ldots, K-1\}$ we set $m^v_k = \dim \frac{W^v_{(\mathbf{i}_k)}}{W^v_{(\mathbf{i}_{k+1})}}$, and set $m^v_K = 1$. Then by Lemma 3.16, m^v_k does not depend on $\{P_{v,1}, \ldots, P_{v,n}\}$ and k, but on $\sigma(\mathbf{i}_k)$. Hence, we set $m_k = m^v_k$. We also note that

$$m_k = d^n \tag{3.18}$$

for all k with $L - d\sigma(\mathbf{i}_k) \geq nd$ (it is equivalent to $\sigma(\mathbf{i}_k) \leq \frac{L}{d} - n$).

From the above filtration, we may choose a basis $\{\psi^v_1, \cdots, \psi^v_u\}$ of V_L such that

$$\{\psi^v_{u-(m_s+\cdots+m_K)+1}, \ldots, \psi^v_u\}$$

is a basis of $W^v_{(\mathbf{i}_s)}$. For each $k \in \{1, \ldots, K\}$ and $l \in \{u - (m_k + \cdots + m_K) + 1, \ldots, u - (m_{k+1} + \cdots + m_K)\}$, we may write

$$\psi^v_l = P^{i_{1k}}_{v,1} \cdots P^{i_{nk}}_{v,n} h_l, \quad \text{where } (i_{1k}, \ldots, i_{nk}) = (\mathbf{i}_k), h_l \in W^v_{L-d\sigma(\mathbf{i}_k)}.$$

Then we have

$$\begin{aligned} |\psi^v_l(\mathbf{f}(z))| &\leq |P_{v,1}(\mathbf{f}(z))|^{i_{1k}} \cdots |P_{v,n}(\mathbf{f}(z))|^{i_{nk}} |h_l(\mathbf{f}(z))| \\ &\leq c_{v,l} |P_{v,1}(\mathbf{f}(z))|^{i_{1k}} \cdots |P_{v,n}(\mathbf{f}(z))|^{i_{nk}} ||\mathbf{f}(z)||^{L-d\sigma(\mathbf{i}_k)} \\ &= c_{v,l} \left(\frac{|P_{v,1}(\mathbf{f}(z))|}{||\mathbf{f}(z)||^d}\right)^{i_{1k}} \cdots \left(\frac{|P_{v,n}(\mathbf{f}(z))|}{||\mathbf{f}(z)||^d}\right)^{i_{nk}} ||\mathbf{f}(z)||^L, \end{aligned}$$

where $c_{v,l} \in \mathcal{C}_{\mathcal{Q}}$, which does not depend on f and z. Taking the product of the both sides of the above inequalities over all l and then taking logarithms, we obtain

$$\begin{aligned} \log \prod_{l=1}^{u} |\psi^v_l(\mathbf{f}(z))| \leq \sum_{k=1}^{K} m_k &\left(i_{1k} \log \frac{|P_{v,1}(\mathbf{f}(z))|}{||\mathbf{f}(z)||^d} + \cdots + i_{nk} \log \frac{|P_{v,n}(\mathbf{f}(z))|}{||\mathbf{f}(z)||^d} \right) \\ &+ uL \log ||\mathbf{f}(z)|| + \log c_I(z), \end{aligned} \tag{3.19}$$

where $c_v = \prod_{l=1}^{u} c_{v,l} \in \mathcal{C}_{\mathcal{Q}}$, which does not depend on f and z.

For each integer l $(0 \leq l \leq \frac{L}{d})$, we set $m(l) = m_k$, where k is an index such that $\sigma(\mathbf{i}_k) = l$. Since m_k only depends on $\sigma(\mathbf{i}_k)$, the above definition of $m(l)$ is well defined. We see that

$$\sum_{k=1}^{K} m_k i_{sk} = \sum_{l=0}^{\frac{L}{d}} \sum_{k|\sigma(\mathbf{i}_k)=l} m(l) i_{sk} = \sum_{l=0}^{\frac{L}{d}} m(l) \sum_{k|\sigma(\mathbf{i}_k)=l} i_{sk}.$$

Note that, by the symmetry $(i_1, \ldots, i_n) \to (i_{\sigma(1)}, \ldots, i_{\sigma(n)})$ with $\sigma \in S(n)$, $\sum_{k|\sigma(\mathbf{i}_k)=l} i_{sk}$ does not depend on s. We set

$$A := \sum_{k=1}^{K} m_k i_{sk}, \quad \text{which is independent of } s \text{ and } v.$$

Hence, (3.19) gives

$$\log\prod_{l=1}^{u}|\psi_l^v(\mathbf{f}(z))|\le A\left(\log\prod_{i=1}^{n}\frac{|P_{v,i}(\mathbf{f}(z))|}{||\mathbf{f}(z)||^d}\right)+uL\log||\mathbf{f}(z)||+\log c_v(z),$$

i.e.,

$$A\left(\log\prod_{i=1}^{n}\frac{||\mathbf{f}(z)||^d}{|P_{v,i}(\mathbf{f}(z))|}\right)\le\log\prod_{l=1}^{u}\frac{||\mathbf{f}(z)||^L}{|\psi_l^v(\mathbf{f}(z))|}+\log c_v(z),$$

Combining the above inequality with (3.15), we obtain that

$$\log\prod_{i=1}^{q}\frac{||\mathbf{f}(z)||^d}{|Q_i(\mathbf{f}(z))|}\le\frac{\Delta_{\mathcal{Q}}}{A}\log\prod_{l=1}^{u}\frac{||\mathbf{f}(z)||^L}{|\psi_l^v(\mathbf{f}(z))|}+\log c_0,\tag{3.20}$$

where c_0 is a function in $C_{\mathcal{Q}}$. We now write

$$\psi_l^v=\sum_{J\in\mathcal{T}_L}c_{lJ}^v x^J\in V_L,\quad c_{lJ}^v\in\mathcal{K}_{\mathcal{Q}},$$

where $\mathcal{T}_L$ is the set of all $(n+1)$-tuples $J=(i_0,\ldots,i_n)$ with $\sum_{s=0}^n j_s=L$, $x^J=x_0^{j_0}\cdots x_n^{j_n}$ and $l\in\{1,\ldots,u\}$. For each l, we fix an index $J_l^v\in J$ such that $c_{lJ_l^v}^v\not\equiv 0$. Define

$$\mu_{lJ}^v=\frac{c_{lJ}^v}{c_{lJ_l^v}^v},\ \ J\in\mathcal{T}_L.$$

Set $\Phi=\{\mu_{lJ}^v; I\subset\{1,\ldots,q\},\sharp I=n,1\le l\le M,J\in\mathcal{T}_L\}$. Note that $1\in\Phi$. Let $B=\sharp\Phi$. We see that $B\le u\binom{q}{n}(\binom{L+n}{n}-1)=\binom{L+n}{n}(\binom{L+n}{n}-1)\binom{q}{n}$. For each positive integer l, we denote by $\mathcal{L}(\Phi(l))$ the linear span over $\mathbb{C}$ of the set

$$\Phi(l)=\{\gamma_1\cdots\gamma_l;\gamma_i\in\Phi\}.$$

It is easy to see that

$$\dim\mathcal{L}(\Phi(l))\le\sharp\Phi(l)\le\binom{B+l-1}{B-1}.$$

We may choose a positive integer p such that

$$p\le p_0:=\left[\frac{B-1}{\log(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}})}\right]^2\ \text{and}\ \frac{\dim\mathcal{L}(\Phi(p+1))}{\dim\mathcal{L}(\Phi(p))}\le 1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}}.$$

Indeed, if $\dfrac{\dim\mathcal{L}(\Phi(p+1))}{\dim\mathcal{L}(\Phi(p))}>1+\dfrac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}}$ for all $p\le p_0$, we have

$$\dim\mathcal{L}(\Phi(p_0+1))\ge(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}})^{p_0}.$$

Therefore, we have

$$\begin{aligned}\log(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}}) &\le \frac{\log\dim\mathcal{L}(\Phi(p_0+1))}{p_0} \le \frac{\log\binom{B+p_0}{B-1}}{p_0}\\ &= \frac{1}{p_0}\log\prod_{i=1}^{B-1}\frac{p_0+i+1}{i} < \frac{(B-1)\log(p_0+2)}{p_0}\\ &\le \frac{B-1}{\sqrt{p_0}} \le \frac{(B-1)\log(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}})}{B-1}\\ &= \log(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}}).\end{aligned}$$

This is a contradiction.

We fix a positive integer p satisfying the above condition. Put $s=\dim\mathcal{L}(\Phi(p))$ and $t=\dim\mathcal{L}(\Phi(p+1))$. Let $\{b_1,\dots,b_t\}$ be an $\mathbb{C}$-basis of $\mathcal{L}(\Phi(p+1))$ such that $\{b_1,\dots,b_s\}$ be a $\mathbb{C}$-basis of $\mathcal{L}(\Phi(p))$.

For each $l\in\{1,\dots,u\}$, we set

$$\tilde{\psi}_l^v=\sum_{J\in\mathcal{T}_L}\mu_{lJ}^v x^J.$$

For each $J\in\mathcal{T}_L$, we consider homogeneous polynomials $\phi_J(x_0,\dots,x_n)=x^J$. Let F be a holomorphic map of $\mathbb{C}$ into $\mathbb{P}^{tu-1}(\mathbb{C})$ with a reduced representation $\mathbf{F}=(hb_i\phi_J(\mathbf{f}))_{1\le i\le t,J\in\mathcal{T}_L}$, where h is a nonzero meromorphic function on $\mathbb{C}$ with $\|\ T_h(r)=o(T_f(r))$. Since f is assumed to be $\mathcal{P}_c^1$-algebraically non-degenerate, F is $\mathcal{P}_c^1$-linearly non-degenerate. We also see that there exist nonzero functions $c_1,c_2\in\mathcal{C}_{\mathcal{Q}}$ such that

$$c_1|h|.\|\mathbf{f}\|^L\le\|\mathbf{F}\|\le c_2|h|.\|\mathbf{f}\|^L.$$

For each $l\in\{1,\dots,u\},1\le i\le s$, we consider the linear form L_{il}^v in x^J such that

$$hb_i\tilde{\psi}_l^v(\mathbf{f})=L_{il}^v(\mathbf{F}).$$

Since f is $\mathcal{P}_c^1$-algebraically non-degenerate, one has that $\{b_i\tilde{\psi}_l^v(\mathbf{f});1\le i\le s,1\le l\le u\}$ is linearly independent over $\mathbb{C}$, and so is $\{L_{il}^v(\mathbf{F});1\le i\le s,1\le l\le u\}$. This yields that $\{L_{il}^v;1\le i\le s,1\le l\le u\}$ is linearly independent over $\mathbb{C}$.

For every point z which is neither zero nor pole of any $hb_i\psi_l^v(\mathbf{f})$, we see that

$$\begin{aligned}s\log\prod_{l=1}^u\frac{||\mathbf{f}(z)||^L}{|\psi_l^v(\mathbf{f}(z))|}&=\log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{||\mathbf{F}(z)||}{|hb_i\psi_l^v(\mathbf{f}(z))|}+\log c_3(z)\\&=\log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{||\mathbf{F}(z)||\cdot||L_{il}^v||}{|L_{il}^v(\mathbf{F}(z))|}+\log c_4(z),\end{aligned}$$

where c_3, c_4 are nonzero functions in $\mathcal{C}_{\mathcal{Q}}$, not depend on f and v, but on $\{Q_i\}_{i=1}^q$. Combining this inequality and (3.20), we obtain that

$$\log\prod_{i=1}^q\frac{||\mathbf{f}(z)||^d}{|Q_i(\mathbf{f}(z))|}\le\frac{\Delta_{\mathcal{Q}}}{sA}\left(\max_v\log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{||\mathbf{F}(z)||\cdot||L_{il}^v||}{|L_{il}^v(\mathbf{F}(z))|}+\log c_4(z)\right)+\log c_0(z),\tag{3.21}$$

for all z outside an discrete subset of $\mathbb{C}$.

Since $\mathbf{F}$ is $\mathcal{P}_c^1$-linearly nondegenerate, $C(F)\not\equiv 0$. By Lemma 2.9, we have

$$\left\|\int_0^{2\pi}\max_v\left\{\log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{||\mathbf{F}(z)||\cdot||L_{il}^v||}{|L_{il}^v(\mathbf{F}(z))|}\right\}\frac{d\theta}{2\pi}\le tuT_F(r)-N_{C(F)}(r)+o(T_f(r)).\right.\tag{3.22}$$

Integrating both sides of (3.21) and using (3.22), we obtain that

$$\left\|\; qdT_f(r)-\sum_{i=1}^q N_{Q_i(f)}(r)\le\frac{tu\Delta_{\mathcal{Q}}}{sA}T_F(r)-\frac{\Delta_{\mathcal{Q}}}{sA}N_{C(F)}(r)+o(T_f(r)).\right.\tag{3.23}$$

We now estimate the quantity $\sum_{i=1}^q N_{Q_i(f)}(r)-\frac{\Delta_{\mathcal{Q}}}{sA}N_{C(F)}(r)$. Fix a point $z_1\in\mathbb{C}$ which is neither zero nor pole of h and of any nonzero coefficients of Q_i's. Suppose that $\nu_{Q_1(\mathbf{f})}^{[tu-1,c]}(z)\ge\nu_{Q_2(\mathbf{f})}^{[tu-1,c]}(z)\ge\cdots\ge\nu_{Q_q(\mathbf{f})}^{[tu-1,c]}(z)$ and ζ_1 is the identity mapping (i.e., $\zeta_1(i)=i$ for all $i=1,\dots,q$). One has $\nu_{Q_j(\mathbf{f})}^{[tu-1,c]}(z)=0$ for all $j\ge l_1$. Then by Lemma 2.6 we have

$$\begin{aligned}\sum_{i=1}^q\nu_{Q_i(\mathbf{f})}^{[tu-1,c]}(z_1)&\le\sum_{j=1}^{l_1}\nu_{Q_j(\mathbf{f})}^{[tu-1,c]}(z_1)\le\sum_{j=1}^n(t_{1,j+1}-t_{1,j})\nu_{Q_{t_{1,j}}(\mathbf{f})}^{[tu-1,c]}(z_1)\\&\le\Delta_{\mathcal{Q}}\sum_{j=1}^n\nu_{Q_{t_{1,j}}(\mathbf{f})}^{[tu-1,c]}(z_1)\le\Delta_{\mathcal{Q}}\sum_{j=1}^n\nu_{P_{1,j}(\mathbf{f})}^{[tu-1,c]}(z_1).\end{aligned}$$

Similarly as in the proof of Theorem 1.1, we have

$$\begin{aligned}\nu_{C(F)}(z_1)&\ge\sum_{\substack{1\le l\le u\\1\le i\le s}}\min_{0\le w\le tu-1}\nu^0_{\overline{L_{il}^1(\mathbf{F})}^{[w]}}(z_1)\ge s\sum_{1\le l\le u}\nu_{\tilde{\psi}_l^v(\mathbf{f})}^{[tu-1,c]}(z_1)\\&\ge s\sum_{\sigma(\mathbf{i}_k)\le L/d}\prod_{j=1}^n\nu_{P_{vj}^{i_{jk}}(\mathbf{f})}^{[tu-1,c]}(z_1)\ge s\sum_{k|L-d\sigma(\mathbf{i}_k)\ge 0}\prod_{j=1}^n i_{jk}\nu_{P_{vj}(\mathbf{f})}^{[tu-1,c]}(z_1)\\&=As\prod_{j=1}^n\nu_{P_{vj}(\mathbf{f})}^{[tu-1,c]}(z_1),\end{aligned}$$

where $\mathbf{i}_k=(i_{1k},\dots,i_{nk})$. Thus

$$\sum_{i=1}^q\nu_{Q_i(\mathbf{f})}(z_1)-\frac{1}{As}\nu_{C(F)}(z_1)\le\sum_{i=1}^q\nu_{Q_i(\mathbf{f})}(z_1)-\sum_{i=1}^q\nu_{Q_i(\mathbf{f})}^{[tu-1,c]}(z_1)=\sum_{i=1}^q\tilde{\nu}_{Q_i(\mathbf{f})}^{[tu-1,c]}(z_1).$$

This implies that

$$\| \sum_{i=1}^{q} N_{Q_i(\mathbf{f})}(r) - \frac{1}{As} N_{C(F)}(r) \le \sum_{j=1}^{n} \tilde{N}^{[tu-1,c]}_{P_{vj}(\mathbf{f})}(r) + o(T_f(r)). \tag{3.24}$$

Now we give some estimates for A, t and s. We use some following estimate from [16]:

$$\begin{aligned} &A \ge d^n \binom{\frac{L}{d}}{n+1} \quad \text{(see [16, page 622, line 1-2])} \\ &(1+x)^n \le 1 + (n+1)x \text{ for all } x \in \left(1, \frac{1}{(n+1)^2}\right) \quad \text{(see [16, ineq (3.18)]).} \end{aligned} \tag{3.25}$$

We chose $L = \lceil n+1+2\Delta_{\mathcal{Q}}(n+1)^3\epsilon^{-1}\rceil d$. Then L is divisible by d and we have

$$\frac{(n+1)d}{L-(n+1)d} = \frac{(n+1)d}{2\Delta_{\mathcal{Q}}(n+1)^3\epsilon^{-1}d} \le \frac{1}{2(n+1)^2}. \tag{3.26}$$

Therefore, using (3.25) and (3.26) we have

$$\begin{aligned} \frac{uL}{dA} &\le \frac{\binom{L+n}{n}L}{d^{n+1}\binom{\frac{L}{d}}{n+1}} = \frac{L\cdot(L+1)\cdots(L+n)}{1\cdot 2\cdots n} \Big/ \frac{(L-nd)\cdot(L-(n-1)d)\cdots L}{1\cdot 2\cdots (n+1)} \\ &= (n+1)\prod_{i=1}^{n} \frac{L+i}{L-(n-i+1)d} < (n+1)\Big(\frac{L}{L-(n+1)d}\Big)^n \\ &= (n+1)\left(1+\frac{(n+1)d}{L-(n+1)d}\right)^n < (n+1)\left(1+\frac{(n+1)^2d}{2\Delta_{\mathcal{Q}}(n+1)^3 I(\epsilon^{-1})d}\right) \\ &\le (n+1) + \frac{(n+1)^3 d}{2(n+1)^3\Delta_{\mathcal{Q}}\epsilon^{-1}} \le n+1+\frac{\epsilon}{2\Delta_{\mathcal{Q}}}. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{tuL}{dAs} &\le \left(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}}\right)\left(n+1+\frac{\epsilon}{2\Delta_{\mathcal{Q}}}\right) \\ &\le n+1+\frac{\epsilon}{2\Delta_{\mathcal{Q}}}+\frac{\epsilon}{3\Delta_{\mathcal{Q}}}+\frac{\epsilon}{6\Delta_{\mathcal{Q}}} = n+1+\frac{\epsilon}{\Delta_{\mathcal{Q}}}. \end{aligned} \tag{3.27}$$

Combining (3.23) and (3.27), we get

$$\| \; (q - \Delta_{\mathcal{Q}}(n+1) - \epsilon)\, T_f(r) \le \sum_{i=1}^{q} \frac{1}{d} \tilde{N}^{[tu-1,c]}_{Q_i(f)}(r) + o(T_f(r)). \tag{3.28}$$

Here we note that:

- $L := \lceil n+1+2\Delta_{\mathcal{Q}}(n+1)^3\epsilon^{-1}\rceil d$,
- $p_0 := \left[\dfrac{B-1}{\log(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}})}\right]^2 \le \left[\dfrac{\binom{L+n}{n}(\binom{L+n}{n}-1)\binom{q}{n}-1}{\log(1+\frac{\epsilon}{3(n+1)\Delta_{\mathcal{Q}}})}\right]^2$,
- $tu \le \binom{L+n}{n}\binom{B+p}{B-1} \le \binom{L+n}{n}p^{B-1} \le \binom{L+n}{n}p_0^{\binom{L+n}{n}(\binom{L+n}{n}-1)\binom{q}{n}-2} = L_0.$

By these estimates and by (3.28), we obtain

$$\| \ (q - \Delta_{\mathcal{Q}}(n+1) - \epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d} \tilde{N}^{[L_0-1]}_{Q_i(f)}(r) + o(T_f(r)).$$

The theorem is proved. □

Remark 3.29. If all hypersurfaces Q_i $(1 \leq i \leq q)$ are fixed then $t = s = 1$. Choosing $L = \lceil n+1+\Delta_{\mathcal{Q}}(n+1)^3\epsilon^{-1}\rceil d$, we have the estimate

$$\begin{aligned}
&\frac{(n+1)d}{L-(n+1)d} \leq \frac{(n+1)d}{\Delta_{\mathcal{Q}}(n+1)^3\epsilon^{-1}d} \leq \frac{1}{(n+1)^2},\\
&\qquad \frac{uL}{dA} \leq (n+1)\left(1+\frac{(n+1)d}{L-(n+1)d}\right)^n < (n+1)\left(1+\frac{(n+1)^2d}{\Delta_{\mathcal{Q}}(n+1)^3\epsilon^{-1}d}\right)\\
&\qquad\qquad \leq (n+1) + \frac{(n+1)^3d}{(n+1)^3\Delta_{\mathcal{Q}}\epsilon^{-1}} \leq n+1+\frac{\epsilon}{\Delta_{\mathcal{Q}}},
\end{aligned}$$

$$\begin{aligned}
u &= \binom{L+n}{n} \leq e^n\left(1+\frac{L}{n}\right)^n \leq e^n\left(\frac{n+(n+1)d}{n}+\frac{\lceil\Delta_{\mathcal{Q}}(n+1)^3(\epsilon^{-1})\rceil d}{n}\right)^n\\
&= e^n\lceil\Delta_{\mathcal{Q}}(n+1)^2\epsilon^{-1}\rceil^n d^n\left(1+\frac{1}{n}+\frac{n+(n+1)d}{n\lceil\Delta_{\mathcal{Q}}(n+1)^2\epsilon^{-1}\rceil d}\right)^n\\
&\leq e^n\lceil\Delta_{\mathcal{Q}}(n+1)^2\epsilon^{-1}\rceil^n d^n\cdot\left(1+\frac{1}{n}+\frac{2}{n(n+1)}\right)^n\\
&\leq e^n\lceil\Delta_{\mathcal{Q}}(n+1)^2\epsilon^{-1}\rceil^n d^n\cdot\left(1+\frac{2}{n}\right)^n \leq e^{n+2}\lceil\Delta_{\mathcal{Q}}(n+1)^2\epsilon^{-1}\rceil^n d^n
\end{aligned}$$

Therefore, from the proof of Theorem 1.3, we get the following theorem.

Theorem 3.30. *Let f be a $\mathcal{P}^1_c$-algebraic nondegenerate holomorphic curve of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ with hyperorder $\varsigma(f) < 1$. Let $\mathcal{Q} = \{Q_i\}_{i=1}^q$ be a family of q hypersurfaces with the distributive constant $\Delta_{\mathcal{Q}}$. Let $d = lcm(\deg Q_1, \dots, \deg Q_q)$. Then for any $\epsilon > 0$,*

$$\| \ (q - \Delta_{\mathcal{Q}}(n+1) - \epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{\deg Q_i} \tilde{N}^{[L_j-1,c]}_{Q_i(f)}(r) + o(T_f(r)),$$

where $L_j = \frac{\deg Q_j L_0}{d} - \left[\frac{\deg Q_j}{d}\right]$ and $L_0 = e^{n+2}\lceil\Delta_{\mathcal{Q}}(n+1)^2\epsilon^{-1}\rceil^n d^n$.

We see that in the case of holomorphic curves into $\mathbb{P}^N(\mathbb{C})$, the truncation level in this result is better than that in Theorem 1.1.

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