ON ABSOLUTE AND QUANTITATIVE SUBSPACE THEOREMS

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Abstract. The Absolute Subspace Theorem, a vast generalization and a quantitative improvement of Schmidt's Subspace Theorem, was first established by Evertse-Schlickewei and then strengthened remarkably by Evertse-Ferretti. We study quantitative generalizations and extensions of Subspace Theorems in various contexts. We establish a generalization of Evertse-Ferretti's Absolute Subspace Theorem for hyperplanes in general position. We obtain improved (non-absolute) Quantitative Subspace Theorems for hyperplanes in general position and in subgeneral position. We show a Semi-quantitative Subspace Theorem for hyperplanes in non-subdegenerate position.

1. Introduction

To set the stage for our discussion, let us first introduce some definitions and notations concerning normalized absolute values of algebraic numbers (Section 1.1), heights and twisted heights (Section 1.2). These basic materials are beautifully written in the classics [20, 22, 11, 1, 24]. We will briefly survey and (slightly) extend the Absolute Subspace Theorem of Evertse-Ferretti in Section 1.3. We will then formulate several new Subspace Theorems in Sections 1.3, 1.4, 1.5, and 1.6.

1.1. **Absolute values.** Throughout the paper, we work over a fixed algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers $\mathbb{Q}$. Let $\Sigma_{\mathbb{Q}}$ denote the set of places of $\mathbb{Q}$; we have $\Sigma_{\mathbb{Q}} = \{\infty\} \cup \{\text{prime numbers}\}$. The standard absolute values on $\mathbb{Q}$ are the usual Euclidean absolute value $|\cdot|_\infty$ and the p-adic absolute values $|\cdot|_p$ which satisfy $|p|_p = \frac{1}{p}$ for each prime number p.

Let $F \subset \overline{\mathbb{Q}}$ be a number field, i.e. F is both a subfield of $\overline{\mathbb{Q}}$ and a finite extension of $\mathbb{Q}$. Let Σ_F denote the set of places of F; write $\Gamma_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. For each place $v \in \Sigma_F$ which lies over $p \in \Sigma_{\mathbb{Q}}$, put $d(v|p) = \frac{[F_v:\mathbb{Q}_p]}{[F:\mathbb{Q}]}$ and let $\|\cdot\|_v$ denote the *normalized absolute value* whose restriction to $\mathbb{Q}$ is $|\cdot|_p^{d(v|p)}$. The Product Formula asserts that

$$\prod_{v\in\Sigma_F} \|x\|_v = 1 \text{ for } x \in F^\times.$$

Suppose that F/E is a finite extension of number fields and that $v \in \Sigma_F$ extends $u \in \Sigma_E$; define the *local degree fraction* $d(v|u) = \frac{[F_v:E_u]}{[F:E]}$. Then

$$\|x\|_v = \|x\|_u^{d(v|u)} \text{ for } x \in E.$$

Furthermore, for each $u \in \Sigma_E$, If $S \subset \Sigma_E$, put $S^F = \{v \in \Sigma_F : v|u \text{ for some } u \in S\}$.

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Putting infinite places and finite places on equal footing, for a place $v \in \Sigma_F$ one defines [9, pp. 234-235]

$$\begin{aligned} \kappa(v) &= \frac{1}{[F:\mathbb{Q}]} && \text{if } v \text{ is real infinite,} \\ \kappa(v) &= \frac{2}{[F:\mathbb{Q}]} && \text{if } v \text{ is complex infinite,} \\ \kappa(v) &= 0 && \text{if } v \text{ is finite.} \end{aligned}$$

One has

$$\sum_{v \in \Sigma_F} \kappa(v) = \sum_{v|\infty} \kappa(v) = 1. \tag{1.1}$$

Absolute values of number fields can be extended (non-canonically) to their algebraic closure as follows. For $v \in \Sigma_F$, let $\overline{F_v}$ be the algebraic closure of the completion F_v. The absolute value $\|\cdot\|_v$ has a unique extension, denoted by $\|\cdot\|'_v$, from F_v to $\overline{F_v}$. We may choose and henceforth fix an embedding τ_v over F of $\overline{\mathbb{Q}}$ into $\overline{F_v}$, setting

$$\|x\|_v = \|\tau_v(x)\|'_v \text{ for } x \in \overline{\mathbb{Q}}.$$

Suppose that F/E is a Galois extension of number fields and that $v \in \Sigma_F$ extends $u \in \Sigma_E$. Then there exists an automorphism $\tau_{v|u} \in \mathrm{Gal}(F/E)$ such that

$$\|x\|_v = \|\tau_{v|u}(x)\|_u^{d(v|u)} \text{ for } x \in F.$$

1.2. **Heights and twisted heights.** Let us recall some notions of heights in projective spaces.

If $\mathbf{x}' = (x_0, \ldots, x_n) \in F^{n+1}$, we put $\|\mathbf{x}'\|_v = \max_{0 \le i \le n} \|x_i\|_v$ for $v \in \Sigma_F$, defining the *multiplicative Weil height* $H(\mathbf{x}') = \prod_{v \in \Sigma_F} \|\mathbf{x}'\|_v$ and the *logarithmic Weil height* $h(\mathbf{x}') = \log H(\mathbf{x}')$. If $\mathbf{x}' \in \overline{\mathbb{Q}}^{n+1}$, we may choose a number field F containing all coordinates of $\mathbf{x}'$ and define the multiplicative and logarithmic Weil heights of $\mathbf{x}'$ accordingly; the heights of $\mathbf{x}' \in \overline{\mathbb{Q}}^{n+1}$ does not depend on the choice of a number field containing its coordinates. Hence, H and h extend to functions on $\overline{\mathbb{Q}}^{n+1}$, called the *multiplicative absolute Weil height* and the *logarithmic absolute Weil height* respectively. Moreover, it follows from the Product Formula that $H(\alpha\mathbf{x}') = H(\mathbf{x}')$ and $h(\alpha\mathbf{x}') = h(\mathbf{x}')$ for $\alpha \in \overline{\mathbb{Q}}^{\times}$ and $\mathbf{x}' \in \overline{\mathbb{Q}}^{n+1}$. Thus, H and h descend to functions on $\mathbb{P}^n(\overline{\mathbb{Q}})$.

If $L(X_0, \ldots, X_n) = a_0X_0 + \cdots + a_nX_n \in \overline{\mathbb{Q}}[X_0, \ldots, X_n]$ is a linear form, its *multiplicative homogeneous height* is $H^*(L) = H(\mathbf{a}_L)$ where $\mathbf{a}_L = (1, a_1, \ldots, a_n)$, and its *logarithmic homogeneous height* is $h^*(L) = \log H^*(L)$.

If $\mathbf{x}' = (x_0, \ldots, x_n) \in \overline{\mathbb{Q}}^{n+1}$ and $\sigma \in \Gamma_F$, let us write

$$\mathbf{x}'^{\sigma} = \sigma(\mathbf{x}') := (\sigma(x_0), \ldots, \sigma(x_n)).$$

This action of Γ_F on $\overline{\mathbb{Q}}^{n+1}$ induces an obvious action of Γ_F on $\mathbb{P}^n(\overline{\mathbb{Q}})$.

We now recall the notion of a twisted height [6, Section 2.2].

Definition 1.1 (Twisted Height). *Let F be a number field. Let n be a positive integer and $\widetilde{L} = (L_i^{(v)} : 0 \le i \le n, v \in \Sigma_F)$ be a system of linear forms $L_i^{(v)} \in F[x_0, \ldots, x_n]$. Let $\widetilde{\gamma} = (\gamma_{iv} : 0 \le i \le n, v \in \Sigma_F)$ be a tuple of real numbers.*

(1) *The pair $(\widetilde{L}, \widetilde{\gamma})$ is called a* twisting datum over F *if it satisfies the following conditions:*
  (i) *the set $\bigcup_{v\in\Sigma_F}\{L_i^{(v)} : 0 \le i \le n\}$ is finite;*
  (ii) *for all but finitely many $v \in \Sigma_F$, one has $\gamma_{0v} = \cdots = \gamma_{nv} = 0$;*
  (iii) *for each $v \in \Sigma_F$, the linear forms $L_0^{(v)}, \ldots, L_n^{(v)}$ are linearly independent over F.*

(2) *Let K/F be a finite extension of number fields. If $w \in \Sigma_K$ extends $v \in \Sigma_F$ and $0 \le i \le n$, set*
$$L_i^{(w)} = L_i^{(v)}, \quad \gamma_{iw} = d(w|v) \cdot \gamma_{iv}.$$
Put
$$\widetilde{L}^K = (L_i^{(w)} : 0 \le i \le n, w \in \Sigma_K),$$
$$\widetilde{\gamma}^K = (\gamma_{iw} : 0 \le i \le n, w \in \Sigma_K).$$
We call the pair $(\widetilde{L}^K, \widetilde{\gamma}^K)$ the induction *of the pair $(\widetilde{L}, \widetilde{\gamma})$ from F to K. It is apparent that if $(\widetilde{L}, \widetilde{\gamma})$ is a twisting datum over F, then $(\widetilde{L}^K, \widetilde{\gamma}^K)$ is a twisting datum over K.*

(3) *Suppose that $(\widetilde{L}, \widetilde{\gamma})$ is a twisting datum over F. Let $Q \in \mathbb{R}_{>0}$. Define the* twisted height $H_{\widetilde{L},\widetilde{\gamma},Q}(\mathbf{x})$ *of $\mathbf{x} \in \mathbb{P}^n(F)$ by writing $\mathbf{x} = (x_0 : \ldots : x_n)$ with $x_i \in F$ for $0 \le i \le n$ and setting*
$$H_{\widetilde{L},\widetilde{\gamma},Q}(\mathbf{x}) = \prod_{v\in\Sigma_F} \max_{0\le i\le n} \left(\|L_i^{(v)}(x_0, \ldots, x_n)\|_v Q^{\gamma_{iv}} \right);$$
by the Product Formula, this does not depend on the choice of coordinates of $\mathbf{x}$.

The twisted height $H_{\widetilde{L},\widetilde{\gamma},Q}$ can be extended from $\mathbb{P}^n(F)$ to $\mathbb{P}^n(\overline{\mathbb{Q}})$ as follows. Let $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$. There is a finite extension K/F such that $\mathbf{x} \in \mathbb{P}^n(K)$; write $\mathbf{x} = (x_0 : \ldots : x_n)$ with $x_i \in K$ for $0 \le i \le n$. Consider $L_i^{(w)} \in \widetilde{L}^K$ and $\gamma_{iw} \in \widetilde{\gamma}^K$ for $0 \le i \le n$ and $w \in \Sigma_K$ where $(\widetilde{L}^K, \widetilde{\gamma}^K)$ is the induction of $(\widetilde{L}, \widetilde{\gamma})$ from F to K. Set
$$H_{\widetilde{L},\widetilde{\gamma},Q}(\mathbf{x}) = \prod_{w\in\Sigma_K} \max_{0\le i\le n} \left(\|L_i^{(w)}(x_0, \ldots, x_n)\|_w Q^{\gamma_{iw}} \right);$$
by the Product Formula, this does not depend on the choice of coordinates of $\mathbf{x}$. Furthermore, $H_{\widetilde{L},\widetilde{\gamma},Q}(\mathbf{x})$ is independent of the choice of the number field K for which $\mathbf{x} \in \mathbb{P}^n(K)$. Thus the twisted height $H_{\widetilde{L},\widetilde{\gamma},Q}$ is well-defined as a function on $\mathbb{P}^n(\overline{\mathbb{Q}})$.

Remark 1.2. *By* [6, Lemma 4.1], *the twisted height is Galois-invariant: if $(\widetilde{L}, \widetilde{\gamma})$ is a twisting datum over F, $Q \in \mathbb{R}_{>0}$, $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and $\sigma \in \Gamma_F$, then $H_{\widetilde{L},\widetilde{\gamma},Q}(\mathbf{x}^\sigma) = H_{\widetilde{L},\widetilde{\gamma},Q}(\mathbf{x})$.*

Definition 1.3 (Parallelepiped). *Let F be a number field; let $S \subset \Sigma_F$ be a finite set of places of F. Let n be a positive integer and $\widetilde{L}_S = (L_i^{(v)} : 0 \le i \le n, v \in S)$ be a system* *of linear forms* $L_i^{(v)} \in F[x_0, \dots, x_n]$*. Let* $\widetilde{\gamma}_S = (\gamma_{iv} : 0 \le i \le n, v \in S)$ *be a tuple of real numbers. Put*

$$\Delta_v = \| \det(L_0^{(v)}, \dots, L_n^{(v)}) \|_v^{\frac{1}{n+1}}.$$

Define the parallelepiped $\Pi_F(\widetilde{L}_S, \widetilde{\gamma}_S)$ *to be the set of* $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ *which satisfies the system of inequalities*

$$\max_{\sigma \in \Gamma_F} \frac{\|L_i^{(v)}(\mathbf{x}^\sigma)\|_v}{\|\mathbf{x}^\sigma\|_v} \le \Delta_v H(\mathbf{x})^{-\gamma_{iv}} \tag{1.2}$$

for all $0 \le i \le n$ *and all* $v \in S$*. Here the ratios of absolute values of points in projective spaces are given by*

$$\frac{\|L_i^{(v)}(\mathbf{y})\|_v}{\|\mathbf{y}\|_v} := \frac{\|L_i^{(v)}(y_0, \dots, y_n)\|_v}{\|(y_0, \dots, y_n)\|_v}$$

where $\mathbf{y} = [y_0 : \dots : y_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$*; this ratio is independent of the choice of coordinates of* $\mathbf{y}$*.*

1.3. **Absolute Subspace Theorem.** In 1972, Wolfgang M. Schmidt discovered his celebrated Subspace Theorem [19], which is a beautiful generalization of Roth's theorem [17] in Diophantine approximation. Schmidt established the first quantitative version of his Subspace Theorem in [21]. Schmidt's foundational works opened a totally new field of explorations and has been an inspiration for many generations of mathematicians. In their breakthrough works [7, 8], Evertse and Schlickewei established several 'absolute' generalizations of the Schmidt's Subspace Theorem: the Twisted Height Theorem, the Absolute Parametric Subspace Theorem, and the Absolute Subspace Theorem. Evertse and Ferretti established impressive generalizations and improvements of these results in a series of papers [4, 5, 6]. We refer the readers to the beautiful and comprehensive surveys of Evertse-Schlickewei [7], Evertse [10] and Bugeaud [2].

The Twisted Height Theorem says that there are a finite number of proper linear subspaces of $\mathbb{P}^n(\overline{\mathbb{Q}})$ such that for each sufficiently large Q, the points in $\mathbb{P}^n(\overline{\mathbb{Q}})$ with small Q-twisted heights are contained in one of these subspaces.

Theorem 1.4 (Twisted Height Theorem)**.** [6, Theorem 2.1] *Let* E *be a number field. Let* $\widetilde{L} = (L_i^{(u)} : 0 \le i \le n, u \in \Sigma_E)$ *be a system of linear forms in* $E[x_0, \dots, x_n]$*. Let* $Q \in \mathbb{R}_{>0}$ *and* $\widetilde{\zeta} = (\zeta_{iu} : 0 \le i \le n, u \in \Sigma_E)$ *be a tuple of real numbers.*

Suppose that $(\widetilde{L}, \widetilde{\zeta})$ *is a twisting datum over* E*. Write*

$$\{L_1, \dots, L_r\} = \bigcup_{u \in \Sigma_E} \{L_i^{(u)} : 0 \le i \le n\} \tag{1.3}$$

and let $R \ge r$ *be arbitrary. Put*

$$H_{\widetilde{L}} = \prod_{u \in \Sigma_E} \max_{1 \le i_0 < \dots < i_n \le r} \| \det(L_{i_0}, \dots, L_{i_n}) \|_u,$$

$$\Delta_u = \| \det(L_0^{(u)}, \dots, L_n^{(u)}) \|_u^{\frac{1}{n+1}}.$$

Suppose further that

$$\sum_{i=0}^{n} \zeta_{iu} = 0 \text{ for } u \in \Sigma_E, \tag{1.4}$$

$$\sum_{u\in\Sigma_E} \min_{0\le i\le n} \zeta_{iu} \ge -1. \tag{1.5}$$

Let $0 < \delta < 1$*. Then there are proper linear subspaces* $T_1, \dots, T_t$ *of* $\mathbb{P}^n(\overline{\mathbb{Q}})$ *defined over* E*, with*

$$t \le t_0(n, R, \delta) := 10^6 4^{n+1}(n+1)^{10}\delta^{-3}\log\left(3\delta^{-1}R\right)\log\left(\delta^{-1}\log(3R)\right) \tag{1.6}$$

satisfying the following property: if

$$Q \ge Q_0(n, R, H_{\widetilde{L}}, \delta) := \max\left(H_{\widetilde{L}}^{\frac{1}{R}}, (n+1)^{\frac{1}{\delta}}\right), \tag{1.7}$$

then there is $1 \le s \le t$ *such that*

$$\left\{\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}}) : H_{\widetilde{L},\widetilde{\zeta},Q}(\mathbf{x}) \le \frac{\prod_{u\in\Sigma_E}\Delta_u}{Q^\delta}\right\} \subset T_s.$$

The Absolute Parametric Subspace Theorem says that each parallelepiped is contained in a finite union of proper linear subspaces of $\mathbb{P}^n(\overline{\mathbb{Q}})$. Its formulation in [6, Theorem 3.1] does not apply directly to our problem because its hypotheses are a bit restricted. Theorem 1.5 below is a (slightly) extended version of [6, Theorem 3.1] to accommodate our situation.

Theorem 1.5 (Absolute Parametric Subspace Theorem)**.** *Let* E *be a number field of degree* d*; let* S *be a finite set of places of* E*. Let* $\widetilde{M_S} = (M_i^{(u)} : 0 \le i \le n, u \in S)$ *be a system of linear forms* $M_i^{(u)} \in \overline{\mathbb{Q}}[x_0, \dots, x_n]$*. Let* $\widetilde{\eta}_S = (\eta_{iu} : 0 \le i \le n, u \in S)$ *be a tuple of real numbers. Put*

$$R := \#\left(\bigcup_{u\in S}\{M_i^{(u)} : 0 \le i \le n\}\right). \tag{1.8}$$

Let $D > 0$ *be such that for all* $0 \le i \le n$ *and* $u \in S$*,*

$$[E(M_i^{(u)}) : E] \le D. \tag{1.9}$$

Let $H^* > 0$ *be such that for all* $0 \le i \le n$ *and* $u \in S$*,*

$$H^*(M_i^{(u)}) \le H^*. \tag{1.10}$$

Suppose that for each $u \in S$*, the linear forms* $M_0^{(u)}, \dots, M_n^{(u)}$ *are linearly independent over* $\overline{\mathbb{Q}}$*.*

Suppose further that there are constants $0 < \delta < 1$ *and* $0 < \iota < 1$ *such that*

$$\sum_{u\in S}\sum_{i=0}^{n} \eta_{iu} = n + 1 + \delta, \tag{1.11}$$

$$\sum_{u\in S}\min_{0\le i\le n}\eta_{iu}\ge\frac{n+1+\delta}{n+1}-\frac{1}{\iota}. \tag{1.12}$$

Then there exist proper linear subspaces $T_1,\dots,T_{t_1}$ of $\mathbb{P}^n(\overline{\mathbb{Q}})$ defined over E, with

$$t_1=t_1(n,R,D,\delta,\iota):=10^6 4^{n+1}(n+1)^{13}(\delta\iota)^{-3}\log^2\left(\frac{6RD(n+1)}{\delta\iota}\right) \tag{1.13}$$

satisfying the following property: if $\mathbf{x}\in\Pi_E(\widetilde{M}_S,\widetilde{\eta}_S)$ and

$$H(\mathbf{x})\ge H_1(n,D,H^*,\delta,\iota):=\max\left((n+1)^{\frac{\iota}{4D}}(H^*)^{\frac{\iota}{2}},(n+1)^{\frac{n+1}{\delta}}\right), \tag{1.14}$$

then

$$\mathbf{x}\in\bigcup_{j=1}^{t_1}T_j.$$

Corollary 1.6. *Theorem 1.5 holds true if the hypothesis* (1.12) *is replaced by*

$$\sum_{u\in\Sigma_E}\min_{0\le i\le n}\eta_{iu}\ge-\frac{\delta}{n+1}, \tag{1.15}$$

the number of subspaces (1.13) *is replaced by*

$$t\le t_1'(n,R,D,\delta):=10^6 4^{n+1}(n+1)^{10}\left(\frac{n+1+2\delta}{\delta}\right)^3\log^2\left(\frac{12R^2D}{\delta}\right), \tag{1.16}$$

and the height lower bound (1.14) *is replaced by*

$$H(\mathbf{x})\ge H_1'(n,D,H^*,\delta):=\max\left((n+1)^{\frac{1}{4D}}(H^*)^{\frac{1}{2}},(n+1)^{\frac{n+1}{\delta}}\right). \tag{1.17}$$

Proof. Corollary 1.6 follows immediately from Theorem 1.5 on taking $\iota:=\frac{n+1}{n+1+2\delta}<1$. □

Corollary 1.7. *Theorem 1.5 holds true if the hypothesis* (1.12) *is replaced by*

$$\sum_{u\in\Sigma_E}\min_{0\le i\le n}\eta_{iu}\ge-(n+1), \tag{1.18}$$

the number of subspaces (1.13) *is replaced by*

$$t\le t_1''(n,R,D,\delta):=10^6 4^{n+1}(n+1)^{10}\delta^{-3}\log^2\left(\frac{12R^3D}{\delta}\right), \tag{1.19}$$

and the height lower bound (1.14) *is replaced by*

$$H(\mathbf{x})\ge H_1''(n,H^*,\delta):=\max\left(2(H^*)^{\frac{1}{2(n+3)}},(n+1)^{\frac{n+1}{\delta}}\right). \tag{1.20}$$

Proof. Corollary 1.7 follows immediately from Theorem 1.5 on taking $\iota:=\frac{1}{n+3}$ and noting that $n^{\frac{1}{n}}<2$ for any positive integer n. □

The Absolute Subspace Theorem says that there are a finite number of linear subspaces such that the points in $\mathbb{P}^n(\overline{\mathbb{Q}})$ satisfying an inequality of the Schmidt-type are contained in the union of these subspaces. The following theorem is a more precise version of [6, Corollary 3.2]; the implicit height bound H_0 in [6, Corollary 3.2] is made explicit here.

Theorem 1.8 (Absolute Subspace Theorem). *Let E be a number field of degree d; let S be a finite set of places of E of cardinality $|S| = s$. Let $\widetilde{N}_S = (N_i^{(u)} : 0 \le i \le n, u \in S)$ be a system of linear forms in $\overline{\mathbb{Q}}[x_0, \dots, x_n]$. Suppose that for each $u \in S$, the set $\{N_0^{(u)}, \dots, N_n^{(u)}\}$ is linearly independent over $\overline{\mathbb{Q}}$. Put*

$$R := \# \left(\bigcup_{u \in S} \{N_i^{(u)} : 0 \le i \le n\} \right). \tag{1.21}$$

Let $D > 0$ be such that for all $0 \le i \le n$ and $u \in S$,

$$[E(N_i^{(u)}) : E] \le D. \tag{1.22}$$

Let $H^* > 0$ be such that for all $0 \le i \le n$ and $u \in S$,

$$H^*(N_i^{(u)}) \le H^*. \tag{1.23}$$

Let $0 < \delta < 1$. Then there are proper linear subspaces $T_1, \dots, T_{t_2}$ of $\mathbb{P}^n(\overline{\mathbb{Q}})$ defined over E, with

$$\begin{aligned} t_2 &= t_2(n, s, R, D, \delta) \\ &:= \left(4e(n+1)^2\delta^{-1}\right)^{(n+1)s} 10^6 4^{n+\frac{7}{2}} (n+1)^{10} \delta^{-3} \log^2 \left(\frac{24R^3D}{\delta} \right), \end{aligned} \tag{1.24}$$

satisfying the following property. The set of solutions $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ of the inequalities

$$\prod_{u \in S} \prod_{i=0}^{n} \max_{\sigma \in \Gamma_E} \frac{\|N_i^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u} \le \frac{\prod_{u \in S} \|\det(N_0^{(u)}, \dots, N_n^{(u)})\|_u}{H(\mathbf{x})^{n+1+\delta}}$$

and

$$H(\mathbf{x}) \ge H_2(n, H^*, \delta) := \max\left(2(H^*)^{\frac{1}{2(n+3)}}, (n+1)^{\frac{2(n+1)}{\delta}}\right) \tag{1.25}$$

is contained in

$$\bigcup_{j=1}^{t_2} T_j.$$

Remark 1.9. *Theorem 1.8 of Evertse-Schlickewei gives an upper bound for the number of subspaces containing 'large' solutions. The first estimate of Schmidt* [21] *for the number of subspaces containing large solutions was $2^{2^{27n\delta^{-2}}}$, doubly exponential in $n\delta^{-2}$ and only for $|S| = 1$. Evertse-Schlickewei* [8, Theorem 3.1] *remarkably improved this bound be exponential in n^2 and $ns(\log n + \log \delta^{-1})$. The best known estimate* (1.24) *of Evertse-Ferretti is only exponential in $ns(\log n + \log \delta^{-1})$.*

We now introduce a variant of local Weil functions, which will be utilized to formulate a useful consequence of the Absolute Subspace Theorem.

Definition 1.10. *Let $F \subset \overline{\mathbb{Q}}$ be a number field. Let $L \in F[x_0, \dots, x_n]$ be a linear form. Let $\mathbf{x} = [x_0 : \dots : x_n] \in \mathbb{P}^n(F)$. For a place $v \in \Sigma_F$, define*

$$\lambda_v(\mathbf{x}, L) = \log \frac{(n+1)^{\kappa(v)} \|L\|_v \|\mathbf{x}\|_v}{\|L(\mathbf{x})\|_v},$$

so that $\lambda_v(\mathbf{x}, L) \geq 1$ *(see* (2.4)*). Here for* $L = \sum_{i=0}^n a_i x_i \in F[x_0, \ldots, x_n]$, *the absolute value* $\|L\|_v$ *is given by*

$$\|L\|_v = \max_{0 \leq i \leq n} \|a_i\|_v.$$

The following result is a non-absolute analog, formulated in terms of local Weil functions, of Theorem 1.8.

Corollary 1.11 (Quantitative Subspace Theorem)**.** *Let* F *be a number field; let* S *be a finite set of places of* F *of cardinality* $|S| = s$. *Let* $\widetilde{L}_S = (L_i^{(v)} : 0 \leq i \leq n, v \in S)$ *be a system of linear forms in* $F[x_0, \ldots, x_n]$. *Suppose that for each* $v \in \Sigma_F$, *the set* $\{L_0^{(v)}, \ldots, L_n^{(v)}\}$ *is linearly independent over* F. *Put*

$$R := \# \left(\bigcup_{v \in S} \{L_i^{(v)} : 0 \leq i \leq n\} \right). \tag{1.26}$$

Let $h^* > 0$ *be such that for all* $0 \leq i \leq n$ *and* $v \in S$,

$$h^*(L_i^{(v)}) \leq h^*. \tag{1.27}$$

Let $0 < \delta < 1$. *Then there are proper linear subspaces* $T_1, \ldots, T_{t_{\text{basic}}}$ *of* $\mathbb{P}^n(F)$, *with*

$$\begin{aligned} t_{\text{basic}} &= t_{\text{basic}}(n, s, R, \delta) \\ &:= \left(4e(n+1)^2\delta^{-1}\right)^{(n+1)s} 10^6 4^{n+\frac{7}{2}} (n+1)^{10} \delta^{-3} \log^2\left(\frac{24R^3}{\delta}\right), \end{aligned} \tag{1.28}$$

satisfying the following property. Every $\mathbf{x} \in \mathbb{P}^n(F) \backslash \bigcup_{j=1}^{t_{\text{basic}}} T_j$ *satisfies either*

$$\sum_{v \in S} \sum_{i=0}^{n} \lambda_v(\mathbf{x}, L_i^{(v)}) < (n+1+\delta)h(\mathbf{x}) + K_{\text{basic}}, \tag{1.29}$$

where

$$K_{\text{basic}} = K_{\text{basic}}(n, s, h^*) := s(n+1)h^* + \left(\frac{s}{2} + 1\right)(n+1)\log(n+1), \tag{1.30}$$

or

$$h(\mathbf{x}) < h_{\text{basic}}(n, h^*, \delta) := \max\left(\frac{1}{2(n+3)}(h^* + \log 2), \frac{2(n+1)}{\delta}\log(n+1)\right). \tag{1.31}$$

1.4. **Hyperplanes in general position.** In this section, we consider the context of hyperplanes in general position, in which the number of hyperplanes may be larger than the number of variables.

Definition 1.12. *Let* k *be a field. Let* $q \geq n$ *be positive integers. A system of linear forms* $(L_i : 0 \leq i \leq q)$ *in* $k[x_0, \ldots, x_n]$ *is said to be in* general position *if for each subset* $I \subseteq \{0, 1, \ldots, q\}$ *of cardinality* $n+1$, *the set* $\{L_i : i \in I\}$ *is linearly independent over* k.

In the literature, the Absolute Subspace Theorem has not been generalized for hyperplanes in general position. This paper offers the following effective result.

Theorem 1.13 (Absolute Subspace Theorem for Hyperplanes in General Position)**.** *Let E be a number field of degree d; let S be a finite set of places of E of cardinality $|S| = s$. Let $q \geq n$ be positive integers. Let $\widetilde{N}_S = (N_i^{(u)} : 0 \leq i \leq q, u \in S)$ be a system of linear forms in $\overline{\mathbb{Q}}[x_0, \ldots, x_n]$ such that for each $u \in S$, the linear forms $(N_i^{(u)} : 0 \leq i \leq q)$ are in general position. Put*

$$R := \# \left(\bigcup_{u \in S} \{N_i^{(u)} : 0 \leq i \leq q\} \right). \tag{1.32}$$

Let $D > 0$ be such that for all $0 \leq i \leq q$ and $u \in S$,

$$[E(N_i^{(u)}) : E] \leq D. \tag{1.33}$$

Let $H^* > 0$ be such that for all $0 \leq i \leq q$ and $u \in S$,

$$H^*(N_i^{(u)}) \leq H^*. \tag{1.34}$$

Let $0 < \delta < 1$. Then there are proper linear subspaces $T_1, \ldots, T_{t_2}$ of $\mathbb{P}^n(\overline{\mathbb{Q}})$ defined over E, with

$$t_2 = t_2(n, s, R, D, \delta)$$

given by (1.24), satisfying the following property. The set of solutions $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ of the inequalities

$$\prod_{u \in S} \prod_{i=0}^{q} \max_{\sigma \in \Gamma_E} \frac{\|N_i^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u} \leq \frac{\prod_{u \in S} \left(\min_{0 \leq i_0 < \cdots < i_n \leq q} \| \det(N_{i_0}^{(u)}, \ldots, N_{i_n}^{(u)}) \|_u \right)}{H(\mathbf{x})^{n+1+2\delta}} \tag{1.35}$$

and

$$H(x) \geq H_2(n, H^*, \delta),$$

where $H_2(n, H^*, \delta)$ is defined by (1.25), and

$$H(\mathbf{x}) \geq \left(\frac{(n+1)^{\frac{n+2}{2}} \left((n+1)^{\frac{1}{2}} H^* \right)^{s(n+1)D^{n+1}}}{\prod_{u \in S} \left(\min_{0 \leq i_0 < \cdots < i_n \leq q} \prod_{t=0}^{n} \min(1, \|N_{i_t}^{(u)}\|_u) \right)} \right)^{\frac{q-n}{\delta}}, \tag{1.36}$$

is contained in

$$\bigcup_{j=1}^{t_2} T_j.$$

The following consequence is a non-absolute version of Theorem 1.13, formulated in terms of local Weil functions.

Corollary 1.14 (Quantitative Subspace Theorem for Hyperplanes in General Position)**.** *Let F be a number field; let S be a finite set of places of F of cardinality $|S| = s$. Let $q \geq n$ be positive integers. Let $\widetilde{L}_S = (L_i^{(v)} : 0 \leq i \leq q, v \in S)$ be a system of linear forms in $F[x_0, \ldots, x_n]$ such that for each $v \in S$, the linear forms $(L_i^{(v)} : 0 \leq i \leq q)$ are in general position. Suppose also that for all $0 \leq i \leq q$ and $v \in S$,*

$$\|L_i^{(v)}\|_v \geq 1. \tag{1.37}$$

Put

$$R := \#\left(\bigcup_{v\in S}\{L_i^{(v)} : 0 \le i \le q\}\right). \tag{1.38}$$

Let $h^* > 0$ *be such that for all* $0 \le i \le q$ *and* $v \in S$,

$$h^*(L_i^{(v)}) \le h^*. \tag{1.39}$$

Let $0 < \delta < 1$. *Then there are proper linear subspaces* $T_1, \ldots, T_{t_{\text{gen}}}$ *of* $\mathbb{P}^n(F)$, *with*

$$\begin{aligned} t_{\text{gen}} &= t_{\text{gen}}(n, s, R, \delta) \\ &:= \left(4e(n+1)^2\delta^{-1}\right)^{(n+1)s} 10^6 4^{n+\frac{7}{2}}(n+1)^{10}\delta^{-3}\log^2\left(\frac{24R^3}{\delta}\right), \end{aligned} \tag{1.40}$$

satisfying the following property. Every $\mathbf{x} \in \mathbb{P}^n(F)\setminus\bigcup_{j=1}^{t_{\text{gen}}} T_j$ *satisfies either*

$$\sum_{v\in S}\sum_{i=0}^{q}\lambda_v(\mathbf{x}, L_i^{(v)}) < (n+1+3\delta)h(\mathbf{x}) \tag{1.41}$$

or

$$\begin{aligned} h(\mathbf{x}) &< h_{\text{gen}}(n, q, s, h^*, \delta) \\ &:= \frac{(q-n+1)(n+2)(s+1)}{\delta}\left(\log(n+1) + h^*\right). \end{aligned} \tag{1.42}$$

1.5. **Hyperplanes in subgeneral position.** One can relax the notion of hyperplanes in general position as follows.

Definition 1.15. *Let* k *be a field. Let* $q \ge m \ge n$ *be positive integers. A system of linear forms* $(L_i : 0 \le i \le q)$ *in* $k[x_0, \ldots, x_n]$ *is said to be in* m-subgeneral position *if for each subset* $I \subseteq \{0, 1, \ldots, q\}$ *of cardinality* $m+1$, *the vector subspace spanned by the coefficient vectors of the linear forms* $\{L_i : i \in I\}$ *is the whole space* k^{n+1}.

The Quantitative Subspace Theorem can be further generalized for hyperplanes in subgeneral position.

Theorem 1.16 (Quantitative Subspace Theorem for Hyperplanes in Subgeneral Position)**.** *Let* F *be a number field; let* S *be a finite set of places of* F *of cardinality* $|S| = s$. *Let* $q \ge m \ge n$ *be positive integers; suppose that* $q > 2m - n$.

Let $\widetilde{L} = (L_i : 0 \le i \le q)$ *be a system of linear forms in* $F[x_0, \ldots, x_n]$ *which are in* m*-subgeneral position. Suppose also that for all* $0 \le i \le q$ *and* $v \in S$,

$$\|L_i\|_v \ge 1. \tag{1.43}$$

Let $h^* > 0$ *be such that for all* $0 \le i \le q$,

$$h^*(L_i) \le h^*. \tag{1.44}$$

Let $0 < \delta < 1$. Then there are proper linear subspaces $T_1, \dots, T_{t_{\text{subgen}}}$ of $\mathbb{P}^n(F)$, with

$$t_{\text{subgen}} = t_{\text{subgen}}(n, q, \delta, s)$$
$$:= \left(4e(n+1)^2\delta^{-1}\right)^{(n+1)s} 10^6 4^{n+\frac{7}{2}} (n+1)^{10} \delta^{-3} \log^2\left(\frac{24q^3}{\delta}\right), \tag{1.45}$$

satisfying the following property. Every $\mathbf{x} \in \mathbb{P}^n(F) \setminus \bigcup_{j=1}^{t_{\text{subgen}}} T_j$ satisfies either

$$\sum_{v \in S} \sum_{i=0}^{q} \lambda_v(\mathbf{x}, L_i) < (2m+1-n)\left(1 + \frac{\delta}{n+1}\right) h(\mathbf{x}) + K_{\text{subgen}} \tag{1.46}$$

where

$$K_{\text{subgen}} = K_{\text{subgen}}(n, m, q, s, h^*)$$
$$:= \frac{(2m-n+1)(q-m+1)}{n+1}\left(s(n+1)h^* + \frac{s(n+1)+2n+4}{2}\log(n+1)\right) \tag{1.47}$$

or

$$h(\mathbf{x}) < h_{\text{subgen}}(n, q, s, h^*, \delta)$$
$$:= \max\left(\frac{1}{2(n+3)}(h^* + \log 2), \frac{2(n+1)}{\delta}\log(n+1)\right). \tag{1.48}$$

Remark 1.17. *Qualitative but non-quantitative Subspace Theorems for hyperplanes in subgeneral position were proved by Ru and Wong* [18, Theorem 3.5]*, and by Vojta* [23, Theorem 1.2]*. To our best knowledge, there is only one Quantitative Subspace Theorem in this context which was due to Hirata-Kohno* [12, Theorem 4.1]*. The bound therein for the number of subspaces containing large solutions is doubly exponential in n and exponential in δ^{-2}. Proposition 5.1 is an improvement of* [12, Theorem 4.1] *in terms of the number of the exceptional subspaces; the bound here is only exponential in $ns(logn + \log\delta^{-1})$. Theorem 1.16, rather than Proposition 5.1, is a more useful formulation because it does not require the quite intricate Nochka's weights.*

1.6. **Hyperplanes in non-subdegenerate position.** Let us recall some notions of linear algebra which pertain to Subspace Theorems.

Definition 1.18. *Let k be a field and V be a k-vector space.*

1. *For a subset $\mathcal{S}$ of V, denote by $(\mathcal{S})_k$ the k-vector subspace of V spanned by $\mathcal{S}$.*
2. *A nonempty subset $\mathcal{R}$ of V is said to be* non-degenerate *if $(\mathcal{R})_k = V$ and for every nonempty proper subset $\mathcal{S}$ of $\mathcal{R}$, the intersection $(\mathcal{S})_k \cap (\mathcal{R}\backslash\mathcal{S})_k$ contains a nonzero element of $\mathcal{R}$.*
3. *A nonempty subset $\mathcal{R}$ of V is said to be* non-subdegenerate *if $(\mathcal{R})_k = V$ and for every nonempty proper subset $\mathcal{S}$ of $\mathcal{R}$ one has*
$$(\mathcal{S})_k \cap (\mathcal{R}\backslash\mathcal{S})_k \neq \{0\}.$$
4. *A system of linear forms $L_1, \dots, L_q \in k[x_0, \dots, x_n]$ is said to be* non-degenerate *(resp.* non-subdegenerate*) if the set of its coefficient vectors is non-degenerate (resp. non-subdegenerate) in k^{n+1}.*

(5) *A nonempty subset $\mathcal{R}$ of V is said to be* minimally dependent (over k) *if it is linearly dependent over k and every nonempty proper subset $\mathcal{S} \subset \mathcal{R}$ is linearly independent over k.*

Remark 1.19. *We give an example of a minimally dependent subset. Let V be a k-vector space. Let $v_1, \ldots, v_r$ be linearly independent vectors in V; let $c_1, \ldots, c_r \in k \setminus \{0\}$ and put $v_0 = c_1 v_1 + \cdots + c_r v_r$. Then the set $\{v_0, v_1, \ldots, v_r\}$ is minimally dependent over k.*

Let $\mathcal{L} = \{L_1, \ldots, L_q\}$ be a non-subdegenerate system of linear forms in $k[x_0, \ldots, x_n]$. By definition, there is a family $\widetilde{\mathcal{L}}$ of less than 2^{q-1} hyperplanes which contains $\mathcal{L}$ and satisfies that for each nonempty proper subset $\mathcal{L}_1 \subset \mathcal{L}$, there exists a hyperplane $L \in \widetilde{\mathcal{L}}$ such that $L \in (\mathcal{L}_1)_k \cap (\mathcal{L} \setminus \mathcal{L}_1)_k$. Such a family $\widetilde{\mathcal{L}}$ is called a *conjunction* of $\mathcal{H}$; in particular $q \leq |\widetilde{\mathcal{L}}| < 2^{q-1}$. Note that, if $\mathcal{L}$ is non-degenerate then $\widetilde{\mathcal{L}} = \mathcal{L}$ is a conjunction of $\mathcal{L}$.

Chen and Ru [3] introduced the notion of non-degenerate hyperplanes and proved a Subspace Theorem for such hyperplanes. The result of [3] was improved considerably by Liu in [13]. However, we see that the non-degenerate requirement that the intersection $(\mathcal{S})_k \cap (\mathcal{R} \setminus \mathcal{S})_k$ contains a nonzero element of $\mathcal{R}$ is difficult to verify in practice. Recently, in order to improve the second main theorem for moving targets with truncated counting function, Quang in [16] introduced the notion of non-subdegeneracy as above. Motivated by the method of Liu [13] and Quang [16], Hiep [15] obtained a qualitative Subspace Theorem for a system of hyperplanes in non-subdegenerate position. However, in [15] the finite set of exceptional subspaces was not quantified explicitly. Our next theorem generalizes and improves the above-mentioned results of Chen-Ru, Liu and Hiep to the setting of non-subdegenerate family of hyperplanes, the number of exceptional subspaces being estimated explicitly.

Theorem 1.20 (Semi-quantitative Subspace Theorem for Hyperplanes in Non-subdegenerate Position)**.** *Let F be a number field; let S be a finite set of places of F.*

Let $q \geq n$ be positive integers. Let $\mathcal{L} = (L_i : 0 \leq i \leq q)$ be a non-subdegenerate system of linear forms in $F[x_0, \ldots, x_n]$ and $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$ be a conjunction of $\mathcal{L}$. In particular, $q \leq |\widetilde{\mathcal{L}}| < 2^{q-1}$.

Let $0 < \delta < 1$. Then every $\mathbf{x} \in \mathbb{P}^n(F)$ which is outside of the union of hyperplanes in $\widetilde{\mathcal{L}}$ satisfies

$$\sum_{v \in S} \sum_{i=0}^{q} \lambda_v(\mathbf{x}, L_i) < (q - 1 + \delta) h(\mathbf{x}) + O(1). \tag{1.49}$$

1.7. The remainder of the paper is organized as follows.

In Section 2, we collect preparatory tools concerning absolute values, heights, and Nochka's weights. Those preliminaries will be utilized in later sections to establish various subspace theorems.

Section 3 contains the proofs of the Absolute Parametric Subspace Theorem (Theorem 1.5), the Absolute Subspace Theorem (Theorem 1.8) and a basic Quantitative Subspace Theorem (Corollary 1.11). In Section 3.1, we deduce Theorem 1.5 from Theorem 1.4. In Section 3.2, we deduce Theorem 1.8 as a consequence of Corollary 1.7, thereby deriving Corollary 1.11.

In Section 4, we work with hyperplanes in general position. We establish a generalized Absolute Subspace Theorem (Theorem 1.13) and also derive an analogous (non-absolute) Quantitative Subspace Theorem (Corollary 1.14), formulated in terms of local Weil functions, in this context.

Section 5 studies hyperplanes in subgeneral position. In this setting, we first invoke the existence of Nochka's weights (Theorem 2.3) to show a preliminary form (Proposition 5.1) and then establish a Quantitative Subspace Theorem (Theorem 1.16).

Finally, in Section 6 we prove a Semi-quantitative Subspace Theorem for hyperplanes in non-subdegenerate position (Theorem 1.20).

2. Preliminaries

2.1. **Inequalities of absolute values and heights.** In the literature there are several normalizations of absolute values and heights which are different from ours. It is useful to be aware of these different choices. In this section, we work with a fixed number field $F \subset \overline{\mathbb{Q}}$.

If $\mathbf{x}' = (x_0, \ldots, x_n) \in \overline{\mathbb{Q}}^{n+1}$ and $v \in \Sigma_F$, one defines [6, Section 6.1]

$$\|\mathbf{x}'\|_{v,2} = \left(\sum_{i=0}^{n} \|x_i\|_v^{\frac{2}{\kappa(v)}} \right)^{\frac{\kappa(v)}{2}} \qquad \text{if } v \text{ is infinite,}$$

$$\|\mathbf{x}'\|_{v,2} = \|\mathbf{x}'\|_v \qquad \text{if } v \text{ is finite.}$$

It is plain that

$$\|\mathbf{x}'\|_v \leq \|\mathbf{x}'\|_{v,2} \leq (n+1)^{\frac{\kappa(v)}{2}} \|\mathbf{x}'\|_v. \tag{2.1}$$

If the number field F contains all the coordinates of $\mathbf{x}'$, the height

$$H_2(\mathbf{x}') = \prod_{v \in \Sigma_F} \|\mathbf{x}'\|_{v,2}$$

does not depend on the choice of F and hence is a well-defined function on $\overline{\mathbb{Q}}^{n+1}$. By the Product Formula, it descends to a height function, also denoted by H_2, on $\mathbb{P}^n(\overline{\mathbb{Q}})$. Moreover, for $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ one has [6, equation (6.3) p. 535]

$$(n+1)^{-\frac{1}{2}} H_2(\mathbf{x}) \leq H(\mathbf{x}) \leq H_2(\mathbf{x}). \tag{2.2}$$

For $v \in \Sigma_F$ and $x_1, \ldots, x_n \in F$, there holds the triangle inequality [6, equation (6.2) p. 534]

$$\|x_1 + \cdots + x_n\|_v \leq n^{\kappa(v)} \max_{1 \leq i \leq n} \|x_i\|_v. \tag{2.3}$$

If, furthermore, $a_1, \ldots, a_n \in F$, one has the Cauchy-Schwarz inequality, which follows from [6, equation (6.4) p. 535] and (2.1),

$$\|a_1 x_1 + \cdots + a_n x_n\|_v \leq n^{\kappa(v)} \max_{1 \leq i \leq n} \|a_i\|_v \max_{1 \leq i \leq n} \|x_i\|_v. \tag{2.4}$$

We will need to compare the absolute value of the determinant of a matrix with the absolute values of its rows or columns. Let $L_1, \ldots, L_n$ be linear forms in $\overline{\mathbb{Q}}[x_1, \ldots, x_n]$ and $v \in \Sigma_F$. On the one hand, the Hadamard inequality gives an upper bound [6, p. 531]

$$\|\det(L_1, \ldots, L_n)\|_v \le n^{\frac{n\kappa(v)}{2}} \prod_{i=1}^{n} \max(1, \|L_i\|_v). \tag{2.5}$$

On the other hand, if the linear forms are linearly independent, then we have a lower bound given by the following result.

Lemma 2.1. *Let $F \subset \overline{\mathbb{Q}}$ be a number field and $v \in \Sigma_F$. Let $L_1, \ldots, L_n$ be linear forms in $\overline{\mathbb{Q}}[x_1, \ldots, x_n]$. Suppose that $[F(L_i) : F] \le D$ and $H(L_i) \le H$. If $L_1, \ldots, L_n$ are linearly independent over $\overline{\mathbb{Q}}$, then*

$$\frac{\|\det(L_1, \ldots, L_n)\|_v}{\prod_{i=1}^{n} \|L_i\|_v} \ge \left(Hn^{\frac{1}{2}}\right)^{-nD^n}. \tag{2.6}$$

Proof. This is immediate on combining [9, Lemma 2 p. 234] with (2.2). □

The Hadamard inequality (2.5) and Lemma 2.6 can be generalized for exterior products as follows. For $1 \le k \le n$, let $L_1, \ldots, L_k$ be linear forms in $\overline{\mathbb{Q}}[x_1, \ldots, x_n]$ and $v \in \Sigma_F$. Write the coefficient vector of L_i as $(a_{i1}, \ldots, a_{in})$. Put $N = \binom{n}{k}$ and let $C(n, k)$ be the sequence of k-element subsets of $\{1, 2, \ldots, n\}$, ordered arbitrarily; the length of $C(n, k)$ is N. The l^{th}-subset $\{l_1 < l_2 < \cdots < l_k\}$ in this sequence gives rise to the determinant

$$A_l := \det(a_{i,l_j})_{i,j=1,\ldots,k} \qquad (1 \le l \le N).$$

The exterior product of $L_1, \ldots, L_k$ is given by

$$L_1 \wedge \ldots \wedge L_k := (A_1, \ldots, A_N).$$

One has [6, equation (6.7) p. 536]

$$\|L_1 \wedge \ldots \wedge L_k\|_{v,2} \le \prod_{i=1}^{k} \|L_i\|_{v,2}. \tag{2.7}$$

Hence, by (2.1),

$$\|L_1 \wedge \ldots \wedge L_k\|_v \le n^{\frac{k\kappa(v)}{2}} \prod_{i=1}^{k} \|L_i\|_v. \tag{2.8}$$

The following lower bounds generalize Lemma 2.1.

Lemma 2.2. *Let $F \subset \overline{\mathbb{Q}}$ be a number field and $v \in \Sigma_F$. Let $L_1, \ldots, L_k$ be linear forms in $\overline{\mathbb{Q}}[x_1, \ldots, x_n]$. Suppose that $[F(L_i) : F] \le D$ and $H(L_i) \le H$. If $L_1, \ldots, L_k$ are linearly independent over $\overline{\mathbb{Q}}$, then*

$$\frac{\|L_1 \wedge \ldots \wedge L_k\|_{v,2}}{\prod_{i=1}^{k} \|L_i\|_{v,2}} \ge \left(Hn^{\frac{1}{2}}\right)^{-kD^k}. \tag{2.9}$$

Consequentially,

$$\frac{\|L_1 \wedge \ldots \wedge L_k\|_v}{\prod_{i=1}^{k} \|L_i\|_v} \ge \left(Hn^{\frac{1}{2}}\right)^{-kD^k} \binom{n}{k}^{-\frac{\kappa(v)}{2}}. \tag{2.10}$$

Proof. The inequality (2.9) is [9, Lemma 2 p. 234]. The inequality (2.10) follows immediately from (2.9) and (2.1). □

2.2. **Nochka's weights.** We recall the notion and the existence of Nochka's weights, which are crucial for reducing the setting of subgeneral position to that of general position. The following result summarizes the main properties of these weights (cf. [14, Theorem 4.1.10 and Lemma 4.1.17]).

Theorem 2.3. *Let k be a field. Let $q \geq m \geq n$ be positive integers; suppose that $q > 2m - n$. Let $\mathcal{L} = (L_i : 0 \leq i \leq q)$ be a system of linear forms in $k[x_0, \ldots, x_n]$ in m-subgeneral position. Then there exist $\omega_j \in \mathbb{Q}$ $(0 \leq j \leq q)$ satisfying the following properties:*

(i) $0 < \omega_j \leq 1$ *for* $0 \leq j \leq q$.

(ii) *Let $R \subseteq \{0, 1, \ldots, q\}$ be a subset with cardinality $0 < |R| \leq N + 1$. Suppose the vector subspace of k^{n+1} spanned by the linear forms L_j with $j \in R$ has dimension r. Then*

$$\sum_{j \in R} \omega_j \leq r.$$

(iii) *Let $\Lambda_j \geq 1$ $(0 \leq j \leq q)$ be arbitrary reals. Let $R \subseteq \{0, 1, \ldots, q\}$ be a subset with cardinality $0 < |R| \leq m + 1$. Suppose the vector subspace $\mathcal{L}_R$ of k^{n+1} spanned by the linear forms L_j with $j \in R$ has dimension r. Then there exist $j_1, \ldots, j_r \in R$ such that $\mathcal{L}_R$ is spanned by $L_{j_1}, \ldots, L_{j_r}$ and that*

$$\prod_{j \in R} \Lambda_j^{\omega_j} \leq \prod_{i=1}^{r} \Lambda_{j_i}.$$

(iv) *Put* $\tilde{\omega} = \max_{0 \leq j \leq q} \omega_j$. *Then*

$$\frac{n+1}{2m - n + 1} \leq \tilde{\omega} \leq \frac{n}{m}.$$

(v) *One has*

$$\sum_{j=0}^{q} \omega_j = \tilde{\omega}(q - 2m + n) + (n + 1).$$

2.3. **Non-subdegeneracy.** The following lemma was essentially proved in [16, Lemma 3.2]. For the sake of clarity and completeness, we rewrite both the statement and the proof below.

Lemma 2.4. *Let k be a field and $q \geq 1$ an integer. Let $\mathcal{L} = \{L_0, L_1, \ldots, L_q\}$ be a non-subdegenerate system of linear forms in $k[x_0, \ldots, x_n]$. For a subset $I \subseteq \{0, 1, \ldots, q\}$, write $\mathcal{L}_I = \{L_i : i \in I\}$. Then there exist subsets $I_0, I_1, \ldots, I_p$ of $\{0, 1, \ldots, q\}$ with $1 \leq p \leq q$ satisfying the following conditions:*

(i) $I_0 = \{0\}$ *and* $I_i \cap I_j = \emptyset$ *for* $i \neq j$;

(ii) *for every* $0 \leq j \leq p$, *the set* $\mathcal{L}_{I_j}$ *is linearly independent over* k;

(iii) $\left(\bigcup_{j=1}^{p} \mathcal{L}_{I_j} \right)_k = k^{n+1}$;

(iv) *for every* $1 \le t \le p$*, the vector space*

$$(\mathcal{L}_{I_t})_k \cap (\mathcal{L}_{I_0} \cup \mathcal{L}_{I_1} \cup \cdots \cup \mathcal{L}_{I_{t-1}})_k$$

is one-dimensional;

(v) *for every* $1 \le t \le p$*, there exist*

$$c_{t\alpha} \in k \qquad (\alpha \in \bigcup_{j=1}^{t} I_j)$$

such that $c_{t\alpha} \neq 0$ *when* $\alpha \in I_t$ *and that*

$$\sum_{\alpha \in I_0 \cup I_1 \cup \cdots \cup I_t} c_{t\alpha} L_\alpha = 0.$$

Proof. Put $J = \{0, 1, \ldots, q\}$ and set $I_0 = \{0\}$. We shall construct the subsets $I_1, \ldots, I_p$ recursively as follows.

Step 1: Since

$$(\mathcal{L}_{I_0})_k \cap (\mathcal{L}_{J \setminus I_0})_k \neq \{0\},$$

there exist a positive integer r together with r indices $1 \le i'_1 < \cdots < i'_r \le q$ such that

$$L_0 \in (L_{i'_1}, \ldots, L_{i'_r})_k.$$

Let r_1 be the smallest such integer r and suppose that $1 \le i_1 < \cdots < i_{r_1} \le q$ satisfies

$$L_0 \in (L_{i_1}, \ldots, L_{i_{r_1}})_k.$$

Set $I_1 = \{i_1, \ldots, i_{r_1}\}$. It is evident that

$$(\mathcal{L}_{I_0})_k \cap (\mathcal{L}_{I_1})_k$$

is the one-dimensional vector space spanned by L_0. Because of the smallest property of r_1, the set $\mathcal{L}_{I_1}$ is linearly independent, and there exist $c_{1\alpha} \in k^\times$ $(\alpha \in I_0 \cup I_1)$ such that

$$\sum_{\alpha \in I_0 \cup I_1} c_{1\alpha} L_\alpha = 0.$$

Step 2: Suppose that for some $t \ge 1$, the subsets $I_0, I_1, \ldots, I_t$ have been constructed such that they satisfy all the conditions of the lemma except possibly for (iii). Put

$$J_t = I_0 \cup I_1 \cup \cdots \cup I_t.$$

If $(\mathcal{L}_{J_t})_k = k^{n+1}$, then $(\mathcal{L}_{I_1 \cup \cdots \cup I_t})_k = k^{n+1}$. Therefore, the subsets $I_0, I_1, \ldots, I_t$ satisfy all the requirements and the construction is complete.

Now suppose that $(\mathcal{L}_{J_t})_k$ is a proper vector subspace of $(\mathcal{L})_k = k^{n+1}$. In particular, J_t is a proper subset of J. By definition of non-subdegeneration, we have that

$$(\mathcal{L}_{J_t})_k \cap (\mathcal{L}_{J \setminus J_t})_k \neq \{0\}.$$

Then there exist a positive integer r together with r indices $1 \le i'_1 < \cdots < i'_r \le q$ in $J \setminus J_t$ such that

$$(\mathcal{L}_{J_t})_k \cap (L_{i'_1}, \ldots, L_{i'_r})_k \neq \{0\}.$$

Let r_{t+1} be the smallest such integer r and suppose that $1 \le i_1 < \cdots < i_{r_{t+1}} \le q$ in $J \setminus J_t$ satisfy

$$(\mathcal{L}_{J_t})_k \cap (L_{i_1}, \ldots, L_{i_{r_{t+1}}})_k \neq \{0\}.$$

Set $I_{t+1} = \{i_1, \ldots, i_{r_{t+1}}\}$.

It follows from the smallest property of r_{t+1} that the set $\mathcal{L}_{I_{t+1}}$ is linearly independent. We claim that the vector space

$$(\mathcal{L}_{J_t})_k \cap (\mathcal{L}_{I_{t+1}})_k$$

is one-dimensional. Indeed, suppose by contradiction that there exist two linearly independent linear forms H and K in this vector space. Write

$$H = \sum_{i \in I_{t+1}} a_i L_i \in (\mathcal{L}_{J_t})_k,$$

$$K = \sum_{i \in I_{t+1}} b_i L_i \in (\mathcal{L}_{J_t})_k,$$

where $a_i, b_i \in k$. By the smallest property of r_{t+1}, all the coefficients a_i and b_i are nonzero. On fixing any $\alpha \in I_{t+1}$, we infer from the linear independence of H and K that

$$b_\alpha H - a_\alpha K = \sum_{\substack{i \in I_{t+1} \\ i \neq \alpha}} (b_\alpha a_i - a_\alpha b_i) L_i \in (\mathcal{L}_{J_t})_k.$$

is a nonzero vector contradicting the smallest property of r_{t+1}. The claim follows.

It also follows from the smallest property of r_{t+1} that there exist elements

$$c_{t+1,\alpha} \in k \qquad (\alpha \in J_t \cup I_{t+1})$$

such that $c_{t+1,\alpha} \neq 0$ whenever $\alpha \in I_{t+1}$ and that

$$\sum_{\alpha \in J_t \cup I_{t+1}} c_{t+1,\alpha} L_\alpha = 0.$$

Finally, we loop through Step 2. The algorithm must stop at some point because we are working in a finite-dimensional vector space. The lemma is proved. □

3. Proofs of Evertse-Ferretti's Absolute Subspace Theorem

3.1. **The Absolute Parametric Subspace Theorem.** In this section we deduce the Absolute Parametric Subspace Theorem (Theorem 1.5) as a consequence of the Twisted Height Theorem (Theorem 1.4).

Proof of Theorem 1.5. Let E be a number field; let S be a finite set of places of E. Let

$$\widetilde{M}_S = (M_i^{(u)} : 0 \leq i \leq n, u \in S)$$

be a system of linear forms $M_i^{(u)} \in \overline{\mathbb{Q}}[x_0, \ldots, x_n]$. Let

$$\widetilde{\eta}_S = (\eta_{iu} : 0 \leq i \leq n, u \in S)$$

be a tuple of real numbers. Let $\delta > 0$. Suppose that for each $u \in S$, the linear forms $M_0^{(u)}, \ldots, M_n^{(u)}$ are linearly independent over E. Suppose further that there are constants $0 < \delta < 1$ and $0 < \iota < 1$ such that

$$\sum_{u \in S} \sum_{i=0}^{n} \eta_{iu} = n + 1 + \delta, \tag{3.1}$$

$$\sum_{u\in S}\min_{0\le i\le n}\eta_{iu}\ge\frac{n+1+\delta}{n+1}-\frac{1}{\iota}. \tag{3.2}$$

Let $\mathbf{x}\in\mathbb{P}^n(\overline{\mathbb{Q}})$ be a point in the parallelepiped $\Pi_E(\widetilde{M}_S,\widetilde{\eta}_S)$. We proceed to construct a twisted height estimate for $\mathbf{x}$ in four steps.

Step 1: We extend $(\widetilde{M}_S,\widetilde{\eta}_S)$ to a twisting datum over E by defining for $0\le i\le n$ and $u\in\Sigma_E\backslash S$ the linear forms $M_i^{(u)}:=X_i$ and the exponents $\eta_{iu}=0$. By construction the pair $(\widetilde{M},\widetilde{\eta})$ given by

$$\widetilde{M}=(M_i^{(u)}:0\le i\le n,u\in\Sigma_E)$$
$$\widetilde{\eta}=(\eta_{iu}:0\le i\le n,u\in\Sigma_E)$$

is a twisting datum over E.

Step 2: Let F/E be a finite Galois extension such that F contains all coefficients of the linear forms $M_i^{(u)}$ for $0\le i\le n$ and $u\in\Sigma_E$. Recall that every absolute value of a number field extends to $\overline{\mathbb{Q}}$. Recall also that $S^F=\{v\in\Sigma_F:v|u\text{ for some }u\in S\}$. If $v\in\Sigma_F$ extends $u\in\Sigma_E$, then there is an automorphism $\tau_{v|u}\in\mathrm{Gal}(F/E)$ such that $\|\alpha\|_v=\|\tau_{v|u}(\alpha)\|_u^{d(v|u)}$ for $\alpha\in F$.

We now define the twisting datum of our targeted twisted height. For $0\le i\le n$ and $v\in\Sigma_F$, set

$$L_i^{(v)}:=\tau_{v|u}^{-1}(M_i^{(u)}), \tag{3.3}$$

$$\zeta_{iv}:=\iota\cdot d(v|u)\left(\eta_{iu}-\frac{1}{n+1}\sum_{j=0}^{n}\eta_{ju}\right). \tag{3.4}$$

It is plain that the pair $(\widetilde{L},\widetilde{\zeta})$ given by

$$\widetilde{L}:=(L_i^{(v)}:0\le i\le n,v\in\Sigma_F)$$
$$\widetilde{\zeta}:=(\zeta_{iv}:0\le i\le n,v\in\Sigma_F)$$

is a twisting datum over F. By the defining formulas (1.8) and (1.9) we have that

$$\#\left(\bigcup_{v\in\Sigma_F}\{L_i^{(v)}:0\le i\le n\}\right)\le n+1+RD. \tag{3.5}$$

Step 3: Let us fix an automorphism $\sigma\in\Gamma_E$. Let K/E be a finite Galois extension such that K contains F and all coordinates of $\mathbf{x}^\sigma$. Let $(\widetilde{L}^K,\widetilde{\zeta}^K)$ be the induction from F to K of $(\widetilde{L},\widetilde{\zeta})$. By (3.3), if $0\le i\le n$ and $w\in\Sigma_K$ extends $u\in\Sigma_E$, then $L_i^{(w)}=\tau_{w|u}^{-1}(M_i^{(u)})$.

If $0\le i\le n$ and $w\in S^K$, then since $\mathbf{x}\in\Pi_E(\widetilde{M}_S,\widetilde{\eta}_S)$ we have

$$\frac{\|L_i^{(w)}(\mathbf{x}^\sigma)\|_w}{\|\mathbf{x}^\sigma\|_w}=\left(\frac{\|(\tau_{w|u}L_i^{(w)})(\mathbf{x}^{\tau_{w|u}\sigma})\|_u}{\|\mathbf{x}^{\tau_{w|u}\sigma}\|_u}\right)^{d(w|u)}=\left(\frac{\|M_i^{(u)}(\mathbf{x}^{\tau_{w|u}\sigma})\|_u}{\|\mathbf{x}^{\tau_{w|u}\sigma}\|_u}\right)^{d(w|u)}$$
$$\le\left(\Delta_uH(\mathbf{x})^{-\eta_{iu}}\right)^{d(w|u)}=\Delta_wH(\mathbf{x})^{-\eta_{iw}}. \tag{3.6}$$

On the other hand, if $0 \le i \le n$ and $w \in \Sigma_K \backslash S^K$, then by the construction in Step 1 we have

$$\frac{\|L_i^{(w)}(\mathbf{x}^\sigma)\|_w}{\|\mathbf{x}^\sigma\|_w} = \frac{\|X_i(\mathbf{x}^\sigma)\|_w}{\|\mathbf{x}^\sigma\|_w} \le 1 = \Delta_w H(x)^{-\eta_{iw}}. \tag{3.7}$$

Step 4: Next we define the twisting parameter

$$Q := H(\mathbf{x})^{\frac{1}{\iota}}. \tag{3.8}$$

We consider each factor in the twisted height

$$H_{\widetilde{L},\widetilde{\zeta},Q}(\mathbf{x}^\sigma) = \prod_{w \in \Sigma_K} \max_{0 \le i \le n} \left(\|L_i^{(w)}(\mathbf{x}^\sigma)\|_w Q^{\zeta_{iw}} \right).$$

Assume first that $w \in S^K$. By (3.4) and (3.6), we have

$$\max_{0 \le i \le n} \left(\frac{\|L_i^{(w)}(\mathbf{x}^\sigma)\|_w}{\|\mathbf{x}^\sigma\|_w} Q^{\zeta_{iw}} \right) \le \Delta_w H(\mathbf{x})^{-\frac{d(w|u)}{n+1}\sum_{j=0}^n \eta_{ju}}. \tag{3.9}$$

Assume now that $w \in \Sigma_K \backslash S^K$. By (3.4) and and (3.7), we have

$$\max_{0 \le i \le n} \left(\frac{\|L_i^{(w)}(\mathbf{x}^\sigma)\|_w}{\|\mathbf{x}^\sigma\|_w} Q^{\zeta_{iw}} \right) \le \Delta_w. \tag{3.10}$$

Combining (3.9) and (3.10) with Galois-invariance $H(\mathbf{x}^\sigma) = H(\mathbf{x})$ of the height, we derive that

$$H_{\widetilde{L},\widetilde{\zeta},Q}(\mathbf{x}^\sigma) \le \left(\prod_{w \in \Sigma_K} \Delta_w \right) H(\mathbf{x}^\sigma)^{1 - \frac{1}{n+1}\sum_{u \in \Sigma_E}\sum_{j=0}^n \eta_{ju}}.$$

It follows from the hypotheses (3.1) and (3.8) that

$$H_{\widetilde{L},\widetilde{\zeta},Q}(\mathbf{x}^\sigma) \le \left(\prod_{w \in \Sigma_K} \Delta_w \right) H(\mathbf{x}^\sigma)^{-\frac{\delta}{n+1}} = \left(\prod_{w \in \Sigma_K} \Delta_w \right) Q^{-\frac{\delta\iota}{n+1}}. \tag{3.11}$$

We have thus established our targeted twisted height estimate.

We now put

$$\{L_1, \ldots, L_{R'}\} := \bigcup_{w \in \Sigma_K} \{L_i^{(w)} : 0 \le i \le n\}.$$

By (3.5) we have $R' \le n + 1 + RD \le R(D+1) \le 2RD$. We need to estimate

$$H_{\widetilde{L}} := \prod_{w \in \Sigma_K} \max_{1 \le i_0 < \cdots < i_n \le R'} \|\det(L_{i_0}, \ldots, L_{i_n})\|_w.$$

We have that [4, (5.10) page 531]

$$H_{\widetilde{L}} \le (n+1)^{\frac{n+1}{2}} (H^*)^{DR}. \tag{3.12}$$

We are in a position to apply Theorem 1.4 to the twisting datum $(\widetilde{L}, \widetilde{\zeta})$ over F and to the twisted height $H_{\widetilde{L},\widetilde{\zeta},Q}$. We need to verify the conditions of the exponents

$$\sum_{i=0}^n \zeta_{iv} = 0 \text{ for } v \in \Sigma_F, \tag{3.13}$$

$$\sum_{v\in\Sigma_F}\min_{0\le i\le n}\zeta_{iv}\ge -1. \tag{3.14}$$

The condition (3.13) is apparent from (3.4), whereas the requirement (3.14) is immediate from (3.2) and (3.4). Note also that

$$\begin{aligned}
Q_0\left(n,2RD,H_{\widetilde{L}},\frac{\delta\iota}{n+1}\right) &= \max\left(H_{\widetilde{L}}^{\frac{1}{2RD}},(n+1)^{\frac{n+1}{\delta\iota}}\right)\\
&\le \max\left(\left((n+1)^{\frac{n+1}{2}}(H^*)^{DR}\right)^{\frac{1}{2RD}},(n+1)^{\frac{n+1}{\delta\iota}}\right)\\
&\le \max\left((n+1)^{\frac{1}{4D}}(H^*)^{\frac{1}{2}},(n+1)^{\frac{n+1}{\iota\delta}}\right)\\
&=: H_1(n,D,H^*,\delta,\iota)^{\frac{1}{\iota}}.
\end{aligned}$$

Therefore, there are proper linear subspaces $T_1,\ldots,T_{t_1}$ of $\mathbb{P}^n(\overline{\mathbb{Q}})$ defined over E, with $t_1=t_1(n,R,\delta,\iota)$ given by

$$\begin{aligned}
t_0\left(n,2RD,\frac{\delta\iota}{n+1}\right) &\le 10^6 4^{n+1}(n+1)^{13}(\delta\iota)^{-3}\log^2\left(\frac{6RD(n+1)}{\delta\iota}\right)\\
&=: t_1(n,R,D,\delta,\iota),
\end{aligned}$$

satisfying the following property: if

$$H(\mathbf{x})\ge H_1(n,D,H^*,\delta,\iota),$$

then

$$\begin{aligned}
Q=H(\mathbf{x})^{\frac{1}{\iota}} &\ge H_1(n,D,H^*,\delta,\iota)^{\frac{1}{\iota}}\\
&\ge Q_0\left(n,2RD,H_{\widetilde{L}},\frac{\delta\iota}{n+1}\right),
\end{aligned}$$

and so there exists $1\le s\le t_1$ such that $\mathbf{x}^\sigma\in T_s$ for every $\sigma\in\Gamma_E$. Put

$$T'_j=\bigcap_{\sigma\in\Gamma_E}\sigma(T_j)\quad (1\le j\le t_1);$$

these are linear subspaces of $\mathbb{P}^n(\overline{\mathbb{Q}})$ defined over E. Thus the parallelepiped $\Pi_E(\widetilde{M}_S,\widetilde{\eta}_S)$ is contained in a finite union

$$\bigcup_{j=1}^{t_1}T'_j$$

of linear subspaces of $\mathbb{P}^n(\overline{\mathbb{Q}})$ defined over E. Theorem 1.5 is proved. □

3.2. **The Absolute Subspace Theorem.** In this section we deduce the Absolute Subspace Theorem (Theorem 1.8) as a consequence of the Absolute Parametric Subspace Theorem (Theorem 1.5).

Proof of Theorem 1.8. We will deduce Theorem 1.8 from Corollary 1.7 in the same way as [8, Section 21]. We will only sketch the proof and refer the reader there for further details.

We partition the set of solutions $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ into two subsets M_1 and M_2 as follows. The set M_1 consists of those $\mathbf{x}$ such that

$$\max_{\sigma\in\Gamma_F} \|N_{i_0}^{(u_0)}(\mathbf{x}^\sigma)\|_{u_0} = 0$$

for some $0 \le i_0 \le n$ and some $u_0 \in S$. The set M_2 consists of those $\mathbf{x}$ which do not belong to M_1.

The set M_1 lies in a union of at most

$$(n+1)s \tag{3.15}$$

proper linear subspaces of $\mathbb{P}^n(\overline{\mathbb{Q}})$ which are defined over F.

The set M_2 can be partitioned into a disjoint union of two subsets M_{21} and M_{22}. The set M_{21} consists of those $\mathbf{x} \in M_2$ such that there exists an index $0 \le i_0 \le n$ with

$$\prod_{u\in S} \max_{\sigma\in\Gamma_F} \frac{\|N_i^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u} \le H(\mathbf{x})^{-(n+1+\delta)}.$$

The set M_{22} is the complement of M_{21} in M_2.

The set M_{21} is covered by [8, Equations (20.9) and (21.29)]

$$(n+1)(4e(n+1)\delta^{-1})^s t_1''\left(n,(n+1)s,D,\frac{\delta}{2}\right) \tag{3.16}$$

proper linear subspaces.

The set M_{22} is covered by [8, Equations (20.9) and (21.41)]

$$(4e(n+1)^2\delta^{-1})^{(n+1)s} t_1''\left(n,(n+1)s,D,\frac{\delta}{2}\right) \tag{3.17}$$

proper linear subspaces.

We add up (3.15), (3.16) and (3.17), to get the total number of proper linear subspaces needed.

$$\begin{aligned}
(n+1)s &+ (n+1)(4e(n+1)\delta^{-1})^s t_1''\left(n,(n+1)s,D,\frac{\delta}{2}\right)\\
&+ (4e(n+1)^2\delta^{-1})^{(n+1)s} t_1''\left(n,(n+1)s,D,\frac{\delta}{2}\right)\\
&< 2\left(4e(n+1)^2\delta^{-1}\right)^{(n+1)s} t_1''\left(n,(n+1)s,D,\frac{\delta}{2}\right)\\
&< \left(4e(n+1)^2\delta^{-1}\right)^{(n+1)s} 10^6 4^{n+\frac{7}{2}}(n+1)^{10}\delta^{-3}\log^2\left(\frac{24R^3D}{\delta}\right)\\
&=: t_2(n,R,D,\delta,s).
\end{aligned}$$

The height lower bound $H_2(n,H^*,\delta)$ is computed by

$$H_2(n,H^*,\delta) := H_1''(n,R,D,H^*,\delta/2).$$

The theorem follows. □

Proof of Corollary 1.11. By Theorem 1.8, there exist proper linear subspaces $T_1, \ldots, T_{t_{\text{basic}}}$, where t_{basic} is given by (1.28), of $\mathbb{P}^n(F)$ satisfying the following condition. If $\mathbf{x} \in \mathbb{P}^n(F)\setminus \bigcup_{j=1}^{t_{\text{basic}}} T_j$ has

$$h(\mathbf{x}) \geq h_{\text{basic}}(n, h^*, \delta),$$

where h_{basic} is given by (1.31), then

$$\prod_{v\in S}\prod_{i=0}^{n} \frac{\|L_i^{(v)}(\mathbf{x})\|_v}{\|\mathbf{x}\|_v} > \frac{\prod_{v\in S}\|\det(L_0^{(v)}, \ldots, L_n^{(v)})\|_v}{H(\mathbf{x})^{n+1+\delta}}.$$

It follows that

$$\prod_{v\in S}\prod_{i=0}^{n} \frac{(n+1)^{\kappa(v)}\|L_i^{(v)}\|_v\|\mathbf{x}\|_v}{\|L_i^{(v)}(\mathbf{x})\|_v} < H(\mathbf{x})^{n+1+\delta}\prod_{v\in S}\frac{(n+1)^{(n+1)\kappa(v)}\prod_{i=0}^n\|L_i^{(v)}\|_v}{\|\det(L_0^{(v)}, \ldots, L_n^{(v)})\|_v}.$$

On applying Lemma 2.1, we infer that

$$\prod_{v\in S}\prod_{i=0}^{n} \frac{(n+1)^{\kappa(v)}\|L_i^{(v)}\|_v\|\mathbf{x}\|_v}{\|L_i^{(v)}(\mathbf{x})\|_v} < H(\mathbf{x})^{n+1+\delta}(n+1)^{n+1}\left(H^*(n+1)^{\frac{1}{2}}\right)^{s(n+1)}.$$

Taking logarithm yields the inequality (1.29). The corollary is proved. □

4. The Absolute Subspace Theorem for Hyperplanes in General Position

In this section, we show an effective upper bound for the number of exceptional subspaces when the hyperplanes are in general position. We first need a result which quantifies how nonzero the image of a nonsingular transformation is.

Lemma 4.1. *Let F be a number field and $v \in \Sigma_F$ one of its places. Let $L_0, \ldots, L_n \in \overline{\mathbb{Q}}[x_0, \ldots, x_n]$ be linear forms which are $\overline{\mathbb{Q}}$-linearly independent. Let $\mathbf{x} = [x_0 : \ldots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$.*

Let $D > 0$ and $H^* \geq 1$ be any parameters such that for all $0 \leq i \leq q$,

$$[F(L_i) : F] \leq D,$$
$$H^*(L_i) \leq H^*.$$

For any $\sigma \in \Gamma_F$, one has

$$\max_{0\leq i\leq n}\frac{\|L_i(\mathbf{x}^\sigma)\|_v}{\|\mathbf{x}^\sigma\|_v} \geq \frac{\prod_{i=0}^n \min(1, \|L_i\|_v)}{(n+1)^{\frac{(n+2)\kappa(v)}{2}}\left((n+1)^{\frac{1}{2}}H^*\right)^{(n+1)D^{n+1}}}. \tag{4.1}$$

and

$$\max_{0\leq i\leq n}\frac{\|L_i(\mathbf{x}^\sigma)\|_v}{(n+1)^{\kappa(v)}\|L_i\|_v\|\mathbf{x}^\sigma\|_v} \geq \frac{\prod_{i=0}^n \min(1, \|L_i\|_v)}{(n+1)^{\frac{(n+4)\kappa(v)}{2}}\left((n+1)^{\frac{1}{2}}H^*\right)^{(n+1)D^{n+1}}}. \tag{4.2}$$

Proof. Let $\mathcal{L}$ be the $(n+1)\times(n+1)$ matrix with the linear form L_i on the i^{th}-row. Let $\sigma\in\Gamma_F$ be arbitrary and write $y_i = L_i(\mathbf{x}^\sigma)$. We appeal to Cramer's rule in linear algebra to write

$$x_c^\sigma = \sum_{i=0}^{n} k_{ci} y_i \tag{4.3}$$

where $k_{ci} = (-1)^{c+i}\cdot\frac{\det\mathcal{L}^{(i,c)}}{\det\mathcal{L}}$. Here $\mathcal{L}^{(i,c)}$ is the (i,c)-minor of $\mathcal{L}$, i.e. the matrix $\mathcal{L}$ with the i^{th}-row and c^{th}-column removed.

By (2.3), for $u\in\Sigma_F$ and $0\le c\le n$ we have

$$\begin{aligned}\|x_c^\sigma\|_v &\le (n+1)^{\kappa(v)} \max_{0\le i\le n}\left(\|k_{ci}\|_v\|y_i\|_v\right)\\ &= (n+1)^{\kappa(v)}\|k_{ci_0}\|_v\|y_{i_0}\|_v \end{aligned}\tag{4.4}$$

for some $0\le i_0\le n$. The Hadamard inequality (2.5) implies that

$$\|\det\mathcal{L}^{(i_0,c)}\|_v \le n^{\frac{n\kappa(v)}{2}} \prod_{\substack{i=0\\ i\ne i_0}}^{n} \max(1,\|L_i\|_v). \tag{4.5}$$

It follows from Lemma 2.1 that

$$\|\det\mathcal{L}\|_v \ge \left((n+1)^{\frac{1}{2}}H^*\right)^{-(n+1)D^{n+1}} \prod_{0\le i\le n}\|L_i\|_v. \tag{4.6}$$

On combining (4.4), (4.5), and (4.6), we deduce that

$$\begin{aligned}\frac{\|y_{i_0}\|_v}{\|\mathbf{x}^\sigma\|_v} &\ge \frac{1}{(n+1)^{\frac{(n+2)\kappa(v)}{2}}\left((n+1)^{\frac{1}{2}}H^*\right)^{(n+1)D^{n+1}}}\cdot\frac{\prod_{0\le i\le n}\|L_i\|_v}{\prod_{\substack{0\le i\le n\\ i\ne i_0}}\max(1,\|L_i\|_v)}\\ &\ge \frac{\prod_{i=0}^{n}\min(1,\|L_i\|_v)}{(n+1)^{\frac{(n+2)\kappa(v)}{2}}\left((n+1)^{\frac{1}{2}}H^*\right)^{(n+1)D^{n+1}}},\end{aligned}$$

and, similarly, that

$$\begin{aligned}\frac{\|y_{i_0}\|_v}{(n+1)^{\kappa(v)}\|L_{i_0}\|_v\|\mathbf{x}^\sigma\|_v} &\ge \frac{1}{(n+1)^{\frac{(n+4)\kappa(v)}{2}}\left((n+1)^{\frac{1}{2}}H^*\right)^{(n+1)D^{n+1}}}\cdot\frac{\prod_{\substack{0\le i\le n\\ i\ne i_0}}\|L_i\|_v}{\prod_{\substack{0\le i\le n\\ i\ne i_0}}\max(1,\|L_i\|_v)}\\ &\ge \frac{\prod_{i=0}^{n}\min(1,\|L_i\|_v)}{(n+1)^{\frac{(n+4)\kappa(v)}{2}}\left((n+1)^{\frac{1}{2}}H^*\right)^{(n+1)D^{n+1}}}.\end{aligned}$$

The lemma follows. □

We are ready to prove the Absolute Subspace Theorem for hyperplanes in general position.

Proof of Theorem 1.13. Put $Q = \{0, 1, \dots, q\}$. Let $\mathbf{x} = [x_0 : \dots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$ satisfy the inequality

$$\prod_{u\in S}\prod_{i=0}^{q}\max_{\sigma\in\Gamma_E}\frac{\|N_i^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u} \leq \frac{\prod_{u\in S}\left(\min_{0\leq i_0<\cdots<i_n\leq q}\|\det(N_{i_0}^{(u)},\dots,N_{i_n}^{(u)})\|_u\right)}{H(\mathbf{x})^{n+1+2\delta}}. \tag{4.7}$$

For $u \in S$, we reorder the indices in Q, say $j_0(u), \dots, j_q(u)$, such that

$$\max_{\sigma\in\Gamma_E}\frac{\|N_{j_0(u)}^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u} \leq \cdots \leq \max_{\sigma\in\Gamma_E}\frac{\|N_{j_q(u)}^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u}.$$

Set $J(u) = \{j_0(u), \dots, j_n(u)\}$.

Applying Lemma 4.1, we see that

$$\max_{\sigma\in\Gamma_E}\frac{\|N_{j_n(u)}^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u} \geq C(u). \tag{4.8}$$

where

$$C(u) = \frac{\prod_{i=0}^{n}\min(1, \|N_{j_i(u)}^{(u)}\|_u)}{(n+1)^{\frac{(n+2)\kappa(u)}{2}}\left((n+1)^{\frac{1}{2}}H^*\right)^{(n+1)D^{n+1}}}.$$

It follows from (4.8) and our reordering that

$$\prod_{j\in Q\setminus J(u)}\max_{\sigma\in\Gamma_E}\frac{\|N_j^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u} \geq C(u)^{q-n}. \tag{4.9}$$

Combining (4.7) and (4.9), we deduce that

$$\begin{aligned}\prod_{u\in S}\prod_{j\in J(u)}\max_{\sigma\in\Gamma_E}\frac{\|N_j^{(u)}(\mathbf{x}^\sigma)\|_u}{\|\mathbf{x}^\sigma\|_u} &\leq \frac{\prod_{u\in S}\left(\|\det(N_j^{(u)} : j\in J(u))\|_u C(u)^{n-q}\right)}{H(\mathbf{x})^{n+1+2\delta}} \\ &\leq \frac{\prod_{u\in S}\|\det(N_j^{(u)} : j\in J(u))\|_u}{H(\mathbf{x})^{n+1+\delta}},\end{aligned} \tag{4.10}$$

provided that

$$H(\mathbf{x}) \geq \left(\prod_{u\in S}C(u)\right)^{\frac{n-q}{\delta}};$$

this is satisfied if

$$H(\mathbf{x})^{\frac{\delta}{q-n}} \geq \frac{(n+1)^{\frac{n+2}{2}}\left((n+1)^{\frac{1}{2}}H^*\right)^{|S|(n+1)D^{n+1}}}{\prod_{u\in S}\left(\min_{0\leq i_0<\cdots<i_n\leq q}\prod_{t=0}^{n}\min(1, \|N_{i_t}^{(u)}\|_u)\right)}.$$

We can now apply Theorem 1.8 to conclude the proof. ∎

Proof of Corollary 1.14. By Theorem 1.13, there exist proper linear subspaces $T_1, \dots, T_{t_{\mathrm{gen}}}$, where t_{gen} is given by (1.40), of $\mathbb{P}^n(F)$ satisfying the following condition. If $\mathbf{x} \in \mathbb{P}^n(F) \setminus \bigcup_{j=1}^{t_{\mathrm{gen}}} T_j$ has

$$h(\mathbf{x}) \geq \max\left(\frac{1}{2(n+3)}(h^* + \log 2), \frac{2(n+1)}{\delta}\log(n+1)\right) \tag{4.11}$$

and

$$h(\mathbf{x}) \geq \frac{(q-n)\left(n+2+s(n+1)\right)}{2\delta} \cdot \log(n+1) + \frac{(q-n)s(n+1)h^*}{\delta}, \tag{4.12}$$

then

$$\prod_{v\in S}\prod_{i=0}^{q} \frac{\|L_i^{(v)}(\mathbf{x})\|_v}{\|\mathbf{x}\|_v} > \frac{\prod_{v\in S}\left(\min_{0\leq i_0<\cdots<i_n\leq q}\|\det(L_{i_0}^{(v)}, \dots, L_{i_n}^{(v)})\|_v\right)}{H(\mathbf{x})^{n+1+2\delta}}.$$

Observe that the right hand sides of both (4.11) and (4.12) are smaller than

$$\frac{(q-n+1)(n+2)(s+1)}{\delta}\left(\log(n+1) + h^*\right).$$

Suppose now that $\mathbf{x} \in \mathbb{P}^n(F) \setminus \bigcup_{j=1}^{t_{\mathrm{gen}}} T_j$ has

$$h(\mathbf{x}) > \frac{(q-n+1)(n+2)(s+1)}{\delta}\left(\log(n+1) + h^*\right), \tag{4.13}$$

so that both (4.11) and (4.12) are satisfied. It follows that

$$\prod_{v\in S}\prod_{i=0}^{q} \frac{(n+1)^{\kappa(v)}\|L_i^{(v)}\|_v\|\mathbf{x}\|_v}{\|L_i^{(v)}(\mathbf{x})\|_v} < H(\mathbf{x})^{n+1+2\delta}\prod_{v\in S}\frac{(n+1)^{(q+1)\kappa(v)}\prod_{i=0}^{q}\|L_i^{(v)}\|_v}{\min_{0\leq i_0<\cdots<i_n\leq q}\|\det(L_{i_0}^{(v)}, \dots, L_{i_n}^{(v)})\|_v}$$

On applying Lemma 2.1, we deduce that

$$\prod_{v\in S}\prod_{i=0}^{q} \frac{(n+1)^{\kappa(v)}\|L_i^{(v)}\|_v\|\mathbf{x}\|_v}{\|L_i^{(v)}(\mathbf{x})\|_v} < H(\mathbf{x})^{n+1+2\delta}(n+1)^{q+1}\left(H^*(n+1)^{\frac{1}{2}}\right)^{s(n+1)}(H^*)^{s(q-n)}.$$

Taking logarithm yields

$$\sum_{v\in S}\sum_{i=0}^{n} \lambda_v(\mathbf{x}, L_i^{(v)}) < (n+1+2\delta)h(\mathbf{x}) + K_{\mathrm{gen}},$$

where

$$K_{\mathrm{gen}} := s(q+1)h^* + \left(\frac{s(n+1)}{2} + q + 1\right)\log(n+1). \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$K_{\mathrm{gen}} < \delta h(\mathbf{x}).$$

Thus

$$\sum_{v\in S}\sum_{i=0}^{q} \lambda_v(\mathbf{x}, L_i^{(v)}) < (n+1+3\delta)h(\mathbf{x}).$$

The corollary is proved. □

5. The Quantitative Subspace Theorem for Hyperplanes in Subgeneral Position

The goal of this section is to prove Theorem 1.16, which provides an effective upper bound for the number of exceptional subspaces when the hyperplanes are in subgeneral position. The following notations and assumptions will be fixed throughout this section. Let F be a number field; let S be a finite set of places of F of cardinality $|S| = s$. Let $q \geq m \geq n$ be positive integers; suppose that $q > 2m - n$. Let $(\omega_i \in \mathbb{Q} \cap (0,1] : 0 \leq i \leq q)$ be a system of Nochka's weights as in Theorem 2.3; set $\tilde{\omega} = \max_{0 \leq i \leq q} \omega_i$.

The following proposition, which is of its own interest, is a preliminary form of Theorem 1.16.

Proposition 5.1. *Let $(L_i^{(v)} : 0 \leq i \leq q, v \in S)$ be a system of linear forms in $F[x_0, \ldots, x_n]$. Suppose that for each $v \in S$, the linear forms $(L_i^{(v)} : 0 \leq i \leq q)$ are in m-subgeneral position. Suppose also that for all $0 \leq i \leq q$ and $v \in S$,*

$$\|L_i^{(v)}\|_v \geq 1. \tag{5.1}$$

Put

$$R := \# \left(\bigcup_{v \in S} \{L_i^{(v)} : 0 \leq i \leq q\} \right). \tag{5.2}$$

Let $h^* > 0$ be such that for all $0 \leq i \leq q$ and $v \in S$,

$$h^*(L_i^{(v)}) \leq h^*. \tag{5.3}$$

Let $0 < \delta < 1$. Then there are proper linear subspaces $T_1, \ldots, T_{t_{\mathrm{subgen}}}$ of $\mathbb{P}^n(F)$, with

$$\begin{aligned} t_{\mathrm{subgen}} &= t_{\mathrm{subgen}}(n, s, R, \delta) \\ &:= \left(4e(n+1)^2\delta^{-1}\right)^{(n+1)s} 10^6 4^{n+\frac{7}{2}} (n+1)^{10} \delta^{-3} \log^2 \left(\frac{24R^3}{\delta} \right), \end{aligned} \tag{5.4}$$

satisfying the following property. Every $\mathbf{x} \in \mathbb{P}^n(F) \setminus \bigcup_{j=1}^{t_{\mathrm{subgen}}} T_j$ satisfies either

$$\sum_{v \in S} \sum_{i=0}^{q} \omega_i \lambda_v(\mathbf{x}, L_i^{(v)}) < (n+1+\delta) h(\mathbf{x}) + K'_{\mathrm{subgen}} \tag{5.5}$$

where

$$\begin{aligned} K'_{\mathrm{subgen}} &= K'_{\mathrm{subgen}}(n, m, q, s, h^*) \\ &:= (q - m + 1) \left(s(n+1)h^* + \frac{s(n+1)}{2} \log(n+1) + (n+2) \log(n+1) \right) \end{aligned} \tag{5.6}$$

or

$$\begin{aligned} h(\mathbf{x}) &< h_{\mathrm{subgen}}(n, h^*, \delta) \\ &:= \max \left(\frac{1}{2(n+3)} (h^* + \log 2), \frac{2(n+1)}{\delta} \log(n+1) \right). \end{aligned} \tag{5.7}$$

Proof. Let $\mathbf{x} = [x_0 : \ldots : x_n] \in \mathbb{P}^n(F)$. For $0 \le i \le q$ and $v \in S$, write

$$\Lambda_i^{(v)} := \frac{(n+1)^{\kappa(v)} \|L_i^{(v)}\|_v \|\mathbf{x}\|_u}{\|L_i^{(v)}(\mathbf{x})\|_v} \ge 1$$
$$\lambda_i^{(v)} := \log \Lambda_i^{(v)} \ge 0.$$

Put $Q = \{0, 1, \ldots, q\}$. For each $v \in S$, we reorder the indices in Q, say $j_0(v), \ldots, j_q(v)$, such that

$$\Lambda_{j_0(v)}^{(v)} \ge \Lambda_{j_1(v)}^{(v)} \cdots \ge \Lambda_{j_q(v)}^{(v)}.$$

Let $J(v) := \{j_0(v), \ldots, j_m(v)\}$.

Put $H^* := e^{h^*}$. It follows from Lemma 4.1, the m-subgeneral position hypothesis and the assumption $\|L_i^{(v)}\|_v \ge 1$ that, for each $v \in S$,

$$\Lambda_{j_m(v)}^{(v)} = \min_{j \in J(v)} \Lambda_j^{(v)} \le D(v) \tag{5.8}$$

where

$$D(v) = (n+1)^{\frac{(n+4)\kappa(v)}{2}} \left((n+1)^{\frac{1}{2}} H^*\right)^{n+1}.$$

It follows from (5.8) and our reordering that

$$\prod_{j \in Q \setminus J(v)} \Lambda_j^{(v)} \le D(v)^{q-m}. \tag{5.9}$$

We now invoke the properties of the Nochka's weights $\omega_i \in \mathbb{Q} \cap (0, 1]\,(0 \le i \le q)$ as elucidated by Theorem 2.3. By the properties (i) and (iii) therein, for each $v \in S$ there exists a subset $J'(v) := \{j_0'(v), j_1'(v), \ldots, j_n'(v)\} \subset J(v)$ such that the linear forms $\{L_j^{(v)} : j \in J'(v)\}$ are linearly independent over F and that

$$\prod_{j \in J(v)} \left(\Lambda_j^{(v)}\right)^{\omega_j} \le \prod_{j \in J'(v)} \Lambda_j^{(v)}. \tag{5.10}$$

It follows from (5.9) and (5.10) that

$$\begin{aligned}\prod_{v \in S} \prod_{i=0}^{q} \left(\Lambda_i^{(v)}\right)^{\omega_i} &\le \left(\prod_{v \in S} \prod_{j \in J(v)} \left(\Lambda_j^{(v)}\right)^{\omega_i}\right) \left(\prod_{v \in S} \prod_{j \in Q \setminus J(v)} \Lambda_j^{(v)}\right) \\ &\le \left(\prod_{v \in S} \prod_{j \in J'(v)} \Lambda_j^{(v)}\right) \left(\prod_{v \in S} D(v)\right)^{q-m}.\end{aligned} \tag{5.11}$$

Now suppose further that

$$\begin{aligned}h(\mathbf{x}) &\ge h_{\text{subgen}}(n, h^*, \delta) := h_{\text{basic}}(n, h^*, \delta) \\ &= \max\left(\frac{1}{2(n+3)}(h^* + \log 2), \frac{2(n+1)}{\delta} \log(n+1)\right).\end{aligned}$$

On applying Corollary 1.11, we infer that there exist proper linear subspaces $T_1, \ldots, T_{t_{\text{subgen}}}$ of $\mathbb{P}^n(F)$, with

$$\begin{aligned} t_{\text{subgen}} &:= t_{\text{basic}}(n, s, R, \delta) \\ &= \left(4e(n+1)^2\delta^{-1}\right)^{(n+1)s} 10^6 4^{n+\frac{7}{2}}(n+1)^{10}\delta^{-3}\log^2\left(\frac{24R^3}{\delta}\right), \end{aligned}$$

satisfying the following property. If $\mathbf{x} \in \mathbb{P}^n(F)\setminus\bigcup_{j=1}^{t_{\text{subgen}}} T_j$, then

$$\sum_{v\in S}\sum_{j\in J'(v)} \lambda_j^{(v)} < (n+1+\delta)h(\mathbf{x}) + K_{\text{basic}} \tag{5.12}$$

where

$$K_{\text{basic}} := s(n+1)h^* + \left(\frac{s}{2}+1\right)(n+1)\log(n+1).$$

Combining (5.11) and (5.12), we deduce that

$$\sum_{v\in S}\sum_{i=0}^{q} \omega_i\lambda_i^{(v)} < (n+1+\delta)h(\mathbf{x}) + K'_{\text{subgen}} \tag{5.13}$$

where

$$\begin{aligned} K''_{\text{subgen}} &:= K_{\text{basic}} + (q-m)\left(s(n+1)h^* + \frac{s(n+1)}{2}\log(n+1) + \frac{n+4}{2}\log(n+1)\right) \\ &\leq (q-m+1)\left(s(n+1)h^* + \frac{s(n+1)}{2}\log(n+1) + (n+2)\log(n+1)\right) \\ &=: K'_{\text{subgen}}. \end{aligned}$$

The proposition follows. □

We are in a position to prove the Quantitative Subspace Theorem for hyperplanes in subgeneral position.

Proof of Theorem 1.16. Let $\mathbf{x} = [x_0 : \ldots : x_n] \in \mathbb{P}^n(F)$. For $0 \leq i \leq q$ and $v \in S$, write

$$\begin{aligned} \Lambda_i^{(v)} &:= \frac{(n+1)^{\kappa(v)}\|L_i\|_v\|\mathbf{x}\|_u}{\|L_i(\mathbf{x})\|_v} \geq 1 \\ \lambda_i^{(v)} &:= \log\Lambda_i^{(v)} \geq 0. \end{aligned}$$

Put $Q = \{0, 1, \ldots, q\}$.

We make two observations. First, for $0 \leq i \leq q$,

$$\sum_{v\in S}\lambda_i^{(v)} \leq \sum_{v\in\Sigma_F}\lambda_i^{(v)} \leq h(\mathbf{x}) + h^* + \log(n+1). \tag{5.14}$$

Next, by the properties (i) and (iii) of the Nochka's weights (Theorem 2.3) we have that

$$\sum_{i=0}^{q}\left(1 - \frac{\omega_i}{\tilde{\omega}}\right) = 2m + 1 - n - \frac{n+1}{\tilde{\omega}}. \tag{5.15}$$

Now suppose further that

$$h(\mathbf{x}) \geq h_{\text{subgen}}(n, h^*, \delta) = \max\left(\frac{1}{2(n+3)}(h^* + \log 2), \frac{2(n+1)}{\delta}\log(n+1)\right).$$

On applying Proposition 5.1 with

$$L_i^{(v)} := L_i \qquad (0 \leq i \leq q, v \in S),$$

we infer that there exist proper linear subspaces $T_1, \ldots, T_{t_{\text{subgen}}}$ of $\mathbb{P}^n(F)$ satisfying the following property. If $\mathbf{x} \in \mathbb{P}^n(F) \backslash \bigcup_{j=1}^{t_{\text{subgen}}} T_j$, then

$$\begin{aligned}
\sum_{i=0}^{q}\sum_{v\in S}\lambda_i^{(v)} &= \sum_{i=0}^{q}\sum_{v\in S}\left(1 - \frac{\omega_i}{\tilde{\omega}}\right)\lambda_i^{(v)} + \frac{1}{\tilde{\omega}}\sum_{i=0}^{q}\sum_{v\in S}\omega_i\lambda_i^{(v)} \\
&< (2m+1-n)\left(h(\mathbf{x}) + h^* + \log(n+1)\right) - \frac{n+1}{\tilde{\omega}}\left(h^* + \log(n+1)\right) \\
&\quad + \frac{1}{\tilde{\omega}}\left(\sum_{i=0}^{q}\sum_{v\in S}\omega_i\lambda_i^{(v)} - (n+1)h(\mathbf{x})\right) \\
&\leq \left(2m+1-n+\frac{\delta}{\tilde{\omega}}\right)h(\mathbf{x}) + (2m+1-n)\left(h^* + \log(n+1)\right) \\
&\quad + \frac{n+1}{\tilde{\omega}}\left(\frac{K'_{\text{subgen}}}{n+1} - h^* - \log(n+1)\right).
\end{aligned}$$

Since $\frac{n+1}{\tilde{\omega}} \leq 2m+1-n$, we derive that

$$\begin{aligned}
\sum_{i=0}^{q}\sum_{v\in S}\lambda_i^{(v)} &< \left(2m+1-n+\frac{\delta}{\tilde{\omega}}\right)h(\mathbf{x}) + \frac{2m+1-n}{n+1}K'_{\text{subgen}} \\
&\leq (2m+1-n)\left(1+\frac{\delta}{n+1}\right)h(\mathbf{x}) + \frac{2m+1-n}{n+1}K'_{\text{subgen}}.
\end{aligned}$$

The theorem is proved. □

6. The Quantitative Subspace Theorem for Non-subdegenerate Families of Hyperplanes

This section is devoted to the proof of Theorem 1.20. We start with a result on the valuations of the coefficients in a linear relation.

Lemma 6.1. *Let $K_0, \ldots, K_t \in F^n \setminus \{0\}$ be nonzero vectors which satisfy*

$$\sum_{i=0}^{t} c_i K_i = 0 \quad (c_i \in F)$$

Let $H > 0$ be such that $H(K_i) \leq H$ for all $0 \leq i \leq t$. Suppose that $c_0 \neq 0$ and that $K_1, \ldots, K_t$ are F-linearly independent. If $v \in \Sigma_F$, then

$$H^{-t-1}t^{-\frac{t\kappa(v)}{2}} \leq \frac{\max_{1\leq i\leq t}\|c_i\|_v}{\|c_0\|_v} \leq H^{t+1}t^{\frac{t\kappa(v)}{2}}. \tag{6.1}$$

Proof. Let $1 \leq r \leq t$ be arbitrary; we want to show that

$$H^{-t-1}t^{-\frac{t\kappa(v)}{2}} \leq \|d_r\|_v \leq H^{t+1}t^{\frac{t\kappa(v)}{2}}, \tag{6.2}$$

whence deducing (6.1).

For $1 \leq j \leq t$, put $d_j = -\frac{c_j}{c_t}$, so that

$$K_0 = \sum_{j=1}^{t} d_j K_j.$$

For $0 \leq j \leq t$, write $K_j = (a_{1j}, \ldots, a_{nj})$. By hypothesis, there exist

$$1 \leq i_1 < \cdots < i_t \leq n$$

satisfying the following condition. For $0 \leq j \leq t$, put $K_j' = (a_{i_1 j}, \ldots, a_{i_t j}) \in F^t$. Then $K_1', \ldots, K_t'$ are F-linearly independent. Furthermore, we have

$$K_0' = \sum_{j=1}^{t} d_j K_j.$$

Without loss of generality, we may assume that $\|K_i'\|_v \geq 1$ for all $0 \leq i \leq t$, because otherwise we would just multiply all the K_i by an appropriate scalar in $F^\times$. We then derive (6.2) by combining Cramer's formula for d_r with (2.8) and Lemma 2.2. The lemma is proved. □

From a minimally dependent set of hyperplanes, one can derive a Schmidt-type inequality which will be applied recursively for a non-subdegenerate system of hyperplanes.

Lemma 6.2. *Let $K_0, \ldots, K_t \in F[x_0, \ldots, x_n]$ be linear forms. Suppose that the set $\{K_i : 0 \leq i \leq t\}$ is minimally dependent over F.*

Suppose that for all $0 \leq i \leq t$ and $v \in S$,

$$h^*(K_i) \leq h^*. \tag{6.3}$$

Let $0 < \delta < 1$. Then there are proper linear subspaces $T_1, \ldots, T_{t_{\text{mindep}}}$ of $\mathbb{P}^n(F)$, with

$$\begin{aligned} t_{\text{mindep}}(t, s, \delta) &:= t_{\text{gen}}(t-1, s, t+1, \delta) \\ &= \left(4et^2\delta^{-1}\right)^{ts} 10^6 4^{t+\frac{5}{2}} t^{10} \delta^{-3} \log^2\left(\frac{24(t+1)^3}{\delta}\right), \end{aligned} \tag{6.4}$$

satisfying the following property. Every $\mathbf{x} \in \mathbb{P}^n(F) \setminus \bigcup_{j=1}^{t_{\text{mindep}}} T_j$ satisfies either

$$\sum_{v \in S} \sum_{i=0}^{t} \log \frac{\max_{1 \leq j \leq t} \|K_j(\mathbf{x})\|_v}{\|K_i(\mathbf{x})\|_v} < (t + 3\delta) h_{\mathbb{P}_F^{t-1}}(K_1(\mathbf{x}) : \ldots : K_t(\mathbf{x})) + k_{\text{mindep}}, \tag{6.5}$$

where

$$k_{\text{mindep}}(t, s, h^*) = s(t+1)h^* - \kappa(S)\left(\frac{t}{2} + 1\right) \log t, \tag{6.6}$$

or

$$\begin{aligned} h(\mathbf{x}) &< h_{\mathrm{mindep}}(t,s,h^*,\delta) := h_{\mathrm{gen}}(t-1,t,s,h^*,\delta) \\ &= \frac{2(t+1)(s+1)}{\delta}\left(\log t + h^*\right). \end{aligned} \tag{6.7}$$

Consequentially, there is a finite collection of proper linear subspaces of $\mathbb{P}^n(F)$ *such that any* $\mathbf{x} \in \mathbb{P}^n(F)$ *outside of the union of these subspaces satisfies*

$$\sum_{v\in S}\sum_{i=0}^{t} \log \frac{\max_{0\le j\le t}\|K_j(\mathbf{x})\|_v}{\|K_i(\mathbf{x})\|_v} < (t+\delta)h_{\mathbb{P}^t_F}(K_0(\mathbf{x}):\ldots:K_t(\mathbf{x})) + O(1). \tag{6.8}$$

Proof. We write the linear dependency of the K_i as

$$c_0K_0 + c_1K_1 + \cdots + c_tK_t = 0$$

with $c_i \in F\setminus\{0\}$ for $0 \le i \le t$. On multiplying by a nonzero constant in F if necessary, we may assume that for all $v \in S$ and $0 \le i \le t$, $\|c_i\|_v \ge 1$.

In $\mathbb{P}^{t-1}(F)$ with coordinates $Y_1,\ldots,Y_t$, consider the following system of hyperplanes:

$$\begin{aligned} N_i^{(v)} &:= c_iY_i && (v\in S, 1\le i\le t),\\ N_0^{(v)} &:= c_1Y_1+\cdots+c_tY_t && (v\in S). \end{aligned}$$

It is clear that for each $v \in S$, the hyperplanes $\{N_i^{(v)} : 0 \le i \le t\}$ are in general position. By virtue of Corollary 1.14, there are proper linear subspaces $U_1,\ldots,U_{t_{\mathrm{mindep}}}$ of $\mathbb{P}^{t-1}(F)$, with

$$\begin{aligned} t_{\mathrm{mindep}}(t,s,\delta) &:= t_{\mathrm{gen}}(t-1,s,t+1,\delta) \\ &= \left(4et^2\delta^{-1}\right)^{ts} 10^6 4^{t+\frac{5}{2}} t^{10}\delta^{-3}\log^2\left(\frac{24(t+1)^3}{\delta}\right), \end{aligned}$$

satisfying the following property. Let $\mathbf{x} \in \mathbb{P}^n(F)$ be arbitrary and set $\mathbf{y} = (K_1(\mathbf{x}) : \ldots : K_t(\mathbf{x})) \in \mathbb{P}^{t-1}(F)$. If $\mathbf{y} \in \mathbb{P}^{t-1}(F)\setminus\bigcup_{j=1}^{t_{\mathrm{mindep}}} U_j$ and

$$\begin{aligned} h(\mathbf{y}) &> h_{\mathrm{gen}}(t-1,t,s,h^*,\delta) \\ &= \frac{2(t+1)(s+1)}{\delta}\left(\log t + h^*\right) =: h_{\mathrm{mindep}}(t,s,h^*,\delta), \end{aligned}$$

then

$$\sum_{v\in S}\sum_{i=0}^{t} \lambda_v(\mathbf{y}, N_i^{(v)}) < (t+3\delta)h_{\mathbb{P}^{t-1}_F}(\mathbf{y}). \tag{6.9}$$

We now consider 2 cases. First, if $1 \le i \le t$ and $v \in S$, then

$$\begin{aligned} \lambda_v(\mathbf{y}, N_i^{(v)}) &= \log \frac{t^{\kappa(v)}\max_{1\le j\le t}\|K_j(\mathbf{x})\|_v\|c_i\|_v}{\|c_iK_i(\mathbf{x})\|_v} \\ &= \log\frac{\max_{1\le j\le t}\|K_j(\mathbf{x})\|_v}{\|K_i(\mathbf{x})\|_v} + \kappa(v)\log t. \end{aligned}$$

Secondly, for $i = 0$ and $v \in S$, it follows from Lemma 6.1 that

$$\begin{aligned}\lambda_v(\mathbf{y}, N_0^{(v)}) &= \log \frac{t^{\kappa(v)}\,(\max_{1\le j\le t}\|K_j(\mathbf{x})\|_v)\,(\max_{1\le j\le t}\|c_j\|_v)}{\|\sum_{j=1}^t c_jK_j(\mathbf{x})\|_v}\\ &= \log\frac{\max_{1\le j\le t}\|K_j(\mathbf{x})\|_v}{\|K_0(\mathbf{x})\|_v} + \log\frac{\max_{1\le j\le t}\|c_j\|_v}{\|c_0\|_v} + \kappa(v)\log t\\ &\ge \log\frac{\max_{1\le j\le t}\|K_j(\mathbf{x})\|_v}{\|K_0(\mathbf{x})\|_v} + \kappa(v)\log t - (t+1)h^* - \frac{\kappa(v)}{2}t\log t.\end{aligned}$$

Therefore, by (6.9),

$$\sum_{v\in S}\sum_{i=0}^t \log\frac{\max_{1\le j\le t}\|K_j(\mathbf{x})\|_v}{\|K_i(\mathbf{x})\|_v} < (t+3\delta)h_{\mathbb{P}_F^{t-1}}(\mathbf{y}) + s(t+1)h^* - \kappa(S)\left(\frac{t}{2}+1\right)\log t.$$

Thus (6.5) follows.

The qualitative statement (6.8) is an immediate consequence of (6.5). □

With the notations as given in the hypotheses of Theorem 1.20, we apply Lemma 2.4 with $k = F$ to obtain subsets $I_0, I_1, \ldots, I_p \subset \{0, 1, \ldots, q\}$ satisfying the conditions (i)−(v) therein. For $0 \le r \le p$, define $J_r = \cup_{i=0}^r I_i$. Put $u_r := |I_r|$ and $v_r := |J_r| - 1$. Then

$$u_0 = 1,\ v_0 = 0,\ u_r \ge 2 \ \text{ for } \ r \ge 1,$$

and

$$v_r = \sum_{i=1}^r u_i \quad (1 \le r \le p).$$

By reindexing the linear forms, we can write

$$\begin{aligned} I_0 &= \{L_0\}\\ I_r &= \{L_{u_1+\cdots+u_{r-1}+1}, L_{u_1+\cdots+u_{r-1}+2}, \ldots, L_{u_1+\cdots+u_r}\}\\ &= \{L_{v_{r-1}+1}, L_{v_{r-1}+2}, \ldots, L_{v_r}\} \quad (1 \le r \le p).\end{aligned}$$

Furthermore, for each $1 \le r \le p$, there exist 'linking scalars'

$$c_{r,\alpha} \in F \qquad (\alpha \in \bigcup_{i=1}^r I_i)$$

such that $c_{r,\alpha} \ne 0$ when $\alpha \in I_r$ and that

$$\sum_{\alpha\in I_0\cup I_1\cup\cdots\cup I_r} c_{r,\alpha}L_\alpha = 0.$$

For a subset $I \subseteq \{0, 1, \ldots, q\}$ and $\mathbf{x} \in \mathbb{P}^n(F)$, we denote

$$\mathbf{x}^I = (L_i(\mathbf{x}) : i \in I) \in \mathbb{P}^{|I|-1}(F)$$

whenever the right hand side is defined. In particular, $\mathbf{x}^I$ is defined if the linear forms $(L_i : i \in I)$ are linearly independent.

Lemma 6.3. *Let $0 < \delta < 1$. Suppose that $1 \leq r \leq p$. Then there are proper linear subspaces $T_1, \dots, T_{t_r}$ of $\mathbb{P}^n(F)$ satisfying the following property. Every $\mathbf{x} \in \mathbb{P}^n(F) \setminus \bigcup_{j=1}^{t_r} T_j$ satisfies*

$$\sum_{v \in S} \sum_{j=0}^{v_r} \log \frac{\max_{0 \leq i \leq v_r} \|L_i(\mathbf{x})\|_v}{\|L_j(\mathbf{x})\|_v} < (v_r + \delta) h_{\mathbb{P}^{v_r}(F)}(\mathbf{x}^{J_r}) + O(1). \tag{6.10}$$

Proof. We proceed by induction. The base case $r = 1$ is none but Lemma 6.2.

Suppose that the lemma holds for $1 \leq r \leq p$. If $r = p$, there is nothing more to prove. Hence we may assume $1 \leq r < p$ and our task is to show that the lemma holds for $r + 1$ in place of r.

Let $\mathcal{S}_r$ denote the left hand side of (6.10). Let $c_{i,\alpha}$ be the 'linking scalars' as above. Put

$$L := \sum_{\alpha \in I_{r+1}} c_{r+1,\alpha} L_\alpha.$$

By induction hypothesis, there is a finite collection of proper linear subspaces of $\mathbb{P}^n(F)$ outside of which every $\mathbf{x} \in \mathbb{P}^n(F)$ satisfies

$$\mathcal{S}_r = \sum_{\substack{v \in S \\ j \in J_r}} \log \frac{\|\mathbf{x}^{J_r}\|_v}{\|L_j(\mathbf{x})\|_v} < (v_r + \delta) h_{\mathbb{P}^{v_r}(F)}(\mathbf{x}^{J_r}) + O(1). \tag{6.11}$$

Also by Lemma 6.2, there is another finite collection of proper linear subspaces of $\mathbb{P}^n(F)$ outside of which every $\mathbf{x} \in \mathbb{P}^n(F)$ satisfies

$$\sum_{\substack{v \in S \\ j \in I_{r+1}}} \log \frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|L_j(\mathbf{x})\|_v} + \sum_{v \in S} \log \frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|L(\mathbf{x})\|_v} < (u_{r+1} + \delta) h_{\mathbb{P}^{u_{r+1}-1}(F)}(\mathbf{x}^{I_{r+1}}) + O(1). \tag{6.12}$$

Now suppose that $\mathbf{x} \in \mathbb{P}^n(F)$ is outside of both the finite collections of proper linear subspaces appearing above. We decompose

$$\begin{aligned} \mathcal{S}_{r+1} = & \sum_{\substack{v \in S \\ j \in J_r}} \log \frac{\|\mathbf{x}^{J_r}\|_v}{\|L_j(\mathbf{x})\|_v} + (v_r + 1) \sum_{v \in S} \log \frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{J_r}\|_v} \\ & + \sum_{\substack{v \in S \\ j \in I_{r+1}}} \log \frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|L_j(\mathbf{x})\|_v} + u_{r+1} \sum_{v \in S} \log \frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{I_{r+1}}\|_v}. \end{aligned} \tag{6.13}$$

It follows from (6.11), (6.12) and (6.13) that

$$\begin{aligned} \mathcal{S}_{r+1} < & (v_r + \delta) h(\mathbf{x}^{J_r}) + (v_r + 1) \sum_{v \in S} \log \frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{J_r}\|_v} - \sum_{v \in S} \log \frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|L(\mathbf{x})\|_v} \\ & + (u_{r+1} + \delta) h(\mathbf{x}^{I_{r+1}}) + u_{r+1} \sum_{v \in S} \log \frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{I_{r+1}}\|_v} + O(1), \end{aligned} \tag{6.14}$$

the heights being taken in appropriate projective spaces.

We first observe that

$$(u_{r+1}+\delta)h(\mathbf{x}^{I_{r+1}}) + u_{r+1}\sum_{v\in S}\log\frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{I_{r+1}}\|_v} < (u_{r+1}+\delta)h(\mathbf{x}^{J_{r+1}}). \tag{6.15}$$

Indeed, the left hand side of (6.15) is

$$= u_{r+1}\sum_{v\in S}\|\mathbf{x}^{J_{r+1}}\|_v + u_{r+1}\sum_{v\notin S}\|\mathbf{x}^{I_{r+1}}\|_v + \delta h(\mathbf{x}^{I_{r+1}}) < (u_{r+1}+\delta)h(\mathbf{x}^{J_{r+1}}).$$

It follows from (6.14) and (6.15) that

$$\begin{aligned}\mathcal{S}_{r+1} < (v_r+\delta)h(\mathbf{x}^{J_r}) + (v_r+1)\sum_{v\in S}\log\frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{J_r}\|_v} - \sum_{v\in S}\log\frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|L(\mathbf{x})\|_v}\\ + (u_{r+1}+\delta)h(\mathbf{x}^{J_{r+1}}) + O(1).\end{aligned} \tag{6.16}$$

We now claim that

$$\sum_{v\in S}\log\frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|L(\mathbf{x})\|_v} \geq \sum_{v\in S}\log\frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{J_r}\|_v} + 0(1). \tag{6.17}$$

Indeed, one can show (6.17) by dividing S into

$$S_{1,\mathbf{x}} = \{v\in S : \|\mathbf{x}^{J_{r+1}}\|_v = \|\mathbf{x}^{J_r}\|_v\}$$

and $S_{2,\mathbf{x}} = S\setminus S_{1,\mathbf{x}}$. First, we assume that $v\in S_{1,\mathbf{x}}$. Then

$$\log\frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{J_r}\|_v} = 0, \log\frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|L(\mathbf{x})\|_v} \geq 0(1).$$

Next, we assume that $v\in S_{2,\mathbf{x}}$, deducing that

$$\log\frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|L(\mathbf{x})\|_v} \geq \log\frac{\|\mathbf{x}^{I_{r+1}}\|_v}{\|\mathbf{x}^{J_r}\|_v} + O(1) = \log\frac{\|\mathbf{x}^{J_{r+1}}\|_v}{\|\mathbf{x}^{J_r}\|_v} + O(1).$$

Combining the two cases, we derive the claim.

By (6.16) and (6.17), we have

$$\mathcal{S}_{r+1} < (v_{r+1}+2\delta)h(\mathbf{x}^{J_{r+1}}) + O(1). \tag{6.18}$$

The induction is complete. □

Proof of Theorem 1.20. Let $0<\delta<1$ be arbitrary. Denote by $\widetilde{\mathcal{L}}_0$ the union of hyperplanes in $\widetilde{\mathcal{L}}$. Put $J = J_p$. Since $\dim_F(L_j : j\in J)_F = n+1$, we have $h(\mathbf{x}) = h(\mathbf{x}^J) + O(1)$. By Lemma 6.3, for every $\mathbf{x}$ in $\mathbb{P}^n(F)$ outside the union of a finite collection $\mathcal{T}_1$ of proper subspaces,

$$\begin{aligned}\sum_{v\in S}\sum_{j\in J}\lambda_v(\mathbf{x}, L_j) &= \sum_{v\in S}\sum_{j\in J}\log\frac{\|\mathbf{x}\|_v}{\|L_j(\mathbf{x})\|_v} + O(1)\\ &= \sum_{v\in S}\sum_{j\in J}\log\frac{\|\mathbf{x}^J\|_v}{\|L_j(\mathbf{x})\|_v} + O(1)\\ &\leq (|J|-1+\delta)h(\mathbf{x}^J) + O(1)\\ &= (|J|-1+\delta)h(\mathbf{x}) + O(1).\end{aligned} \tag{6.19}$$

It follows that

$$\begin{aligned}\sum_{v\in S}\sum_{i=0}^{q}\lambda_v(\mathbf{x},L_i) &\le \sum_{v\in S}\sum_{\substack{0\le i\le q\\ i\notin J}}\lambda_v(\mathbf{x},L_i)+(|J|-1+\delta)h(\mathbf{x})+O(1)\\ &\le (q+\delta)h(\mathbf{x})+O(1).\end{aligned}$$

Consider an arbitrary subspace W in $\mathcal{T}_1$ with $\dim W = n-1$ and $W \not\subseteq L$ for every $L \in \widetilde{\mathcal{L}}$. It is clear that the system of hyperplanes $\mathcal{L}_W = \{L_i|_W, 0 \le i \le q\}$ is non-subdegenerate with $\widetilde{\mathcal{L}}_W = \{L|_W; L \in \widetilde{\mathcal{L}}\}$ being a conjuntion. By Lemma 6.3, the inequality

$$\sum_{v\in S}\sum_{i=0}^{q}\lambda_v(\mathbf{x},L_i) \le (q+\delta)h(\mathbf{x})+O(1),$$

holds for every $\mathbf{x} \in W$ outside $\widetilde{\mathcal{L}}_0$ and outside the union of a finite collection of proper subspaces of W of codimension 1. Applying this argument for all such subspaces W, we infer the inequality

$$\sum_{v\in S}\sum_{i=0}^{q}\lambda_v(\mathbf{x},L_i) \le (q+\delta)h(\mathbf{x})+O(1),$$

for every $\mathbf{x} \in \mathbb{P}^n(k)$ outside $\widetilde{\mathcal{L}}_0$ and outside the union $\mathcal{T}_2$ of a finite collection of subspaces of codimension at least 2 in $\mathbb{P}^n(F)$.

Continuing this process, after n steps, we have that the inequality holds for every $\mathbf{x} \in \mathbb{P}^n(F)$ outside $\mathcal{L}_0$ and outside the union of finitely many points in $\mathbb{P}^n(F)$. Since the heights of finite points is bounded above and hence can be absorbed by $O(1)$, we have

$$\sum_{v\in S}\sum_{i=0}^{q}\lambda_v(\mathbf{x},L_i) \le (q+\delta)h(\mathbf{x})+O(1),$$

for every $\mathbf{x} \in \mathbb{P}^n(F)$ outside $\mathcal{L}_0$. The theorem is proved. □

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References

1. Enrico Bombieri and Walter Gubler, *Heights in Diophantine geometry*, New Math. Monogr., vol. 4, Cambridge: Cambridge University Press, 2006.
2. Yann Bugeaud, *Quantitative versions of the subspace theorem and applications*, J. Théor. Nombres Bordx. **23** (2011), no. 1, 35–57.
3. Z. Chen and M. Ru, *Integer solutions to decomposable form inequalities*, J. Number Theory **115** (2005), no. 1, 58–70.
4. J.-H. Evertse and R. G. Ferretti, *Diophantine inequalities on projective varieties*, Int. Math. Res. Not. (2002), no. 25, 1295–1330.
5. ______, *A generalization of the Subspace Theorem with polynomials of higher degree*, Diophantine approximation, Dev. Math., vol. 16, SpringerWienNewYork, Vienna, 2008, pp. 175–198.

6. ———, *A further improvement of the quantitative subspace theorem*, Ann. of Math. (2) **177** (2013), no. 2, 513–590.
7. J.-H. Evertse and H. P. Schlickewei, *The Absolute Subspace Theorem and linear equations with unknowns from a multiplicative group*, Number theory in progress, Vol. 1 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999, pp. 121–142.
8. ———, *A quantitative version of the Absolute Subspace Theorem*, J. Reine Angew. Math. **548** (2002), 21–127.
9. Jan-Hendrik Evertse, *An improvement of the quantitative subspace theorem*, Compositio Math. **101** (1996), no. 3, 225–311.
10. ———, *On the quantitative subspace theorem*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **377** (2010), no. Issledovaniya po Teorii Chisel. 10, 217–240, 245.
11. M. Hindry and J. H. Silverman, *Diophantine geometry. An introduction*, Grad. Texts Math., vol. 201, New York, NY: Springer, 2000.
12. Noriko Hirata-Kohno, *An application of quantitative subspace theorem*, Analytic number theory, Kyoto: Kyoto University, Research Inst. for Math. Sciences, 1994, pp. 81–87 (English).
13. Yuancheng Liu, *On the problem of integer solutions to decomposable form inequalities*, Int. J. Number Theory **4** (2008), no. 5, 859–872.
14. J. Noguchi and J. Winkelmann, *Nevanlinna theory in several complex variables and Diophantine approximation*, Grundlehren Math. Wiss., vol. 350, Tokyo: Springer, 2014 (English).
15. Duc Hiep Pham, *Schmidt's subspace theorem for non-subdegenerate families of hyperplanes*, Int. J. Number Theory **18** (2022), no. 3, 557–574.
16. Si Duc Quang, *Degeneracy second main theorem for meromorphic mappings and moving hypersurfaces with truncated counting functions and applications*, Int. J. Math. **31** (2020), no. 6, 18, Id/No 2050045.
17. Klaus F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20 (English).
18. M. Ru and P.-M. Wong, *Integral points of $\mathbb{P}^n$ minus $2n+1$ hyperplanes in general position*, Invent. Math. **106** (1991), no. 1, 195–216.
19. Wolfgang M. Schmidt, *Norm form equations*, Ann. of Math. (2) **96** (1972), 526–551.
20. ———, *Diophantine approximation*, Lecture Notes in Mathematics, vol. 785, Springer, Berlin, 1980.
21. ———, *The subspace theorem in Diophantine approximations*, Compositio Math. **69** (1989), no. 2, 121–173. MR 984633
22. ———, *Diophantine approximations and Diophantine equations*, Lect. Notes Math., vol. 1467, Berlin etc.: Springer-Verlag, 1991.
23. Paul Vojta, *On the Nochka-Chen-Ru-Wong proof of Cartan's conjecture*, J. Number Theory **125** (2007), no. 1, 229–234.
24. ———, *Diophantine approximation and Nevanlinna theory*, Arithmetic geometry. Lectures given at the C.I.M.E summer school, Cetraro, Italy, September 10–15, 2007., Berlin: Springer, 2011, pp. 111–224.

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