Modeling and analysis of some turbulence models. mini-course Vietnam Institute for Advanced Study in Mathematics (VIASM) - May 13-23, 2025- Abstract

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The aim of this course is to:

1. Explain how to derive from the Navier-Stokes equations the turbulence model that couples the mean velocity and the mean pressure to the turbulent kinetic energy (TKE), a model called the NSTKE model,

2. Provide some mathematical tools to carry out its mathematical analysis and present some recent results regarding the existence of weak solutions.

To be more specific, let $\mathbf{v}$ and p denote the velocity and pressure of the fluid, respectively. The pair $(\mathbf{v}, p)$ solves the Navier-Stokes equations (NSE),

$$\begin{cases} \mathbf{v}_t + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \\ \operatorname{div} \mathbf{v} = 0. \end{cases} \tag{1}$$

According to Reynolds decomposition, $\mathbf{v}$ and p are decomposed as the sum of their mean and fluctuation (cf. [2]),

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad \text{and} \quad p = \overline{p} + p',$$

where the averaging filter is linear, commutes with any differential operator (namely $D\overline{\psi} = \overline{D\psi}$), and is idempotent (i.e., $\overline{\overline{\psi}} = \overline{\psi}$). From these assumptions, one obtains the relation

$$\overline{\mathbf{v} \otimes \mathbf{v}} = \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} - \boldsymbol{\sigma}^{(\mathrm{R})},$$

where the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ is given by

$$\boldsymbol{\sigma}^{(\mathrm{R})} := -\overline{\mathbf{v}' \otimes \mathbf{v}'}.$$

Therefore, applying the mean operator to the NSE (1) yields

$$\begin{cases} \overline{\mathbf{v}}_t + \operatorname{div}(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} - \operatorname{div} \boldsymbol{\sigma}^{(\mathrm{R})} + \nabla \overline{p} = \overline{\mathbf{f}}, \\ \operatorname{div} \overline{\mathbf{v}} = 0. \end{cases} \tag{2}$$

From here, the challenge is to determine the Reynolds stress. Following the Boussinesq assumption, we assume that it is dissipative and proportional to the deformation tensor $D\overline{\mathbf{v}} = \frac{1}{2}(\nabla \overline{\mathbf{v}} + \nabla \overline{\mathbf{v}}^T)$, namely,

$$\boldsymbol{\sigma}^{(\mathrm{R})} = \nu_{\mathrm{turb}} D\overline{\mathbf{v}}, \tag{3}$$

leading to the further challenge of determining the coefficient ν_{turb}, called the "eddy viscosity."
The model we consider assumes that ν_{turb} depends on the turbulent kinetic energy

$$k = \frac{1}{2}\overline{|\mathbf{v}'|^2},$$

which, roughly speaking, measures the deviation of $\mathbf{v}$ from its mean $\overline{\mathbf{v}}$. From dimensional analysis, we find that ν_{turb} must take the form

$$\nu_{\text{turb}} = C\ell\sqrt{k},$$

where ℓ is the Prandtl mixing length, and C is a constant typically determined from experimental data.
In the first part of the course, we will explain how to derive, from the standard assumptions about homogeneous isotropic turbulence, the following equation for the turbulent kinetic energy k:

$$k_t + \overline{\mathbf{v}} \cdot \nabla k - \operatorname{div}(\mu_{\text{turb}}(k)\nabla k) = \nu_{\text{turb}}(k)|D\overline{\mathbf{v}}|^2 - \frac{k\sqrt{k}}{\ell}.$$

This leads to the following coupled system:

$$\begin{cases} \overline{\mathbf{v}}_t + \operatorname{div}(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \operatorname{div}((\nu + \nu_{\text{turb}}(k)D\overline{\mathbf{v}}) + \nabla \overline{p} = \overline{\mathbf{f}}, \\ k_t + \overline{\mathbf{v}} \cdot \nabla k - \operatorname{div}(\mu_{\text{turb}}(k)\nabla k) = \nu_{\text{turb}}(k)|D\overline{\mathbf{v}}|^2 - \dfrac{k\sqrt{k}}{\ell}, \\ \operatorname{div}\overline{\mathbf{v}} = 0. \end{cases} \tag{4}$$

In the second part of the course, we will perform a mathematical analysis of System (4) with various boundary conditions, starting from the initial results in [3] to the most recent results in [1].

References

[1] C. Amrouche, G. Leloup, and R. Lewandowski. TKE model involving the distance to the wall. Part 1: the relaxed case. *Journal of Mathematical Fluid Mechanics*, 26:paper number 58, 2024.

[2] T. Chacòn-Rebollo and R. Lewandowski. *Mathematical and Numerical Foundations of Turbulence Models and Applications*. Modeling and Simulation in Science, Engineering and Technology. Springer New York, 2014.

[3] R. Lewandowski. The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy viscosity. *Nonlinear Anal.*, 28(2):393–417, 1997.