Quantum topology, hyperbolic geometry, and connections between the two

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Outline.

1. Introduction: the Kashaev Volume Conjecture

2. Some hyperbolic geometry

3. The quantum Teichmüller space and its representations

4. Invariants of surface diffeomorphisms

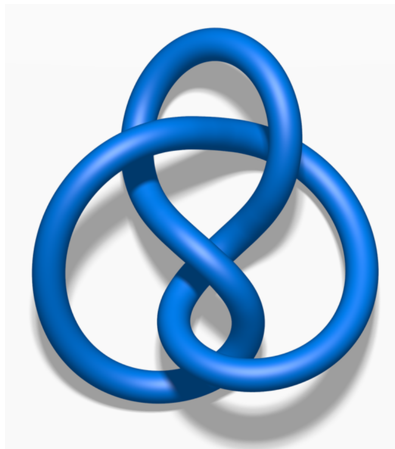
5. Asymptotics of quantum invariants

Part I. The Kashaev Volume Conjecture in knot theory

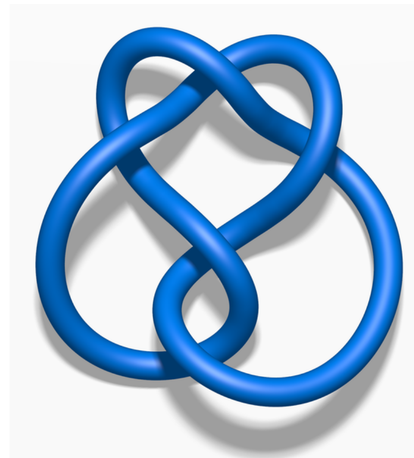
Knots in space. Knot theory is the study of the following problem.

Problem. Given two simple closed curves K_1, K_2 in the 3–dimensional space $\mathbb{R}^3$, is there a diffeomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that:

- it sends K_1 to K_2 in the sense that $\varphi(K_1) = K_2$;
- it is *isotopic to the identity* in the sense that there exists a continuous family of diffeomorphisms $(\varphi_t)_{0 \leqslant t \leqslant 1}$ with $\varphi_0 = \varphi$ and $\varphi_1 = \mathrm{Id}_{\mathbb{R}^3}$.



The knot 4_1



The knot 5_2

In general, the difficult part is to show that K_1 and K_2 are *not* isotopic in the above sense. This is usually done by assigning to knots certain *invariants*, namely quantities that are unchanged under isotopy of the knots. If two knots have different invariants, this guarantees that they are not isotopic. (The converse is unfortunately not true)

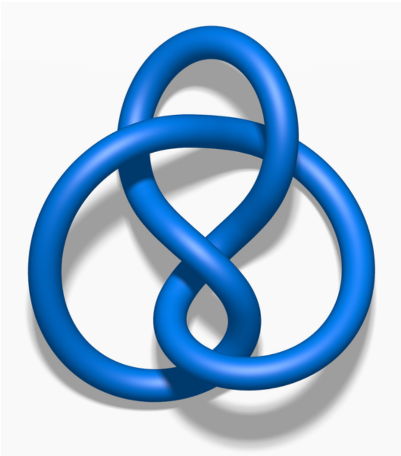
We will encounter two very different methods to show that two knots K_1, K_2 are not isotopic.

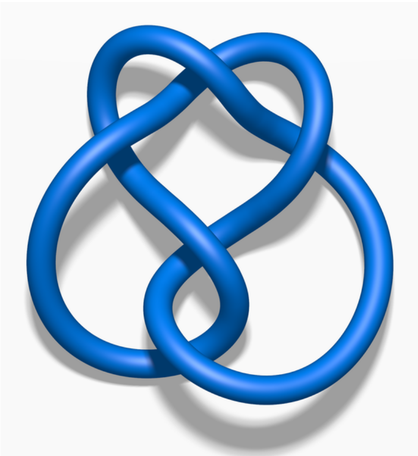
- quantum topology, which is very algebraic and combinatorial;
- hyperbolic geometry, which is very geometric and analytic.

Quantum topology. Provides invariants based on the representation theory of quantum groups. There is no need to know what they are. We will just say that these invariants usually are polynomials with integer coefficients.

A fundamental example for us is the n–th Jones polynomial $J_n^K(q) \in \mathbb{Z}[q]$ of the knot K, defined for every positive integer n.

Example.





$$J_1^{4_1}(q) = q^2 - q + 1 - q^{-1} + q^{-2} \qquad J_1^{5_2}(q) = q^{-1} - q^{-2} + 2q^{-3} - q^{-4} + q^{-5} - q^{-6}$$

$$J_n^{4_1}(q) = \sum_{k=0} q^{-k(n+1)} \prod_{l=1}^{k} (1 - q^{n+l+1})(1 - q^{n-l+1})$$

The colored Jones polynomials of a knot are computed from a picture of the knot, using an explicit combinatorial recipe.

Hyperbolic geometry. This approach is based on the fundamental theorem of Thurston. It is now useful to consider the 3–dimensional sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Theorem (Thurston, Mostow)**.** *For "most" knots $K \subset \mathbb{R}^3 \subset S^3$, the knot complement $S^3 - K$ admits a unique complete riemannian metric d_{hyp} of constant sectional curvature -1.*

The exceptions where the theorem does not apply are very rare. They admit canonical splittings into links (= knots with several components) where the theorem applies.

The metric d_{hyp} is the *hyperbolic metric* of the knot complement $S^3 - K$.

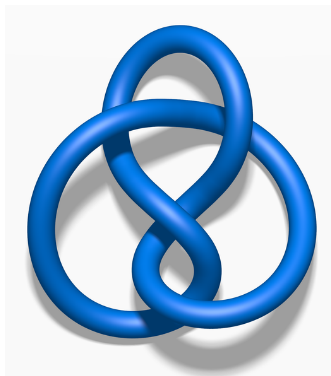
The uniqueness property means that any two such metrics d_{hyp}, d'_{hyp} are isometric, in the sense that there exists a diffeomorphism $\varphi\colon S^3 - K \to S^3 - K$ such that $d'_{\mathrm{hyp}}\big(\varphi(x), \varphi(y)\big) = d_{\mathrm{hyp}}(x, y)$ for every x, $y \in S^3 - K$.

In particular, any quantity that can be read from the metric d_{hyp}, for instance its total volume $\mathrm{vol}_{\mathrm{hyp}}(S^3 - K)$, is an invariant of the knot.

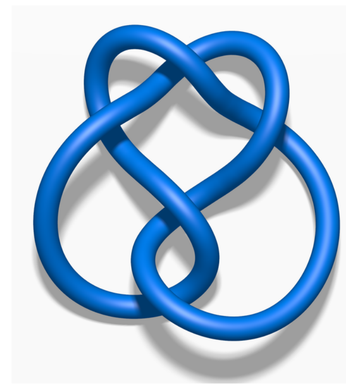
The proof of Thurston's theorem is grounded in analysis, and involves finding a fixed point for a certain complex analytic map. In particular, it is an abstract existence result, and does not provide an algorithm to explicitly find the hyperbolic metric d_{hyp}.

There are geometric techniques to explicitly construct hyperbolic metrics on specific examples, but no general methods to do so. The free software `SnapPy` is very good at doing this, and computing data associated to the hyperbolic metric d_{hyp}.

Example.

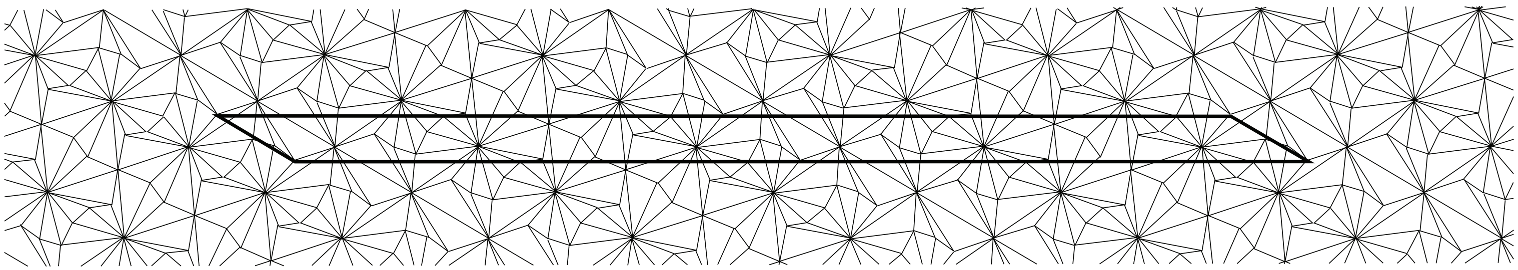


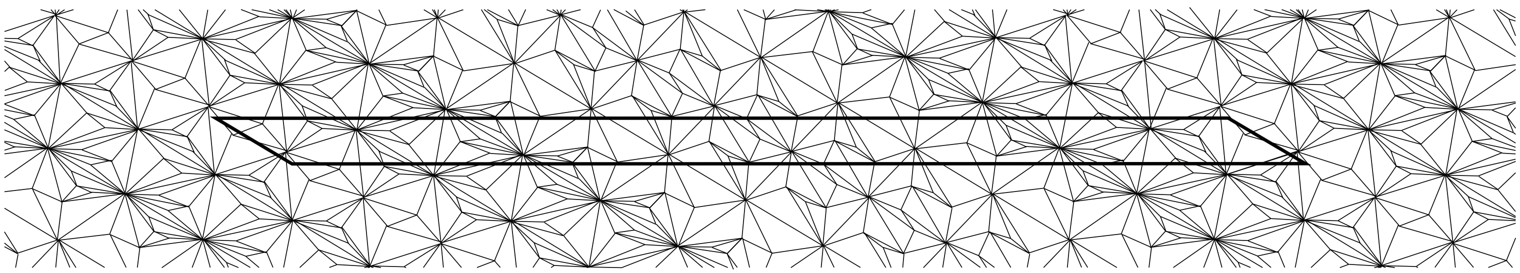
$\text{vol}_{\text{hyp}}(S^3 - 4_1) \approx 2.02988\ldots$



$\text{vol}_{\text{hyp}}(S^3 - 5_1) \approx 2.82812\ldots$

Another type of invariant for d_{hyp} is the Ford-Riley domain (called cusp triangulation in `SnapPy`)





This is a very powerful invariant, because two hyperbolic knots are the same *if and only if* their Ford-Riley domain (together with their gluing information) are combinatorially the same.

The Kashaev Volume Conjecture. Let $J_n^K(q)$ be the n–th colored Jones polynomial of a knot K.

The Kashaev Volume Conjecture states that, for $q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$, the quantity $J_n^K\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\big)$ grows exponentially with n, and that its rate of exponential growth is related to the volume of the hyperbolic metric d_{hyp} as follows.

Conjecture (Kashaev Volume Conjecture)**.**

$$\lim_{n\to\infty} \frac{1}{n} \log \left| J_n^K\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\big) \right| = \frac{1}{2\pi} \mathrm{vol}_{\mathrm{hyp}}(S^3 - K).$$

The exciting feature of this conjecture is that it connects two very different aspects of knot theory.

The history behind it is that Rinat Kashaev had introduced a “new” knot invariant for which, by several heuristic arguments, he had made the above conjecture. Shortly thereafter, Hitoshi Murakami and Jun Murakami proved that this invariant was actually equal to $\widetilde{J}_n^K\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\big)$.

In this minicourse, we will discuss a closely related conjecture.

Part II. Some hyperbolic geometry

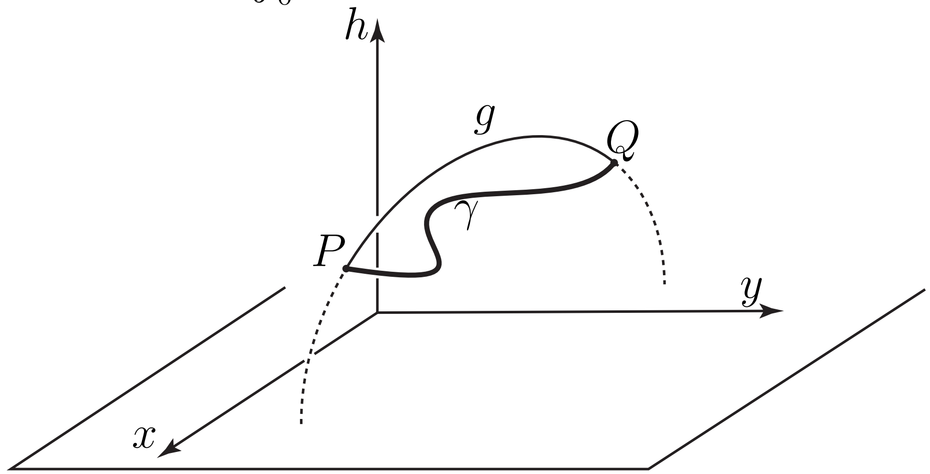
We will discuss a method to explicit construct a hyperbolic metric on a knot complement.

The hyperbolic space. We begin with a few basic facts about hyperbolic geometry. In $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, the 3–*dimensional hyperbolic space* is

$$\mathbb{H}^3 = \{(z, h); z \in \mathbb{C}, 0 < h < +\infty\}$$

endowed with the metric d_{hyp} that, at the point (z, h), is $\frac{1}{h}$ times the euclidean metric. Namely, if $\gamma\colon [0,1] \to \mathbb{H}^3$ is a curve in $\mathbb{H}^3$ parametrized by $\gamma(t) = \big(x(t) + \mathrm{i}y(t), h(t)\big)$, its *hyperbolic length* is

$$\ell_{\mathrm{hyp}}(\gamma) = \int_0^1 \frac{1}{h}\sqrt{x'(t)^2 + y'(t)^2 + h'(t)^2}\, dt.$$



The *hyperbolic distance* between two points P, $Q \in \mathbb{H}^3$ is then defined as the minimum hyperbolic length of all curves going from P to Q

$$d_{\mathrm{hyp}}(P, Q) = \min\{\ell_{\mathrm{hyp}}(\gamma); \gamma \text{ goes from } P \text{ to } Q\}.$$

The minimum is realized by a *hyperbolic geodesic* g, which is the unique circle arc going from P to Q that is contained in a circle that is perpendicular to the plane $\mathbb{C} \times 0$ (and in particular has its center in $\mathbb{C} \times 0$).

We will only use a few facts in hyperbolic geometry.

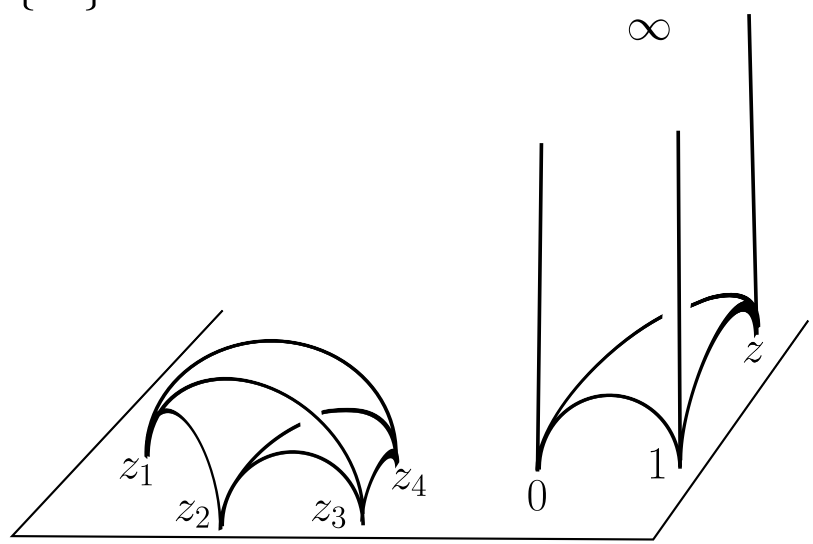
Isometries. The first one is that every orientation-preserving isometry $\mathbb{H}^3 \to \mathbb{H}^3$ continuously extends to the boundary $\mathbb{C} \cup \{\infty\}$ of $\mathbb{H}^4 \subset \mathbb{R}^3$, and acts there by the linear fractional map

$$z \mapsto \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$. (There is an explicit formula for the extension of $z \mapsto \frac{az+b}{cz+d}$ to an isometry of $\mathbb{H}^3$, but it is ugly.)

This identifies the group of orientation-preserving isometries of $\mathbb{H}^3$ to the *projective linear group* $\mathrm{PSL}_2(\mathbb{C})/\{\pm\mathrm{Id}\}$.

Ideal tetrahedra. We will also need to know something about *ideal tetrahedra*, namely tetrahedra in $\mathbb{H}^3$ with straight (= geodesic) edges, flat faces, and whose vertices are on the boundary at infinity $\mathbb{C} \cup \{\infty\}$.



If Δ is an ideal tetrahedron with vertices z_1, z_2, z_3, $z_4 \in \mathbb{C} \cup \{\infty\}$, and if Δ' s another ideal tetrahedron with vertices z_1', z_2', z_3', z_4', there exists an orientation-preserving isometry sending z_1 to z_1', z_2 to z_2', z_3 to z_3' and z_4 to z_4' if and only if the corresponding *cross ratios*

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)} = \frac{(z_1' - z_3')(z_2' - z_4')}{(z_1' - z_2')(z_3' - z_4')}$$

are equal.

The cross-ratio $z = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)} \in \mathbb{C} - \{0,1\}$ is the *shape parameter* of Δ along the edge z_1z_4. From the case where $z_1 = 0$, $z_2 = 1$, $z_3 = z$ and $z_4 = \infty$, we see the argument $\arg z$ is equal to the dihedral angle of Δ along the edge z_1z_4.

Ideal tetrahedra as building blocks of hyperbolic manifold. Let $\overline{M}$ be a 3–dimensional manifold whose boundary consists of tori. Let M be its interior, and let $\widehat{M}$ be obtained from $\overline{M}$ by collapsing each boundary component to a single point.

A (topological) *ideal triangulation* of M is a triangulation of $\widehat{M}$ whose vertices are exactly the points coming from the boundary components of $\overline{M}$. Each tetrahedron of the triangulation of $\widehat{M}$ fives a tetrahedrom with its vertices removed in M, which is homeomorphic to any ideal tetrahedron in the hyperbolic space $\mathbb{H}^3$

For instance, every knot complement $S^3 - K$ admits such an ideal triangulation. The case of the figure-eight knot 4_1, which admits an ideal triangulation with only two ideal tetrahedra, is particularly famous.

Proposition. *Let the 3–manifold M come with a topological ideal triangulation with ideal tetrahedra Δ_1, Δ_2, ..., Δ_n. Suppose that we can find identifications of Δ_1, Δ_2, ..., Δ_n with ideal hyperbolic tetrahedra such that:*

(1) *for each edge of the ideal triangulation, the dihedral angles of the hyperbolic ideal tetrahedra occurring around that edge are all nonnegative and add up to 2π;*
(2) *for each edge of the ideal triangulation, the product of the angles of the hyperbolic ideal tetrahedra occurring around that edge is equal to 1;*
(3) *a condition at the vertices of the triangulation of $\widehat{M}$ that we will neglect in these lectures.*

Then, the hyperbolic metrics of the hyperbolic ideal tetrahedra can be glued together to give a complete metric d_∞ on M that is locally isometric to the metric of $\mathbb{H}^3$.

Note that the first two conditions are related, but not equivalent. They guarantee that the metric is locally isometric to that of $\mathbb{H}^3$.

The third condition is equivalent to the completeness of the metric d_{hyp}.

For the figure-eight knot complement $S^3 - 4_1$, the ideal triangulation consists of two ideal tetrahedra with all shape parameters equal to $\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$.

The Ford-Riley domain of a hyperbolic knot complement has something to do with such a decomposition into hyperbolic ideal tetrahedra.

Part III. The quantum Teichmüller space

Enhanced homomorphisms $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$. Let S be the surface obtained from a compact surface with boundary $\overline{S}$ of genus g by removing p punctures $v_1, v_2, \ldots, v_p$.

An *enhanced homomorphism* from $\pi_1(S)$ to $\mathrm{PSL}_2(\mathbb{C})$ is the data of a group homomorphism $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ together the choice, for every puncture v, of a point in $\mathbb{C}\cup\infty$ that is fixed by the action of the element $r(\gamma_v) \in \mathrm{PSL}_2(\mathbb{C})$, where γ_v is a loop in S going counterclockwise once around the puncture v.

Because the element $\gamma_v \in \pi_1(S)$ is only defined once we have chosen a path going from γ_v to the base point x_0 used in the definition of $\pi_1(S) = \pi_1(S; x_0)$, we need to be careful in the definition. A *peripheral subgroup* of the fundamental group $\pi_1(S)$ is a cyclic subgroup generated by an element $\gamma_v \in \pi_1(S)$ defined in such a way. In particular, a puncture v corresponds to many peripheral subgroups, one for each homotopy class of paths joining that puncture to the based point used in the definition of $\pi_1(S)$, which are all conjugate to each other.

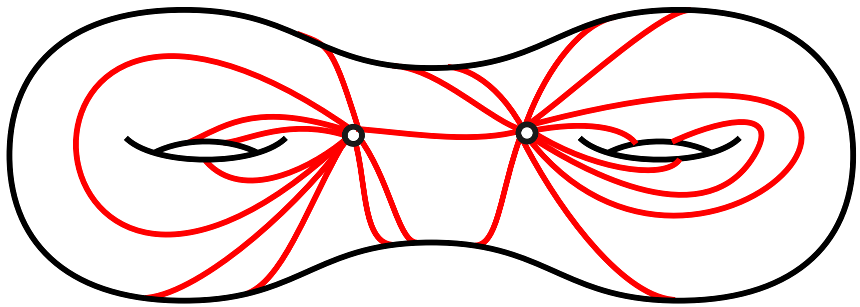
Let Π denote the set of peripheral subgroups of $\pi_1(S)$. An *enhanced homomorphism* from $\pi_1(S)$ to $\mathrm{PSL}_2(\mathbb{C})$ is the data (r, ξ) of a group homomorphism $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ together with a map $\xi\colon \Pi \to \mathbb{C} \cup \{\infty\}$ that is r*–equivariant* in the sense that

$$r(\gamma\pi\gamma^{-1}) = r(\gamma)\big(\xi(\pi)\big)$$

for every $\pi \in \Pi$ and $\gamma \in \pi_1(S)$. In particular, this implies that $\xi(\pi) \in \mathbb{C} \cup \{\infty\}$ is fixed by $r(\gamma) \in \mathrm{PSL}_2(\mathbb{C})$ for every γ in the peripheral subgroup π.

Note that, for a peripheral subgroup π associated to the puncture v with $r(\gamma_v) \neq 1$, this property “almost” determines $\xi(\pi)$ since the fixed point set of $r(\gamma_v)$ then consists of at most 2 points.

Ideal triangulations. An *ideal triangulation* of the surface S is a triangulation where all vertices are at the punctures. They always exist, at least when the surface has at least one puncture and is different from the once- and twice-punctured sphere.

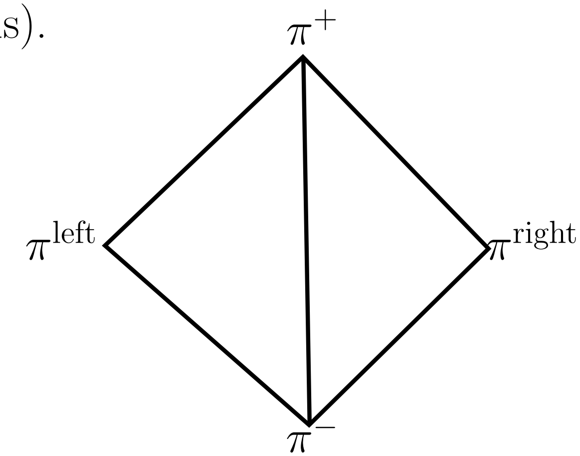


If the surface S has Euler characteristic $\chi(S) = 2-2g-p$, an ideal triangulation has $-3\chi(S)$ edges and $-2\chi(S)$ faces.

Shearbend coordinates for enhanced homomorphisms. Let λ be an ideal triangulation of the surface S, and consider an enhanced homomorphism (r, ξ), consisting of the homomorphism $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ and of the enhancement $\xi\colon \Pi \to \mathbb{C} \cup \{\infty\}$.

The enhanced homomorphism (r, ξ) is *adapted* to the ideal triangulation τ if $\xi(\pi) \neq \xi(\pi')$ whenever the two peripheral subgroups π, $\pi' \in \Pi$ are connected by an edge of λ (use a path connected the edge to the base point to make sense of this).

If (r, ξ) is adapted to λ, let e be an edge of λ with an arbitrary orientation, and let T^{left} and T^{right} be the two faces that are adjacent to e. Let π^+, π^-, π^{left}, $\pi^{\mathrm{left}} \in \Pi$ be the peripheral subgroups respectively associated to the positive endpoint of e, the negative endpoint of e, the third vertex of T^{left} and the third vertex of T^{right} (again, use a path connected the edge to the base point to make sense of this).



The *shearbend parameter* of (r, ξ) along the edge e is

$$x_e = -\frac{\big(\xi(\pi^-) - \xi(\pi^{\mathrm{left}})\big)\big(\xi(\pi^+) - \xi(\pi^{\mathrm{right}})\big)}{\big(\xi(\pi^-) - \xi(\pi^{\mathrm{right}})\big)\big(\xi(\pi^+) - \xi(\pi^{\mathrm{left}})\big)} \in \mathbb{C}^\times = \mathbb{C} - \{0\},$$

namely minus the crossratio of the points $\xi(\pi^+)$, $\xi(\pi^-)$, $\xi(\pi^{\mathrm{left}})$, $\xi(\pi^{\mathrm{right}}) \in \mathbb{C} \cup \{\infty\}$.

By r–equivariance of the enhancement $\xi\colon \Pi \to \mathbb{C} \cup \{\infty\}$ and by invariance of crossratios under the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{C} \cup \{\infty\}$, the shearbend parameter

$$x_e = -\frac{\big(\xi(\pi^-) - \xi(\pi^{\mathrm{left}})\big)\big(\xi(\pi^+) - \xi(\pi^{\mathrm{right}})\big)}{\big(\xi(\pi^-) - \xi(\pi^{\mathrm{right}})\big)\big(\xi(\pi^+) - \xi(\pi^{\mathrm{left}})\big)},$$

is independent of the path used to join e to the base point. The formula is also independent of the orientation of e.

Note that the shearbend parameter is also equal to minus the shape parameter of the ideal tetrahedron with vertices $\xi(\pi^+)$, $\xi(\pi^-)$, $\xi(\pi^{\mathrm{left}})$, $\xi(\pi^{\mathrm{right}}) \in \mathbb{C} \cup \{\infty\} = \partial_\infty \mathbb{H}^3$, along the edge $\xi(\pi^-)\xi(\pi^+)$.

The group $\mathrm{PSL}_2(\mathbb{C})$ acts on homomorphisms $r\colon \pi(S) \to \mathrm{PSL}_2(\mathbb{C})$ by conjugation, and on enhancements $\xi\colon \Pi \to \mathbb{C} \cup \{\infty\}$ by its action on $\mathbb{C} \cup \{\infty\}$. For an ideal triangulation λ, consider the space

$$\mathcal{H}^\lambda_{\mathrm{PSL}_2(\mathbb{C})}(S) = \big\{\text{enhanced homomorphisms } (r,\xi) \text{ adapted to } \lambda\big\}/\mathrm{PSL}_2(\mathbb{C})$$

Recall that an ideal triangulation λ of the surface S has $-3\chi(S) = 6g - 6 + 3p$ edges.

Proposition. *The shearbend coordinates $x_e \in \mathbb{C}^\times = \mathbb{C} - \{0\}$ associated to the edges e of an ideal triangulation λ define a homeomorphism*

$$\varphi_\lambda \colon \mathcal{H}^\lambda_{\mathrm{PSL}_2(\mathbb{C})}(S) \to (\mathbb{C}^\times)^{-3\chi(S)}.$$

This is proved by explicitly defining an inverse map. The key ingredient is that, given any two triples of distinct points of $\mathbb{C} \cup \{\infty\}$, there is a unique element of $\mathrm{PSL}_2(\mathbb{C})$ sending the first triple to the second one. One first uses this property to progressively reconstruct the enhancement $\xi \colon \Pi \to \mathbb{C} \cup \{\infty\}$ from its shearbend parameters. Then, the same property shows that this enhancement is r–equivariant for some homomorphism $r \colon \pi(S) \to \mathrm{PSL}_2(\mathbb{C})$.

As one lets λ range over all ideal triangulations of S, consider the space

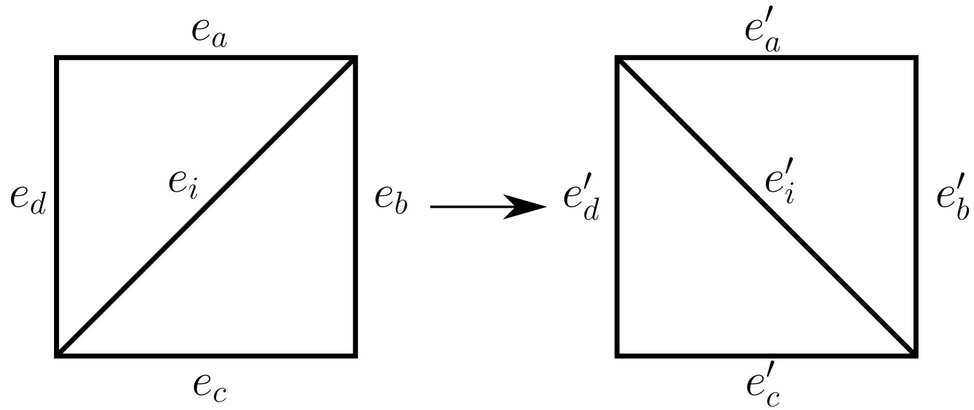
$$\mathcal{H}_{\mathrm{PSL}_2(\mathbb{C})}(S) = \bigcap_{\text{ideal triangulations } \lambda} \mathcal{H}^\lambda_{\mathrm{PSL}_2(\mathbb{C})}(S)$$

of all enhanced $\mathrm{PSL}_2(\mathbb{C})$–characters (r, ξ) that are adapted to all ideal triangulations. It covers a large part of the space of all enhanced $\mathrm{PSL}_2(\mathbb{C})$–homomorphisms. For instance, it covers all enhanced characters (r, ξ) where the homomorphism r comes from a hyperbolic metric on $S \times {]-\infty, +\infty[}$, a fundamental object in 2– and 3–dimensional hyperbolic geometry.

We can think of the maps φ_λ as charts of an atlas for this space $\mathcal{H}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. We will now investigate the coordinate changes on the overlap of two such charts.

Diagonal exchanges. Let λ be an ideal triangulation of the surface S, with edges e_1, e_2, $\dots$, e_n for $n = -3\chi(S)$.

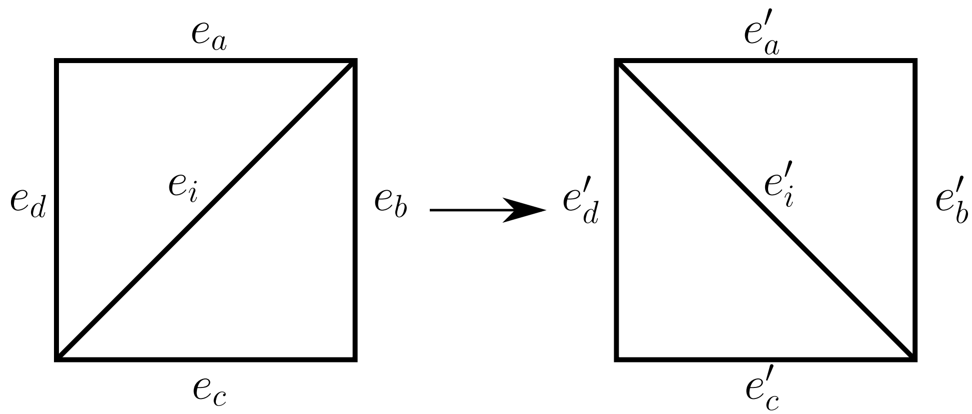
The two faces adjacent to the edge e_i form a square Q, possibly with pairs of sides glued together. Let λ' be the ideal triangulation obtained from λ by replacing the edge e_i with the other diagonal e_i' of Q. We then say that λ' is obtained from λ by performing a *diagonal exchange*, or a *flip*, at its i–th edge.



Fact. *Any two ideal triangulations can be joined by a sequence of diagonal changes.*

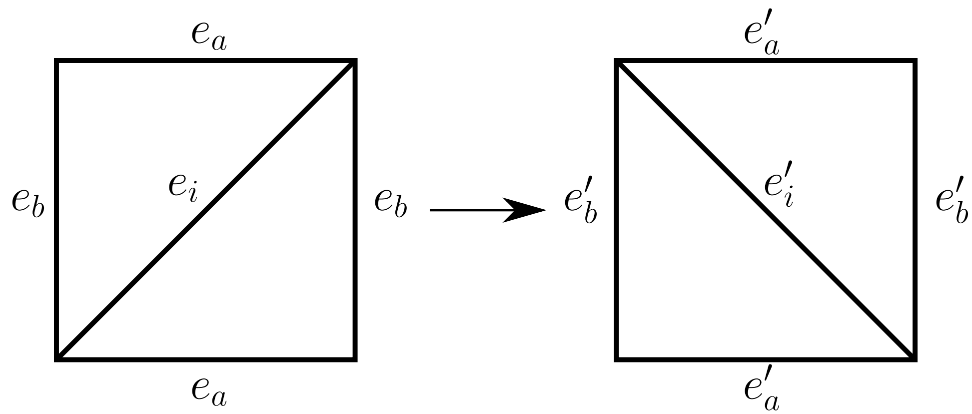
Proposition. *Let the enhanced homomorphism (r, ξ) be adapted to λ, and correspond to shearbend parameters x_1, x_2, $\dots x_n \in \mathbb{C}^\times$ associated to the edges e_1, e_2, $\dots$, e_n of λ. If $e_i \neq -1$, then (r, ξ) is also adapted to the ideal triangulation λ' obtained by performing a diagonal exchange along the edge e_i, and the shearbend parameters x_1', x_2', $\dots x_n' \in \mathbb{C}^\times$ of (r, ξ) along the edges e_1', e_2', $\dots$, e_n' of λ' are computed from by explicit formulas.*

Fundamental example. The sides e_a, e_b, e_c, e_d of the square Q are all distinct.



$$x'_j = \begin{cases} x_i^{-1} & \text{if } j = i \\ x_j\big(1 + x_i\big) & \text{if } j = a \text{ or } c \\ x_j\big(1 + x_i^{-1}\big)^{-1} & \text{if } j = b \text{ or } d \\ x_j & \text{if } j \neq i, a, b, c, d \end{cases}$$

EXTREME SPECIAL EXAMPLE. $e_a = e_c$, $e_b = e_d$ and S is the one-puncture torus.



$$x'_j = \begin{cases} x_i^{-1} & \text{if } j = i \\ x_j\big(1 + x_i\big)^2 & \text{if } j = a \\ x_j\big(1 + x_i^{-1}\big)^{-2} & \text{if } j = b \end{cases}$$

OTHER CASES. Similar formulas.

Summary.

The space

$$\mathcal{H}_{\mathrm{PSL}_2(\mathbb{C})}(S) = \bigcap_{\text{ideal triangulations } \lambda} \mathcal{H}^{\lambda}_{\mathrm{PSL}_2(\mathbb{C})}(S)$$

of all enhanced $\mathrm{PSL}_2(\mathbb{C})$–characters (r, ξ) that are adapted to all ideal triangulations comes with an atlas of charts

$$\varphi_\lambda \colon \mathcal{H}^{\lambda}_{\mathrm{PSL}_2(\mathbb{C})}(S) \to (\mathbb{C}^\times)^n$$

with $n = -3\chi(S)$, indexed by all ideal triangulations λ, and the coordinate changes between these charts are given by certain explicit *rational functions*.

In particular, $\mathcal{H}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ has a well-defined *birational structure*.

Note that each chart $\varphi_\lambda \colon \mathcal{H}^{\lambda}_{\mathrm{PSL}_2(\mathbb{C})}(S) \to (\mathbb{C}^\times)^n$ comes with a preferred family

$$\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}]$$

consisting of Laurent polynomials in variables X_i associated to the edges of the ideal triangulation λ.

We will now define a quantum (noncommutative) version of this construction.

The Chekhov-Fock algebra of an ideal triangulation. Let λ be an ideal triangulation, with edges e_1, e_2, …, e_n with $n = -3\chi(S) = 6g - 6 + 3p$.

As we go counterclockwise around a puncture, we see a succession of edges. Define

$$\begin{aligned} a_{ij} = \#\ &\text{times } e_i \text{ comes immediately before } e_j \text{ counterclockwise} \\ &- \#\ \text{times } e_i \text{ comes immediately before } e_j \text{ counterclockwise.} \end{aligned}$$

Note that $a_{ij} \in \{0, \pm 1, \pm 2\}$, and that $a_{ji} = -a_{ij}$.

The *Chekhov-Fock algebra* $\mathcal{T}^q_\lambda(S) = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_e^{\pm 1}]^q_\lambda$ of the ideal triangulation λ is the algebra of Laurent polynomials in variables X_1, X_2, …, X_n where the multiplication is twisted by the property that the variables X_i do not commute any more, but satisfy instead the q–commutativity relation

$$X_i X_j = q^{2a_{ij}} X_j X_i.$$

Example. The one-puncture torus $S_{1,1}$.

The Chekhov-Fock algebra $\mathcal{T}^q_\lambda(S_{1,1}) = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]^q_\lambda$ is defined by the relations

$$X_1X_2 = q^4 X_2 X_1 \qquad X_2 X_3 = q^4 X_3 X_2 \qquad X_3 X_1 = q^4 X_1 X_3.$$

Example. The 3-puncture sphere $S_{0,3}$.

The Chekhov-Fock algebra $\mathcal{T}^q_\lambda(S_{0,3}) = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]^q_\lambda$ is defined by the relations

$$X_1X_2 = X_2 X_1 \qquad X_2 X_3 = X_3 X_2 \qquad X_3 X_1 = X_1 X_3.$$

Namely, $\mathcal{T}^q_\lambda(S_{0,3})$ is the usual commutative algebra of Laurent polynomials in 3 variables.

The Chekhov-Fock coordinate changes. The elements of the Chekhov-Fock algebra are noncommuting "functions" on a coordinate chart φ_λ.

The *Chekhov-Fock coordinate changes* describe how to go from one chart to another. As in the commutative case, these need to involve rational functions.

Let $\widehat{\mathcal{T}}^q_\lambda(S) = \mathbb{C}(X_1, X_2, \dots, X_e)^q_\lambda$ be the *fraction algebra* of the Chekhov-Fock algebra $\mathcal{T}^q_\lambda(S) = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_e^{\pm 1}]^q_\lambda$. Namely, $\widehat{\mathcal{T}}^q_\lambda(S)$ consists of rational fractions in the variables X_1, X_2, ..., X_e, and are manipulated following the q–commutativity rule $X_i X_j = q^{2a_{ij}} X_j X_i$.

BEWARE! A rational fraction can be written as a right fraction PQ^{-1} of two non-commuting polynomials P, $Q \in \mathbb{C}[X_1, X_2, \dots, X_e]^q_\lambda$, or as a left fraction $Q'^{-1}P'$, but these two decompositions can be very different. This makes the addition of such fractions, by reduction to a common denominator quite tricky.

EXERCISE. In an algebra where $XY = qYX$, write $(X+1)^{-1} + (Y+1)^{-1}$ in the form of a single fraction PQ^{-1}, with P and Q two polynomials in X and Y. Unlike in the commutative case, the common denominator Q needs to have degree at least 3, and there are many such common denominators of degree 3.

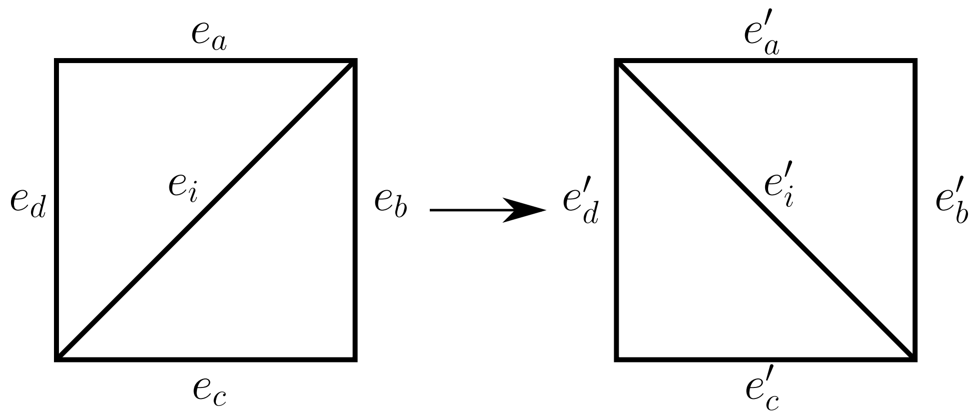
Proposition (Chekhov-Fock)**.** *There exists a unique family of algebra homomorphisms* $\Phi^q_{\lambda\lambda'}\colon \widehat{\mathcal{T}}^q_{\lambda'}(S) \to \widehat{\mathcal{T}}^q_\lambda(S)$*, indexed by all pairs of ideal triangulations* λ*,* λ'*, such that:*

- *for any three ideal triangulations* λ*,* λ'*,* λ''*,*

$$\Phi^q_{\lambda\lambda''} = \Phi^q_{\lambda\lambda'} \circ \Phi^q_{\lambda'\lambda''}$$

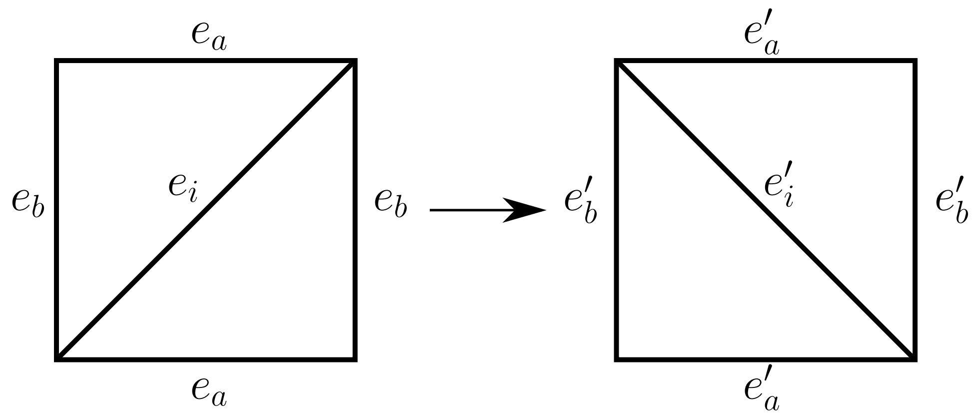
- *when* λ *and* λ' *differ by a diagonal exchanges,* $\Phi^q_{\lambda\lambda'}$ *is given by the formulas of the next slides.*

FUNDAMENTAL CASE. The sides e_a, e_b, e_c, e_d of the square Q are all distinct.



$$\Phi^q_{\lambda\lambda'}(X'_j) = \begin{cases} X_i^{-1} & \text{if } j = i \\ X_j\big(1 + q^{-1}X_i\big) & \text{if } j = a \text{ or } c \\ X_j\big(1 + q^{-1}X_i^{-1}\big)^{-1} & \text{if } j = b \text{ or } d \\ X_j & \text{if } j \neq i, a, b, c, d \end{cases}$$

EXTREME SPECIAL CASE. $e_a = e_c$, $e_b = e_d$ and S is the one-puncture torus.

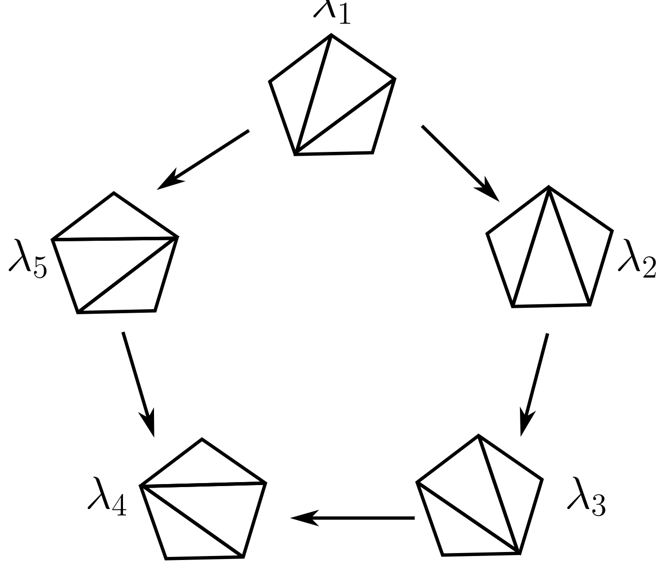


$$X'_j = \begin{cases} X_i^{-1} & \text{if } j = i \\ X_j\big(1 + q^{-1}X_i\big)\big(1 + q^{-3}X_i\big) & \text{if } j = a \\ X_j\big(1 + q^{-1}X_i^{-1}\big)^{-1}\big(1 + q^{-3}X_i^{-1}\big)^{-1} & \text{if } j = b \end{cases}$$

OTHER CASES. Similar formulas.

Remark. The uniqueness of the coordinate changes $\Phi^q_{\lambda\lambda'}$ is clear, since any two ideal triangulations can be joined by a sequence of diagonal exchanges.

What is difficult is to show the existence. In particular, the $\Phi^q_{\lambda\lambda'}$ satisfy the following *pentagon relation*.



$$\Phi^q_{\lambda_4\lambda_3} \circ \Phi^q_{\lambda_3\lambda_2} \circ \Phi^q_{\lambda_2\lambda_1} = \Phi^q_{\lambda_4\lambda_5} \circ \Phi^q_{\lambda_5\lambda_1}$$

Remark. Hua Bai showed that the family of isomorphisms $\Phi^q_{\lambda\lambda'} \colon \widehat{\mathcal{T}}^q_{\lambda'}(S) \to \widehat{\mathcal{T}}^q_{\lambda}(S)$ is, up to uniform rescaling, the only one that satisfies the property that $\Phi^q_{\lambda\lambda''} = \Phi^q_{\lambda\lambda'} \circ \Phi^q_{\lambda'\lambda''}$ and a certain naturality condition.

In other words, the formulas for the case of the diagonal exchange are essentially unique.

Definition. The *quantum Teichmüller space* $\mathcal{T}_S^q$ is (the "atlas" consisting of) the family of the Chekhov-Fock algebras $\mathcal{T}_\lambda^q(S) = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_e^{\pm 1}]_\lambda^q$ and of the Chekhov-Fock coordinate changes $\Phi_{\lambda\lambda'}^q\colon \widehat{\mathcal{T}}_{\lambda'}^q(S) \to \widehat{\mathcal{T}}_\lambda^q(S)$, as λ, λ' range over all ideal triangulations of the surface S.

Representations of the quantum Teichmüller space. A *representation* of the Chekhov-Fock algebra $\mathcal{T}_\lambda^q(S) = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_e^{\pm 1}]_\lambda^q$ over the finite-dimensional vector space V is an algebra homomorphism $\rho_\lambda\colon \mathcal{T}_\lambda^q(S) \to \mathrm{End}(V)$ from $\mathcal{T}_\lambda^q(S)$ to the algebra $\mathrm{End}(V)$ of linear maps $V \to V$.

A *representation of the quantum Teichmüller space* is a family of representations $\rho_\lambda\colon \mathcal{T}_\lambda^q(S) \to \mathrm{End}(V)$ that are compatible with the coordinate changes $\Phi_{\lambda\lambda'}^q$ in the following sense: For every $X' \in T_\lambda^q(S)$, its image $\Phi_{\lambda\lambda'}^q(X') \in \widehat{\mathcal{T}}_\lambda^q(S)$ can be written as fractions

$$\Phi_{\lambda\lambda'}^q(X') = PQ^{-1} = Q'^{-1}P'$$

with P, Q, P', $Q' \in \mathcal{T}_\lambda^q(S)$ such that $\rho_\lambda(Q)$ and $\rho_\lambda(Q') \in \mathrm{End}(V)$ are invertible, and such that

$$\rho_{\lambda'}(X') = \rho_\lambda(P)\rho_\lambda((Q)^{-1} = \rho_\lambda(Q')^{-1}\rho_\lambda(P').$$

In other words, we are trying to say that $\rho_{\lambda'} = \rho \circ \Phi_{\lambda\lambda'}^q$, but we need to be careful because $\Phi_{\lambda\lambda'}^q$ is valued in fractions.

Irreducible representations of the Chekhov Fock algebra $\mathcal{T}^q_\lambda(S)$. Consider the Chekhov-Fock algebra $\mathcal{T}^q_\lambda(S) = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_e^{\pm 1}]^q_\lambda$ of the ideal triangulation λ. This is what is known as a *quantum torus*, so its representation theory is relatively simple.

To have finite-dimensional representations, we need the quantum parameter q to be a root of unity, so assume that q^2 is a primitive N–root of unity. We will restrict attention to the case where N is odd, as the statements are then simpler.

Note that $X_i^N X_j = q^{2a_{ij}N} X_j X_i^N = X_j X_i^N$, so every X_i^N is central.

Similarly, for every puncture v, let e_{i_1}, e_{i_2}, …, e_{i_m} be the edges of λ that lead to v (where an edge occurs twice in this list when both of its ends lead to v), and define

$$H_v = q^{-\sum_{k<l} a_{i_k i_l}} X_{i_1} X_{i_2} \ldots X_{i_m}$$

The q factor is specially designed so that this element is independent of the order in which we take the e_{i_k}. (This is called the Weyl ordering.)

A case-by-case analysis shows that this H_v is central in $\mathcal{T}^q_\lambda(S)$.

There is an additional, more global, central element $H = q^{-\sum_{k<l} a_{kl}} X_1 X_2 \ldots X_n$ but we will not need it in this case where N is odd.

Recall that $n = -3\chi(S)$ is the number of edges of the ideal triangulation λ, and that p is the number of punctures.

Theorem. *Suppose that q^2 is a primitive n–root of unity.*

(1) *For any irreducible representation $\rho_\lambda\colon \mathcal{T}^q_\lambda(S) \to \mathrm{End}(V)$, there exists nonzero numbers $x_1, x_2, \ldots, x_n, h_1, h_2, \ldots, h_p \in \mathbb{C}$ such that*

$$\rho(X_i^N) = x_i \mathrm{Id}_V \quad \textit{and} \quad \rho(H_{v_j}) = h_i \mathrm{Id}_V.$$

(2) *Two irreducible representations ρ_λ, $\rho'_\lambda\colon \mathcal{T}^q_\lambda(S) \to \mathrm{End}(V)$ are isomorphic if and only if they are associated to the same numbers $x_1, x_2, \ldots, x_n, h_1, h_2, \ldots, h_p \in \mathbb{C}$ as above.*

(3) *A set of numbers $x_1, x_2, \ldots, x_n, h_1, h_2, \ldots, h_p \in \mathbb{C}$ is associated to an irreducible representation if and only if the following property holds: If $e_{i_1}, e_{i_2}, \ldots, e_{i_m}$ are the edges leading to the puncture v_j, then $h_j^N = x_{i_1} x_{i_2} \ldots x_{i_m}$.*

Example. The one-puncture torus. Recall that $\mathcal{T}^q_\lambda(S^{1,1}) = \mathbb{C}[X_1, X_2, X_3]^q_\lambda$ with the relations that $X_i X_{i+1} = q^4 X_{i+1} X_i$ for every i. Given nonzero x_1, x_2, x_3, $h_v \in \mathbb{C}$ with $h_v^N = x_1^2 x_2^2 x_3^2$, choose arbitrary N–roots $x_1^{\frac{1}{N}}$, $x_2^{\frac{1}{N}}$, and let k_v be the unique square root of h_v such that $k_v^N = x_1 x_2 x_3$ (remember that N is odd). Then, for every irreducible representation $\rho_\lambda \colon \mathcal{T}^q_\lambda(S) \to \mathrm{End}(V)$ with $\rho(X_1^N) = x_1 \mathrm{Id}_V$, $\rho(X_2^N) = x_2 \mathrm{Id}_V$, $\rho(X_3^N) = x_3 \mathrm{Id}_V$, and $\rho(H_v) = h_v \mathrm{Id}_V$, there is a basis $\{v_1, v_2, \dots, v_N\}$ for V such that

$$\begin{aligned}
\rho_\lambda(X_1)(v_i) &= x_1^{\frac{1}{N}} q^{4i} v_i \\
\rho_\lambda(X_2)(v_i) &= x_2^{\frac{1}{N}} q^{-2i} v_{i+1} \\
\rho_\lambda(X_3)(v_i) &= k_v x_1^{-\frac{1}{N}} x_2^{-\frac{1}{N}} q^{-2i} v_{i-1}.
\end{aligned}$$

(Note that $\rho_\lambda(H) = k_v \mathrm{Id}_V$ for the global central element $H = q^{-2} X_1 X_2 X_3$.)

Compatibility with diagonal exchanges. This is a key property that will enable us to connect the representation theory of the quantum Teichmüller space to the geometry of enhanced $\mathrm{PSL}_2(\mathbb{C})$–homomorphisms.

Proposition. *Suppose that the ideal triangulation λ' is obtained from λ by a diagonal exchange along the edge e_i, and let the irreducible representation*

$$\rho_\lambda \colon \mathcal{T}^q_\lambda(S) \to \mathrm{End}(V)$$

be associated to the edge weights $x_1, x_2, \ldots, x_n \in \mathbb{C}^\times$ and the puncture weights $h_1, h_2, \ldots, h_p \in \mathbb{C}^\times$. Suppose in addition that $x'_1, x'_2, \ldots, x'_n \in \mathbb{C}^\times$ are the edge weights for λ' that define the same enhanced $\mathrm{PSL}_2(\mathbb{C})$–character (r, ξ) as the edge weights $x_1, x_2, \ldots, x_n \in \mathbb{C}^\times$.

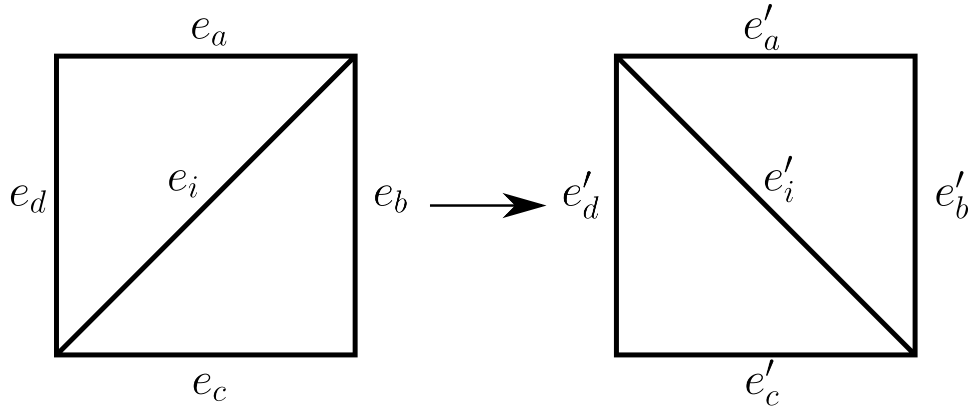
Then, the representation

$$\rho_{\lambda'} = \rho_\lambda \circ \Phi^q_{\lambda\lambda'} \colon \mathcal{T}^q_{\lambda'}(S) \to \mathrm{End}(V)$$

makes sense, is irreducible, and is classified by the edge weights $x'_1, x'_2, \ldots, x'_n \in \mathbb{C}^\times$ and the same puncture weights $h_1, h_2, \ldots, h_p \in \mathbb{C}^\times$.

Let us see this in the

FUNDAMENTAL EXAMPLE. The sides e_a, e_b, e_c, e_d of the square Q are all distinct.



$$x'_j = \begin{cases} x_i^{-1} & \text{if } j = i \\ x_j\big(1 + x_i\big) & \text{if } j = a \text{ or } c \\ x_j\big(1 + x_i^{-1}\big)^{-1} & \text{if } j = b \text{ or } d \\ x_j & \text{if } j \neq i, a, b, c, d \end{cases}$$

In particular, $x_i \neq -1$.

$$\Phi^q_{\lambda\lambda'}(X'_j) = \begin{cases} X_i^{-1} & \text{if } j = i \\ X_j\big(1 + q^{-1}X_i\big) = \big(1 + qX_i\big)X_j & \text{if } j = a \text{ or } c \\ X_j\big(1 + q^{-1}X_i^{-1}\big)^{-1} = \big(1 + qX_i^{-1}\big)^{-1}X_j & \text{if } j = b \text{ or } d \\ X_j & \text{if } j \neq i, a, b, c, d \end{cases}$$

We need to show that $\rho_\lambda\big(1+q^{\pm1}X_i\big) \in \mathrm{End}(V)$ is invertible, so that

$$\rho_{\lambda'} = \rho_\lambda \circ \Phi^q_{\lambda\lambda'} \colon \mathcal{T}^q_{\lambda'}(S) \to \mathrm{End}(V)$$

makes sense. (Note that $\big(1+q^{\pm1}X_i^{-1}\big)^{-1} = q^{\mp1}X_i\big(1+q^{\mp1}X_i\big)^{-1}$.)

For instance, consider $\rho_\lambda\big(1+qX_i\big)$ and note that

$$\begin{aligned}
\rho_\lambda\big(1+qX_i\big) \circ \rho_\lambda(1 - qX_i + q^2X_i^2 - q^3X_i^3 + \cdots - q^{N-1}X_i^{N-1}) \\
&= \rho_\lambda\Big(\big(1+qX_i\big)\big(1 - qX_i + q^2X_i^2 - q^3X_i^3 + \cdots - q^{N-1}X_i^{N-1}\big)\Big) \\
&= \rho_\lambda\big(1+q^NX_i^N\big) = \rho_\lambda(1) + \rho_\lambda X_i^N) = \mathrm{Id}_V + x_i\mathrm{Id}_V
\end{aligned}$$

is invertible, since $x_i \neq -1$.

Therefore, $\rho_\lambda\big(1+qX_i\big)$ is surjective, and therefore invertible.

We also need to show that $\rho_{\lambda'}(X_j'^N) = x_j'\mathrm{Id}_V$ for every j, and $\rho_{\lambda'}(H'_{v_k}) = h_k\mathrm{Id}_V$ for every puncture v_k. For instance, let's do this for $\rho_{\lambda'}(X_a'^N)$.

Then

$$\rho_{\lambda'}(X_a'^N) = \rho_\lambda \circ \Phi^q_{\lambda\lambda'}(X_a'^N) = \rho_\lambda\big((X_i + qX_iX_a)^N\big).$$

Note that $X_i(qX_iX_a) = q^{-2}(qX_iX_a)X_i$. We then use the

Lemma (Quantum Frobenius Formula)**.** *Suppose that $AB = q^{-2}BA$ and that q^2 is a primitive N–root of unity. Then,*

$$(A+B)^N = A^N + B^N.$$

Proof. This is based on the *Quantum Binomial Formula* which states that, for every q,

$$(A+B)^N = \sum_{k-0}^{N} \binom{k}{N}_{q^2} A^k B^{n-k}$$

for the *quantum binomial coefficients*

$$\binom{k}{N}_{q^2} = \frac{(N)_{q^2}(N-1)_{q^2}\dots(N-k+1)_{q^2}}{(k)_{q^2}(k-1)_{q^2}\dots(1)_{q^2}}$$

defined by the *quantum integers*

$$(j)_{q^2} = \frac{1-q^{2j}}{1-q} = 1 + q^2 + q^4 + \dots + q^{2(j-1)}.$$

If q^2 is a primitive N–root of unity, $(N)_{q^2} = 0$ and $(j)_{q^2} \neq 0$ whenever $0 < j < N$. Therefore,

$$\binom{k}{N}_{q^2} = \begin{cases} 0 & \text{if } 0 < k < N \\ 1 & \text{if } k = 0, N. \end{cases}$$

□

Going back to

$$\begin{aligned}
\rho_{\lambda'}(X_a'^N) &= \rho_\lambda \circ \Phi^q_{\lambda\lambda'}(X_a'^N) = \rho_\lambda\big((X_i + qX_iX_a)^N\big) \\
&= \rho_\lambda\big(X_i^N\big) + \rho_\lambda\big((qX_iX_a)^N\big) \\
&= \rho_\lambda\big(X_i^N\big) + q^{N^2}\rho_\lambda\big(X_i^N\big)\rho_\lambda\big(X_a^N\big) \\
&= x_i\mathrm{Id}_V + x_ix_a\mathrm{Id}_V = x_a'\mathrm{Id}_V,
\end{aligned}$$

which is what we wanted.

The other cases are similar.

Representations of the quantum Teichmüller space and enhanced $\mathrm{PSL}_2(\mathbb{C})$–homomorphisms. Let $(r, \xi) \in \mathcal{H}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ be an enhanced $\mathrm{PSL}_2(\mathbb{C})$–character that is adapted to all ideas triangulations λ.

For an ideal triangulation λ and a puncture v of the surface S, let e_{i_1}, e_{i_2}, $\dots$, e_{i_m} be the edges of λ that emanate from v, and let $x_{i_1}, x_{i_2}, \dots, x_{i_m} \in \mathbb{C}^\times$ be the corresponding shear-bend parameters of (r, ξ).

Lemma. *The product*

$$\widehat{h}_v = x_{i_1} x_{i_2} \dots x_{i_m} \in \mathbb{C}^\times$$

is independent of the ideal triangulation λ, and depends only on the enhanced character (r, ξ).

This is proved by a boring case-by-case analysis for diagonal exchanges.

Theorem. *Let q be a primitive N–root of unity with N odd.*

The classification theorem for irreducible representations of Chekhov-Fock algebras provides a one-to-one correspondence between:

- *irreducible representations $\rho\colon \mathcal{T}^q(S) \to \mathrm{End}(V)$ of the quantum Teichmüller space, considered up to isomorphism;*
- *enhanced $\mathrm{PSL}_2(\mathbb{C})$–characters $(r,\xi) \in \mathcal{H}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ adapted to all ideal triangulations together with, for every puncture v, a choice of an N–root h_v for the invariant $\widehat{h}_v$ associated to v by (r,ξ).*

In addition, $\dim V = N^{3g-3+p}$ if S has genus g and $p \geqslant 1$ punctures.

This follows from the compatibility property of diagonal exchanges with the classification theorem for irreducible representations of Chekhov-Fock algebras.

Part IV. Invariants of surface diffeomorphisms

Enhanced $\mathrm{PSL}_2(\mathbb{C})$**–characters fixed by a surface diffeomorphism.** Let S be an oriented surface with genus g and with $p \geqslant 1$ punctures, and let $\varphi.: S \to S$ be an orientation-preserving diffeomorphism.

Then φ acts on the space $\mathcal{H}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ of all enhanced $\mathrm{PSL}_2(\mathbb{C})$–characters (r, ξ) that are adapted to all ideal triangulations. There exists many elements $(r, \xi) \in \mathcal{H}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ that are fixed by this action of φ.

Example 1. Let

$$M_\varphi = S \times [0,1]/(x,1) \sim (\varphi(x), 0)$$

be the *mapping torus* of φ. Thurston's Geometrization Theorem provides a hyperbolic metric d_∞ (for most diffeomorphisms φ), and the monodromy of this metric gives a homomorphism $r_{\mathrm{hyp}}: \pi_1(M_\varphi) \to \mathrm{PSL}_2(\mathbb{C})$, whose restriction to $r_{\mathrm{hyp}}: \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ admits a unique enhancement ξ_{hyp}. The fundamental group of M_φ is an HNN extension

$$\pi_1(M_\varphi) = \big\{\pi_1(S), t;\ t\gamma t^{-1} = \varphi_*(\gamma), \forall \gamma \in \pi_1(S)\big\},$$

and the homomorphisms r_{hyp}, $r_{\mathrm{hyp}} \circ \varphi_*: \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ are therefore conjugate by $r(t) \in \mathrm{PSL}_2(\mathbb{C})$. As a consequence, the enhanced character $(r_{\mathrm{hyp}}, \xi_{\mathrm{hyp}})$ is invariant under the action of φ.

An invariant. Example 2. If p' is the number of orbits of the action of φ on the (finite) set of punctures of S, Thurston's Hyperbolic Dehn Surgery Theorem provides a complex p'–dimensional family of homomorphisms $r: \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ near r_{hyp}, and therefore many other φ–invariant enhanced $\mathrm{PSL}_2(\mathbb{C})$:–characters (r, ξ).

Example 3. There are many more, including some examples where $r: \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ does not lift to a φ–invariant homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$.

Suppose that we are given:

- an orientation-preserving diffeomorphism $\varphi\colon S \to S$;
- a φ–invariant enhanced $\mathrm{PSL}_2(\mathbb{C})$–character (r, ξ);
- a primitive N–root of unity q with N odd;
- φ–invariant choices of N–roots $h_v = \widehat{h}_v^{\frac{1}{N}}$ for the puncture weights $\widehat{h}_v \in \mathbb{C}^\times$ defined by (r, ξ).

The classification theorem for representations of the quantum Teichmüller space associates an irreducible representation $\rho\colon \mathcal{T}^q(S) \to \mathrm{End}(\mathbb{C}^{2g-3+p})$.

The data is φ–invariant $\Longrightarrow$ the output ρ is φ–invariant *up to an isomorphism* $\Lambda^q_{\varphi,r,\xi}\colon \mathbb{C}^{3g-3+p} \to \mathbb{C}^{3g-3+p}$.

Because ρ is irreducible, Schur's Lemma $\Longrightarrow$ $\Lambda^q_{\varphi,r,\xi}$ is unique up to conjugation and scalar multiplication by a nonzero number.

To partially neutralize the rescaling, choose $\Lambda^q_{\varphi,r,\xi}$ so that its determinant is 1.

Lemma. $\Lambda^q_{\varphi,r,\xi}\colon \mathbb{C}^{3g-3+p} \to \mathbb{C}^{3g-3+p}$ *is unique up to conjugation and rescaling by a root of unity of order* $3g-3+p$.

In particular, $\left|\mathrm{Trace}\,\Lambda^q_{\varphi,r,\xi}\right|$ *depends only on the data of the diffeomorphism* φ*, the enhanced character* (r,ξ)*, the quantum parameter* q *and the puncture roots* $h_v = \widehat{h}_v^{\frac{1}{N}} \in \mathbb{C}^\times$.

How to compute $\Lambda^q_{\varphi,r,\xi}$. To "parametrize" the quantum Teichmüller space $\mathcal{T}^q(S)$, fix a single chart $\mathcal{T}^q_{\lambda_0}(S)$ associated to an ideal triangulation λ_0. The diffeomorphism φ then induces an "obvious" isomorphism $\Psi^q_\varphi \colon \mathcal{T}^q_{\lambda_0}(S) \to \mathcal{T}^q_{\varphi(\lambda_0)}(S)$, sending the generator of $\mathcal{T}^q_{\lambda_0}(S)$ associated to an edge e of λ_0 to the generator of $\mathcal{T}^q_{\varphi(\lambda_0)}(S)$ associated to $\varphi(e)$. Then, all the required properties can be expressed in terms of the composition

$$\Phi^q_{\lambda_0\varphi(\lambda_0)} \circ \Psi^q_\varphi \colon \widehat{\mathcal{T}}^q_{\lambda_0}(S) \to \widehat{\mathcal{T}}^q_{\lambda_0}(S)$$

with the Chekhov-Fock coordinate change isomorphism.

In particular, the enhanced $\mathrm{PSL}_2(\mathbb{C})$–character (r,ξ) can be described by its shear-bend coordinates, and it is invariant under the action of φ if and only if these coordinates are fixed by $\Phi^1_{\lambda_0\varphi(\lambda_0)} \circ \Psi^1_\varphi$

To compute $\Phi^q_{\lambda_0\varphi(\lambda_0)}$, connect λ_0 to $\varphi(\lambda_0)$ by a sequence ideal triangulations λ_0, λ_1, λ_2, $\dots$, $\lambda_{k_0} = \varphi(\lambda_0)$ such that each λ_k is obtained from λ_{k-1} by a diagonal exchange. Then, $\Phi^q_{\lambda_0\varphi(\lambda_0)}$ is the composition of the $\Phi^q_{\lambda_{k-1}\lambda_k}$.

This reduces the problem to finding explicit isomorphisms between two representations ρ_k, and $\rho_{k-1}\circ\Phi^q_{\lambda_{k-1}\lambda_k}$ of the Chekhov-Fock algebra $\mathcal{T}^q_{\lambda_k}$ that are abstractly known to be isomorphic. This can usually be done "by hand".

In particular because, up to diffeomorphism of S, there are only finitely many pairs (λ, e) where e is an edge of the ideal triangulation λ, this computation can be preprocessed by first making the computations in this bounded number of cases.

Part V. Asymptotics of $\left|\operatorname{Trace} \Lambda^q_{\varphi,r,\xi}\right|$

Asymptotics. Suppose that we are given:

- an orientation-preserving diffeomorphism $\varphi\colon S \to S$;
- a φ–invariant enhanced $\mathrm{PSL}_2(\mathbb{C})$–character (r, ξ).

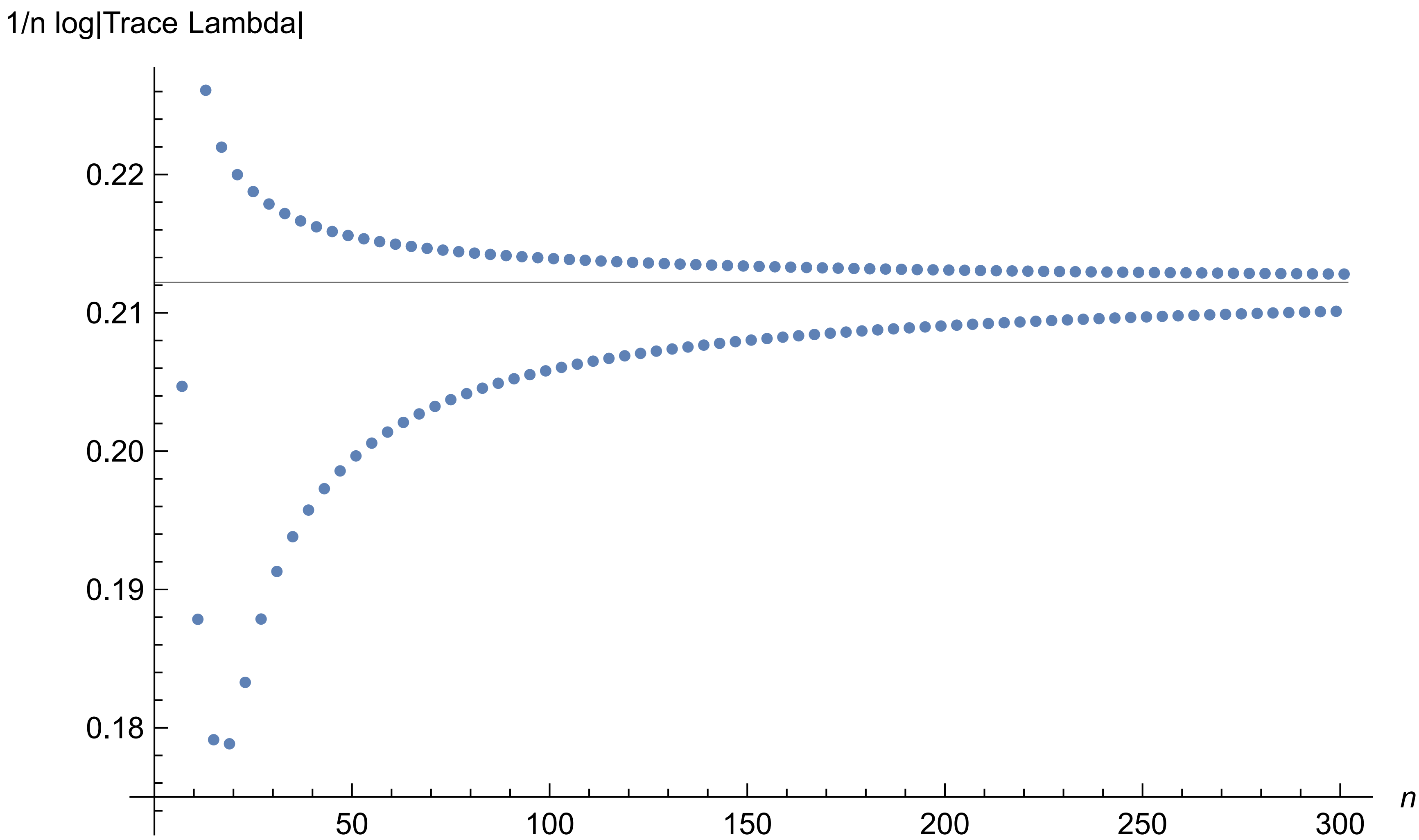
Choose φ–invariant logarithms $\log \widehat{h}_v$ for the puncture weights $\widehat{h}_v \in \mathbb{C}^\times$ defined by (r, ξ).

Then, for every odd N, we make the following systematic choices:

- the primitive N–root of unity $q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}}$;
- the N–roots $h_v = \mathrm{e}^{\frac{1}{N}\log \widehat{h}_v}$ for the puncture weights $\widehat{h}_v$.

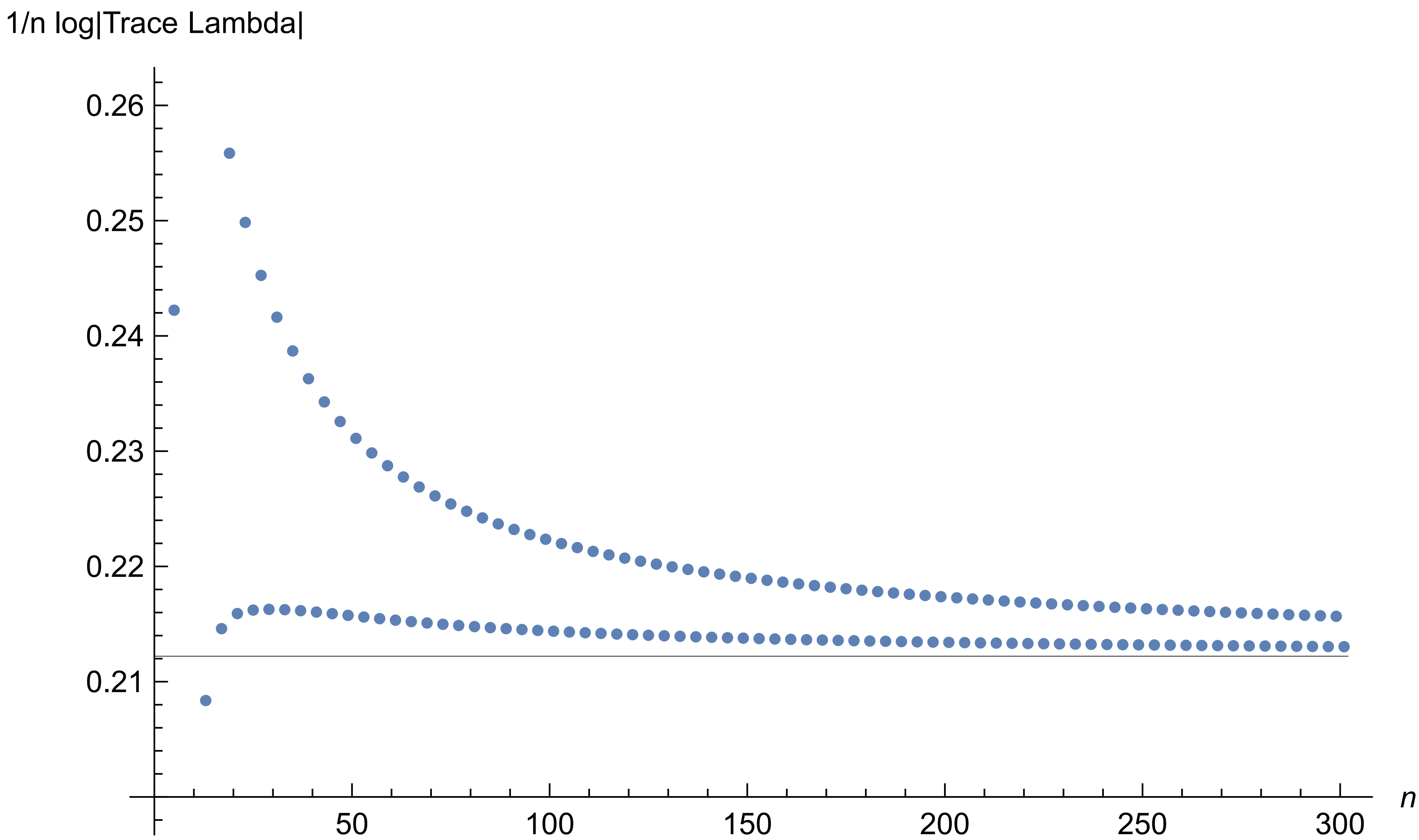
Then, for every N odd, we can plot $\frac{1}{N} \log \left| \mathrm{Trace}\, \Lambda^q_{\varphi, r, \xi} \right|$

EXAMPLE 1. For a specific $\varphi\colon S_{1,1} \to S_{1,1}$ of the 1–puncture torus $S_{1,1}$, and for some φ–invariant enhanced $\mathrm{PSL}_2(\mathbb{C})$–character (r_1, ξ_1),



limit $\approx 0.21221243\ldots$

Example 2. For the same $\varphi\colon S_{1,1} \to S_{1,1}$ of the 1–puncture torus $S_{1,1}$, and a different φ–invariant enhanced $\mathrm{PSL}_2(\mathbb{C})$–character (r_2, ξ_2),



limit $\approx 0.21221243\ldots$

What is $0.21221243\ldots$**?**

Consider the mapping torus M_φ of the same diffeomorphism $\varphi\colon S_{1,1} \to S_{1,1}$ of the 1–puncture torus $S_{1,1}$.

Its complete hyperbolic metric has volume

$$\mathrm{vol}_{\mathrm{hyp}} M_\varphi \approx 2.66674004\ldots$$

$$\frac{1}{4\pi}\mathrm{vol}_{\mathrm{hyp}} M_\varphi \approx 0.21221243\ldots$$

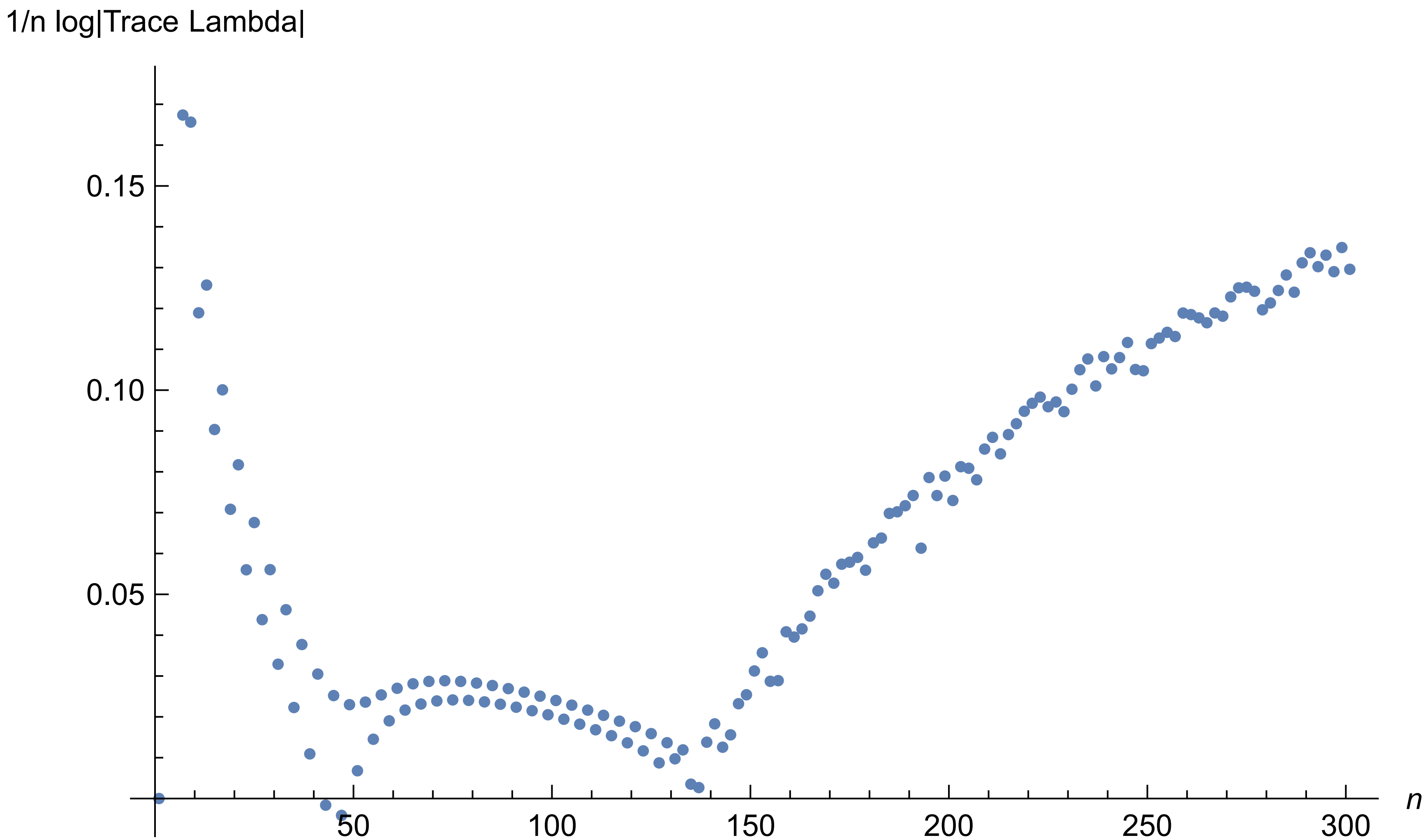
Conjecture. *Given a surface diffeomorphism* $\varphi\colon S \to S$*, a* φ*–invariant enhanced* $\mathrm{PSL}_2(\mathbb{C})$*–character* (r,ξ) *and, for each* N *odd, the quantum parameter* $q = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}$ *and the* N*–roots* $h_v = \mathrm{e}^{\frac{1}{N}\log \widehat{h}_v}$ *for the puncture weights* $\widehat{h}_v$ *defined by* (r,ξ)*, let* $\Lambda^q_{\varphi,r,\xi} \in \mathrm{End}(\mathbb{C}^{3g-3+p})$ *be the associated isomorphism between* ρ *and* $\rho\circ\varphi_*$*, normalized so that* $\det \Lambda^q_{\varphi,r,\xi} = 1$*.* Assume, in addition, that $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ lifts to a φ–invariant homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. *Then,*

$$\lim_{N\to\infty} \frac{1}{N} \log \left| \mathrm{Trace}\, \Lambda^q_{\varphi,r,\xi} \right| = \frac{1}{4\pi}\mathrm{vol}_{\mathrm{hyp}} M_\varphi$$

where $\mathrm{vol}_{\mathrm{hyp}} M_\varphi$ *is the volume of the complete hyperbolic metric of the mapping torus* M_φ*.*

Why the hypothesis on the lift to $\mathrm{SL}_2(\mathbb{C})$? Because of the following numerical observation.

EXAMPLE 2. For the same $\varphi\colon S_{1,1} \to S_{1,1}$ of the 1–puncture torus $S_{1,1}$ as before, and another different φ–invariant enhanced $\mathrm{PSL}_2(\mathbb{C})$–character (r_2, ξ_2),



limit = ??? Most likely phenomenon at $N \geqslant 120$: Rounding errors in the new drastic cancellations that occur in this case.

Conjecture. *Given a surface diffeomorphism $\varphi\colon S \to S$, a φ–invariant enhanced* $\mathrm{PSL}_2(\mathbb{C})$*–character (r,ξ) and, for each N odd, the quantum parameter $q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}}$ and the N–roots $h_v = \mathrm{e}^{\frac{1}{N}\log \widehat{h}_v}$ for the puncture weights $\widehat{h}_v$ defined by (r,ξ), let $\Lambda^q_{\varphi,r,\xi} \in \mathrm{End}(\mathbb{C}^{3g-3+p})$ be the associated isomorphism between ρ and $\rho\circ\varphi_*$, normalized so that $\det \Lambda^q_{\varphi,r,\xi} = 1$. Assume, in addition, that $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ DOES NOT to a φ–invariant homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. Then,*

$$\limsup_{N\to\infty} \frac{1}{N} \log \left| \mathrm{Trace}\, \Lambda^q_{\varphi,r,\xi} \right| < \frac{1}{4\pi} \mathrm{vol}_{\mathrm{hyp}} M_\varphi$$

where $\mathrm{vol}_{\mathrm{hyp}} M_\varphi$ is the volume of the complete hyperbolic metric of the mapping torus M_φ.

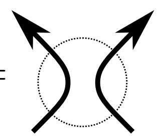
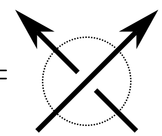
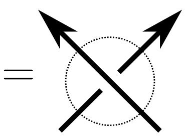
Theorem (FB + Helen Wong + Tian Yang)**.** *The two conjectures are true for many diffeomorphisms of the one-puncture torus.*

Part VI. Exercises, possible discussion topics, references

Part I. The Kashaev Volume Conjecture in knot theory The n–th Jones polynomial is usually defined through the representation theory of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$ (no need to know what this is), but the case $n = 1$ can be given a more elementary description.

In this description, the Jones polynomial function is the unique assignment of a Laurent polynomial $J_1^K(q) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ to each oriented link (= knot with several components) K in $\mathbb{R}^3$ such that:

- $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_1^{K_0}(q) = q^{-\frac{1}{2}}J_1^{K_+}(q) - q^{\frac{1}{2}}J_1^{K_-}(q)$ whenever K_0, K_+ and K_- coincide everywhere except on a small ball where

$K_0 =$  , $K_+ =$  , $K_- =$  .

- $J_1^K(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{k-1}$ if K is the trivial link consisting of k unknotted and unlinked circles.

Use this formulation to recover the expressions of $J_1^{4_1}(q)$ and $J_1^{5_2}(q)$ in Page 5.

Part II. Some hyperbolic geometry

1. The software `SnapPy` works on a large number of platforms, and is freely downloadable at `https://snappy.computop.org/`

Install it on your computer and play with it!

Part III. The quantum Teichmüller space

1. Solve the exercise mentioned in Page 27:

In an algebra where $XY = qYX$, write $(X+1)^{-1} + (Y+1)^{-1}$ in the form of a single fraction PQ^{-1}, with P and Q two polynomials in X and Y. Unlike in the commutative case, the common denominator Q needs to have degree at least 3, and there are many such common denominators of degree 3. Then, rewrite the same sum as a left fraction $Q'^{-1}P'$.

2. Rigorously prove the result mentioned in Page 34, on the explicit description of irreducible representations of the Chekhov-Fock algebra of the one-puncture torus.

3. If, in the formulas of Page 34, we take different N–roots $x_1^{\frac{1}{N}}$ and $x_2^{\frac{1}{N}}$ but keep the same puncture weight h_v with $h_v^N = x_1^2 x_2^2 x_3^2$, we abstractly know that the resulting new representation is abstractly isomorphic to the first one. Compute an explicit isomorphism.

Part IV. Invariants of surface diffeomorphisms

Part V. Asymptotics of $\left|\mathrm{Trace}\,\Lambda^q_{\varphi,r,\xi}\right|$

Possible references

- Rinat Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275.
- Francis Bonahon, *Low-dimensional geometry: from euclidean surfaces to hyperbolic knots*, Student Mathematical Library, vol. 49, American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2009, IAS/Park City Mathematical Subseries.
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- Francis Bonahon and Xiaobo Liu, *Representations of the quantum Teichmüller space and invariants of surface diffeomorphisms*, Geom. Topol. **11** (2007), 889–937.

- Xiaobo Liu, *The quantum Teichmüller space as a noncommutative algebraic object*, J. Knot Theory Ramifications **18** (2009), no. 5, 705–726.
- Francis Bonahon, Helen Wong, and Tian Yang, *Asymptotics of quantum invariants of surface diffeomorphisms I: conjecture and algebraic computations*, `arXiv:2112.12852`, 2021.
- Francis Bonahon, Helen Wong, and Tian Yang, *Asymptotics of quantum invariants of surface diffeomorphisms II: the figure-eight knot complement*, `arXiv:2203.05730`, 2022.