LECTURE NOTES ON TQFTS- VIASM 2025

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Contents

1. General Overview

Warning. These notes are really in a draft version and might contain many typos and mistakes. They are for the eyes of the participants to the Summer School on Quantum Topology 2025 at VIASM Hanoi only ! If you're reading these notes and you find some typos or mistakes please do not hesitate to inform me. Also the bibliography is incomplete and, mainly in the introduction I should add many more references to important papers and works existing in the litterature.

In these lecture notes we will define the notion of TQFT and provide different examples of their construction, typically in dimension $(3, 2)$ (aka $2 + 1$) and $(4, 3)$ (aka $3 + 1$). As we will see, a TQFT is a functor from a source category of manifolds and their cobordisms to a target category which can be Vect, GrVect, $H - Mod$ (for some algebra H) but sometimes is more involved as $H - Bimods$. So for the time being, a general definition of what is a TQFT is the following:

Rough definition 1.1. Let Cob be a category whose objects are smooth n-dimensional manifolds (of some sort) endowed with a monoidal structure (tensor product) and a braiding, and $\mathcal{C}$ a target braided category. A TQFT is a braided monoidal functor $Z : \mathrm{Cob} \to \mathcal{C}$.

The above definition becomes correct if one specifies the categories Cob, $\mathcal{C}$ and the structure of the monoidal functor Z (as we will see later on, a monoidal functor is not just a functor but a pair given by the functor and a suitable natural transformation...). In particular we will see various versions of Cob which are pertinent:

(1) Cob (plain vanilla), whose objects are n-dimensional closed oriented smooth manifolds and morphisms are their cobordisms up to diffeomorphism. The monoidal structure is disjoint union.
(2) Cob^{nc} (non compact), the subcategory of Cob in which each connected component of a morphism must have non-empty negative boundary.
(3) Cob_∂ (with boundary), whose objects are n-dimensional compact oriented smooth manifolds with boundary S^{n-1}, and morphisms are connected cobordisms with a side boundary $S^{n-1}\times[-1, 1]$. The monoidal structure is induced by "glueing along a pant", or equivalently by boundary connected sum along S^{n-1}.
(4) In dimension $2 + 1$, for each of the above, their "version extended with signature" denoted respectively $\mathrm{Cob}^\sigma, \mathrm{Cob}^{\sigma,nc}, \mathrm{Cob}^\sigma_\partial$, in which the objects are decorated by a suitable lagrangian subspace of the H_1 and the cobordism carry an integer which sums up under composition with a specific rule given by the Maslov index.
(5) In dimension $2 + 1$ $\mathrm{Cob}^{nc,\sigma,\omega}$, whose objects are further decorated by cohomology classes with coefficients in some abelian group (e.g. $\mathbb{C}/2\mathbb{Z}$).

A TQFT starting from a category as above "without the σ" will be called "anomaly free", else "with anomaly".

Similarly the possible target categories are varied and we shall mainly consider the following:

(1) $Vect$, the category of vector spaces with its standard monoidal structure and symmetry
(2) $GrVect$, the category of graded vector spaces with its standard monoidal structure and supersymmetry
(3) $H-mod$ (or H-comod) the category of finite dimensional modules (resp. co-modules) over a Hopf algebra H with suitable properties
(4) Bim_H a category whose objects are algebra-comodules over a Hopf algebra H and morphism their bimodules up to isomorphism.

To build examples of TQFTs, various strategies have been invented, depending on Cob and on $\mathcal{C}$:

- Present Cob (for instance as the category "generated by a single object with suitable morphisms satisfying some relations). Then to get Z it is sufficient to find objects with similar properties in $\mathcal{C}$. This is the idea underlying for instance the cobordism hypothesis. But we will see it in action in easier and more concrete cases.
- Present the category Cob by a list of generating morphisms and their relations. We will apply this strategy to get $2+1$ and $3+1$-dimensional TQFTs out of modular categories and more in general categories with modified trace and "chromatic morphisms".
- Use the "universal construction" to extend a quantum invariant (i.e. the value of Z only on the endomorphisms of the empty manifold) to a whole TQFT.

So after a series of recalls on category theory and Hopf algebras, here is a table of what we are going to study (if time permits !) where the numbers indicate the order we will follow, but I actually do not expect we will be able to see $4), 5)$ and 6).

SourceTarget	Vec	GrVec	H-mod	Bim_H
Cob	1) Dim 1+1 2) Dim2+1 TV,CGPV 3)Dim 3+1 CKY,CGHP			
Cob^{nc}	2) Dim2+1 CGPV 3)Dim 3+1 CGHP			
Cob$_\partial$				5) CL-CF
Cob$^\sigma$	2') WRT			
Cob$^{\sigma,nc}$		6) BCGP		
Cob$^\sigma_\partial$			4) KL	

1.1. **Higher categories, extended TQFTs and the cobordism hypothesis.** Although the previous section summarizes what we are going to deal with, it is important to be aware of a much larger framework which allows to encapsulate most of the concepts we are going to provide. It is that of the "cobordism hypothesis", due to Baez and Dolan [4] and of which a detailed sketch of proof has been provided by Lurie [37]. The idea behind this result, is that if one allows to cut the manifolds not only along hypersurfaces but along hypersurfaces in hypersurfaces and so on, then he will be able to decompose each n-manifold in basic objects as points, arcs connecting them, discs etc etc. These can be though of as an example of a n-category, in which the points are 0-morphisms, the arcs generate the 1-morphisms and so on.

Although according to my poor knowledge there is not yet a complete agreement on the notion of the algebraic notion of n-category , the notion of (∞, n) category has been formalised: roughly speaking in an ∞-category there are morphisms of all orders but their composition is not well defined, rather it is defined only up to higher morphisms. Also in an (∞, n)-category the morphisms higher than n are to be invertible.

There is a natural way of associating to a standard category $\mathcal{C}$ an ∞ category: the nerve $N(\mathcal{C})$ whose objects and 1-morphisms are those of C and whose n morphisms are sequences of n-composable morphisms in $\mathcal{C}$. These simplicial spaces turn out to be a model for $(\infty, 1)$ categories (which are typically defined as simplicial sets satisfying the Segal condition, which by

Grothendieck's nerve theorem is satisfied iff the simplicial set is the nerve of any category). The definition of (∞, n) is more involved (it is a n-fold complete Segal space, but this means nothing without a proper definition...). What we will recall though, is that there is one 4-category called $BrTens$ given by Brochier, Jordan, Safronov and Snyder, [8] whose:

0 -morphisms are braided k-linear categories
1 -morphisms are monoidal $k$linear categories seen as central bimodules categories
2 -morphisms are k-linear categories seen as bimodules categories
3 -morphisms are functors between categories
4 -morphisms are natural transformations

The statement of the cobordism hypothesis is that if one wants to "represent" the fully extended category Cob_n (i.e. the one containing the point as 0-morphism etc.etc.) then it will be sufficient to find in the target n-category a "fully dualisable" object, namely a 0-object coming with a dual object, and duality morphisms satisfying the standard duality conditions up to higher morphisms, together with duality datas also for these morphisms, all the way up...

Whatever this means, [8] showed that there are plenty of fully dualisable objects in $BrTens$: all the modular categories are ! Therefore to each such category there should be a fully extended 4-TQFT, which, in particular, associates numbers to closed 4-manifolds, vector spaces to 3-manifolds, categories to surfaces etc etc...

Warning. Actually the previous statements are true for the category Cob^{fr} of framed cobordisms. If one wants to actually consider Cob as we do, he/she should find a is fully dualisable $SO(n)$-homotopy fixed point object. We will not consider this major difficulty here.

We will see in this course what is supposed the "tip" of such a TQFT, (conjecturally): we will indeed build a family of $(4, 3)$-TQFT out of any modular category with suitable structure (this construction is taken from [12] and generalises to the non semi-simple setting the Crane-Yetter construction).

In a previous work Douglas, Schommer-Pries and Snyder [16], build a 3-category $Tens$ whose objects are monoidal categories, morphisms are bimodule categories etc..: it was shown in [8] that this category is $Mor_{BrTens}(1, 1)$. Always by the cobordism hypothesis, a fully dualisable object of Tens can be used to obtain a fully extended 3-TQFT. How does this morally relate to the previous construction ? Given a closed 3-manifold M, pick two 4-manifolds $W_\pm$ whose boundaries are $\pm M$ respectively. Then one can consider the closed 4-manifold $W_+ \cup_M W_-$ and consider it as a 4-manifold where one applies the trivial TQFT (associated to $1 \in BrTens$) except that there is a "topological defect" along M^3 and the defect is encoded by a category $\mathcal{C} \in Ob(Tens)$ (seen as a bimodule over the trivial braided category $1 = Vect_k$). The resulting invariant is morally the "Turaev-Viro" invariant associated to $\mathcal{C}$, which should be actually extendable until the point. We will indeed see the tip of this construction by providing a $(3, 2)$-TQFT associated to each spherical (non necessarily semi-simple) category $\mathcal{C}$ (this construction is taken from [13] and encompasses previous fundamental constructions due to Turaev-Viro and Barrett-Westbury in the semi-simple case).

If one looks closely to the previous framework he/she will be surprised because to a braided tensor category is associated a 4-TQFT and not a 3-TQFT as it is usually the case for the Reshetikhin-Turaev-Witten theories. This has been explained by Kevin Walker, and developed by Freed, Teleman and many others. The crux of the idea is the following (my apologies for over simplifying): as before let M be a closed 3-manifold colored by a braided tensor category $\mathcal{C}$ but now color W_+ with the trivial object and W_- with the category $\mathcal{C}$. From this point of view the color of M is to be seen as a bimodule category from $1 \to \mathcal{C}$. If before the choice of $W_\pm$ did not matter in the computation of the invariant of W because they were "colored by the trivial object

$1 \in BrTens$", here W_- is colored by a non trivial theory and so its topology will have to be taken into account. Fortunately though, the theory associated to $\mathcal{C}$ is supposed to be invertible, i.e. the invariant of each closed 4-manifold is an invertible scalar and the vector space associated to each M^3 is a 1-dimensional vector space. From this point of view the inclusion of $Vect$ into $\mathcal{C}$ given by the left monoidal action of $Vect$ on $\mathcal{C}$ provides a trivialisation of such a line bundle making it k and the application of the TQFT associated to the braided modular category $\mathcal{C}$ to the cobordism W_- gives a linear map to k, so a scalar which is then $RTW(M)$ (but depends on W_-, so this notation is abusive for the moment). In order to have a theory which only depends on M and not on the choice of W_- it is necessary to decorate M with enough information to be able to "forget" W_- and replace it with this sufficient data. For semisimple modular categories (Crane-Yetter theories) this datum is basically the signature of W_-. One can further play with a decomposition of W_- into two sub 4-manifolds $W' \circ W''$ whose boundary splits M into two 3-manifolds with boundary $M' \circ M''$ glued along a surface S, to understand how this extends to a $(3,2)$ TQFT. In this case, the question is : what structure should one put on S in order to "remember" enough of W' and W'' to be able to compute when glueing M' and M" the signature of W ? By Wall's signature formula, the answer is a lagrangian subspace of $H_1(S,\mathbb{R})$. This explains why in the WRT theories one has to deal with the anomalies associated to signatures. Actually in Walker, Freed and Teleman's wording, the Crane-Yetter theory is the anomaly for the WRT theory, or, better, the WRT theory is a "boundary condition" for the Crane-Yetter theory. From this point of view, the anomalous non semi-simple $(3,2)$-TQFTs associated to non-semisimple modular categories in [21] should be the boundary conditions to the $(4,3)$-theories from [12] we are going to build in this course. If time permits in this course we will construct WRT theories and their non semi-simple version (aka DGGPR theories from [21]) but without taking care of the anomaly problem: we will therefore obtain projective TQFTs and not proper TQFTs.

As stated above, associated to an invertible object of $BrTens$ should be a fully extended 4-TQFT. In particular this theory should associate to each 3-manifold a vector space and to each surface a category. If time permits, we will see that this is indeed the case, or, more precisely, that associated to some modular categories is indeed associated a category for each surface (the category of modules over its stated skein algebra) and a vector space to each 3-manifold (its stated skein module). To be more precise we will make this construction work for the subcategory Cob_∂ above. This is the content of the recent paper [9], which extends previous work of [10], [33],[34].

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2. The categories Cob and $\mathrm{Cob}^{\mathsf{nc}}$

From now on, all manifolds will be smooth compact and oriented and all the maps will be smooth unless explicitly stated the contrary.

Definition 2.1. Two diffeomorphisms between manifolds $f, g : M \to N$ are :

- *homotopic* : if there exists a map $h : M \times [0,1] \to N$ such that $h|_{M\times\{0\}} = f$ and $h|_{M\times\{1\}} = g$.
- *pseudo-isotopic*: if there exists an embedding $h : M\times[0,1] \to N\times[0,1]$ such that $h|_{M\times\{0\}} = f \times \{0\}$, $h|_{M\times\{1\}} = g \times \{1\}$.
- *isotopic*: if there exists an embedding $h : M\times[0,1] \to N\times[0,1]$ such that $h|_{M\times\{0\}} = f\times\{0\}$, $h|_{M\times\{1\}} = g \times \{1\}$ and for each t, $h_t := h|_{M\times\{t\}} \subset N \times \{t\}$.

Remark 2.2. Clearly isotopy $\implies$ pseudo-isotopy $\implies$ homotopy. The reverse implications are false in general in dimensions ≥ 3 (see for instance [20] for an example in dimension 3 of maps which are pseudo-isotopic but non isotopic). On contrast, in dimension 2 they are all true: this is the content of Baer's theorem.

Definition 2.3. The category Cob is the category whose objects are the $n-1$-dimensional manifolds (which typically we will denote with the letters Σ) and whose morphisms are 5-uples $\mathrm{Mor}(\Sigma_-, \Sigma_+) = \{(W, \partial_+ W, f_+, \partial_- W, f_-)\}/\sim$ where

(1) W is a n-manifold,
(2) $\partial W = \partial_- W \sqcup \partial_+ W$ (oriented with the outward vector first convention),
(3) $f_- : \Sigma_- \to \partial W_-$ (resp. $f_+ : \Sigma_+ \to \partial W_+$) are diffeomorphisms which reverse (resp. preserve) the orientation,

and we say that two 5-uples $(W, \partial_+ W, f_+, \partial_- W, f_-)$ and $(W', \partial_+ W', f_+, \partial_- W', f_-)$ are equivalent ($\sim$) if there exists an orientation preserving diffeomorphism $\psi : W \to W'$ such that:

$$\psi(\partial_+ W) = \partial_+ W', \qquad f'_+ = \psi \circ f_+, \qquad \psi(\partial_- W) = \partial_- W', \qquad f'_- = \psi \circ f_-.$$

The composition of cobordisms :

$$\begin{aligned} \mathcal{W}_1 = (W_1, \partial_+ W_1, f_+, \partial_- W_1, f_-) \in \mathrm{Mor}(\Sigma_-, \Sigma) \text{ and} \\ \mathcal{W}_2 = (W_2, \partial_+ W_2, g_+, \partial_- W_2, g_-) \in \mathrm{Mor}(\Sigma, \Sigma_+) \text{ is defined as} \\ \mathcal{W}_2 \circ \mathcal{W}_1 = (W_2 \sqcup_{g_- \circ f_+^{-1}} W_1, \partial_+ W_2, g_+, \partial_- W_1, f_-) \in \mathrm{Mor}(\Sigma_-, \Sigma_+), \text{ where} \\ W_2 \sqcup_{g_- \circ f_+^{-1}} W_1 := (W_1 \sqcup W_2)/\{x \sim y \iff x \in \partial_- W_2, y \in \partial_+ W_1 \text{ and } x = g_- \circ f_+^{-1}(y)\}. \end{aligned}$$

Observe that the identity morphism Id_Σ is $(\Sigma \times [-1,1], \Sigma \times \{-1\}, Id, \Sigma \times \{1\}, Id)$. More in general if $f \in \mathsf{Diff}_+(\Sigma)$ then we define the cobordism $C_f := (\Sigma \times [-1,1], \Sigma \times \{-1\}, f, \Sigma \times \{1\}, Id)$: the following holds :

Lemma 2.4. (1) *The semigroup* $\mathrm{Mor}(\emptyset, \emptyset)$ *is the abelian semigroup freely generated by oriented diffeomorphism classes of connected* $n+1$*-manifolds. Its only inversible element is the class of the empty manifold.*
(2) *For each* Σ *the map* $\mathsf{Diff}_+(\Sigma) \ni f \to C_f \in \mathrm{Mor}(\Sigma, \Sigma)$ *is a homomorphism whose kernel is* $\{f |\ f$ *is pseudo-isotopic to the identity*$\}$.

Proof. **1)**.The fact that $\mathrm{Mor}(\emptyset, \emptyset)$ is a semigroup is true in general, furthermore, by definition of the composition of two cobordisms, if those cobordisms have empty boundary, their composition is the diffeomorphism class of their disjoint union. The identity cobordism is $\emptyset \times [-1,1] = \emptyset$ and it is invertible.

2). We need to prove that $C_f \circ C_g = C_{f \circ g}$. By definition the cobordism C_g can be also represented as $(\Sigma \times [-1,1], \Sigma \times \{-1\}, f \circ g, \Sigma \times \{1\}, f)$ (indeed the diffeomorphism f can be extended to the whole C_g via $f \times Id$). Now it becomes evident that the composition of the two cobordisms the composition $C_f \circ C_g$ is the cobordism $(\Sigma \times [-1,3], \Sigma \times \{3\}, Id, \Sigma \times \{-1\}, f \circ g) = C_{f \circ g}$. The cobordism C_f is equivalent to the cobordism $C_{Id} = Id_\Sigma$ iff there exists a diffeomorphism $\phi : \Sigma \times [-1,1] \to \Sigma \times [-1,1]$ such that

$$\phi(x, 1) = (x, 1) \text{ and } \phi(f(x), -1) = (x, -1)\ \forall x \in \Sigma.$$

Up to a reparametrization of the $[-1,1]$ factor this is precisely saying that f is pseudo-isotopic to Id (see Definition 2.1). $\boxed{2.4}$

The category Cob has naturally much more structure than what was given above. Observe first that a monoidal structure in Cob is given by the disjoint union : $\Sigma_1 \otimes \Sigma_2 := \Sigma_1 \sqcup \Sigma_2$, and the unit object id is the empty manifold $\emptyset$. Furthermore, the natural diffeomorphisms $\Sigma_1 \sqcup \Sigma_2 \to \Sigma_2 \sqcup \Sigma_1$ induce a symmetry on the monoidal structure: Cob is then a symmetric monoidal category.

Observe furthermore Cob is a rigid category: each object Σ has a left and right dual object $\overline{\Sigma}$ which is the same manifold with the opposite orientation and there are morphisms $\eta : \mathbb{I} \to \Sigma \otimes \overline{\Sigma}$ (defined as $\eta := (\Sigma \times [-1,1], \Sigma \times \{\pm 1\}, Id \sqcup Id, \emptyset, \emptyset))$ and $\epsilon : \overline{\Sigma} \otimes \Sigma \to \mathbb{I}$ (defined as $\epsilon := (\Sigma \times [-1,1], \emptyset, \emptyset, \Sigma \times \{\pm 1\}, Id \sqcup Id))$ which satisfy the triangle identities (namely

$$(\Sigma \longrightarrow \mathbb{I} \otimes \Sigma \xrightarrow{\eta \otimes \mathrm{id}} \Sigma \otimes \overline{\Sigma} \otimes \Sigma \xrightarrow{\mathrm{id} \otimes \epsilon} \Sigma) = Id_\Sigma \text{ , and } (\Sigma \longrightarrow \Sigma \otimes \mathbb{I} \xrightarrow{\mathrm{id} \otimes \eta} \Sigma \otimes \overline{\Sigma} \otimes \Sigma \xrightarrow{\epsilon \otimes \mathrm{id}} \Sigma) = Id_\Sigma.)$$

Let $\mathrm{Cob}^{\mathsf{nc}}$ be the largest subcategory of Cob such that each component of every cobordism has a nonempty source, and we also include the empty object in Cob. The category $\mathrm{Cob}^{\mathsf{nc}}$ is a symmetric monoidal subcategory of Cob, but notice it is not rigid.

Definition 2.5. A *non-compact (2+1)-TQFT* is a symmetric monoidal functor $\mathrm{Cob}^{\mathsf{nc}} \to \mathrm{Vect}_{\mathbb{K}}$. A non-compact (2+1)-TQFT is *finite dimensional* if it takes values in the subcategory of finite dimensional vector spaces.

2.1. **Presenting** Cob **and** $\mathrm{Cob}^{\mathsf{nc}}$ **(following Juhasz).** In [23], Juhász gives a presentation of Cob whose generators $\{e_{\Sigma,\mathbb{S}}, e_d\}$ are indexed by framed k-spheres $\mathbb{S}$ in a $(n-1)$-manifold Σ and diffeomorphisms $d\colon \Sigma \to \Sigma'$ between $(n-1)$-manifolds, see Section 2.2. These generators correspond to $k+1$-handles and mapping cylinders that we now describe.

Let Σ be an oriented $(n-1)$-manifold. For $k \in \{0,1,2,\ldots n-1\}$, a *framed k-sphere* in Σ is an orientation reversing embedding $\mathbb{S}\colon S^k \times D^{2-k} \hookrightarrow \Sigma$. Then we can perform surgery on Σ along $\mathbb{S}$ by removing the interior of the image of $\mathbb{S}$ and gluing in $D^{k+1} \times S^{1-k}$, getting a well defined topological manifold $\Sigma(\mathbb{S})$ which, using the framing of the sphere, can be endowed with a canonical smooth structure. The associated oriented cobordism $(\Sigma \times [0,1]) \cup_{\mathbb{S}} (D^{k+1} \times D^{2-k})$ represents a morphism $W(\mathbb{S})$ in Cob from $\Sigma \to \Sigma(\mathbb{S})$. Juhász considers two additional types of framed sphere, namely $\mathbb{S} = 0$ and $\mathbb{S} = \emptyset$, where $\Sigma(0) = \Sigma \sqcup S^{n-1}$ and $\Sigma(\emptyset) = \Sigma$ with associated the cobordisms $W(0) = \Sigma \times [-1,1] \sqcup D^n \colon \Sigma \to \Sigma(0)$ and $W(\emptyset) = \Sigma \times [-1,1]\colon \Sigma \to \Sigma(\emptyset)$.

Finally, recall that any orientation preserving diffeomorphism $d\colon \Sigma \to \Sigma'$ between closed oriented $(n-1)$-manifolds gives rise to the morphism $c_d\colon \Sigma \to \Sigma'$ in Cob represented by the cylindrical cobordism whose underlying manifold is $\Sigma \times [0,1]$ with boundary $(-\Sigma \times \{0\}) \sqcup (\Sigma \times \{1\})$ parameterized by $(x,0) \mapsto x$ and $(x,1) \mapsto d(x)$ for all $x \in \Sigma$. In Juhász's presentation, the formal generators $e_{\Sigma,\mathbb{S}}$ and e_d correspond to the above cobordisms $W(\mathbb{S})$ and c_d respectively.

The generators of $\mathrm{Cob}^{\mathsf{nc}}$ are the same with exception of those associated with the formal spheres $\mathbb{S} = 0$ since the cobordisms $W(0)$ do not belong to $\mathrm{Cob}^{\mathsf{nc}}$.

2.2. **Juhász's presentation of** Cob **and** Cob'**.** Following [23], we consider the subcategory Cob' of cobordism such that each component of every cobordism has a nonempty source and nonempty target. Here, we consider the empty surface as an object of Cob'.

Let $\mathcal{G}$ be the directed graph described as follows. The vertices are closed oriented surfaces. There are two kinds of edges of $\mathcal{G}$. First, for each orientation preserving diffeomorphism $d\colon \Sigma \to \Sigma'$ between closed oriented surfaces, there is an edge e_d going from Σ to Σ'. Second, for each framed sphere $\mathbb{S}$ in a closed oriented surface Σ, there is an edge $e_{\Sigma,\mathbb{S}}$ from Σ to $\Sigma(\mathbb{S})$. Let $\mathcal{G}^{\mathsf{nc}}$ (resp. $\mathcal{G}'$) be the subgraph of $\mathcal{G}$ obtained by removing the empty surface and the edges $e_{\Sigma,\mathbb{S}}$ where $\mathbb{S} = 0$ (resp. where $\mathbb{S} = 0$ or $\mathbb{S}$ is a framed 2-sphere). Denote by $\mathcal{F}(\mathcal{G})$ (resp. $\mathcal{F}(\mathcal{G}^{\mathsf{nc}})$, resp. $\mathcal{F}(\mathcal{G}')$) the free categories generated by $\mathcal{G}$ (resp. $\mathcal{G}^{\mathsf{nc}}$, resp. $\mathcal{G}'$).

In [23, Definition 1.4], Juhász considers a set of relations $\mathcal{R}$ in $\mathcal{F}(\mathcal{G})$ which we recall now. If w and w' are words consisting of composable arrows, then we write $w \sim w'$ if $w = w'$ is a relation in $\mathcal{R}$.

(R1) For composable diffeomorphisms d and d' between closed oriented surfaces, we have the relation $e_{d \circ d'} \sim e_d \circ e_{d'}$. We also have the relations $e_{\Sigma,\emptyset} \sim e_{\mathrm{id}_\Sigma}$ and $e_d \sim e_{\mathrm{id}_\Sigma}$ if $d \colon \Sigma \to \Sigma$ is a diffeomorphism isotopic to the identity.

(R2) Let $d \colon \Sigma \to \Sigma'$ be an orientation preserving diffeomorphism between closed oriented surfaces and $\mathbb{S}$ be a framed sphere in Σ. Consider the framed sphere $\mathbb{S}' = d \circ \mathbb{S}$ in Σ' and denote by $d^{\mathbb{S}} \colon \Sigma(\mathbb{S}) \to \Sigma'(\mathbb{S}')$ the induced diffeomorphism. Then the commutativity of the following diagram defines a relation:

$$\begin{tikzcd} \Sigma \arrow[r, "e_{\Sigma,\mathbb{S}}"] \arrow[d, "e_d"'] & \Sigma(\mathbb{S}) \arrow[d, "e_{d^{\mathbb{S}}}"] \\ \Sigma' \arrow[r, "e_{\Sigma',\mathbb{S}'}"] & \Sigma'(\mathbb{S}') \end{tikzcd}$$

(R3) Let $\mathbb{S}, \mathbb{S}'$ be disjoint framed sphere in an oriented surface Σ. Notice that $\Sigma(\mathbb{S})(\mathbb{S}') = \Sigma(\mathbb{S}')(\mathbb{S})$ and denote this surface by $\Sigma(\mathbb{S}, \mathbb{S}')$. The commutativity of the following diagram defines a relation:

$$\begin{tikzcd} \Sigma \arrow[r, "e_{\Sigma,\mathbb{S}}"] \arrow[d, "e_{\Sigma,\mathbb{S}'}"'] & \Sigma(\mathbb{S}) \arrow[d, "e_{\Sigma(\mathbb{S}),\mathbb{S}'}"] \\ \Sigma(\mathbb{S}') \arrow[r, "e_{\Sigma(\mathbb{S}'),\mathbb{S}}"] & \Sigma(\mathbb{S}, \mathbb{S}') \end{tikzcd}$$

(R4) Let $\mathbb{S}$ be a framed k-sphere in an oriented surface Σ and $\mathbb{S}'$ a framed k'-sphere in $\Sigma(\mathbb{S})$. If the attaching sphere $\mathbb{S}'(S^{k'} \times \{0\}) \subset \Sigma(\mathbb{S})$ intersects the belt sphere $\{0\} \times S^{-k+1} \subset \Sigma(\mathbb{S})$ once transversely, then there is a diffeomorphism (well defined up to isotopy) $\phi \colon \Sigma \to \Sigma(\mathbb{S}, \mathbb{S}')$ (see [23, Definition 2.17]) and the following is a relation:

$$e_{\Sigma(\mathbb{S}),\mathbb{S}'} \circ e_{\Sigma,\mathbb{S}} \sim e_\phi.$$

(R5) For each be a framed k-sphere $\mathbb{S}$ in an oriented surface Σ, there is a relation $e_{\Sigma,\mathbb{S}} \sim e_{\Sigma,\bar{\mathbb{S}}}$, where the framed k-sphere $\bar{\mathbb{S}} \colon S^k \times D^{2-k} \hookrightarrow \Sigma$ is defined by $\bar{\mathbb{S}}(x, y) = \mathbb{S}(r_{k+1}(x), r_{2-k}(y))$ for any $x \in S^k \subset \mathbb{R}^{k+1}$ and $y \in D^{2-k} \subset \mathbb{R}^{2-k}$, with $r_m(x_1, x_2, \ldots, x_m) = (-x_1, x_2, \ldots, x_m)$.

Let $\mathcal{R}^{\mathsf{nc}}$ and $\mathcal{R}'$ be the subset of relations involving only edges in $\mathcal{G}^{\mathsf{nc}}$ and $\mathcal{G}'$ respectively.

Following [23, Definition 1.5], let $c \colon \mathcal{G} \to \mathrm{Cob}$ be the map which is the identity on vertices, assigns the cylindrical cobordism c_d to the generator e_d associated to a diffeomorphism d, and assigns the cobordism $W(\mathbb{S})$ to the edge $e_{\Sigma,\mathbb{S}}$. This extends to a symmetric strict monoidal functor $c \colon \mathcal{F}(\mathcal{G}) \to \mathrm{Cob}$. Recall that given a category $\mathcal{F}$ and a set of relations $\sim$ on its morphisms, the quotient category $\mathcal{F}/\sim$ has the same objects as $\mathcal{F}$ and equivalence classes of morphisms of $\mathcal{F}$ as morphisms. The following was proved by Juhász [23, Theorem 1.7]:

Theorem 2.6. *The functor $c \colon \mathcal{F}(\mathcal{G}) \to \mathrm{Cob}$ induces isomorphisms of symmetric monoidal categories*

$$\mathcal{F}(\mathcal{G})/\mathcal{R} \to \mathrm{Cob} \quad \textit{and} \quad \mathcal{F}(\mathcal{G}')/\mathcal{R}' \to \mathrm{Cob}'.$$

In [13] the same statement was proved for $c \colon \mathcal{F}(\mathcal{G}^{\mathsf{nc}}) \to \mathrm{Cob}^{\mathsf{nc}}$ and relations $\mathcal{R}^{\mathsf{nc}}$.

2.3. **The universal construction.** Let us agree on the following:

Definition 2.7. A functor (not necessarily monoidal) $Z : \mathrm{Cob} \to \mathrm{Vect}$ is *non-degenerate* (or *cobordism generated*) if for each Σ it holds

$$Z(\Sigma) = \mathsf{span}_{\mathbb{C}}\{Z(\mathrm{Mor}(\emptyset, \Sigma)\}.$$

Proposition 2.8 (Universal construction, [5])**.** *Let $Z : \mathrm{Mor}(\emptyset, \emptyset) \to \mathbb{C}$ be a diffeomorphism invariant of oriented $n+1$-manifolds which is multiplicative under disjoint union. There exists a unique non-degenerate functor $Z : \mathrm{Cob} \to Vect_{\mathbb{C}}$ whose restriction to $\mathrm{Mor}(\emptyset, \emptyset)$ is Z.*

Proof. Define $V(\Sigma) := span\{Mor(\emptyset, \Sigma)\}$ and $V'(\Sigma) := span\{Mor(\Sigma, \emptyset)\}$. Define a pairing $\langle \cdot, \cdot \rangle : V'(\Sigma) \otimes V(\Sigma) \to \mathbb{C}$ by extending linearly the bracket defined on the bases as $\langle M_2, M_1 \rangle = Z(M_2 \circ M_1)$. Let then $Z(\Sigma) := V(\Sigma)/\{v \in V(\Sigma) | \langle w, v \rangle = 0 \forall w \in V'(\Sigma)\}$ and similarly let $Z'(\Sigma) := V'(\Sigma)/\{w \in V'(\Sigma) | \langle w, v \rangle = 0 \forall v \in V(\Sigma)\}$. It is straightforward to check that this defines a functor into Vect which by construction is non-degenerate. $\boxed{2.8}$

In general the following holds :

Theorem 2.9 (Turaev, Theorem 3.7 [47])**.** *If Z_1 and Z_2 are two TQFT whose invariants of closed manifolds coincide and if Z_1 is non-degenerate, then Z_1 and Z_2 are isomorphic.*

Corollary 2.10. *If Z is a degenerate TQFT the result of the universal construction on Z is a functor but not a TQFT.*

3. Warm up : $(2, 1)$-TQFTs

3.1. **Frobenius algebras.**

Definition 3.1 (Frobenius algebra object)**.** A Frobenius algebra object A in a monoidal category $\mathcal{C}$ is a quintuple $(A, \mu, 1, \Delta, \epsilon)$ where :

(1) $\mu : A \otimes A \to A$ is associative (i.e. $\mu \circ (\mu \otimes Id) = \mu \circ (Id \otimes \mu)$)
(2) $1 \in \mathrm{Mor}(1, A)$ is such that $\mu \circ (1 \otimes Id) = Id = \mu \circ (Id \otimes 1)$;
(3) $\Delta : A \to A \otimes A$ is co-associative (i.e. $\Delta \otimes Id \circ \Delta = Id \otimes \Delta \circ \Delta$);
(4) $\epsilon : A \to 1$ is a co-unit i.e. it is such that $\epsilon \otimes Id \circ \Delta = Id = Id \otimes \epsilon \circ \Delta$.
(5) The Frobenius Law holds : $\Delta \circ \mu = (Id \otimes \mu) \circ (\Delta \otimes Id) = (\mu \otimes Id) \circ (Id \otimes \Delta)$.

Furthermore, if $\mathcal{C}$ is symmetric with symmetry s we say that A is commutative if it holds $\mu \circ s = \mu$, cocommutative if $s \circ \Delta = \Delta$.

Let $\mathbb{S}_n$ be the n-dimensional sphere seen as the round unit sphere in $\mathbb{R}^{n+1}$ and oriented as the outside of the round unit radius ball $\mathcal{B}_n$ of center the origin. Let $1 \in \mathrm{Mor}(\emptyset, \mathbb{S}_n)$ be the cobordism represented by B_n and let μ be the $n+1$ cobordism from $\mathbb{S}_n \otimes \mathbb{S}_n \to \mathbb{S}_n$ formed by the "pant" i.e. the complement of two disjoint copies of the round ball of radius 1 whose centers are in coordinates $(\pm 2, 0, \cdots, 0) \in \mathbb{R}^{n+1}$ inside the round ball of radius 4 and center the origin (the boundary components of μ are to identified with $\mathbb{S}_n$ by means of the obvious compositions of translations and positive homotheties). Similarly let Δ, ϵ be the $n+1$-cobordisms obtained by reversing the orientations of μ and 1 respectively.

Lemma 3.2. $(\mathbb{S}_n, \mu, 1, \Delta, \epsilon)$ *is a commutative Frobenius algebra object in* Cob_n.

Let's observe first that if $n = 2$ then each object of Cob is a tensor product of circles and so to know a TQFT it is sufficient to know $Z(\mathbb{S}^1)$ which is a Frobenius algebra:

Theorem 3.3 (Dijkgraaf? Abrams?, well detailed by Kock [19])**.** *A 1+1-TQFT is uniquely determined by the commutative Frobenius algebra structure of $Z(\mathbb{S}^1)$. Reciprocally, given a commutative finite dimensional Frobenius algebra A there exists a unique TQFT Z such that $Z(\mathbb{S}^1) = A$.*

Exercise 3.4. Let A be a commutative Frobenius algebra. Prove that then the bilinear form $\langle x, y\rangle := \epsilon(xy)$ is non-degenerate and satisfies $\langle xy, z\rangle = \langle x, yz\rangle$, $\forall x, y, z \in A$. Reciprocally prove that if A is a commutative, unital algebra equipped with a non-degenerate form having these properties then A is a Frobenius algebra.

We will use extensively the following exercise in what follows :

Exercise 3.5. Let A be a commutative Frobenius algebra and fix a basis x_i of A as a $\mathbb{C}$-vector space; let $x_i^* \in A$ be the element defined so that $\epsilon(x_i^* x_j) = \delta_{i,j}$ and finally let $\theta = \sum_i x_i x_i^*$. If Z is a $1+1$-TQFT such that $Z(\mathbb{S}^1) = A$ then the value of Z on a closed surface of genus $g \geq 0$ is $\epsilon(\theta^g)$. In particular its value on $\mathbb{S}^1 \times \mathbb{S}^1$ is $\dim_{\mathbb{C}}(A)$.

From now on let us fix the following notation (for this section). Let $\Sigma_{g,h} := \Sigma_g \sqcup \Sigma_h$ and $Y_k = \Sigma_k \setminus D^2$.

Exercise 3.6. Let A be the de Rham cohomology of your favorite compact complex manifold where ϵ is given by integrating on the fundamental class. It is a commutative Frobenius algebra. In particular for $\mathbb{CP}^1$ one gets the algebra $\mathbb{C}[X]/X^2$ which is at the base of the construction of Khovanov homology. Notice that $\Delta(1) = 1 \otimes x + x \otimes 1$ and $\Delta(x) = x \otimes x$ and that these values can be computed starting from the ϵ form (evaluation on the fundamental cycle of $\mathbb{CP}^1$). The associated TQFT evaluates each sphere to 0 each torus to 2 and each other connected surface to 0. Apply the universal construction and show that it yields a TQFT (i.e. a monoidal functor).

Solution 3.7. Let Σ_g be the complement of a disc in a genus g oriented surface. If we apply the universal construction we immediately see that $Z(\mathbb{S}^1) = span_{\mathbb{C}}\{\Sigma_0, \Sigma_1\}$ and it is not difficult to realize that the vectors $\Sigma_{i,j}, Y_k, i, j, k \in \{0, 1\}$ generate $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$ but they are not independent as the coupling matrix (i.e. expressing $\epsilon \circ m$) written in the basis $\Sigma_{0,0}, \Sigma_{1,0}, \Sigma_{0,1}, \Sigma_{1,1}, Y_0, Y_1$ is :

$$\begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

whose rank is 4. Actually as the rank of the first 4×4 minor is 4 the vectors $\Sigma_{i,j}$ form a basis of $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$. More in general it is not difficult to check that $Z(\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1)$ is $Z(\mathbb{S}^1) \otimes \cdots \otimes Z(\mathbb{S}^1)$ and thus Z is a TQFT. Indeed, denoting Σ^i the cobordism from $\mathbb{S}^1$ to $\emptyset$ represented by a genus g surface with one boundary component, then one can verify that $Id_{\mathbb{S}^1} = \frac{1}{2}\big(\Sigma_0 \circ \Sigma^1 + \Sigma_1 \circ \Sigma^0\big)$. This allows to split any cobordism from $\emptyset$ to a $\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1$ into a linear combination of surfaces with only one boundary component thus of elements of $Z(\mathbb{S}^1) \otimes \cdots \otimes Z(\mathbb{S}^1)$.

In the case of the previous exercice, if we apply the universal construction to invariants of the TQFT associated to the Frobenius algebra $H^*(\mathbb{CP}^1)$ we recover the initial TQFT. But this is not always the case as the following examples show.

Exercise 3.8. If $A = H^*(\mathbb{CP}^n)$ what is the value of $\mathbb{Z}(X_g)$ where X_g is the connected surface of genus g?

Solution 3.9. In the Frobenius algebra $\mathbb{C}[x]/x^{n+1}$ we have $\epsilon(x^a) = 0$ unless $a = n$, so that $\theta = \sum_{i=0}^{n} m(x^i \otimes x^{n-i}) = (n+1)x^n$. Hence $\mathbb{Z}(X_g) = 0$ unless $g = 1$ in which case we have $\mathbb{Z}(\mathbb{S}^1 \times \mathbb{S}^1) = n + 1$.

Exercise 3.10. Let $A = H^*(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{C})$ i.e. $A = \mathbb{C}[x,y]/\{x^2, y^2\}$. Then $\theta_A = 4xy$ and $\theta_A^g = 0\ \forall g > 1$ so that $Z_A(\mathbb{S}^2) = 0$, $Z_A(\mathbb{S}^1 \times \mathbb{S}^1) = 4$ and $Z_A(\Sigma_g) = 0\ \forall g > 1$. These values coincide with those of the case $A' = H^*(\mathbb{CP}^3)$. Prove that this TQFT is not isomorphic to the previous one, even though they have the same invariants.

Exercise 3.11. Let Σ_g be the complement of a disc in a genus g oriented surface. If we apply the universal construction to the functor Z of the preceding example then we have $Z(\mathbb{S}^1) = span_{\mathbb{C}}\{\Sigma_0, \Sigma_1\}$, and it is not difficult to check that $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$ is generated by the images through Z of $\Sigma_{0,0}, \Sigma_{1,0}, \Sigma_{0,1}, \Sigma_{1,1}, Y_0, Y_1$ and writing the pairing matrix in the basis $\Sigma_{0,0}, \Sigma_{1,0}, \Sigma_{0,1}, \Sigma_{1,1}, Y_0, Y_1$ we get :

$$\begin{pmatrix} 0 & 0 & 0 & 16 & 0 & 4 \\ 0 & 0 & 16 & 0 & 4 & 0 \\ 0 & 16 & 0 & 0 & 4 & 0 \\ 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 4 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

whose rank is > 4 : then $\dim_{\mathbb{C}}(Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)) > 4$ and so Z is not a TQFT but just a finite functor. (Prove finiteness as an exercice !) Remark furthermore that the so-obtained functor is different from both functors Z_A and $Z_{A'}$ associated to the Frobenius algebras A and A' in the preceding exercices : indeed those functors were by TQFTs (i.e. monoidal) by Theorem 3.3 while Z is not; moreover $\dim_{\mathbb{C}}(Z(\mathbb{S}^1)) = 2, \dim_{\mathbb{C}}(Z_A(\mathbb{S}^1)) = 4 = \dim_{\mathbb{C}}(Z_{A'}(\mathbb{S}^1))$.

Exercise 3.12. Contrast the previous 3 exercices with the statements of Theorem 2.9 and Corollary 2.10.

Exercise 3.13. Let us now go back to the case of general n. Let Z be the multiplicative invariant of n-manifold to be defined on connected ones as $Z(M) = \exp(\chi(M))$ (the Euler caracteristic). Then the universal construction gives for every $\Sigma \in$ Cob that $Z(\Sigma) = \mathbb{C}$ and $Z(W) = \exp(\chi(W) - \chi(\partial W_+))$ for each morphism W.

Exercise 3.14. Let us keep general n. For each connected manifold M let $Z(M) = k^{b_1(M)}$ for some $k \in \mathbb{R} \setminus \{\pm 1\}$ (the exponential of the first Betti number). Applying the universal construction one sees that, with the notation of the preceding example, $\Sigma_g = k^{2g}\Sigma_0$ in $Z(\mathbb{S}^1)$ and that thus $Z(\mathbb{S}^1)$ is one dimensional. Similarly in $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$ it holds $Y_h = k^{2h}Y_0$ and Y_0 and $\Sigma_0 \sqcup \Sigma_0$ are easily seen to be equa so that $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1) = span_{\mathbb{C}}\{\ \Sigma_0 \sqcup \Sigma_0\}$.

Exercise 3.15. Let $n = 2$ and for each connected manifold M let $Z(M) = b_1(M)$ (the first Betti number). Extend this invariant multiplicatively under disjoint unions. Applying the universal construction one sees that $Z(\mathbb{S}^1)$ is 2-dimensional indeed letting Σ_g be the connected genus g surface with one boundary component, the pairing of the TQFT is $Z(\Sigma_g \circ \Sigma_h) = 2(g+h)$ and the infinite dimensional matrix whose $(i,j)^{th}$ entry is $2(i+j)$ has rank 2. In particular the generators of $Z(\mathbb{S}^1) = span_C\{Z(\Sigma_0), Z(\Sigma_1)\}$. If one computes $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$ then he sees that actually its dimension is at least 5 : this shows that Z is only a functor and not a TQFT. Indeed letting $\Sigma_{h,k} := \Sigma_h \sqcup \Sigma_k$ and Y_k the complement of two discs in a genus k surface, then one sees that the coupling between $Z(\mathbb{S}^1 \sqcup \mathbb{S}^1)$ and itself, written in the base $\Sigma_{0,0}, \Sigma_{1,0}, \Sigma_{0,1}, \Sigma_{1,1}, Y_1$ is :

$$\begin{pmatrix} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 4 & 0 & 8 & 2 \\ 4 & 8 & 8 & 16 & 4 \\ 0 & 2 & 2 & 4 & 2 \end{pmatrix}$$

whose rank is 5.

4. Our general strategy to build $(3,2)$ and $(4,3)$-TQFTs

In the next sections we will build families of $(3,2)$ and $(4,3)$-TQFTs depending on an algebraic initial datum namely a unimodular spherical (resp. modular) category $\mathcal{C}$ which will be either semi-simple or not.

At a first approximate level, the final results we will be discussing, are the following (we will see that the hypotheses on $\mathcal{C}$ can be relaxed slightly):

Theorem 4.1 ($(3,2)$-dimensional TQFT, [13])**.** *Let $\mathcal{C}$ be a unimodular spherical finite tensor category endowed with a non degenerate modified trace. Then associated to $\mathcal{C}$ is a $(3,2)$-TQFT $Z_{\mathcal{C}} : \mathrm{Cob}^{\mathsf{nc}} \to \mathrm{Vect}$. Furthermore, it extends to the full category* Cob *(obtaining then a "compact theory") iff $\mathcal{C}$ is semi-simple with non-zero dimension and in this case it coincides with the Barrett-Wesbury (Turaev-Viro) theory associated to $\mathcal{C}$.*

Theorem 4.2 ($(4,3)$-dimensional TQFT, [12])**.** *Let $\mathcal{C}$ be a "chromatic non degenerate" category. Then associated to $\mathcal{C}$ is a $(4,3)$-TQFT $Z_{\mathcal{C}} : \mathrm{Cob}^{\mathsf{nc}} \to \mathrm{Vect}$. Furthermore, it extends to the full category* Cob *(obtaining then a "compact theory") if $\mathcal{C}$ is semi-simple and in this case it coincides with the Crane-Yetter theory associated to $\mathcal{C}$.*

In both the above constructions, the strategy will be first to assign to each object of Cob (i.e. to each $n-1$-manifold) a finite dimensional vector space: it is going to be the space of its "$\mathcal{C}$-admissible skeins". This vector space will naturally come with an action of the group of diffeomorphisms of the manifold up to isotopy. Then for each operation of handle-glueing we will have to describe a linear map on the skein spaces associated to the handle glueing. We will progressively describe it starting from the highest index handles:

- n-handle: Since the vector space associated to the $n-1$-sphere is going to be isomorphic to $\Bbbk$ this will correspond to fixing an element of the dual of this space; algebraically, this will correspond to the datum of a modified trace on the category $\mathcal{C}$.
- $n-1$-handle: This operation will correspond to the ability to "cut" a skein along a $n-2$-dimensional sphere and will be obtained by associating to the modified trace the datum of a suitable map $P \to \Omega_P \in Hom_{\mathcal{C}}(P, 1) \otimes \mathrm{Hom}_{\mathcal{C}}(1, P)$ for each projective object $P \in \mathcal{C}$; the fact that the modified trace is "non-degenerate" will ensure both the existence and naturality of this map.
- $n-2$-handle: This operation will be the crucial one and will be obtained by coloring the cocore of the attached handle by a special skein called "red". Actually the red color is going to be just a placeholder for a skein to be defined using the other skeins in the manifold via an operation called "'red-to-blue". This operation will be based on the hypothesis that $\mathcal{C}$ has a projective generator (which boils down to a kind of finiteness condition on $\mathcal{C}$) and that the projective generator has a special kind of morphism, called the "chromatic morphism". We can prove that such a morphism exists, if $\Bbbk$ is algebraically closed, for unimodular spherical and modular categories (Theorem 1.6 in [13]).

$n-3$-handle In dimension $(3,2)$ the list is over for the non-compact theories and for the compact ones, it is sufficient to associate to the 0-handle the embedding via the empty skein in S^2. In dimension $(4,3)$ we will need another object associated to 1-handles; it is called the "glueing morphism" and we can prove that it exists if $\mathcal{C}$ is "chromatic non-degenerate".

$n-4$-handles In dimension $(4,3)$ the attachment of 0-handles will be possible only for "compact theories" and this will impose the condition of "chromatic compactness" on $\mathcal{C}$.

After associating to each handle-glueing operation a linear map between skein modules, we will have to prove that these operations satisfy Juhasz's relations. The crucial ones will involve the chromatic morphism, but its property are in a sense made so that the handle cancellation work.

In the final section we will also describe how the above strategy can also be used to describe the WRT theories and their non semi-simple generalisation [21] at least if one thinks of it as projective TQFTs, namely TQFTs with values in the category PVect of vector spaces and their linear maps up to non zero scalar. This construction is original and appears here for the first time, although not too difficult to deduce from the previous results.

5. Spherical and modular categories, their modified traces and beyond...

The first goal of this section is to provide the necessary algebraic setup to define spherical and modular categories, which are widely studied objects. In particular its main goal is to provide all the elements to understand the following key definitions:

Definition 5.1 (Spherical tensor category)**.** A *spherical tensor category* (over $\Bbbk$) is a pivotal unimodular finite tensor category $\mathcal{C}$ (over an algebraically closed $\Bbbk$) such that the right m-trace on $\mathsf{Proj}_{\mathcal{C}}$ is also a left m-trace.

Definition 5.2 (Modular tensor category)**.** A *modular tensor category* (over $\Bbbk$) is a ribbon finite tensor category $\mathcal{C}$ (over an algebraically closed $\Bbbk$) which is factorisable (i.e. its transparent objects are direct sums of $\mathbb{1}$).

In particular it can be seen that a modular tensor category is also a spherical tensor category (because it is automatically pivotal and it always has a right m-trace which is also a left one and it is unimodular according to [21] Proposition 2.6).

In the last subsection we will provide explicit examples of such categories which are quite common (despite the long definition...).

The above data are the "standard" input to build respectively $(3,2)$ and $(4,3)$ TQFTs at least when $\mathcal{C}$ is semisimple. But when $\mathcal{C}$ is spherical non-semisimple the associated $(3,2)$ is non-compact. On contrast if $\mathcal{C}$ is modular, the associated $(4,3)$-theory is invertible and compact (both in the semi-simple and non semi-simple case) and, as such, it cannot be very interesting (in particular cannot distinguish exotic pairs of 4-manifolds).

But we want to stress that the constructions in [13] and [12] we will be based on, go beyond these algebraic data. Indeed the input data for the $(3,2)$-theories of [13] are so-called "chromatic categories" which are strictly more general than spherical tensor categories:

Definition 5.3. A *chromatic category* (over a non necessarily algebraically closed field $\Bbbk$) is a pivotal $\Bbbk$-category $\mathcal{C}$ endowed with a non-degenerate m-trace on $\mathsf{Proj}_{\mathcal{C}}$ such that:

- any non zero morphism to the unit object $\mathbb{1}$ is an epimorphism,
- there exists a "chromatic map" for a nonzero projective generator.

Remark 5.4. With respect to a spherical tensor category, a chromatic category is not necessarily abelian and the field is not necessarily algebraically closed.

Similarly the input data for the non-compact $(4,3)$-theories of [12] are ribbon chromatic categories which satisfy the requirement of being "chromatic non-degenerate" (i.e. a suitable endomorphism of the projective cover of $\mathbb{1}$ is non zero); these theories can be extended to compact ones if furthermore the input data are "chromatic compact" (i.e. this endomorphism is a non zero multiple of the cutting morphism).

5.1. **Projective objects, covers, and generators.** An object P of a category $\mathcal{C}$ is *projective* if the functor $\mathrm{Hom}_{\mathcal{C}}(P,-)\colon \mathcal{C} \to \mathrm{Set}$ preserves epimorphisms. A category has *enough projectives* if every object has an epimorphism from a projective object onto it.

A *projective cover* of an object X of a category $\mathcal{C}$ is a projective object $P(X)$ of $\mathcal{C}$ together with an epimorphism $p\colon P(X) \to X$ such that if $g\colon P \to X$ is an epimorphism from a projective object P to X, then there exists an epimorphism $h\colon P \to P(X)$ such that $ph = g$. In an abelian category, a projective cover (if it exists) is unique up to a non-unique isomorphism, and a projective cover of a simple object is indecomposable.

By a *generator* of a preadditive category (that is, a category that is enriched over the category of abelian groups), we mean an object G of the category such that any other object X is retract of $G^{\oplus n}$ for some non-negative integer n. A *projective generator* of a preadditive category $\mathcal{C}$ is a generator of the full subcategory of projective objects of $\mathcal{C}$.

5.2. **Linear monoidal categories.** A monoidal category is $\Bbbk$-*linear* if each hom-set carries a structure of a $\Bbbk$-vector space so that the composition and monoidal product of morphisms are $\Bbbk$-bilinear.

By a $\Bbbk$-*category*, we mean a $\Bbbk$-linear monoidal category $\mathcal{C}$ such that the hom-sets in $\mathcal{C}$ are finite dimensional and the $\Bbbk$-algebra map $\Bbbk \to \mathrm{End}_{\mathcal{C}}(\mathbb{1})$, $k \mapsto k\,\mathrm{id}_{\mathbb{1}}$ is an isomorphism, used then to identify $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$.

We say a $\Bbbk$-category that $\mathcal{C}$ is *semisimple* if every object of $\mathcal{C}$ is projective. Note that if $\mathcal{C}$ is abelian, then $\mathcal{C}$ is semisimple (in the above sense) if and only if it is abelian semisimple (in the sense every object is a direct sum of simple objects).

5.3. **Modified traces.** Let $\mathcal{C}$ be a pivotal $\Bbbk$-category. We first recall from the definition of a modified trace on an ideal of $\mathcal{C}$ (see [28, 30] for details).

An object Y of $\mathcal{C}$ is a *retract* of an object X of $\mathcal{C}$ if there are morphisms $r\colon X \to Y$ and $i\colon Y \to X$ such that $ri = \mathrm{id}_Y$. An *ideal* of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ which is

- *closed under monoidal products*: for all $X \in \mathcal{I}$ and $Y \in \mathcal{C}$, we have: $X \otimes Y \in \mathcal{I}$ and $Y \otimes X \in \mathcal{I}$,
- *closed under retracts*: any retract of an object of $\mathcal{I}$ belongs to $\mathcal{I}$.

Recall from [28] that the pivotality of $\mathcal{C}$ implies that any ideal of $\mathcal{C}$ is stable under duality.

Let $\mathcal{I}$ be an ideal of $\mathcal{C}$. A family $\mathsf{t} = \{\mathsf{t}_X\colon \mathrm{End}_{\mathcal{C}}(X) \to \mathbb{K}\}_{X\in\mathcal{I}}$ of $\Bbbk$-linear forms satisfies the

- *cyclicity property* if $\mathsf{t}_X(gf) = \mathsf{t}_Y(fg)$ for all morphisms $f\colon X \to Y$ and $g\colon Y \to X$ with $X, Y \in \mathcal{I}$;
- *right partial trace property* if $\mathsf{t}_{X\otimes Y}(f) = \mathsf{t}_X\big(\mathrm{ptr}_r^Y(f)\big)$ for all $f \in \mathrm{End}_{\mathcal{C}}(X\otimes Y)$ with $X \in \mathcal{I}$;
- *left partial trace property* if $\mathsf{t}_{Y\otimes X}(f) = \mathsf{t}_X\big(\mathrm{ptr}_l^Y(f)\big)$ for all $f \in \mathrm{End}_{\mathcal{C}}(Y \otimes X)$ with $X \in \mathcal{I}$.

A *right m-trace* (respectively *left m-trace*, respectively *m-trace*) on $\mathcal{I}$ is a family $\mathsf{t} = \{\mathsf{t}_X\colon \mathrm{End}_{\mathcal{C}}(X) \to \mathbb{K}\}_{X\in\mathcal{I}}$ of $\Bbbk$-linear forms satisfying the cyclicity and right (respectively left, respectively right and left) partial trace properties.

For example, identifying $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$, the family $\mathrm{tr}_r = \{f \in \mathrm{End}_{\mathcal{C}}(X) \mapsto \mathrm{tr}_r(f) \in \Bbbk\}_{X\in\mathcal{C}}$ is a right m-trace on $\mathcal{C}$ and the family $\mathrm{tr}_l = \{f \in \mathrm{End}_{\mathcal{C}}(X) \mapsto \mathrm{tr}_l(f) \in \Bbbk\}_{X\in\mathcal{C}}$ is a left m-trace on $\mathcal{C}$

called the *categorical left and right traces* of $\mathcal{C}$. If these traces coincide, then $\mathrm{tr} = \mathrm{tr}_r = \mathrm{tr}_l$ is a m-trace on $\mathcal{C}$ called the *categorical trace* of $\mathcal{C}$.

A m-trace t on an ideal $\mathcal{I}$ of $\mathcal{C}$ is *non-degenerate* if for any $X \in \mathcal{I}$, the pairing

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(X, \mathbb{1}) \to \mathbb{K}, \quad u \otimes v \mapsto \mathsf{t}_X(uv)$$

is non-degenerate. Given such a non-degenerate trace t, we set for any $X \in \mathcal{I}$,

$$\Omega_X = \sum_i x^i \otimes x_i \in \mathrm{Hom}_{\mathcal{C}}(X, \mathbb{1}) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X) \quad \text{and} \quad \Lambda^{\mathsf{t}}_X = \sum_i x_i \circ x^i \in \mathrm{End}_{\mathcal{C}}(X), \tag{1}$$

where $\{x^i\}_i$ and $\{x_i\}_i$ are basis of $\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X)$ which are dual with respect to the m-trace t, that is, such that $\mathsf{t}_X(x_i \circ x^j) = \delta_{i,j}$. Clearly, Ω_X and Λ^{t}_X are independent of the choice of such dual basis. The properties of the m-trace t translate to the copairings Ω_X as follows:

Lemma 5.5. *Let $X, Y \in \mathcal{I}$ and $Z \in \mathcal{C}$, and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.*

(a) Duality*: If $\Omega_X = \sum_i x^i \otimes x_i$, then $\Omega_{X^*} = \sum_i (x_i)^* \otimes (x^i)^* \in \mathrm{Hom}_{\mathcal{C}}(X^*, \mathbb{1}) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X^*)$.*

(b) Naturality*: If $\Omega_X = \sum_i x^i \otimes x_i$ and $\Omega_Y = \sum_j y^j \otimes y_j$, then*

$$\sum_i x^i \otimes (f \circ x_i) = \sum_j (y^j \circ f) \otimes y_j \in \mathrm{Hom}_{\mathcal{C}}(X, \mathbb{1}) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, Y).$$

(c) Rotation*: If $\Omega_{X \otimes Z} = \sum_i z^i \otimes z_i$ then $\Omega_{Z \otimes X} = \sum_i \widetilde{z}^i \otimes \widetilde{z}_i$ where*

$$\widetilde{z}^i = \overrightarrow{\mathrm{ev}}_Z(\mathrm{id}_Z \otimes z^i \otimes \mathrm{id}_{Z^*})(\mathrm{id}_{Z \otimes X} \otimes \overleftarrow{\mathrm{coev}}_Z) \quad \text{and} \quad \widetilde{z}_i = (\mathrm{id}_{Z \otimes X} \otimes \overrightarrow{\mathrm{ev}}_Z)(\mathrm{id}_Z \otimes z_i \otimes \mathrm{id}_{Z^*})\, \overleftarrow{\mathrm{coev}}_Z\,.$$

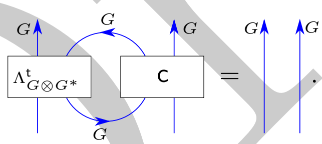
Exercise 5.6. Prove the lemma. The solution is Lemma 1.1 in [13].

5.4. **Chromatic maps.** Let $\mathcal{C}$ be a pivotal $\Bbbk$-category. The full subcategory $\mathsf{Proj}_{\mathcal{C}}$ of projective objects of $\mathcal{C}$ is an ideal of $\mathcal{C}$ (see [28]). Assume that $\mathcal{C}$ is endowed with a non-degenerate m-trace t on $\mathsf{Proj}_{\mathcal{C}}$.

A *chromatic map* for a projective generator G of $\mathcal{C}$ is a map $\mathsf{c} \in \mathrm{End}_{\mathcal{C}}(G \otimes G)$ satisfying

$$(\mathrm{id}_G \otimes \overleftarrow{\mathrm{ev}}_G \otimes \mathrm{id}_G)(\Lambda^{\mathsf{t}}_{V \otimes G^*} \otimes \mathsf{c})(\mathrm{id}_G \otimes \overrightarrow{\mathrm{coev}}_G \otimes \mathrm{id}_G) = \mathrm{id}_{G \otimes G}, \tag{2}$$

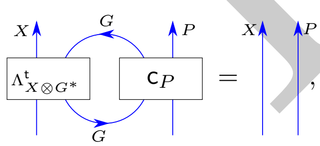
that is,



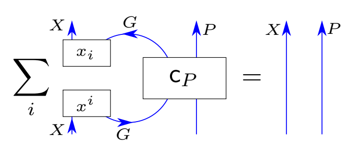
More generally, a *chromatic map based on a projective object* P for a projective generator G is a map $\mathsf{c}_P \in \mathrm{End}_{\mathcal{C}}(G \otimes P)$ such that for all $X \in \mathcal{C}$,

$$(\mathrm{id}_X \otimes \overleftarrow{\mathrm{ev}}_G \otimes \mathrm{id}_P)(\Lambda^{\mathsf{t}}_{X \otimes G^*} \otimes \mathsf{c})(\mathrm{id}_X \otimes \overrightarrow{\mathrm{coev}}_G \otimes \mathrm{id}_P) = \mathrm{id}_{X \otimes P}, \tag{3}$$

that is,



or more explicitly



where $\{x^i\}_i$ and $\{x_i\}_i$ are basis of $\mathrm{Hom}_{\mathcal{C}}(X \otimes G^*, \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes G^*)$ which are dual with respect to the m-trace t.

Clearly, a chromatic map based on G for a projective generator G is a chromatic map for G. Conversely, any chromatic map gives rise to chromatic maps based on projective objects:

Lemma 5.7. *Let $\mathsf{c} \in \mathrm{End}_{\mathcal{C}}(G \otimes G)$ be a chromatic map for a projective generator G of $\mathcal{C}$ and let $P \in \mathsf{Proj}_{\mathcal{C}}$. Pick any non zero morphism $\varepsilon \colon G \to \mathbb{1}$ and a morphism $e_{P,G} \colon P \to G \otimes P$ such that $\mathrm{id}_P = (\varepsilon \otimes \mathrm{id}_P) e_{P,G}$ (such morphisms always exist). Then the map*

$$\mathsf{c}_P = (\mathrm{id}_G \otimes \varepsilon \otimes \mathrm{id}_P)(\mathsf{c} \otimes \mathrm{id}_P)(\mathrm{id}_G \otimes e_{P,G}) \in \mathrm{End}_{\mathcal{C}}(G \otimes P)$$

is a chromatic map based on P for G.

Proof. See Lemma 1.2 in [13]. The existence of chromatic maps does not depend of the choice of the projective generator:

Lemma 5.8. *Let G, G' be projective generator and c_P be a chromatic map based on a projective object P for G. Then there is a finite family $\{\gamma_i \colon G \to G', \delta_i \colon G' \to G\}_i$ of morphisms such that $\sum_i \delta_i \gamma_i = \mathrm{id}_G$ and $\mathsf{c}'_P = \sum_i (\gamma_i \otimes \mathrm{id}_P) \mathsf{c}_P (\delta_i \otimes \mathrm{id}_P)$ is a chromatic map based on P for G'.*

Proof. The existence of $\{\gamma_i, \delta_i\}_i$ comes from the facts that G is a retract of $(G')^{\oplus n}$. To prove that c'_P is a chromatic map, one can precompose $\overleftarrow{\mathrm{ev}}_G$ with $\mathrm{id}_{G^* \otimes G} = \sum_i \mathrm{id}_{G^*} \otimes \delta_i \gamma_i$ in Equation (3), and then slide δ_i using the naturality of $\Lambda^{\mathsf{t}}_{\bullet}$. $\boxed{5.8}$

5.5. **Chromatic categories.**

Definition 5.9 (Chromatic category)**.** A *chromatic category* (over $\Bbbk$) is a pivotal $\mathbb{K}$-category $\mathcal{C}$ endowed with a non-degenerate m-trace on $\mathsf{Proj}_{\mathcal{C}}$ such that:

- any non zero morphism to the unit object $\mathbb{1}$ is an epimorphism,
- there exists a chromatic map for a nonzero projective generator.

Note that Lemmas 5.7 and 5.8 imply that in a chromatic category, there are chromatic maps based at any projective object for any projective generator.

First examples of chromatic categories are given by spherical fusion categories and categories of representations of unimodular and unibalanced finite dimensional Hopf algebras, see the Examples 5.10 and 5.11 below. A large family of chromatic categories is given by the spherical tensor categories over an algebraically closed field, see Theorem 5.18.

A chromatic category is *semisimple* if it is semsimple as a $\mathbb{K}$-category (see Section 5.2) or, equivalently, if the unit object $\mathbb{1}$ is projective. Note that the m-trace t of a semisimple chromatic category is a nonzero multiple of the categorical trace tr. Indeed the partial trace property implies that $\mathsf{t} = \mathsf{t}_{\mathbb{1}}(\mathrm{id}_{\mathbb{1}})\,\mathrm{tr}$, and $\mathsf{t}_{\mathbb{1}}(\mathrm{id}_{\mathbb{1}}) \neq 0$ because t is nonzero.

The *dimension* of a semisimple chromatic category $\mathcal{C}$ is $\dim(\mathcal{C}) = \mathrm{tr}(\mathsf{c}_{\mathbb{1}}) = \frac{\mathsf{t}_G(\mathsf{c}_{\mathbb{1}})}{\mathsf{t}_{\mathbb{1}}(\mathrm{id}_{\mathbb{1}})} \in \Bbbk$ for any chromatic map $\mathsf{c}_{\mathbb{1}}$ based on $\mathbb{1}$ for some projective generator G of $\mathcal{C}$. (This terminology is justified by the last assertion of Example 5.10.) Note that $\dim(\mathcal{C})$ does not depend on the choice of $\mathsf{c}_{\mathbb{1}}$ (see Remark 6.7) but does depend on the m-trace.

Example 5.10. [Semisimple spherical categories] In this example $\Bbbk$ need not be algebraically closed. Let $\mathcal{C}$ be a spherical fusion $\Bbbk$-category. Here, *fusion* means that there is a finite family I of objects of $\mathcal{C}$ such that $\mathbb{1} \in I$, $\mathrm{Hom}_{\mathcal{C}}(i,j) = \delta_{i,j} \Bbbk\, \mathrm{id}_i$ for all $i, j \in I$, and each object of $\mathcal{C}$ is a direct sum of objects in I. (Such fusion categories are in particular semisimple $\Bbbk$-categories in the sense of Section 5.2). Also, *spherical* means that the categorical left and right traces of $\mathcal{C}$ coincide (see Section 5.3). Then any object of $\mathcal{C}$ is projective, the categorical trace tr is non-degenerate, $G = \bigoplus_{i \in I} i$ is a (projective) generator of $\mathcal{C}$, and for any object $P \in \mathcal{C}$,

$$\mathsf{c}_P = \bigoplus_{i \in I} \dim(i)\, \mathrm{id}_i \otimes \mathrm{id}_P$$

is a chromatic map based on P for G, where $\dim(i) = \mathrm{tr}(\mathrm{id}_i) \in \mathbb{K}$. Formally, $\mathsf{c}_P = \mathrm{id}_\Omega \otimes \mathrm{id}_P$, where $\Omega = \bigoplus_{i\in I} \dim(i)\, i$ is the so-called "Kirby color" of $\mathcal{C}$. Consequently, $\mathcal{C}$ (endowed with its categorical trace) is a semisimple chromatic category. Note that the dimension of $\mathcal{C}$ (as a semisimple chromatic category) coincides with its usual definition $\dim(\mathcal{C}) = \sum_{i\in I} \dim(i)^2$ as a spherical fusion category. (This follows from the computation of $\mathrm{tr}(\mathsf{c}_{\mathbb{1}})$ for the above chromatic map based on $\mathbb{1}$.)

Example 5.11. [Spherical categories from Hopf-algebras] In this example $\Bbbk$ need not be algebraically closed. Let H be a finite dimensional Hopf algebra over $\Bbbk$. The category H-mod of finite dimensional (left) H-modules and H-linear homomorphisms is a $\Bbbk$-category. Assume that H is unimodular and unibalanced in the sense of [2], meaning that the square of the antipode S of H is the conjugation by a square root g of the distinguished grouplike element of H. Pick a nonzero right integral $\lambda\colon H \to \Bbbk$ for H. Then H is a projective generator of H-mod, the integral λ determines a non-degenerate m-trace t on $\mathsf{Proj}_{H\text{-mod}}$ characterized by $\mathsf{t}_H(f) = \lambda(gf(1))$ for all $f \in \mathrm{End}_H(H)$, and a chromatic map for H is

$$\mathsf{c}_H\colon \begin{cases} H\otimes H & \to \quad H\otimes H \\ x\otimes y & \mapsto \quad \lambda(S(y_{(1)})gx)\, y_{(2)} \otimes y_{(3)} \end{cases}$$

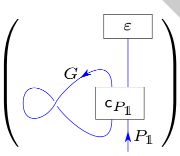
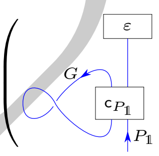
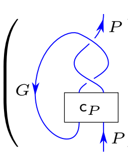
where $y_{(1)}\otimes y_{(2)}\otimes y_{(3)}$ is the double coproduct of y. (This follows from [15, Lemma 6.3] or the more general computations performed in Section 4.3 of [13]) More generally, for any finite dimensional projective H-module P,

$$\mathsf{c}_P = \sum_i (\mathrm{id}_H \otimes g_i)\mathsf{c}_H(\mathrm{id}_H \otimes f_i)\colon H\otimes P \to H \otimes P$$

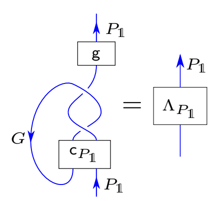
is a chromatic map based on P for H, where $\{f_i\colon P\to H, g_i\colon H\to P\}_i$ is any finite family of H-linear homomorphisms such that $\mathrm{id}_P = \sum_i g_i f_i$. Consequently, H-mod is a chromatic category. In particular, finite dimensional modules over many small versions of (super) quantum groups fit into this setting. Note that H-mod is semisimple (as a chromatic category) if and only if H is semisimple (as an algebra), and if such is the case, then the dimension of H-mod (as a semisimple chromatic category) is equal to $\lambda(1)$ and so is nonzero if and only if H is cosemisimple (by Maschke's theorem for Hopf algebras). Consequently, the chromatic category H-mod is semisimple with nonzero dimension if and only if H is semisimple and cosemsisimple, or equivalently (by [26, Corollary 3.2]) if and only if H is involutory with $\dim_{\Bbbk}(H)1_{\Bbbk} \neq 0$.

5.5.1. *Gluing morphisms.* Let $P_{\mathbb{1}} \in \mathcal{C}$ be the projective cover of $\mathbb{1}$ and $\epsilon : P_{\mathbb{1}} \to \mathbb{1}$ be the associated non zero morphism. The following lemma was proved in [12]:

Lemma 5.12. *There exists scalars $\Delta_+, \Delta_- \in \mathbb{K}$ and a family of $\{\Delta_0^P \in \mathrm{Hom}_{\mathcal{C}}(P,P)\}_{P\in\mathsf{Proj}}$, such that for any chromatic morphisms $\mathsf{c}_{P_{\mathbb{1}}}$, c_P based on $P_{\mathbb{1}}$ and P respectively, one has*

$F\Bigg($  $\Bigg) = \Delta_+\varepsilon$, $F\Bigg($  $\Bigg) = \Delta_-\varepsilon$, *and* $F\Bigg($  $\Bigg) = \Delta_0^P$.

Definition 5.13. A *gluing morphism* is an endomorphism

$\mathsf{g} \in \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$ such that $\mathsf{g}\circ\Delta_0^{P_{\mathbb{1}}} = \Lambda_{P_{\mathbb{1}}}$, i.e.  .

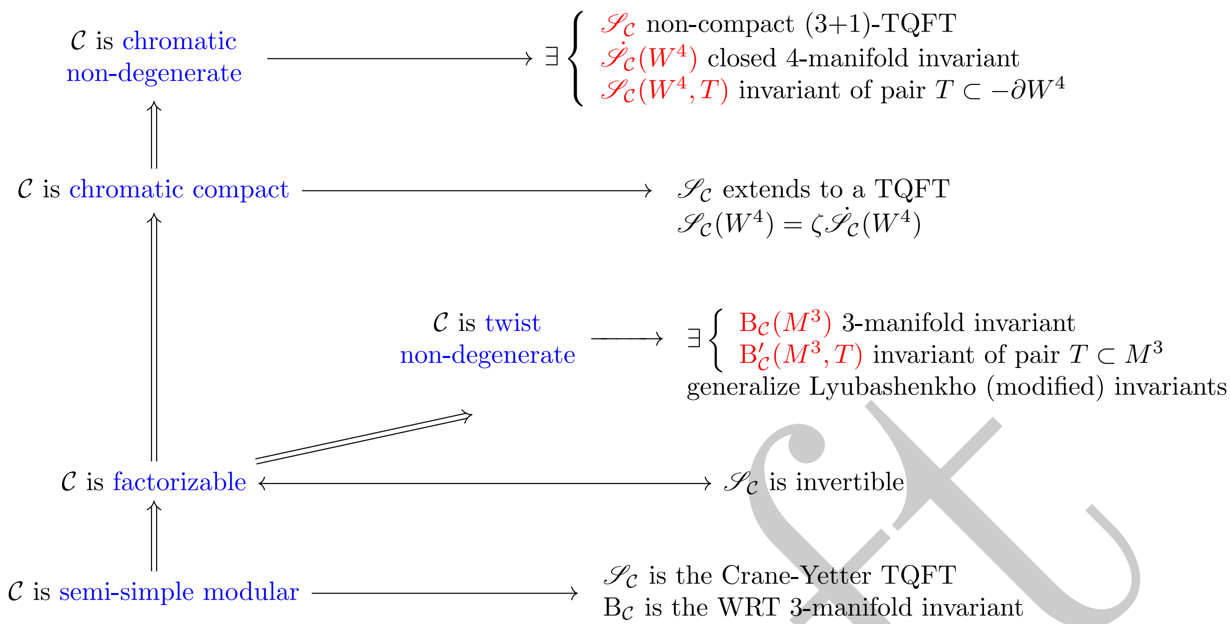


FIGURE 1. This figure represents different properties on a ribbon chromatic category $\mathcal{C}$ and their relationships and corresponding 3-manifold invariants and TQFTs. A category at the tail of a double arrow implies the property at the head of an arrow. For example, chromatic compact implies chromatic non-degenerate. A category at the tail of a single arrow implies the existence of the invariant at the head of the arrow. For example, a chromatic non-degenerate category gives rise the non-compact $(3+1)$-TQFT $\mathscr{S}_{\mathcal{C}}$.

Proposition 5.14 ([12]). *The category $\mathcal{C}$ admits a gluing morphism $\mathsf{g} \in \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$ if and only if $\Delta_0^{P_{\mathbb{1}}} \neq 0$.*

The above proposition prompts the following :

Definition 5.15. We say that

(1) $\mathcal{C}$ is *chromatic non-degenerate* if $\Delta_0^{P_{\mathbb{1}}} \neq 0$, i.e. if $\mathcal{C}$ admits a gluing morphism,
(2) $\mathcal{C}$ is *chromatic compact* if there exists a scalar $\zeta \in \mathbb{K}^*$ such that $\Delta_0^{P_{\mathbb{1}}} = \zeta \Lambda_{P_{\mathbb{1}}}$,
(3) $\mathcal{C}$ is *factorizable* if there exists a scalar $\zeta \in \mathbb{K}^*$ such that for any projective P, $\Delta_0^P = \zeta \Lambda_P$,
(4) $\mathcal{C}$ is *twist non-degenerate* if $\Delta_+ \Delta_- \neq 0$.

The follwing lemma, proved in [12] will be used later on:

Lemma 5.16. *The category $\mathcal{C}$ has a gluing morphism which is an isomorphism of $P_{\mathbb{1}}$ if and only if*

$$\Delta_0^{P_{\mathbb{1}}} = \zeta \Lambda_{P_{\mathbb{1}}} \textit{ for some scalar } \zeta \in \mathbb{K}^* \text{ (i.e. iff } \mathcal{C} \text{ is chromatic compact).}$$

In this case, $\zeta^{-1}\mathrm{id}_{P_{\mathbb{1}}} + n$ is a gluing morphism for any nilpotent $n \in End(P_1)$.

5.6. **Finite tensor categories.** Following [26], a *finite tensor category* (over $\Bbbk$) is a rigid abelian $\Bbbk$-category $\mathcal{C}$ such that:

- every object of $\mathcal{C}$ has finite length,
- the category $\mathcal{C}$ has enough projectives,
- there are finitely many isomorphism classes of simple objects.

Let $\mathcal{C}$ be a finite tensor category. Then the unit object $\mathbb{1}$ of $\mathcal{C}$ is simple (see [26, Theorem 4.3.8]). Also, every simple object of $\mathcal{C}$ has a projective cover, and any indecomposable projective object P of $\mathcal{C}$ has a unique simple subobject, called the *socle* of P (see [26, Remark 6.1.5]). In particular, the socle of the projective cover of the unit object $\mathbb{1}$ is an invertible object called the *distinguished invertible* object of $\mathcal{C}$. Finally $\mathcal{C}$ has a projective generator (for example the direct sum of the projective covers of the elements of a representative set of the isomorphism classes of simple objects).

5.7. **Spherical and modular tensor categories.** A finite tensor category is *unimodular* if its distinguished invertible object (see Section 5.6) is the unit object.

Definition 5.17 (Spherical tensor category)**.** A *spherical tensor category* (over $\Bbbk$) is a pivotal unimodular finite tensor category $\mathcal{C}$ (over $\Bbbk$) such that the right m-trace on $\mathsf{Proj}_{\mathcal{C}}$ (which exists and is unique up to scalar multiple by [29, Corollary 5.6]) is also a left m-trace.

Note that by [SS, Theorem 1.3], this definition agrees with [22, Definition 3.5.2] where the above condition on the right m-trace is replaced by the equality of the square of the pivotal structure with the Radford equivalence. Theorem 1.6 of [13] states the following:

Theorem 5.18. *Any spherical tensor category over an algebraically closed field is a chromatic category.*

Note that the categories of Examples 5.10 and 5.11 are examples of spherical tensor categories when the ground field $\Bbbk$ is algebraic closed. Moreover, a spherical tensor category over an algebraically closed field which is semisimple (as a chromatic category or, equivalently, as an abelian category) is a spherical fusion category (in the sense of Example 5.10).

Finally if $\mathcal{C}$ is ribbon category then it is pivotal and if it is also a unimodular finite tensor category (over $\Bbbk$) then it has a unique (up to scalar) right m-trace on Proj (see [29, Corollary 5.6]); since $\mathcal{C}$ is ribbon this right m-trace is also a left m-trace. Therefore $\mathcal{C}$ is a spherical tensor category.

Definition 5.19 (Modular tensor category)**.** A *modular tensor category* (over $\Bbbk$) is a ribbon finite tensor category $\mathcal{C}$ (over an algebraically closed $\Bbbk$) which is factorisable (i.e. its transparent objects are direct sums of $\mathbb{1}$).

Remark 5.20. The actual definition of factorizability is the equivalence of the Drinfeld center of $\mathcal{C}$ with $\mathcal{C} \boxtimes \mathcal{C}^{op}$ but, by a result of Shimizu [46], this is equivalent to the triviality of the Müger center i.e. of the set of transparent objects as stated above.

5.8. **Concrete examples.** The following constructions are examples of semi-simple modular tensor categories over $\mathbb{C}$.

Example 5.21 (The Ising anyons)**.** The only simple objects are $1, X, Y$ with fusion rules :

$$X^2 = \mathbb{1} \oplus Y, X \otimes Y = Y \otimes X = X, Y^2 = \mathbb{1}.$$

The S-matrix and T matrices are

$$S = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp\frac{\pi i}{8} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The quantum dimensions are $1, \sqrt{2}, 1$ (for $\mathbb{1}, X, Y$ respectively). See [45] (section 5.3.4) for full details.

Example 5.22 (The Fibonacci category)**.** The following MTC is obtained from considering $U_q(\mathfrak{sl}_2)$ at $q = \exp(i\pi/5)$ by taking the subcategory of its modules with integer highest weights (sometimes called the "even part") and semi-simplifying it (see [45] for a general overview of this kind of construction, or the next example). There are two simple objects $\mathbb{1}$ and X satisfying fusion rules: $X \otimes X = \mathbb{1} \oplus X$ and $\mathbb{1}$ being the tensor unit. The S-matrix and T matrices are

$$S = \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & \exp\frac{4\pi i}{5} \end{pmatrix}.$$

The quantum dimensions are 1 and $\frac{1+\sqrt{5}}{2}$ respectively. The F matrix expresses the change of basis between two different basis of the hom spaces (it is basically the $6j$-symbols of the theory). In particular here

$$F_X^{X,X,X} = \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^{-1} & (\frac{1+\sqrt{5}}{2})^{-1/2} \\ (\frac{1+\sqrt{5}}{2})^{-1/2} & -(\frac{1+\sqrt{5}}{2})^{-1} \end{pmatrix}$$

This example is taken from [45].

Example 5.23 ($U_\xi(\mathfrak{sl}_2)$-mod)**.** Recall the definition of $U_q(\mathfrak{sl}_2)$ from the appendix. Remark that $U_q(\mathfrak{sl}_2)$ is infinite dimensional and that its simple modules are infinitely many. So let now ξ be a primitive l^{th}-root of unity and set $l' = l$ if l is odd and $l' = l/2$ else. Let $\overline{U}_\xi(\mathfrak{sl}_2)$ be the quotient of $U_\xi(\mathfrak{sl}_2)$ by the ideal generated by $E^{l'}, F^{l'}, K^l - 1$. This ideal is also a coideal annihilated by unit and counit so that $\overline{U}_\xi(\mathfrak{sl}_2)$ is a finite dimensional Hopf algebra over $\mathbb{C}$. If l is odd, the R matrix is given by :

$$R = l^{-1}\left(\sum_{n=0}^{l-1} \xi^{-\frac{n(n-1)}{2}} E^n \otimes F^n\right)\left(\sum_{\beta,\gamma=0}^{l-1} \xi^{2\beta\gamma} K^\beta \otimes K^\gamma\right)$$

Let $\mu = K^{-1}$. The notice that $\forall x \in \overline{U}_\xi(\mathfrak{sl}_2)$ we have $S^2(x) = \mu x \mu^{-1}$. Letting $R = \sum a_i \otimes b_i$, let $u = \sum S(b_i)a_i$; it turns out that $uxu^{-1} = S^2(x)\ \forall x \in \overline{U}_\xi(\mathfrak{sl}_2)$. Therefore $v = u^{-1}\mu$ is in the center of $\overline{U}_\xi(\mathfrak{sl}_2)$. Furthermore it is invertible and it satisfies $S(v) = v$ and $\Delta(v) = R_{21}R \cdot v \otimes v)$ (where $R_{21} = \sum b_i \otimes a_i$), so that v is the twist of $\overline{U}_\xi(\mathfrak{sl}_2)$.

The simple modules of $\overline{U}_\xi(\mathfrak{sl}_2)$ turn out to be all of "highest weight type" i.e. they contain a vector v_0 such that $Kv_0 = \xi^n v_0, Ev_0 = 0$ and they are $n+1$-dimensional for $n \leq l-1$; we denote them S_n. If we pick $l = 2r$ their fusion rules are known and given by the Clebsch-Gordan coefficients:

$$S_a \otimes S_b = P \oplus \left(\bigoplus_{c=|a-b|\ by\ 2}^{min(r-2,a+b)} S_c\right)$$

where P is some projective module which we do not describe. The S matrix is non-degenerate and is given by (see [47] Lemma 5.2 Chapter XII):

$$S_{i,j} = (-1)^{i+j}[i+1][j+1]\ \forall 0 \leq i, j \leq r-2.$$

where $[n] = \frac{\xi^n - \xi^{-n}}{\xi - \xi^{-1}}$. The twist matrix T is the diagonal one with entries : $T_{ii} = (-1)^i \xi^{i(i+2)/2}$ where we fixed a square root of ξ denoted a in [47] Chapter XII. The category obtained by forgetting the projective parts (i.e. applying "the purification process") in the above decomposition is modular (see [47] Theorem 7.1 Chapter XII).

Notice that if r is odd, then the above decomposition rules show that the $S_{2i}, i < \frac{r-1}{2}$ generate a full subcategory. Example 5.22 was this case with $r = 5$.

Example 5.24. Let $\Bbbk = \mathbb{C}$ and m, n, r be positive integers such that $n|m$ and $r \geq 2$. Let q be a primitive $2r$-th root of unity and choose $q^{\frac{2}{mn}}$ a primitive mnr-th root of unity. Note that $(q^{\frac{2}{mn}})^{\frac{m}{n}}$ is a primitive n^2r-th root of unity. Let

$$H = \mathfrak{u}_q^{m,n}(\mathfrak{sl}_2) = \mathbb{C}\langle E, F, \kappa | E^r = F^r = 0, \kappa^{mnr} = 1, \kappa E = q^{\frac{2}{m}} E\kappa, \kappa F = q^{-\frac{2}{m}} F\kappa, EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\rangle$$

where $K = \kappa^m$. The algebra H can be given the structure of a Hopf algebra with coproduct Δ, counit ε and antipode S defined by

$$\begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes K, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1}, \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF, \\ \Delta(\kappa) &= \kappa \otimes \kappa, & \varepsilon(\kappa) &= 1, & S(\kappa) &= \kappa^{-1}. \end{aligned}$$

Note that H contains a version of the small quantum group at even root of unity as the sub-Hopf-algebra generated be E, F and K. Let $\mathcal{C} = H - mod$ be the category of finite dimensional left H-modules. For $i \in \mathbb{Z}/mnr\mathbb{Z}$, denote $\kappa_i = \dfrac{1}{mnr} \displaystyle\sum_{j=0}^{mnr-1} q^{\frac{-2ij}{mn}} \kappa^j$. Then

$$\kappa\kappa_i = q^{\frac{2i}{mn}} \kappa_i, \quad \kappa_i \kappa_j = \delta_{i,j} \kappa_i, \quad \sum_{i=0}^{mnr-1} \kappa_i = 1, \quad E\kappa_i = \kappa_{i+n} E, \quad \text{and} \quad F\kappa_i = \kappa_{i-n} F.$$

Namely, κ_i acts as the projection on the $q^{\frac{2i}{mn}}$ eigenspace of κ. As proved in [12], the Hopf algebra $H = \mathfrak{u}_q^{m,n}(\mathfrak{sl}_2)$ is ribbon where the R-matrix and twist are given by:

$$R = \left(\sum_{i,j=0}^{mnr-1} q^{\frac{2ij}{n^2}} \kappa_i \otimes \kappa_j \right) \cdot \left(\sum_{k=0}^{r-1} \frac{\{1\}^{2k}}{\{k\}!} q^{\frac{k(k-1)}{2}} E^k \otimes F^k \right),$$

$$\theta = K^{r-1} \sum_{k=0}^{r-1} \frac{\{1\}^{2k}}{\{k\}!} q^{\frac{k(k-1)}{2}} S(F^k) \left(\sum_{i=0}^{mnr-1} q^{\frac{-2i^2}{n^2}} \kappa_i \right) E^k.$$

The cointegral is $\Lambda = c\kappa_0 E^{r-1} F^{r-1}$ for some scalar $c \in \mathbb{K}^\times$ and the right integral is $\lambda(\kappa^j E^n F^k) = \frac{mnr}{c} \delta_{i,m(1-r)} \delta_{n,r-1} \delta_{k,r-1}$. In particular $\lambda(\kappa_i F^{r-1} E^{r-1}) = \frac{1}{c} q^{\frac{2i(r-1)}{n}}$. It was proved in [12] that the category $\mathcal{C} = H - mod$ is chromatic compact. It is factorizable if and only if $m = n$ and both n and r are odd. It is twist degenerate if and only if n is odd and r is a multiple of 4.

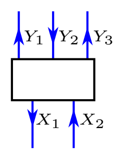
6. Turaev-Viro like theories associated to chromatic categories

We are now ready to detail the construction of a $(3, 2)$-TQFT associated to each chromatic $\Bbbk$-category $\mathcal{C}$. Given an oriented surface Σ , let $\mathcal{S}_{\mathcal{C}}(\Sigma)$ be defined as the vector space of $\mathcal{C}$-colored ribbon graphs (see Subsection 6.1) in Σ containing at least one projective color up to admissible skein relations (see Subsection 6.4). There is a natural action of the mapping class group of Σ on $\mathcal{S}_{\mathcal{C}}(\Sigma)$ so in order to define the TQFT we only need to define the linear maps associated to each type of handle attachment and then to check the relations provided by Juhasz's presentation. This is the plan for this section.

6.1. **Ribbon graphs.** Loosely speaking, a ribbon graph is an oriented compact surface embedded in manifold which is decomposed into elementary pieces: bands, annuli, and coupons, see [47]. A $\mathcal{C}$-coloring of such a graph is a labeling of the core of each band and annuli with an object of $\mathcal{C}$ and a compatible morphism to each coupon. A $\mathcal{C}$*-coloring* of a ribbon graph Γ is a function assigning to every strand of Γ an object of $\mathcal{C}$, called its *color*, and assigning to every coupon Q of Γ a morphism $Q_\bullet \to Q^\bullet$ in $\mathcal{C}$. Here $Q_\bullet$ and $Q^\bullet$ are objects of $\mathcal{C}$ defined as follows. Let us call the endpoints of the arcs of Γ lying on the bottom (respectively, top) base of Q the *inputs* (respectively, *outputs*) of Q. The orientation of the bottom base of Q induced by the orientation of Q determines an order in the set of the inputs. Let $X_i \in \mathcal{C}$ be the color of the arc of Γ adjacent to the i-th input. Set $\varepsilon_i = +$ if this arc is directed toward Q at the i-th input and $\varepsilon_i = -$ otherwise. The orientation of the top base of Q induced by the orientation of Q determines an order in the set of the outputs, and we take the opposite order. Let $Y_j \in \mathcal{C}$ be the color of the arc of Γ adjacent to the j-th output. Set $\nu_j = -$ if this arc is directed toward Q at the j-th output and $\nu_j = +$ otherwise. Then

$$Q_\bullet = X_1^{\varepsilon_1} \otimes \cdots \otimes X_m^{\varepsilon_m} \quad \text{and} \quad Q^\bullet = Y_1^{\nu_1} \otimes \cdots \otimes Y_n^{\nu_n},$$

where m and n are respectively the numbers of inputs and outputs of Q and, as usual, $X^+ = X$ and $X^- = X^*$ for $X \in \mathcal{C}$. For example, the following coupon whose bottom base is the horizontal bottom one



must be colored with a morphism $X_1^* \otimes X_2 \to Y_1 \otimes Y_2^* \otimes Y_3$

6.2. **Invariants of colored ribbon graphs.** To each free end of a $\mathcal{C}$-colored ribbon graph Γ in $\mathbb{R} \times [0,1]$ is associated a signed object consisting of the color of the arc incident to the free end and of a sign ± 1 depending if that arc is directed up or down. Then one can view Γ as a morphism from the sequence of signed objects associated with its bottom free ends (i.e., its free ends in $\mathbb{R} \times \{0\}$) to the sequence of signed objects associated with its top free ends (i.e., its free ends in $\mathbb{R} \times \{1\}$). This defines a monoidal category $\mathrm{Rib}_\mathcal{C}$ whose objects are finite sequences of signed objects, whose morphisms are isotopy classes of $\mathcal{C}$-colored ribbon graph in $\mathbb{R} \times [0,1]$, whose composition is given by putting one $\mathcal{C}$-colored ribbon graph on top of the other, and whose monoidal product is given by concatenation. The graphical calculus of Section 9.6 gives rise to a monoidal functor

$$F\colon \mathrm{Rib}_\mathcal{C} \to \mathcal{C}. \tag{4}$$

If the left and right traces tr_l and tr_l on $\mathcal{C}$ coincide, then F induces an isotopy invariant $F : \mathcal{L} \to \mathrm{End}_\mathcal{C}(\mathbb{1}) = \Bbbk$, where $\mathcal{L}$ is the class of $\mathcal{C}$-colored ribbon graphs in $S^2 = (\mathbb{R} \times]0,1[) \cup \{\infty\}$. This invariant can be renormalized using a modified trace as follows.

Denote by $\mathcal{L}_\mathcal{I}$ the class of $\mathcal{C}$-colored ribbon graphs in S^2 having at least one strand colored with an object in $\mathcal{I}$. In particular, each $\Gamma \in \mathcal{L}_\mathcal{I}$ is the braid closure of some $\mathcal{C}$-colored ribbon graph T_X in $\mathbb{R} \times [0,1]$ with exactly one bottom free end and one top free end both supported by arcs oriented upward and colored by some object $X \in \mathcal{I}$, so that $F(T_X) \in \mathrm{End}_\mathcal{C}(X)$. Then, by [28, Theorem 5], each m-trace t on $\mathcal{I}$ induces an isotopy invariant

$$F'\colon \mathcal{L}_\mathcal{I} \to \Bbbk, \quad \Gamma \mapsto F'(\Gamma) = \mathsf{t}_X\big(F(T_X)\big). \tag{5}$$

6.3. **Admissible graphs.** Let Σ be an oriented surface and $\mathcal{I} \subset \mathcal{C}$ an ideal. An *$\mathcal{I}$-admissible graph* in Σ is a $\mathcal{C}$-colored ribbon graph Γ in Σ with no free ends such that each connected component of Σ contains at least one strand of Γ colored with an object in $\mathcal{I}$.

Given $\mathcal{I}$-admissible graphs $\Gamma_1, \ldots, \Gamma_k$ in Σ and $a_1, \ldots, a_k \in \mathbb{K}$, the linear combination $a_1\Gamma_1 + \cdots + a_n\Gamma_n$ is a *$\mathcal{I}$-skein relation (in Σ)* if there is a coupon Q embedded in Σ and $\mathcal{I}$-admissible graphs $\Gamma'_1, \ldots, \Gamma'_k$ in M such that:

- Γ'_i is isotopic to Γ_i (as a $\mathcal{C}$-colored graph in Σ) for all $1 \leq i \leq k$;
- the Γ'_is coincide outside Q: $\Gamma'_i \cap (\Sigma \setminus Q) = \Gamma'_j \cap (\Sigma \setminus Q)$ for all $1 \leq i, j \leq k$;
- Γ'_i intersects ∂Q only in its bottom and tops bases and transversally along the stands of Γ'_i (so that $\Gamma'_i \cap Q$ can be seen as a $\mathcal{C}$-colored ribbon graph in $\mathbb{R} \times [0, 1]$) for all $1 \leq i \leq k$;
- $a_1 F(\Gamma'_1 \cap Q) + \cdots + a_k F(\Gamma'_k \cap Q) = 0$ (as a morphism in $\mathcal{C}$);
- each Γ'_i has an edge colored by a projective object which is not entirely contained in the coupon Q.

Two linear combinations of $\mathcal{I}$-admissible graphs are *$\mathcal{I}$-skein equivalent* if their difference is an $\mathcal{I}$-skein relation.

6.4. **Admissible skein modules.** The *$\mathcal{I}$-admissible skein module* $\mathscr{S}_{\mathcal{I}}(\Sigma)$ of an oriented surface Σ is the quotient of the $\mathbb{K}$-vector space generated by the $\mathcal{I}$-admissible graphs in Σ by its vector subspace generated by the $\mathcal{I}$-skein relations. The empty graph in Σ is not admissible unless Σ is empty. Then $\mathscr{S}_{\mathcal{I}}(\emptyset)$ is the 1-dimensional vector space generated by the empty graph.

Lemma 6.1. *$\mathscr{S}_{\mathcal{I}}(\Sigma)$ is generated by $\mathcal{I}$-admissible graphs where each strand is colored by an object of $\mathcal{I}$.*

Exercise 6.2. Prove the lemma. A solution is given in Lemma 2.2 of [13].

If $f : \Sigma \to \Sigma'$ is an orientation preserving embedding and Γ is a ribbon graph in Σ, then $f(\Gamma)$ is a ribbon graph in Σ' in an obvious way. Further, if Γ is $\mathcal{C}$-colored, then so if $f(\Gamma)$ (with colors inherited from Γ). An embedding $f : \Sigma \to \Sigma'$ is *admissible* if $f(\Sigma)$ meets every component of Σ' or, equivalently, if $H_0(f)$ is surjective. The image under an admissible orientation preserving embedding f of an $\mathcal{I}$-admissible graph is an $\mathcal{I}$-admissible graph. Clearly, the image under f of a skein relation in Σ is a skein relation in Σ'. Consequently the map $\Gamma \mapsto f(\Gamma)$ induces a $\Bbbk$-linear homomorphism

$$\mathscr{S}_{\mathcal{I}}(f) : \mathscr{S}_{\mathcal{I}}(\Sigma) \to \mathscr{S}_{\mathcal{I}}(\Sigma').$$

Let Emb_2^a be the category whose objects are oriented surfaces and morphisms are isotopy classes of admissible orientation preserving embeddings. This is a monoidal category with disjoint union as monoidal product. Denote by $\mathrm{Vect}_{\mathbb{K}}$ the monoidal category of $\mathbb{K}$-vector spaces and $\Bbbk$-linear homomorphisms.

Theorem 6.3. *Recall, $\mathcal{C}$ is a pivotal $\mathbb{K}$-category. The assignments $\Sigma \mapsto \mathscr{S}_{\mathcal{I}}(\Sigma)$ and $f \mapsto \mathscr{S}_{\mathcal{I}}(f)$ define a monoidal functor*

$$\mathscr{S}_{\mathcal{I}} : \mathrm{Emb}_2^a \to \mathrm{Vect}_{\mathbb{K}}.$$

In particular, this functor provides representations of the mapping class group of surfaces. Moreover, if the ideal $\mathcal{I}$ has a generator (in the sense of Section 5.1), then for any closed oriented surface Σ, the $\mathbb{K}$-vector space $\mathscr{S}_{\mathcal{I}}(\Sigma)$ is finite dimensional.

Proof. The functoriality and monoidality of $\mathscr{S}_{\mathcal{I}}$ are direct consequences of the definitions. Assume that $\mathcal{I}$ has a generator G and let Σ be a closed oriented surface. It is sufficient to prove the last statement of the theorem for Σ a compact connected surface. Consider a cellularization of Σ consisting in a single vertex v, $2g$ closed curves $c_1, \ldots, c_{2g}$ and one disk D. Let Γ be an $\mathcal{I}$-admissible graph in Σ. We can assume that Γ intersects each c_i transversally and that all its strands are $\mathcal{I}$-colored (by Lemma 6.1). By fusing all the strands intersecting each c_i, we obtain that Γ is skein equivalent to an $\mathcal{I}$-colored ribbon graph intersecting each c_i once. Moreover, since G is a generator of $\mathcal{I}$ up to applying some skein relation for each c_i, we can replace Γ with a linear combination of $\mathcal{I}$-colored ribbon graphs intersecting c_i via a single edge colored by the generator $G_i = G$ (here we denote the generator with a subscript i so we can discern which one is associated to c_i). Thus, Γ is skein equivalent to a linear combination of graphs of the form of a bouquet of circles where each arc intersects a single c_i once and is colored by G_i, and these arcs end up in a single coupon contained in the disk D and colored by some $f \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, G_1 \otimes G_2 \otimes G_1^* \otimes G_2^* \otimes \cdots \otimes G_{2g-1} \otimes G_{2g} \otimes G_{2g-1}^* \otimes G_{2g}^*)$. Since this space of homomorphisms is finite dimensional (because $\mathcal{C}$ is a $\Bbbk$-category), we conclude that so is $\mathscr{S}_{\mathcal{I}}(\Sigma)$. $\boxed{6.3}$

In the next theorem, we interpret skein modules of the 2-disk D^2 and the sphere 2-sphere in terms of m-traces. Note that Walker and Reutter announced in [49] a related result.

Theorem 6.4. *Recall, $\mathcal{C}$ is a pivotal $\mathbb{K}$-category. There are canonical $\Bbbk$-linear isomorphisms:*

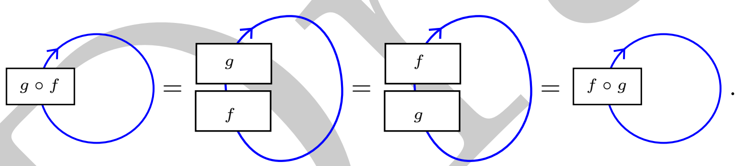
$$\mathscr{S}_{\mathcal{I}}(D^2)^* \cong \{\textit{right m-traces on } \mathcal{I}\} \cong \{\textit{left m-traces on } \mathcal{I}\} \quad \textit{and} \quad \mathscr{S}_{\mathcal{I}}(S^2)^* \cong \{\textit{m-traces on } \mathcal{I}\}.$$

Proof. We limit ourselves to show that each element of $\mathscr{S}_{\mathcal{I}}(D^2)^*$ gives a modified trace on $\mathcal{I}$. For the full proof we invite to check [13].

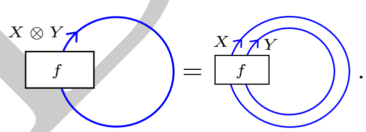
We prove the right version of the first statement of Theorem 6.4 (the left version being analogous). We associate to any $T \in \mathscr{S}_{\mathcal{I}}(D^2)^*$ a family $\mathsf{t}^T = \{\mathsf{t}^T_X \colon \mathrm{End}_{\mathcal{C}}(X) \to \mathbb{K}\}_{X \in \mathcal{I}}$ of linear forms as follows: for any $f \in \mathrm{End}_{\mathcal{C}}(X)$ with $X \in \mathcal{I}$, set

$$\mathsf{t}^T_X(f) = T(O_f)$$

where O_f is the admissible graph in D^2 given by the right closure of the coupon colored with f. Let us prove that t^T is a right m-trace on $\mathcal{I}$. First, since a coupon colored with $f \circ g$ is $\mathcal{I}$-skein equivalent to a coupon colored with f composed with a coupon colored with g, we get that $O_{f\circ g}$ is skein equivalent to $O_{g\circ f}$ via an isotopy which exchanges f and g:



Therefore t^T satisfies the cyclicity property of an m-trace. Next, for any $f \in \mathrm{End}_{\mathcal{C}}(X \otimes Y)$ with $X \in \mathcal{I}$ and $Y \in \mathcal{C}$, the admissible graph O_f is skein equivalent to the closure of a coupon colored with f with two incoming and outgoing arcs colored with X and Y :



This shows that t^T satisfies the right partial trace property of an m-trace. Then the assignment $T \mapsto \mathsf{t}^T$ is a $\Bbbk$-linear homomorphism $\mathscr{S}_{\mathcal{I}}(D^2)^* \to \{\text{right m-traces on } \mathcal{I}\}$. $\boxed{6.4}$

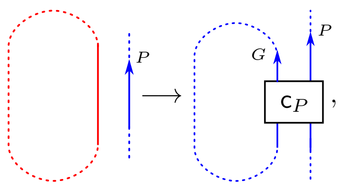
Remark 6.5. Theorems 6.3 and 6.4 have analogue in dimension 3 by assuming that $\mathcal{C}$ is moreover ribbon, by considering the Reshetikhin-Turaev functor F from the category of $\mathcal{C}$-colored ribbon graphs in $\mathbb{R}^2 \times [0,1]$ to $\mathcal{C}$ (see [47]), and by using this functor to define (as above) the skein module

$\mathscr{S}_{\mathcal{I}}(M)$ associated to an oriented compact 3-manifold M. In particular, for the 3-ball B^3 and 3-sphere S^3, there are canonical $\Bbbk$-linear isomorphisms

$$\mathscr{S}_{\mathcal{I}}(B^3)^* \cong \mathscr{S}_{\mathcal{I}}(S^3)^* \cong \{\text{m-traces on } \mathcal{I}\}.$$

These skein modules of 3-manifolds will be used later (as in in [12]) to construct (3+1)-TQFTs.

6.5. **Skein modules elements from bichrome graphs.** In this subsection, we assume that $\mathcal{C}$ is a chromatic category. Following [15], a *bichrome graph* in a closed oriented surface Σ is the disjoint union of an admissible graph in Σ (called the *blue part*) and finitely many pairwise disjoint unoriented embedded circles in Σ (called the *red part*). A *red to blue modification* of a bichrome graph is the modification in an annulus given by



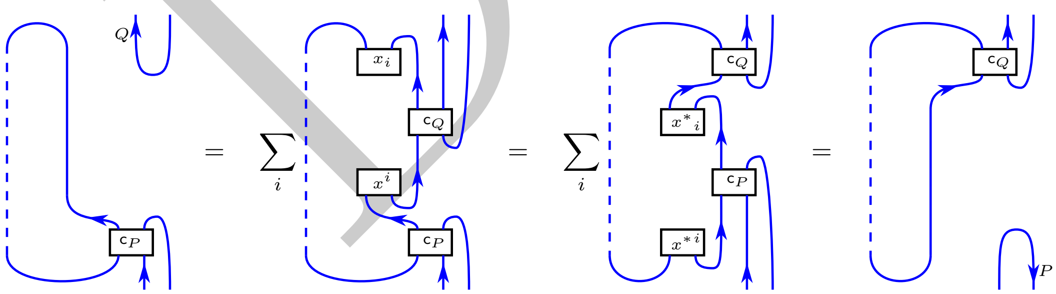
(6)

where c_P is any chromatic map based on a projective object P at a projective generator G of $\mathcal{C}$. Here we allow the P-colored strand to be replaced by several parallel strands with at least one colored by a projective object. Note that if the category $\mathcal{C}$ is spherical fusion, then the red to blue modification amounts to arbitrarily orient the red curve and color it with the Kirby color of $\mathcal{C}$ (see Example 5.10).

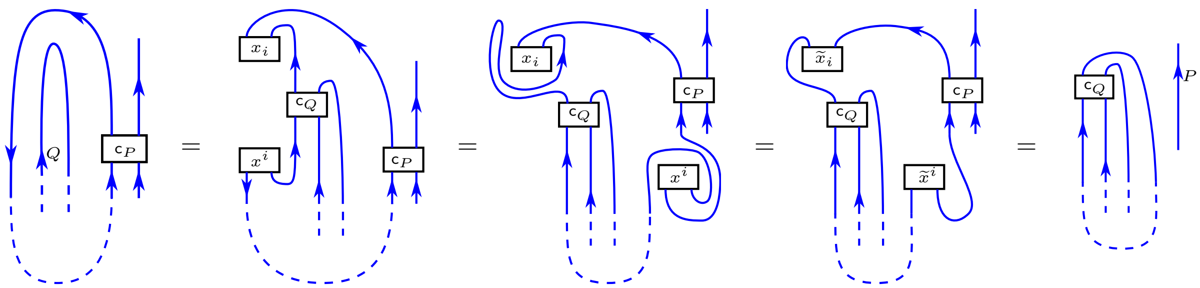
Red to blue modifications transform any bichrome graph into a $\mathsf{Proj}_{\mathcal{C}}$-admissible graph in Σ whose class in the skein module $\mathscr{S}_{\mathsf{Proj}_{\mathcal{C}}}(\Sigma)$ is well-defined:

Lemma 6.6. *Using the red to blue modification, bichrome graphs in Σ represent well defined elements of the skein module $\mathscr{S}_{\mathsf{Proj}_{\mathcal{C}}}(\Sigma)$.*

Proof. To prove the lemma, we show that two red to blue modifications of a red curve at different places with different chromatic maps give skein equivalent diagrams. Let P, Q be projective objects and G, G' be projective generators of $\mathcal{C}$. Pick a chromatic map c_P based on P at G and a chromatic map c_Q based on Q at G'. There are two cases to consider. First, if the two modifications are made on the same side of the red curve, then



where ${x^*}_i$ and ${x^*}^i$ are the dual basis obtained by ${x^*}^i = (x_i)^* \circ (\phi_{G'} \otimes \mathrm{id}_{G^*})$ and ${x^*}_i = (\phi_{G'}^{-1} \otimes \mathrm{id}_{G^*}) \circ (x^i)^*$. Here the first and third equalities follow from (3) and the second equality from isotopying the coupon and applying duality of Lemma 5.5. Second, if the modifications are made on opposite sides of the red curve, then (with implicit summation):



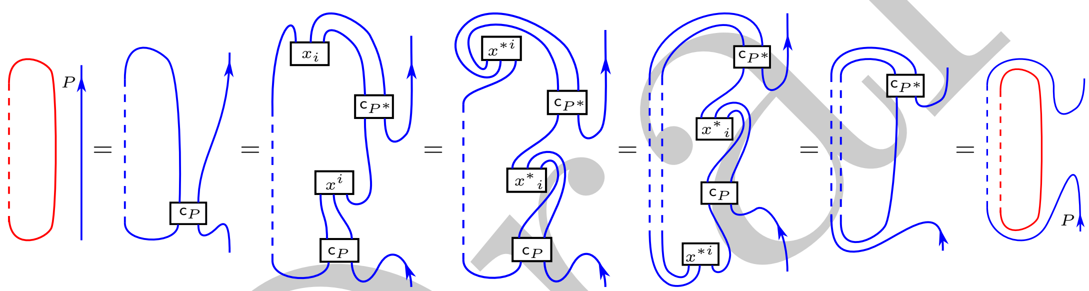
where $\widetilde{x}_i$ and $\widetilde{x}^i$ are the dual basis obtained from x_i and x^i by the rotation property of Lemma 5.5. $\boxed{6.6}$

Remark 6.7. If $\mathcal{C}$ is semisimple, then applying Lemma 6.6 to a red unknot with $P = \mathbb{1}$ implies that $\operatorname{tr}(\mathsf{c}_{\mathbb{1}})$ does not depend of the chromatic map $\mathsf{c}_{\mathbb{1}}$ based on $\mathbb{1}$.

The next lemma shows the usefulness of bichrome graphs.

Lemma 6.8. *A blue strand can be slid over a red curve of an admissible bichrome graph in* $\mathscr{S}_{\mathsf{Proj}_{\mathcal{C}}}(\Sigma)$.

Proof. We first consider the case where we want to slide a strand colored by $P \in \mathsf{Proj}_{\mathcal{C}}$ over a red curve. Then we have the following skein relations:



where $x^*{}_i$ and x^{*i} are the dual basis defined by $x^{*i} = (x_i)^* \circ (\phi_G \otimes \mathrm{id}_{P^* \otimes G^*})$ and $x^*{}_i = (\phi_G^{-1} \otimes \mathrm{id}_{P^* \otimes G^*}) \circ (x^i)^*$. Next, consider the general case where we want to slide a strand colored by $Y \in \mathcal{C}$ over a red curve. Applying the procedure explained in the proof of Lemma 6.1, we can push a strand colored by $P \in \mathsf{Proj}_{\mathcal{C}}$ next to the Y-colored strand. Inserting coupons colored by identities, we replace the Y-colored arc we want to slide by an arc colored by $Y \otimes P \in \mathsf{Proj}_{\mathcal{C}}$ which we then slide over the red curve. By removing then the inserted coupons, we obtain the desired result. $\boxed{6.8}$

6.6. **Construction of the non-compact TQFT.** The admissible skein module functor associated with the ideal $\mathsf{Proj}_{\mathcal{C}}$ of projective objects of $\mathcal{C}$ (see Theorem 6.3) induces (by restriction) a monoidal functor

$$\mathscr{S}_{\mathsf{Proj}_{\mathcal{C}}} \colon \mathbf{Man} \to \mathrm{Vect}_{\mathbb{K}}, \tag{7}$$

where $\mathbf{Man} \subset \mathrm{Cob}^{\mathsf{nc}}$ is the category of closed oriented surfaces and orientation preserving diffeomorphisms. Our goal is to extend it to a functor $\mathscr{S} \colon \mathrm{Cob}^{\mathsf{nc}} \to \mathrm{Vect}_{\mathbb{K}}$. In particular, for any closed oriented surface Σ and any orientation preserving diffeomorphism $d \colon \Sigma \to \Sigma'$ between closed oriented surfaces, we set

$$\mathscr{S}(\Sigma) = \mathscr{S}_{\mathsf{Proj}_{\mathcal{C}}}(\Sigma) \quad \text{and} \quad \mathscr{S}(\Sigma)(e_d) = \mathscr{S}_{\mathsf{Proj}_{\mathcal{C}}}(d) \colon \mathscr{S}(\Sigma) \to \mathscr{S}(\Sigma').$$

We need to assign values to the other generators of $\mathrm{Cob}^{\mathsf{nc}}$. More precisely, given a nonempty closed oriented surface Σ and a framed sphere $\mathbb{S}$ in Σ, we need to assign a $\mathbb{K}$-linear homomorphism $\mathscr{S}(e_{\Sigma,\mathbb{S}})\colon \mathscr{S}(\Sigma) \to \mathscr{S}(\Sigma(\mathbb{S}))$ in the case $\mathbb{S} = \emptyset$ or $\mathbb{S} = \mathbb{S}^k$ is a framed k-sphere with $k \in \{0, 1, 2\}$:

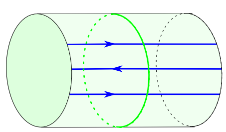
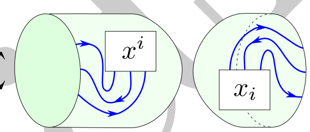
• **Case $\mathbb{S} = \mathbb{S}^2$:** A framed 2-sphere $\mathbb{S}^2$ in Σ determines a spherical component of Σ denoted S^2. Recall from Theorem 6.4 that the m-trace t induces a linear form

$$F'\colon \mathscr{S}(S^2) \to \mathbb{K}. \tag{8}$$

Any admissible graph Γ in Σ decomposes as $\Gamma = \Gamma_1 \sqcup \Gamma_2$ with $\Gamma_1 \subset \Sigma(\mathbb{S}^2)$ and $\Gamma_2 = \Gamma \cap \mathbb{S}^2$. Then the element

$$\mathscr{S}(e_{\Sigma,\mathbb{S}^2})(\Gamma) = F'(\Gamma_2)\Gamma_1 \in \mathscr{S}(\Sigma(\mathbb{S}^2))$$

only depends on the framed sphere $\mathbb{S}^2$ and the class of Γ in $\mathscr{S}(\Sigma)$.
• **Case $\mathbb{S} = \mathbb{S}^1$:** Given a framed 1-sphere $\mathbb{S}^1$ in Σ, let γ be a simple closed curve embedded in Σ so that $\mathbb{S}^1 \simeq \gamma \times [-1, 1]$ in Σ. We fix an orientation and a base point $*$ on γ. Let Γ be an admissible graph in Σ. Isotopying Γ, we can assume that Γ is transverse to $\mathbb{S}^1$ in the sense that $\mathbb{S}^1 \cap \Gamma$ consists in a finite number of portions of edges of Γ in position $\gamma(t_i) \times [-1, 1]$ for $t_i \neq *$ and with at least one intersecting edge colored by a projective object. We define $\mathscr{S}(e_{\Sigma,\mathbb{S}^0})(\Gamma)$ to be the admissible graph in $\Sigma(\mathbb{S}^1)$ obtained from $(\Sigma, \Gamma) \setminus \mathbb{S}^1$ by filling the two attached discs with two coupons colored with dual basis (see Section 5.3):

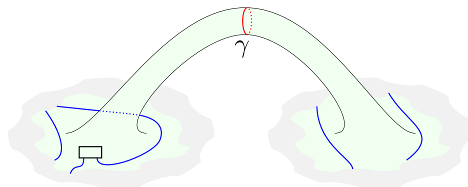
$\Gamma =$  $\mapsto \quad \mathscr{S}(e_{\Sigma,\mathbb{S}^1})(\Gamma) = \sum_i$  . (9)

Lemma 6.9. *The element $\mathscr{S}(e_{\Sigma,\mathbb{S}^1})(\Gamma)$ only depends on the framed sphere $\mathbb{S}^1$ and the class of Γ in $\mathscr{S}(\Sigma)$.*

Proof. If Γ_1 and Γ_2 are isotopic in Σ, with an isotopy where no strand passes through the base point and no coupon passes through γ, then $\mathscr{S}(e_{\Sigma,\mathbb{S}^1})(\Gamma_1)$ and $\mathscr{S}(e_{\Sigma,\mathbb{S}^1})(\Gamma_2)$ are isotopic in $\Sigma(\mathbb{S}^1)$. Assume first that a coupon crosses γ. Assertion (b) of Lemma 5.5 implies that there are two coupons Q_1 and Q_2 such that $F(Q_1) \otimes_{\mathbb{K}} F(Q_2) = 0$, where F is the functor given in (4). Thus one can prove that (see [13]) the difference $\mathscr{S}(e_{\Sigma,\mathbb{S}^1})(\Gamma_1) - \mathscr{S}(e_{\Sigma,\mathbb{S}^1})(\Gamma_2)$ is a sum of skein relations in $\Sigma(\mathbb{S}^1)$. Next, Assertions (a) and (c) of Lemma 5.5 imply respectively that $\mathscr{S}(e_{\Sigma,\mathbb{S}^1})(\Gamma)$ is invariant under the change of the orientation of γ and under the change of the base point on γ. Hence $\mathscr{S}(e_{\Sigma,\mathbb{S}^1})(\Gamma)$ only depends of the isotopy class of Γ in Σ. Since any skein relation in Σ can be isotoped to a skein relation involving a coupon disjoint from $\mathbb{S}^1$, it induces an equivalent skein relation inside $\Sigma(\mathbb{S}^1)$. $\boxed{6.9}$

• **Case $\mathbb{S} = \mathbb{S}^0$:** Consider the disjoint embedded disks D and D' in Σ given by the a framed 0-sphere $\mathbb{S}^0$. Set $\Sigma' = \Sigma \setminus (D \sqcup D')$ and let $C \simeq S^1 \times [0, 1]$ be the cylinder such that $\Sigma(\mathbb{S}^0) = \Sigma' \cup_\partial C$. Set $\gamma = S^1 \times \{\frac{1}{2}\}$ be a red curve inside C. Let Γ be an admissible graph in Σ. Slightly isotopying Γ away from D and D', we obtain an admissible graph Γ' in Σ'. Then $\Gamma' \cup \gamma$ is a bichrome graph in $\Sigma(\mathbb{S}^0)$:

$\Gamma =$  $\rightsquigarrow$ $\Gamma' =$ 

$\Gamma' \cup \gamma =$ 

By Lemma 6.6, the bichrome graph $\Gamma' \cup \gamma$ defines an element in $\mathscr{S}(\Sigma(\mathbb{S}^0))$.

Lemma 6.10. *The element $\mathscr{S}(e_{\Sigma,\mathbb{S}^0})(\Gamma) = \Gamma' \cup \gamma \in \mathscr{S}(\Sigma(\mathbb{S}^0))$ only depends on the framed sphere $\mathbb{S}^0$ and the class of Γ in $\mathscr{S}(\Sigma)$.*

Proof. If Γ'_1 and Γ'_2 are two preimages of Γ isotopic in Σ by an isotopy during which an edge passes over the disk D or D', then by the sliding property of Lemma 6.8, we have $(\Sigma', \Gamma'_1) \cup_\partial (C, \gamma) = (\Sigma', \Gamma'_2) \cup_\partial (C, \gamma) \in \mathscr{S}(\Sigma(\mathbb{S}^0))$. Any isotopy in Σ can be modified so that no coupons of Γ pass through $\mathbb{S}^0$. Finally, any skein relation in Σ is isotopic to a skein relation in a box that does not intersect $\mathbb{S}^0$ which induce a corresponding skein relation between $(\Sigma', \Gamma'_1) \cup_\partial (C, \gamma)$ and $(\Sigma', \Gamma'_2) \cup_\partial (C, \gamma)$ in $\mathscr{S}(\Sigma)$. Remark that interchanging D and D' does not change $\mathscr{S}(e_{\Sigma,\mathbb{S}^0})(\Gamma)$. $\boxed{6.10}$

• **Case $\mathbb{S} = \emptyset$:** We set

$$\mathscr{S}(e_{\Sigma,\emptyset}) = \mathrm{id}_{\mathscr{S}(\Sigma)}.$$

Theorem 6.11. *Recall, $\mathcal{C}$ is a chromatic category. The above assignments define a finite dimensional non-compact (2+1)-TQFT*

$$\mathscr{S} : \mathrm{Cob}^{\mathsf{nc}} \to \mathrm{Vect}_{\Bbbk}.$$

Furthermore $\mathscr{S}$ (uniquely) extends to a genuine (2+1)-TQFT $\mathrm{Cob} \to \mathrm{Vect}_{\Bbbk}$ *if and only if $\mathcal{C}$ is semisimple with nonzero dimension (see Section 5.5).*

We prove Theorem 7.4 in Section 6.7 using the presentation of $\mathrm{Cob}^{\mathsf{nc}}$ given in Section 2.2.

By construction, the non-compact TQFT of Theorem 7.4 extends the skein module functor (7). Also, it follows from the work of Bartlett [3] that if $\mathcal{C}$ is a spherical fusion category with nonzero dimension (see Example 5.10), then the (2+1)-TQFT associated with $\mathcal{C}$ by Theorem 7.4 is isomorphic to the Turaev-Viro TQFT associated with $\mathcal{C}$.

The next corollary is a direct consequence of Theorems 5.18 and 7.4:

Corollary 6.12. *Any spherical tensor category over an algebraically closed field defines a finite dimensional non-compact (2+1)-TQFT.*

The next theorem relates the TQFT $\mathscr{S}$ of with the spherical chromatic invariant $\mathcal{K}_{\mathcal{C}}$ of closed oriented 3-manifolds defined in [15].

Theorem 6.13. *[[13]] Recall, $\mathcal{C}$ is a chromatic category. Let M be a closed connected oriented 3-manifold. Consider $\dot{M} = M \setminus \mathrm{Int}(B^3) \colon S^2 \to \emptyset$ and $\ddot{M} = M \setminus \mathrm{Int}(S^0 \times B^3) \colon S^2 \to S^2$. Then*

$$\mathscr{S}(\dot{M}) = \mathcal{K}_{\mathcal{C}}(M) F',$$

where F' is given by (8)*. In particular, if the m-trace of $\mathcal{C}$ is unique (up to scalar multiple, see* [29]*), then $\dim_{\mathbb{K}}(\mathscr{S}(S^2)) = 1$ and so $\mathscr{S}(\ddot{M}) = \mathcal{K}_{\mathcal{C}}(M)\mathrm{id}_{\mathscr{S}(S^2)}$.*

An easy consequence of the previous theorem is the following:

Corollary 6.14. *If the m-trace of $\mathcal{C}$ is unique (up to scalar multiple), then the 3-manifold invariant $\mathcal{K}_{\mathcal{C}}$ is multiplicative with respect to connected sums.*

Proof. Let M_1, M_2 be closed connected oriented 3-manifolds and denote by $M = M_1 \sharp M_2$ their connected sum. We have: $\ddot{M} = \ddot{M}_1 \circ \ddot{M}_2 \in \mathrm{Cob}^{\mathsf{nc}}$. Then it follows from Theorem 6.13 and the functoriality of $\mathscr{S}$ that $\mathcal{K}_{\mathcal{C}}(\ddot{M})\mathrm{id}_{\mathscr{S}(S^2)} = \mathscr{S}(\ddot{M}) = \mathscr{S}(\ddot{M}_1) \circ \mathscr{S}(\ddot{M}_2) = \mathcal{K}_{\mathcal{C}}(M_1)\mathcal{K}_{\mathcal{C}}(M_2)\mathrm{id}_{\mathscr{S}(S^2)}$. $\boxed{6.14}$

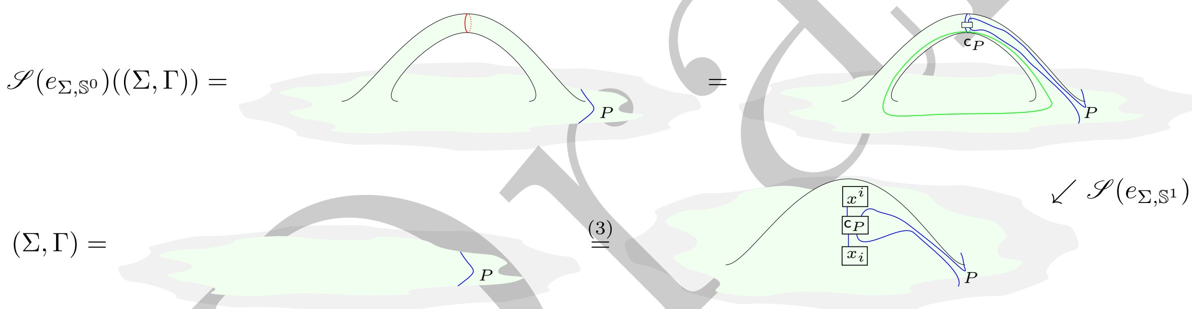
6.7. **Proof of Theorem 7.4.** To prove the first statement of the theorem, we need to show that the relations (R1)-(R5) of Subsection 2.2 are satisfied by $\mathscr{S}$.

(R1) Since $\mathscr{S}\colon$ **Man** $\to \mathrm{Vect}_{\mathbb{K}}$ is functorial we have $\mathscr{S}(e_{d \circ d'}) = \mathscr{S}(e_d) \circ \mathscr{S}(e_{d'})$. Also, since elements of $\mathscr{S}(\Sigma)$ are defined by graphs up to isotopy we clearly have $\mathscr{S}(e_d) = \mathrm{id}$ if d is isotopic to id_Σ.

(R2) Since the construction of the maps $\mathscr{S}(e_{\Sigma,\mathbb{S}})$ are local, they are covariant under diffeomorphisms of the pair $(\Sigma, \mathbb{S})$.

(R3) Again, since the construction of the maps $\mathscr{S}(e_{\Sigma,\mathbb{S}})$ are local, they commute for disjoint framed spheres.

(R4) The 1-2 handle cancellation reduces to the chromatic identity (3) as shown in the following picture:

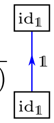


Here, Γ is a skein element in the surface Σ with an edge colored by $P \in \mathsf{Proj}_{\mathcal{C}}$. On the top left we depict the result of a $\mathscr{S}(e_{\Sigma,\mathbb{S}^0})$ move which is cancelled then by a $\mathscr{S}(e_{\Sigma,\mathbb{S}^1})$ (diagonal arrow) where the $\mathbb{S}^1$ is the green curve on the top right hand side. The bottom equality reduces to Equation (3) for $Q = \mathbb{1}$ after rotating the coupons colored with the dual basis and applying the duality property of Lemma 5.5.

The 2-3 handle cancellation reduces to a skein relation which replaces a skein in a disk whose image by F is $f \in \mathrm{Hom}(\mathbb{1}, P)$ by a unique coupon colored by $\sum_i \mathsf{t}_P(f x^i) x_i = f$.

(R5) As stated in the proof of Lemma 6.10 interchanging the disks D and D' does not change the map $\mathscr{S}(e_{\Sigma,\mathbb{S}^0})$. This implies that (R5) is satisfied for any framed 0-sphere. Similarly, in the proof of Lemma 6.9 it is shown that the map $\mathscr{S}(e_{\Sigma,\mathbb{S}^1})$ does not depend on the orientation of γ, implying that (R5) is satisfied for any framed 1-sphere.

We now prove the second statement of the theorem. Assume that $\mathcal{C}$ is semisimple with nonzero dimension (as a chromatic category, see Section 5.5). To extend $\mathscr{S}$ to a (2+1)-TQFT, we first need to assign the value under $\mathscr{S}$ for the generator $e_{\Sigma,0}\colon \Sigma \to \Sigma(0) = \Sigma \cup S^2$ where Σ is an oriented closed surface. Let Γ be an admissible graph in Σ. Consider the graph γ in S^2 defined by

$$\gamma = \frac{1}{\dim(\mathcal{C})}$$  ,

where $\dim(\mathcal{C})$ is the dimension of $\mathcal{C}$. Then

$$\mathscr{S}(e_{\Sigma,0})(\Gamma) = \Gamma \cup \gamma \in \mathscr{S}(\Sigma(0))$$

only depends on the class of Γ in $\mathscr{S}(\Sigma)$. Next we need to verify that the relation (R4) is satisfied for 0-1-handle cancellation: the result of a 0-handle followed by a cancelling 1-handle sends a skein $\Gamma \in \mathscr{S}(\Sigma)$ to the same graph union the graph γ encircled by a red unknot. Now an admissible skein relation replaces the encircled γ with $\frac{1}{\dim(\mathcal{C})} \operatorname{tr}_{\mathcal{C}}(\mathsf{c}_{\mathbb{1}}) = 1$.

Conversely, assume that $\mathcal{C}$ is not semisimple or is semisimple with dimension zero. We will prove that the 3d-pants cobordism $M\colon S^2 \sqcup S^2 \to S^2$ given by a 3-ball minus two smaller 3-balls is sent to 0 by $\mathscr{S}$. As a consequence, since the cobordism M has a right inverse in Cob given by $e_{\mathrm{id}_{S^2}} \sqcup B^3\colon S^2 \to S^2 \sqcup S^2$ and since $\mathrm{id}_{\mathscr{S}(S^2)} \neq 0$, this implies that $\mathscr{S}$ can not be extended to a functor with domain the category Cob. To compute $\mathscr{S}(M)$, we remark that M is given by gluing a unique 1-handle to the cylinder over $S^2 \sqcup S^2$, that is, $M = W(\mathbb{S}^0) = ((S^2 \sqcup S^2) \times [0,1]) \cup_{\mathbb{S}^0} (D^2 \times D^1)$. The the $\mathbb{K}$-linear homomorphism $\mathscr{S}(M)\colon \mathscr{S}(S^2 \sqcup S^2) \to \mathscr{S}(S^2)$ defines a map given by $\Gamma_1 \sqcup \Gamma_2 \mapsto \Gamma$ where Γ is the admissible graph in S^2 represented by a red curve at the equator and the graphs Γ_1 and Γ_2 in the upper and lower hemispheres, respectively. We now consider the two cases. First, if $\mathcal{C}$ is not semisimple, then after making the red circle of Γ blue, we obtain the disjoint union of two admissible graphs in S^2 which is skein equivalent to 0. Indeed, any admissible closed graph is sent to 0 by the functor F (given in (4)) associated to a non-semisimple category. Second, if $\mathcal{C}$ is semisimple with dimension zero, then the unit object $\mathbb{1}$ is projective and it can be used to make the red circle of Γ blue. In this case, Γ becomes skein equivalent to $F(\Gamma_1) \operatorname{tr}(\mathsf{c}_{\mathbb{1}})\Gamma_2 = 0$ because $\operatorname{tr}(\mathsf{c}_{\mathbb{1}}) = 0$ (see Section 5.5).

6.8. **Explicit examples.** If $\mathcal{C}$ is a semi-simple modular tensor category then there is a $(3,2)$ RTW TQFT associated to $\mathcal{C}$ which has an anomaly and which associates to a surface S the skein module of any 3-manifold bounding it. In particular the vector space $\mathcal{S}_{\mathcal{C}}(\Sigma)$ is by definition isomorphic to the so-called "skein algebra" of Σ, i.e. to the vector space associated by the RTW-like TQFT associated to $\mathcal{C}$ to $\Sigma \sqcup \overline{\Sigma}$. Hence it is isomorphic to $End(RTW_{\mathcal{C}}(\Sigma))$ which is a simple algebra. (By this observation one can reconstruct the projective action of the mapping class group on $RTW_{\mathcal{C}}(\Sigma)$.) The action of the mapping class group on the torus is given by the conjugation via the matrices of the action on the RTW TQFT, so in particular the S matrix and the T matrix act by conjugation on any matrix. More in general the Dehn twist along a curve $\gamma \subset \Sigma$ induces a matrix T_γ acting on RTW by left multiplication and on $TV(\Sigma)$ via conjugation; the matrix is obtained via skein calculus by superposing a $+1$-framed Kirby colored skein on γ on each element of a basis of $RTW(\Sigma)$.

In particular in the examples of the Ising anyons, for the torus the TV vector space is $End(\mathbb{C}^3)$, for the Fibonacci it is $End(\mathbb{C}^2)$ and for the level $l = 2r$ $U_q(\mathfrak{sl}_2)$ it is $End(\mathbb{C}^{(r-1)})$, where in each case a basis of the module is given by the simple modules coloring the core of $D^2 \times S^1$ with framing given by a fixed longitude of the torus.

7. Building $(4,3)$-TQFTs

Following the same strategy as the previous section, we will now build a family of $(4,3)$-TQFT. This construction is taken from [12].

We start with the algebraic datum of a ribbon category $\mathcal{C}$. To each closed 3-manifold we will associate its skein vector space $\mathcal{S}_{\mathcal{C}}(M)$ which is naturally acted upon by the mapping class group of the manifold. And to each handle glueing we will associate a linear map between the skein spaces of the corresponding manifolds. Clearly, in order for the skein vector space to make sense $\mathcal{C}$ must be ribbon. Then, as before, we start from the top index handles and they will impose exactly the same conditions on $\mathcal{C}$ as in the previous section: to be able to glue handles of index $4,3,2$ it will have to be a chromatic category. Then glueing of 1-handles will impose a new condition: the existence of a "glueing morphism", which by Proposition 5.14 is equivalent to $\mathcal{C}$ being "chromatic non-degenerate". Chromatic non-degenerate categories will then allow non compact TQFTs in dimension 4. In order to further get compact TQFTs they will have to satisfy a further condition, we called "chromatic compact" (see Definition 5.15). Examples of chromatic compact categories are all the "factorizable" ones (e.g. modular tensor categories). But for factorizable ones we will prove that the associated TQFT is a bit trivial in the sense that it is invertible (i.e. all vector spaces for 3-manifolds are 1-dimensional and all the maps are isomorphisms).

7.1. **Construction of TQFT and 4-dimensional invariants.** As before, we consider the functor $\mathscr{S}_{\mathcal{C}} : \mathbf{Man} \to \mathrm{Vect}$ which associates to each oriented 3-manifold its admissible stated skein space and to each embedding the natural linear map associated to it. We extend it to a functor $\mathscr{S}_{\mathcal{C}} : \mathcal{F}(\mathcal{G}^{\mathsf{nc}}) \to \mathrm{Vect}$ (respectively $\mathscr{S}_{\mathcal{C}} : \mathcal{F}(\mathcal{G}) \to \mathrm{Vect}$ if $\mathcal{C}$ is chromatic compact) by assigning to each $\mathbb{S}$-surgery a linear map between skein modules.

Let M be a closed 3-manifold. For $k = 0, \dots, 4$, recall from Subsection 2.1, the cobordism $W(\mathbb{S}^{k-1})$ which is given by gluing a k-handle on $M \times [-1,1]$. Its domain and target are related by a index k-surgery (along a framed sphere $\mathbb{S}^{k-1}$) which can be described using green circles as follows (in what follows the links L and L' are all green and describe two distinct components of M by surgery):

1. index 0-surgery: $M \to M \sqcup S^3$.
2. index 1-surgery: if the gluing $\mathbb{S}^0$ is not contained in a single component of M: $L \sqcup L' \to L \cup L'$; else : $L \to L \cup O$.
3. index 2-surgery: $L \to L \cup$ "*green knot*" arbitrarily linked with L. Alternatively, since the result of a $\mathbb{S}^1$-surgery on a 3-manifold is invertible by another $\mathbb{S}^1$ surgery, then for a well chosen representation of the domain of $W(\mathbb{S}^1)$, its target can be represented as its domain with a green knot removed.
4. index 3-surgery: if the glueing $\mathbb{S}^2$ disconnects a component of M: $L \cup L' \to L \sqcup L'$ where L and L' live in two different hemispheres of S^3; else: $L \mapsto L \setminus O$ where the green unknot bounds a disc disjoint from the other components.
5. index 4-surgery: $M \sqcup S^3 \to M$.

For $k \in \{0, \dots, 4\}$, given a framed sphere $\mathbb{S}^{k-1}$ in M we define a morphism

$$\chi_{M,\mathbb{S}^{k-1}} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M(\mathbb{S}^{k-1}))$$

which will be assigned to the morphism $\mathscr{S}_{\mathcal{C}}(e_{M,\mathbb{S}^{k-1}})$ as follows.

4-handle: Given a framed sphere $\mathbb{S}^3$ in M, the map $\chi_{M,\mathbb{S}^3}$ corresponding to filling of a 3-sphere of $M = M' \sqcup S^3$ is given by

$$(M, \Gamma) \mapsto F'(\Gamma \cap S^3)(M', \Gamma \cap M') \in \mathscr{S}_{\mathcal{C}}(M').$$

3-handle: Given a framed sphere $\mathbb{S}^2$ in M there exists a *cutting map*:

$$\chi_{M,\mathbb{S}^2} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M(\mathbb{S}^2))$$

sending parallel strands passing through the cutting sphere S^2 to the copairing Ω, see Figure 2. We

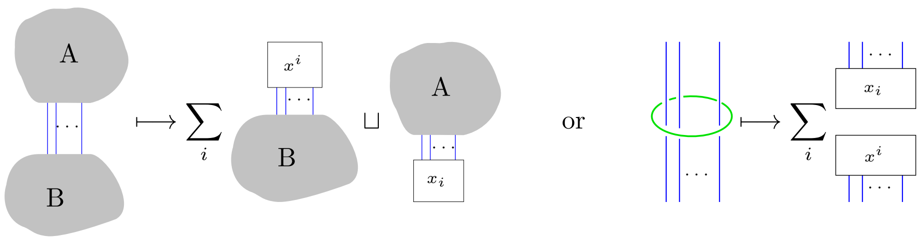


FIGURE 2. The cutting map $\chi_{M,\mathbb{S}^2}$: two representations depending if $\mathbb{S}^2$ is a separating (left) or a non-separating sphere in M (right).

say that the skein is in standard position with respect to $\mathbb{S}^2$ if its intersection consists in n parallel edges in a rectangle (i.e. a disc of the form $\alpha \times [0,1] \subset \mathbb{S}^2 \times [0,1]$ for some simple arc $\alpha \subset \mathbb{S}^2$) with at least one edge colored by a projective module (see Figure 2). We now consider a skein in standard position. Then the image by the RT-functor of this rectangle is the identity of P for some $P \in \mathsf{Proj}$. The cutting map $\chi_{M,\mathbb{S}^2}$ replaces the framed sphere by the sums of graphs in two balls each containing a unique coupon colored with the dual basis of $\mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P)$.

Proposition 7.1. *The linear map $\chi_{M,\mathbb{S}^2}$ is well defined.*

Proof. We refer here to the proof of Lemma 6.9 which is completely similar. The main idea is that the naturality of Ω implies that the images of isotopic skeins are skein equivalent. $\boxed{7.1}$

2-handle: Given a framed sphere $\mathbb{S}^1$ in M there exists a *knot-surgery map*:

$$\chi_{M,\mathbb{S}^1} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M(\mathbb{S}^1))$$

adding a red circle along the meridian of the surgery knot, see the r.h.s. of Figure 3. Let $C =$

FIGURE 3. The knot-surgery map $\chi_{M,\mathbb{S}^1}$, two alternative representations: on the left we choose a representation of M where $\mathbb{S}^1$ is a meridian of a green knot; a presentation for $M(\mathbb{S}^1)$ is then obtained by forgetting the green knot in the presentation of M, but the map on skeins consists of adding a red component along that $\mathbb{S}^1$. On the right, the surgery presentation of $M(\mathbb{S}^1)$ is obtained by adding the green circle (which is $\mathbb{S}^1$) and the map on skeins consists in adding also its red meridian.

$-B^2 \times S^1$ where the sign of B^2 means reversing orientation and $O_r \subset C$ be a red ribbon knot of the form $[-0.1, 0.1] \times \{0\} \times S^1$. Let $\mathbb{S}^1 \simeq S^1 \times B^2$ be a framed knot in M, $M' = M \setminus (S^1 \times B^2)$

and $M'' = M' \cup_\partial C$. Let $\mathscr{S}_\mathcal{C}(M') \xrightarrow{i} \mathscr{S}_\mathcal{C}(M)$ and $\mathscr{S}_\mathcal{C}(M') \xrightarrow{i''} \mathscr{S}_\mathcal{C}(M'')$ be the maps induced by the inclusions. We define $\chi_{M,\mathbb{S}^1}$ to be the map that sends a skein $i(T)$ to $i''(T) \cup O_r$. Observe that this map is defined on all $\mathscr{S}_\mathcal{C}(M)$ because each skein in M can be isotoped off C .

Proposition 7.2. *The linear map $\chi_{M,\mathbb{S}^1}$ is well defined.*

Proof. If $T_1, T_2 \in \mathscr{S}_\mathcal{C}(M')$ are such that $i(T_1) = i(T_2)$ then T_1 and T_2 differ by isotopies in M', slidings through meridian discs of C and skein relations which, up to isotopy, can be supposed to be supported in a box disjoint from C. Then $i''(T_1) \sqcup O_r$ and $i''(T_2) \sqcup O_r$ differ by isotopies in $i''(M')$, skein relations in $i''(M')$ and sliding of edges on the created red component O_r, which by Lemma 6.10 preserves the class in $\mathscr{S}_\mathcal{C}(M'')$. $\boxed{7.2}$

1-handle: Given a framed sphere $\mathbb{S}^0$ in M there exists a *gluing map*:

$$\chi_{M,\mathbb{S}^0} : \mathscr{S}_\mathcal{C}(M) \to \mathscr{S}_\mathcal{C}(M(\mathbb{S}^0))$$

which glues two edges terminating on coupons colored by η and ε by a gluing morphism as represented in Figure 4. Let us describe this morphism in more detail. Let x, y be two distinct points

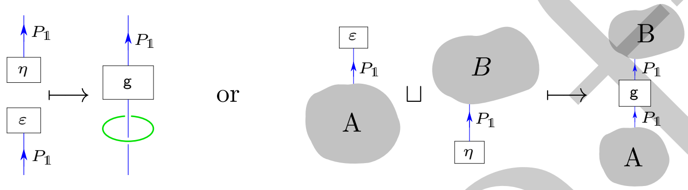


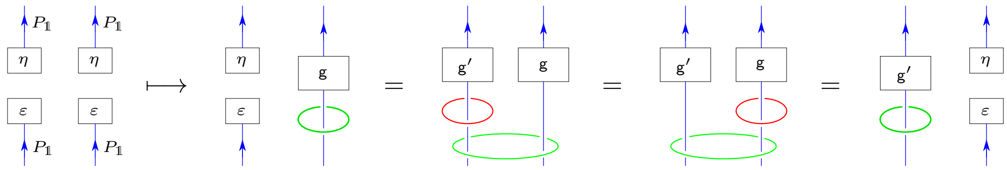
FIGURE 4. The gluing map $\chi_{M,\mathbb{S}^0}$ is depicted by two different representations depending if $\mathbb{S}^0$ is embedded in a unique connected component of M (left) or not (right).

of a 3-manifold M. Let B_x, B_y be neighborhood of x and y both oriented and parameterized by B^3 and let $\mathbb{S}^0$ be the framed 0-sphere $B_x \sqcup B_y$. Let $M' = M \setminus (B_x \sqcup B_y) \overset{i}{\hookrightarrow} M$ be the inclusion and $C \simeq S^2 \times [0,1]$ be the cylinder such that $M(\mathbb{S}^0) = M' \cup_\partial C$. We put in this cylinder a skein Γ_g with a single coupon colored by any gluing morphism g and an incoming and an outgoing edge parallel to $(1,0,0) \times [0,1]$, framed in the direction $(0,0,1)$. We will say that a skein T in M is in good position with respect to $\mathbb{S}^0$ if $B_x \cap T$ consists of a planar ribbon graph in $\mathbb{R}^+ \times \mathbb{R} \times \{0\} \cap B_x$ consisting of a unique edge oriented from $(1,0,0) \in \partial B_x$ towards a coupon colored by ε and if $B_y \cap T$ consists of a planar ribbon graph in $\mathbb{R}^+ \times \mathbb{R} \times \{0\} \cap B_y$ consisting of a unique edge oriented from a coupon colored by η towards $(1,0,0) \in \partial B_y$. The map $\chi_{M,\mathbb{S}^0}$ assigns to a skein T in good position with respect to $\mathbb{S}^0$ the skein $(M', T \cap M') \cup_\partial (C, \Gamma_\mathsf{g})$.

Proposition 7.3. *The linear map $\chi_{M,\mathbb{S}^0}$ is well defined and does not depend on the ordering of $\{x, y\}$ nor on the gluing morphism* g.

Proof. First we note that the admissible skein module is generated by skeins in M where every component of M contains a coupon colored by ε and a coupon colored by η. Indeed, consider a box containing a part of an edge colored by $P \in \mathsf{Proj}$ whose image by the RT-functor is $\overleftarrow{\mathrm{ev}}_P$; up to applying a skein relation one can make appear a coupon colored by $\varepsilon : P_\mathbb{1} \to \mathbb{1}$. Let us choose an isomorphism $\psi : P_\mathbb{1} \to P_\mathbb{1}^*$ normalized so that $\eta^* \circ \psi = \varepsilon$ then a coupon colored by ε is skein equivalent to a graph with two coupons colored by ψ and η. So applying this procedure twice we can ensure the presence of a ϵ-colored coupon and of a η-colored coupon in each connected component of M.

Now, up to isotopy of the skein, the definition of $\chi_{M,\mathbb{S}^0}$ only depends a priori on the choice of the two coupons colored by ε and η, and on the choice of a gluing morphism g: we will now prove independence on these data. Let g' be an other gluing morphism and consider the element obtained by using g' instead of g and two different coupons colored with ε and η. Then we have if $\mathbb{S}^0$ is embedded in a unique connected component,



where the first and last equalities are skein equivalences given by definition of gluing morphisms and the middle one is an isotopy of the red circle in the belt 2-sphere created by gluing the 1-handle. Similarly, if the surgery is connecting two different components of M, the representation of the equivalence is similar without the green circles but with the separating belt 2-sphere represented by the horizontal plane.

The map $\chi_{M,\mathbb{S}^0}$ preserves skein relations as we can always choose coupons ε and η outside a fixed box.

Finally reversing the orientation of the sphere $\mathbb{S}^0$ that is interchanging x and y does not change the map since $\eta = \psi^{-1}\varepsilon^*$, $\varepsilon = \psi\eta^*$ and $\psi^{-1}\mathsf{g}^*\psi$ is also a gluing morphism. $\boxed{7.3}$

0-handle: We only consider 0-handles when $\mathcal{C}$ is chromatic compact and so $\mathsf{g} = \zeta^{-1}\mathrm{id}_{P_{\mathbb{1}}}$ is a gluing morphism. Let $\mathbb{S}^{-1} : \emptyset \hookrightarrow M$ be a framed -1-sphere. Let Γ_0 be the ribbon graph with a unique edge from a coupon colored with η to a coupon colored by ε. Then there exists a *birth map*:

$$\chi_{M,\mathbb{S}^{-1}} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M \sqcup S^3)$$

sending a skein in M to its disjoint union with $(S^3, \zeta\Gamma_0)$, see Figure 5.

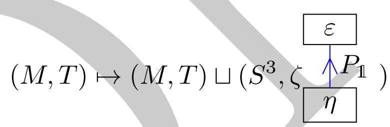


FIGURE 5. The birth map $\chi_{M,\mathbb{S}^{-1}}$ augments a skein by adding a disjoint union of S^3 containing $\zeta\Gamma_0$.

Theorem 7.4. *There exists a unique symmetric monoidal functor*

$$\mathscr{S}_{\mathcal{C}} : \mathsf{ncCob} \to \mathrm{Vect}$$

extending $\mathscr{S}_{\mathcal{C}} : \boldsymbol{Man} \to \mathrm{Vect}$ *such that* $\mathscr{S}_{\mathcal{C}}(e_{\Sigma,\mathbb{S}}) = \chi_{\Sigma,\mathbb{S}}$.
If $\mathcal{C}$ *is chromatic compact, then the functor extends to a symmetric monoidal functor on* Cob*:*

$$\mathscr{S}_{\mathcal{C}} : \mathrm{Cob} \to \mathrm{Vect}.$$

Proof. We only need to prove that the relation (R1)–(R5) are satisfied by $\mathscr{S}_{\mathcal{C}}$.

(R1) Since $\mathscr{S}_\mathcal{C} : \mathbf{Man} \to \mathrm{Vect}$ is functorial we have $\mathscr{S}_\mathcal{C}(e_{d\circ d'}) = \mathscr{S}_\mathcal{C}(e_d) \circ \mathscr{S}_\mathcal{C}(e_{d'})$. Also, since elements of $\mathscr{S}_\mathcal{C}(M)$ are defined by ribbon graphs up to isotopy we clearly have $\mathscr{S}_\mathcal{C}(e_d) = \mathrm{id}$ if d is isotopic to id_Σ.

(R2) Since the construction of the maps $\chi_{M,\mathbb{S}}$ are local, they are covariant under diffeomorphisms of the pair $(M, \mathbb{S})$.

(R3) Again, since the construction of the maps $\chi_{M,\mathbb{S}}$ are local, they commute for disjoint framed spheres.

(R4) The 2-3-handle cancellations reduces to the chromatic identity (3) as shown in Figure 6. Indeed since the attaching framed 2-sphere of the 3-handle intersects the belt circle of the 2-handle once, the attaching circle for the 2-handle bounds a disc in the intermediate 3-manifold. This is why we can represent the green circle in Figure 6 as an unknot.
The 1-2-handle cancellations reduces to the defining property of the gluing map. Indeed the sphere S^2 created by the 1-handle can't be separating since it is intersected once by the attaching $\mathbb{S}^1$ of the 2-handle. This means that we can represent the map $\chi_{M,\mathbb{S}^1}$ as in the left hand-side of Figure 4 and the map $\chi_{M,\mathbb{S}^2}$ is then the left hand-side of Figure 3 turning the green unknot into red.
The 3-4-handle cancellation relies on the fact that evaluating F' on a cut 3-ball is a skein relation.
Finally, in the compact case, the 0-1-handle cancellation is obvious since we can choose $\mathsf{g} = \zeta^{-1}\mathrm{id}_{P_\mathbb{1}}$ as gluing morphism.

(R5) The maps $\chi_{M,\mathbb{S}}$ do not depend on the orientation of $\mathbb{S}$.

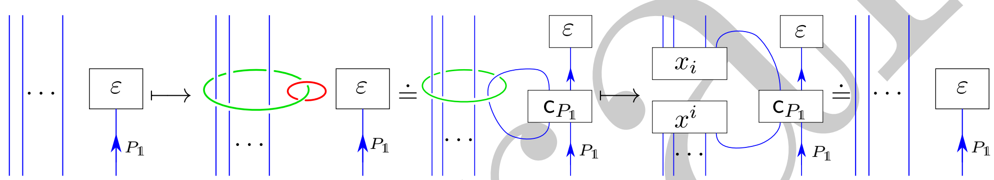


FIGURE 6. The cancellation of a 2-handle by a 3-handle.

$\boxed{7.4}$

7.2. **The four manifold invariant.** We now extract (even in the non-compact case) two scalar invariants of 4-manifolds: $\mathscr{S}_\mathcal{C}(W,T)$ for manifolds with an admissible graph in the boundary and $\dot{\mathscr{S}}_\mathcal{C}(W)$ for connected closed 4-manifolds.

Definition 7.5. Let W be an oriented compact 4-manifolds with no closed components. A $\mathcal{C}$-ribbon graph $T \subset (-\partial W)$ is *admissible* if for each component M of $-\partial W$, $T \cap M$ is admissible i.e. if T represents an admissible skein of $\mathscr{S}_\mathcal{C}(-\partial W)$ (where the minus sign is for opposite orientation). If $T \subset (-\partial W)$ is admissible then define the invariant

$$\mathscr{S}_\mathcal{C}(W,T) = \mathscr{S}_\mathcal{C}(\widetilde{W})(T)$$

where $\widetilde{W}$ is W seen as a cobordism from $-\partial W$ to $\emptyset$.

Definition 7.6. Let W be a connected closed 4-manifold. Define

$$\dot{\mathscr{S}}_\mathcal{C}(W) = \mathscr{S}_\mathcal{C}(\dot{W}, \Gamma_0) \in \mathbb{K}$$

where $\dot{W} = W \setminus B^4$ is a once punctured W.

If $\mathcal{C}$ is chromatic compact, by definition of the maps $\chi_{M,\mathbb{S}^0}$, we have for any closed connected 4-manifold W:

$$\mathscr{S}_{\mathcal{C}}(W) = \zeta \dot{\mathscr{S}}_{\mathcal{C}}(W)\mathrm{id}_{\mathbb{K}}. \tag{10}$$

For example, $\dot{\mathscr{S}}_{\mathcal{C}}(S^4) = 1$ and $\mathscr{S}_{\mathcal{C}}(S^4) = \zeta \mathrm{id}_{\mathbb{K}}$.

7.3. **Properties.** Assume that $\mathcal{C}$ is chromatic non-degenerate so that in particular it has an m-trace t, chromatic morphism c and gluing morphism g.

Proposition 7.7. *Let $\kappa \in \mathbb{K}^*$, then*

(1) $\mathsf{t}' := \kappa\mathsf{t}$ *is a non-degenerate m-trace on* Proj,
(2) *its associated copairing is given by* $\Omega'_P = \frac{1}{\kappa}\Omega_P$, *and* $\Gamma'_0 = \frac{1}{\kappa}\Gamma_0$,
(3) $\mathsf{c}' = \kappa\mathsf{c}$ *is a chromatic morphism associated to* t',
(4) $\mathsf{g}' = \frac{1}{\kappa^2}\mathsf{g}$ *is a gluing morphism, and in the compact case* $\zeta' = \kappa^2\zeta$.

Finally the TQFT $\mathscr{S}'_{\mathcal{C}}$ associated to t' satisfies $\mathscr{S}'_{\mathcal{C}}(W) = \kappa^{\chi(W)}\mathscr{S}_{\mathcal{C}}(W)$ where χ is the Euler characteristic.

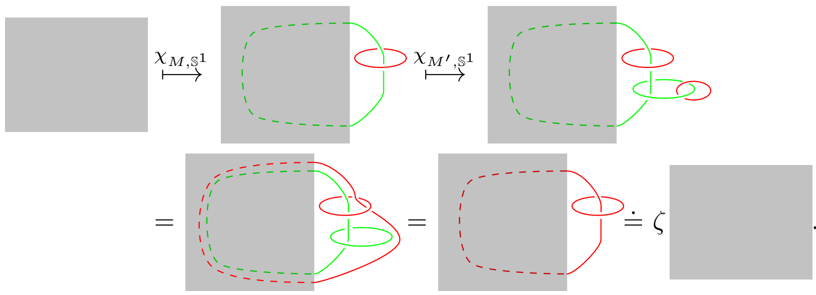
Proof. The first four points are immediate from the definitions. In the compact case, the 0-handle map becomes $\chi'_{M,\mathbb{S}^{-1}} = \kappa\chi_{M,\mathbb{S}^{-1}}$, as $\zeta'\Gamma'_0 = \kappa\zeta\Gamma_0$. The 1-handle map becomes $\chi'_{M,\mathbb{S}^0} = \frac{1}{\kappa}\chi_{M,\mathbb{S}^0}$ as it maps a $\Omega'_{P_{\mathbb{1}}}$ to a g'. The 2-handle map becomes $\chi'_{M,\mathbb{S}^0} = \kappa\chi_{M,\mathbb{S}^0}$ as $\mathsf{c}' = \kappa\mathsf{c}$. The 3-handle map becomes $\chi'_{M,\mathbb{S}^0} = \frac{1}{\kappa}\chi_{M,\mathbb{S}^0}$ as $\Omega'_P = \frac{1}{\kappa}\Omega_P$. The 4-handle map becomes $\chi'_{M,\mathbb{S}^0} = \kappa\chi_{M,\mathbb{S}^0}$ as $\mathsf{t}' = \kappa\mathsf{t}$. Therefore for a 4-bordism W decomposed using n_i i-handles, $0 \le i \le 4$, one has:

$$\mathscr{S}'_{\mathcal{C}}(W) = \kappa^{n_4-n_3+n_2-n_1+n_0}\mathscr{S}_{\mathcal{C}}(W) = \kappa^{\chi(W)}\mathscr{S}_{\mathcal{C}}(W). \qquad \boxed{7.7}$$

Theorem 7.8. *The TQFT $\mathscr{S}_{\mathcal{C}}$ is invertible if and only if $\mathcal{C}$ is factorizable (i.e. the only transparent objects are direct sums of $\mathbb{1}$).*

Proof. First we prove the necessity of the theorem: recall that $\mathscr{S}_{\mathcal{C}}(S^3) \simeq \mathbb{K}$ is generated by the skein (S^3, Γ_0). Let G be a projective generator with a unique indecomposable factor $P_{\mathbb{1}} \xrightarrow{i} G \xrightarrow{p} P_{\mathbb{1}}$. Then by naturality of Λ and since G contains a single copy of P_1, we have $\Lambda_G = i\Lambda_{P_{\mathbb{1}}}p$. Consider the subspace of $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ generated by graphs $\{O_f\}_{f\in \mathrm{End}_{\mathcal{C}}(G)}$ with a unique coupon colored by $f \in \mathrm{End}_{\mathcal{C}}(G)$ and a unique edge of the form $\{pt\} \times S^1$. Consider the two cobordisms $W_2, W_3 : S^2 \times S^1 \to S^3$ given by gluing a 2-handle (resp. a 3-handle) to $S^2 \times S^1 \times [0,1]$ respectively along $\{pt\} \times S^1 \times \{1\}$ and $S^2 \times \{pt\} \times \{1\}$. Then $\mathscr{S}_{\mathcal{C}}(W_2)(O_f) = \mathsf{t}_G(\Delta_0^G f) \in \mathbb{K} \simeq \mathscr{S}_{\mathcal{C}}(S^3)$ and $\mathscr{S}_{\mathcal{C}}(W_3)(O_f) = \mathsf{t}_G(\Lambda_G f) \in \mathbb{K} \simeq \mathscr{S}_{\mathcal{C}}(S^3)$. In particular, for the gluing morphism g, $\mathscr{S}_{\mathcal{C}}(W_2)(O_{i\mathsf{g}p}) = 1 = \mathscr{S}_{\mathcal{C}}(W_3)(O_{\mathrm{id}}) = 1$ so the two maps are non-zero. If $\mathscr{S}_{\mathcal{C}}$ is invertible, $\dim_{\mathbb{K}}(S^2 \times S^1) = 1$ and there exists $\zeta \in \mathbb{K}^*$ such that $\mathscr{S}_{\mathcal{C}}(W_3) = \zeta\mathscr{S}_{\mathcal{C}}(W_2)$. Then for any $f \in \mathrm{End}_{\mathcal{C}}(G)$, $\mathsf{t}_G(\Delta_0^G f) = \mathscr{S}_{\mathcal{C}}(W_3)(O_f) = \zeta\mathscr{S}_{\mathcal{C}}(W_2) = \zeta\mathsf{t}_G(\Lambda_G f)$. Finally, by non-degeneracy of the m-trace, $\Delta_0^G = \zeta\Lambda_G$.

Now we prove the sufficiency of the theorem: we suppose $\mathcal{C}$ is factorizable and we show that for any connected 3-manifold M, $\dim_{\mathbb{K}}(\mathscr{S}_{\mathcal{C}}(M)) = 1$. This is true because the image of any 1-surgery given by a 2-handle can be inverted:

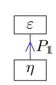


Here the first map is the image of any S^1-surgery on M and the second is the image of an appropriate second S^1-surgery; the first equality is an isotopy in the manifold obtained by sliding the second red curve along the first green curve, the second equality comes from the fact that topologically on the level of the 3-manifolds, a surgery along a meridian of a $\mathbb{S}^1$-surgery component cancels both components and the last equivalence is a skein equivalence due to the factorizability of $\mathcal{C}$.

Then we check that every cobordism induces an isomorphism. It is immediate from the definition that 0-handles and 4-handles are isomorphisms. The proof for 2-handles is given above. Note that a 1-handle followed by a 3-handle glued on the belt sphere created by the 1-handle is a scalar times the identity. Indeed, the 1-handle will introduce a gluing morphism (which is a scalar times the identity of $P_{\mathbb{1}}$ by assumption) from a pair of coupons ε and η. Then the 3-handle will cut it, turning it back to a pair of coupons ε and η. This shows that 1-handles are injective and 3-handles surjective. Because every skein module is 1-dimensional, they are also bijective. $\boxed{7.8}$

Proposition 7.9. *Behavior under connected sums:*

- *The invariant of closed connected 4-manifolds $\dot{\mathscr{S}}_{\mathcal{C}}(W)$ is multiplicative under connected sum.*
- *If W is a closed connected 4-manifold and $W' : M' \to N' \in \mathrm{Cob}^{nc}$ (resp. $W' \in \mathrm{Cob}$ if $\mathcal{C}$ is chromatic compact), both non-empty, then*
$$\mathscr{S}_{\mathcal{C}}(W\#W') = \dot{\mathscr{S}}_{\mathcal{C}}(W)\mathscr{S}_{\mathcal{C}}(W') \in \mathrm{Hom}_{\mathbb{K}}(\mathscr{S}_{\mathcal{C}}(M'), \mathscr{S}_{\mathcal{C}}(N')).$$
- *For non-empty 4-manifolds W, W' containing admissible graphs T, T' in their boundaries,*
$$\mathscr{S}_{\mathcal{C}}(W\#W', T\cup T') = \left| \begin{array}{l} \zeta^{-1}\mathscr{S}_{\mathcal{C}}(W,T)\mathscr{S}_{\mathcal{C}}(W,T) \text{ if } \mathcal{C} \text{ is chromatic compact,} \\ 0 \text{ else;} \end{array}\right.$$
- *If $\mathcal{C}$ is chromatic compact, for two non-empty 4-cobordisms $W : M \to N$ and $W' : M' \to N'$,*
$$\mathscr{S}_{\mathcal{C}}(W\#W') = \zeta^{-1}\mathscr{S}_{\mathcal{C}}(W) \otimes \mathscr{S}_{\mathcal{C}}(W') : \mathscr{S}_{\mathcal{C}}(M) \otimes \mathscr{S}_{\mathcal{C}}(M') \to \mathscr{S}_{\mathcal{C}}(N) \otimes \mathscr{S}_{\mathcal{C}}(N').$$

Proof. The admissible skein module $\mathscr{S}_{\mathcal{C}}(S^3)$ is one dimensional and generated by $\Gamma_0 =$ 

. For a closed connected 4-manifold W the twice punctured cobordism $\mathscr{S}_{\mathcal{C}}(\ddot{W}) : \mathscr{S}_{\mathcal{C}}(S^3) \to \mathscr{S}_{\mathcal{C}}(S^3)$ acts as multiplication by the scalar $\dot{\mathscr{S}}_{\mathcal{C}}(W)$. Composition corresponds to connected sum for the twice-punctured cobordisms, and to multiplication for the scalars. The second point is obtained by adding a cancelling pair of 3 and 4-handles to W'. Then connected sum with W precomposes by $\mathscr{S}_{\mathcal{C}}(\ddot{W})$ before the 4-handle, hence simply multiplies by $\dot{\mathscr{S}}_{\mathcal{C}}(W)$.
Let $P : S^3 \sqcup S^3 \to S^3$ be the three dimensional pair of pants, namely a 3-punctured S^4 which

can be seen as a unique 1-handle. The cobordism $(W\dot{\#}W') : (-\partial W) \sqcup (-\partial W') \to S^3$ factors as $W\dot{\#}W' = P \circ (\dot{W} \sqcup \dot{W}')$.

The map $\mathscr{S}_{\mathcal{C}}(P) : \mathbb{K} \otimes \mathbb{K} = \mathbb{K} \to \mathbb{K}$ is a scalar morphism which sends $\Gamma_0 \otimes \Gamma_0$ to the unique graph with 3 coupons colored by η, g and ε. Since $\varepsilon \circ \mathsf{g} = 0$ unless g is invertible (i.e. $\mathcal{C}$ is chromatic compact by Lemma 5.16), the second case follows. Let's now assume that $\mathcal{C}$ is chromatic compact and let us use $\mathsf{g} = \zeta^{-1}\mathrm{id}_{P_{\mathbb{1}}}$ for the gluing morphism. Then $\mathscr{S}_{\mathcal{C}}(W\#W', T \cup T') = \mathscr{S}_{\mathcal{C}}(W,T)\mathscr{S}_{\mathcal{C}}(W',T')F'(\mathscr{S}_{\mathcal{C}}(P)(\Gamma_0 \otimes \Gamma_0)) = \zeta^{-1}\mathscr{S}_{\mathcal{C}}(W,T)\mathscr{S}_{\mathcal{C}}(W',T')$.

For the last statement, since every object of Cob is dualizable we can suppose that $N = N' = \emptyset$. Then the statement follows from the previous identity since for any $T \otimes T' \in \mathscr{S}_{\mathcal{C}}(M) \otimes \mathscr{S}_{\mathcal{C}}(M') \cong \mathscr{S}_{\mathcal{C}}(-\partial(W \sqcup W'))$, we have $\mathscr{S}_{\mathcal{C}}(W\#W')(T \otimes T') = \mathscr{S}_{\mathcal{C}}(W\#W', T \cup T')$. $\boxed{7.9}$

Proposition 7.10. *The category $\mathcal{C}$ is chromatic compact if and only if $\dot{\mathscr{S}}_{\mathcal{C}}(S^1 \times S^3) \neq 0$.*

Proof. A handle decomposition of the punctured bordism $S^1 \dot{\times} S^3 : S^3 \to \emptyset$ is given by a 1-handle followed by a 3-handle glued on its belt sphere and a closing 4-handle. The skein Γ_0 is sent to a circle with a coupon g in $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ which is then cut into the closure of $\mathsf{g} \circ \Lambda_{P_{\mathbb{1}}}$ in $\mathscr{S}_{\mathcal{C}}(S^3)$. This is non-zero if and only if g is invertible. The statement follows then by Lemma 2.8 in [12] which shows that there exists an invertible glueing morphism for $\mathcal{C}$ iff $\mathcal{C}$ is chromatic compact. $\boxed{7.10}$

Proposition 7.11. *If $\mathcal{C}$ is twist non-degenerate or if $\dot{\mathscr{S}}_{\mathcal{C}}(S^2 \times S^2) \neq 0$ then $\mathscr{S}$ does not distinguish exotic pairs of cobordisms.*

Proof. Since $\dot{\mathscr{S}}_{\mathcal{C}}(\pm\mathbb{CP}^2) = \Delta_{\pm}$, the category is twist non-degenerate if and only if $\dot{\mathscr{S}}_{\mathcal{C}}(\mathbb{CP}^2)\dot{\mathscr{S}}_{\mathcal{C}}(-\mathbb{CP}^2) \neq 0$. Gompf ([31]) showed that two homeomorphic compact orientable 4-manifolds (possibly with boundary) become diffeomorphic after some finite sequence of connected sums with $S^2 \times S^2$; the same is true for connected sums with complex projective planes (or their opposites) since $(S^2 \times S^2)\#\mathbb{CP}^2$ is diffeomorphic to $\mathbb{CP}^2\#\mathbb{CP}^2\#(-\mathbb{CP}^2)$. The statement then follows from Proposition 7.9. $\boxed{7.11}$

7.4. **Examples.** Let $H = \mathfrak{u}_q^{m,n}(\mathfrak{sl}_2)$ be the Hopf algebra of Example 5.24.

Proposition 7.12 ([12], Prop. 6.4)**.** *The category $\mathcal{C} = H - mod$ is chromatic compact. It is factorizable if and only if $m = n$ and both n and r are odd. It is twist degenerate if and only if n is odd and r is a multiple of 4.*

Proposition 7.13 ([12], Prop. 6.5)**.** *For n odd and $4|r$, the (3+1)-TQFT $\mathscr{S}_{\mathcal{C}}$ distinguishes the closed 4-manifolds $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, which have same signature, Euler characteristic and fundamental groups but different spin status. One has:*

$$\mathscr{S}_{\mathcal{C}}(S^2 \times S^2) = \frac{m^3 n \ gcd(nr,2)\{1\}^{8(r-1)}}{c^4 r^2} \quad \textit{and} \quad \mathscr{S}_{\mathcal{C}}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}) = 0$$

We give an example of a category which is chromatic non-degenerate but not chromatic compact, and therefore gives a non-compact TQFT. The example we give is very simple and unlikely to give interesting 4-manifold invariant, but the TQFT already shows some very interesting features. Its associated algebra on $S^2 \times S^1$ is non-semisimple, so it does not fall under Reutter's theorem [40] showing that semi-simple TQFTs cannot detect exotic structures.

In characteristic p, one may find a cocommutative Hopf algebra H which is non-semisimple but such that H^* is semi-simple. This gives a symmetric monoidal, non-semisimple and chromatic non-degenerate category, therefore with non-semisimple Müger center.

Definition 7.14. Let $\mathbb{K}$ be an algebraically closed field of characteristic p, and $H = \mathbb{K}[\mathbb{Z}/p\mathbb{Z}]$. Denote α the generator of $\mathbb{Z}/p\mathbb{Z}$. Let $\mathcal{C} = H\text{-mod}^{fd}$ be the symmetric monoidal category of finite dimensional left H-modules.

Proposition 7.15. *The category $\mathcal{C}$ is chromatic non-degenerate, but not chromatic compact. It gives a non-compact TQFT $\mathscr{S}_{\mathcal{C}}$.*

Proof. The cointegral is $\Lambda = \sum_{i=0}^{p-1} \alpha^i$, and the right integral is $\lambda = 1^*$ in the basis $(1, \alpha, \dots, \alpha^{p-1})$. We observe indeed that $\varepsilon(\Lambda) = p = 0$ whereas $\lambda(1) = 1 \neq 0$, so H^* is semi-simple whereas H is not. One computes the central element $\Delta_0 = \lambda(1)1 = 1 \in H$, thus $\Delta_0^P = \mathrm{id}_P$ for any projective. Therefore, the gluing morphism g is given by $\Lambda_{P_{\mathbb{1}}}$ which is not invertible as $\mathcal{C}$ is non-semisimple. $\boxed{7.15}$

Note that $\mathsf{g} = \Lambda_{P_{\mathbb{1}}}$ means that the 1-handle map does not affect the skein. Similarly, $\mathcal{C}$ being symmetric and $\lambda(1) = 1$ implies that a homotopically-trivial red links can be ignored.
As explained in [40], the vector space $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ has a natural algebra structure induced by the cobordism $\overset{\dots}{S}^3 \times S^1$ where $\overset{\dots}{S}^3$ is the thrice-punctured sphere. Note that this algebra is non-unital as the TQFT is non-compact.

Proposition 7.16. *The non-unital algebra $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ is non-semisimple (i.e. it is non-semisimple if one freely adjoins a unit).*

Proof. For $f : P \to P$ an endomorphism of a projective object, denote O_f the skein $\{pt\} \times S^1 \subseteq S^2 \times S^1$ colored by P with a single coupon f. The skein module of $S^2 \times S^1$ is generated by the O_f's. As the braiding and twist are trivial, the only relation is cyclicity: $O_{f\circ g} = O_{g \circ f}$ for $f : P \to Q$ and $g : Q \to P$. A handle decomposition of $\overset{\dots}{B}^3 \times S^1$ is given by a single 1-handle and a single 2-handle, both of which doesn't affect the skeins. The algebra structure is given by $O_f.O_g = O_{f\otimes g}$.
As H is a projective generator of the category, one can restrict to $P = H$ for the generators of $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$. Furthermore endomorphisms of H are right multiplications by elements of H, so since H is commutative, the cyclic relations are trivial. So $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ is isomorphic to $End_{\mathcal{C}}(H) \simeq H$ as a vector space, with basis the $O_i := O_{-.\alpha^i}$'s. To compute their product, we need to decompose $H \otimes H = \oplus_{k=0}^{p-1} H.(1 \otimes \alpha^k)$. Then $O_i.O_j$ is multiplication by $\alpha^i \otimes \alpha^j$ on $H \otimes H$. It maps $1 \otimes \alpha^k$ to $\alpha^i \otimes \alpha^{k+j}$ which is in the $k + j - i$ summand. We get $O_i.O_j = \sum_{k=0}^{p-1} \delta_{i,j} O_i = p\delta_{i,j} O_i = 0$.
If one freely adjoins a unit to $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ one gets the non-semisimple $(p+1)$-dimensional algebra $\mathbb{K}[O_0, O_1, \dots, O_{p-1}]/(O_i.O_j = 0)$. $\boxed{7.16}$

8. Projective RTW and RDGGP theories

Let us fix a factorizable chromatic category $\mathcal{C}$: by the previous construction associated to it is an invertible $(4, 3)$-TQFT $Z_{\mathcal{C}}$. We now build a different kind of $(3, 2)$ TQFT which we call of RTW type. In particular it will not take values in Vect but in PVect : the category whose objects are finite dimensional vector spaces and morphisms are linear maps up to a non zero scalar. To each surface Σ let's associate the vector space $RT(\Sigma) = \mathscr{S}_{\mathcal{C}}(M_\Sigma)$ where M_Σ is any connected 3-manifold bounding Σ. We have the following :

Lemma 8.1. *If M, M' are connected oriented 3-manifolds bounding Σ, then there exists an isomorphism $\phi : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M')$.*

Proof. It is a standard fact that M' is obtained by surgery on a link L in M. Then the map is obtained by moving each skein in $\mathscr{S}_{\mathcal{C}}(M)$ so that it avoids L and then coloring the meridians of L

in red. Its inverse is given by coloring L in red (the proof of the fact that this is indeed the inverse is in Theorem 7.8). $\boxed{8.1}$

Fix a surface Σ and a connected oriented 3-manifold M_Σ bounding it. We need to assign values to the generators of $\mathrm{Cob}^{\mathsf{nc}}$. In particular, given a framed sphere $\mathbb{S}$ in Σ, we need to assign a $\mathbb{K}$-linear homomorphism $\mathscr{S}(e_{\Sigma,\mathbb{S}})\colon \mathscr{S}(M_\Sigma) \to \mathscr{S}(M_{\Sigma(\mathbb{S})})$. This is quite simple : notice that $e_{\Sigma,\mathbb{S}}$ consists in glueing a $3d$-handle on Σ so that we can let $e_{\Sigma,\mathbb{S}}(M_\Sigma)$ be the 3-manifold obtained by glueing the handle on the boundary of M; furthermore observe that since M_Σ is connected, then also $e_{\Sigma,\mathbb{S}}(M_\Sigma)$ is. So the induced map on the level of skeins is simply induced by the inclusion $i : M_\Sigma \hookrightarrow e_{\Sigma,\mathbb{S}}(M_\Sigma)$. If M'_Σ is another connected 3-manifold bounding Σ and $L \subset M_\Sigma$ be a framed link such that $M' = M(L)$. Then L induces both a linear map $\phi : \mathscr{S}(M) \to \mathscr{S}(M')$ and a linear map $e_{\Sigma,\mathbb{S}}(\phi) : e_{\Sigma,\mathbb{S}}(M_\Sigma) \hookrightarrow e_{\Sigma,\mathbb{S}}(M'_\Sigma)$ induced as in Lemma 8.1. So overall we have the following:

Lemma 8.2. *The following diagram of linear maps commutes:*

$$\begin{tikzcd}
\mathscr{S}(M_\Sigma) \arrow[r, "\mathscr{S}(e_{\Sigma,\mathbb{S}})"] \arrow[d, "\mathscr{S}(e_{\Sigma,\mathbb{S}'})"'] & \mathscr{S}(e_{\Sigma,\mathbb{S}}(M_\Sigma)) \arrow[d, "\mathscr{S}(e_{\Sigma(\mathbb{S}),\mathbb{S}'})"] \\
\mathscr{S}(e_{\Sigma,\mathbb{S}'}(\mathscr{S}(M_\Sigma)) \arrow[r, "\mathscr{S}(e_{\Sigma(\mathbb{S}'),\mathbb{S}})"] & \mathscr{S}(e_{\Sigma,\mathbb{S},\mathbb{S}'}(M_\Sigma))
\end{tikzcd}$$

So the linear map $\mathscr{S}(e_{\Sigma,\mathbb{S}})$ is well defined on $RT(\Sigma)$. Furthermore it is clear that the handle cancellations hold, simply because they hold at the level of the 3-manifold.

The second set of generators of Cob^{nc} is given by the elements e_d associated to diffeomorphisms $d : \Sigma \to \Sigma$. There are two ways to realise the map $\mathscr{S}(e_d) : RT(\Sigma) \to RT(\Sigma)$. Recall that M_Σ by definition comes with an identification $\phi : \partial M_\Sigma \to \Sigma$ (the "boundary parametrization"). The easiest way to describe $\mathscr{S}(e_d)$ is to let M'_Σ be $d \circ M_\Sigma$, i.e. be the 3-manifold M_Σ with the identification of the boundary $\phi' : \partial M_\Sigma \to \Sigma$ given by $d \circ \phi$. Then one gets a linear map $\mathscr{S}(e_d) : \mathscr{S}(M_\Sigma) \to \mathscr{S}(M'_\Sigma)$ given by the identity on skeins (i.e. a skein in M_Σ is by definition of M'_Σ also a skein in M' : it is only the boundary parametrization which has changed). If one wishes, though, to relate $\mathscr{S}(M_\Sigma)$ and $\mathscr{S}(M'_\Sigma)$ via surgery and then consider $\mathscr{S}(e_d)$ and an endomorphism of $\mathscr{S}(M_\Sigma)$ he needs to first set up the following notation.

By Dehn's theorem d is isotopic to a composition of left Dehn twists (or their inverses) along simple closed curves $\gamma_1, \dots \gamma_n$ in Σ. Then let $T_i = T_{\gamma_i}$ be the Dehn twist along γ_i and let $c_i^\pm \subset \Sigma \times [-1, 1]$ be the framed link consisting in the curve $\gamma_i \times \{0\}$ with framing ± 1. Then if $d = T_{i_n}^{\epsilon_n} T_{i_{n-1}}^{\epsilon_{n-1}} \cdots T_{i_1}^{\epsilon_1}$ let $c = c_{i_n}^{\epsilon_n} \sqcup \cdots \sqcup c_{i_1}^{\epsilon_1}$ where the heights of c_{i_n} is the highest and it progressively decreases until c_{i_1}. Let finally $\mathscr{S}(e_d)$ be the map consisting in glueing the cylinder $\Sigma \times [0, 1]$ on M_Σ containing the red link c.

Lemma 8.3. *The linear map $\mathscr{S}(e_d) : \mathscr{S}(M_\Sigma) \to \mathscr{S}(M_\Sigma)$ is well defined up to a scalar.*

Proof. It is sufficient to prove that if c corresponds to a product of Dehn twists which is trivial then the map $\mathscr{S}(e_d)$ is multiplication by a non-zero scalar. Since the 3-manifold M' obtained by surgering along c is diffeomorphic to M by Kirby's theorem one can apply a finite sequence of Kirby moves of type 1 and 2 to reduce c to the empty link. Let $c = c_0 \to c_1 \cdots \to c_k$ be the sequence of framed links in M_Σ obtained along the way. Let $f_i : \mathscr{S}(M_\Sigma) \to \mathscr{S}(M_\Sigma)$ be the maps consisting in glueing $\Sigma \times [-1, 1]$ along the boundary containing a red curve c_i. We claim that $f_i = \lambda_i f_{i+1}$ for each i for some non-zero λ_i. If the move between c_i and c_{i+1} is a Kirby 2 move this is just the fact one can slide on red curves obtaining equivalent skeins. If the move is a positive (resp. negative) Kirby 1 move creating an unknot with framing ϵ then $f_{i+1} = \Delta_\epsilon f_i$ (resp. $f_{i+1} = \Delta_\epsilon^{-1} f_i$). $\boxed{8.3}$

Now that we have defined $\mathscr{S}(e_d)$ we have to check the relations in Juhasz's presentation of Cob (recall Subsection 2.2). As already remarked the handle cancellations (R4) are automatic and it is also clear that (R1) , (R3) and (R5) hold. We need to verify the relations (R2):

$$\begin{tikzcd} \Sigma \arrow[r, "e_{\Sigma,\mathbb{S}}"] \arrow[d, "e_d"'] & \Sigma(\mathbb{S}) \arrow[d, "e_{d^{\mathbb{S}}}"] \\ \Sigma' \arrow[r, "e_{\Sigma',\mathbb{S}'}"] & \Sigma'(\mathbb{S}') \end{tikzcd} \tag{11}$$

This is checked by letting $M'_\Sigma = d \circ M_\Sigma$ and $\mathbb{S}' = e(\mathbb{S})$ so that we clearly have :

$$\begin{tikzcd} \mathscr{S}(M_\Sigma) \arrow[r, "\mathscr{S}(e_{\Sigma,\mathbb{S}})"] \arrow[d, "\mathscr{S}(e_d)"'] & \mathscr{S}(e_{\Sigma,\mathbb{S}}(M_\Sigma)) \arrow[d, "\mathscr{S}(e_{d^{\mathbb{S}}})"] \\ \mathscr{S}(M'_\Sigma) \arrow[r, "\mathscr{S}(e_{\Sigma',\mathbb{S}'})"] & \mathscr{S}(e_{\Sigma,\mathbb{S}'}(M'_\Sigma) = e_{d^{\mathbb{S}}}(e_{\Sigma,\mathbb{S}}(M_\Sigma))) \end{tikzcd}$$

where the equality in the bottom right entry is the key one and corresponds to the fact that $d^{\mathbb{S}}$ is the diffeomorphism which makes the diagram (11) commute.

Overall we get the following:

Theorem 8.4 ("Lazy version" of RTW DGGPR TQFTs)**.** *The functor $\Sigma \to \mathscr{S}(M_\Sigma)$ can be extended to a TQFT valued in PVect.*

Remark 8.5. The above theorem is a lazy version of the stronger results of Reshetikhin-Turaev (in the case of $\mathcal{C}$ semi-simple) and of DGGPR which give TQFTs valued in Vect but from the category of "decorated cobordisms" where surfaces are decorated with Lagrangian subspaces of their H_1 and cobordisms are extended by integers.

9. Appendix : recalls on category theory

9.1. Basic generalities on categories.

Definition 9.1. A category $\mathcal{C}$ is a class of objects and for each pair of objects a class of morphisms (from the source object to the target) such that :

1. Given morphisms $f : A \to B$ and $g : B \to C$ there is a morphism $g \circ f : A \to C$.
2. The above operation is associative in the obvious sense.
3. Each object A has an identity morphism Id_A which is the left and right identity for the previous compositions.

If for each pair of objects the class of morphisms $Hom_{\mathcal{C}}(A, B)$ is a set, then the category is locally small. In what follows by "category" we will mean a locally small one. Sometimes a non locally small category is also called a "metacategory". If the class of objects is a set and the category is locally small, then it is small.

Definition 9.2. A functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are categories is a function assigning to every object c of $\mathcal{C}$ an object $F(c) \in \mathcal{D}$ and to every morphism $f : c \to c'$ in $\mathcal{C}$ a morphism $F(f) : F(c) \to F(c')$ in a way that sends composition of morphisms to composition of morphisms and identity morphisms to identity morphisms.

If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, a natural transformation from F to G is a class of morphisms $n_c : F(c) \to G(c)$ for $c \in Ob(\mathcal{C})$ such that the obvious square diagrams given by all the morphisms $f : c \to c'$ in $\mathcal{C}$ commute. In particular a natural isomorphism is a natural transformation whose maps are all isomorphisms in $\mathcal{D}$.

Given a category $\mathcal{C}$ its opposite $\mathcal{C}^{op}$ has the same objects and morphisms but with reversed source and target and reversed composition. A functor $F : \mathcal{C} \to \mathcal{D}$ is called covariant. One from $\mathcal{C}^{op} \to \mathcal{D}$ is contravariant. The identity functor from $\mathcal{C}^{op} \to \mathcal{C}$ is contravariant.

Example 9.3. The following will be the most interesting categories for us:

(1) Cob. Its objects are n-dimensional closed oriented manifolds. A morphism $W : M_- \to M_+$ is a $n+1$-dimensional manifold (compact oriented smooth) W such that $\partial W = \overline{M_-} \sqcup M_+$. Attention: here we mean that ∂W is decomposed into 2 connected components (possibly empty) denoted $\partial_\pm W$ and by the equality sign we mean that there are fixed diffeomorphisms (parametrizations) $\phi_\pm : \partial_\pm W \to M_\pm$ which preserve for the sign $+$ (resp. reverse for the sign $-$) the orientation. The cobordisms are considered up to orientation preserving diffeomorphism which commutes with the parametrizations (see Exercice below). The composition of two cobordism is given by the cobordism obtained by glueing (see exercice below).
(2) $Vect$. Its objects are vector spaces over a fixed field k and morphisms are linear maps.
(3) Bim. Its objects are algebras; a morphism $B : A_1 \to A_2$ is an (A_2, A_1)-bimodule (i.e. a $A_2 \otimes A_1^{op}$-module), up to isomorphism. The composition of a (A_3, A_2)-bimodule B' with an (A_2, A_1)-bimodule B is $B' \circ B = B' \otimes_{A_2} B$.
(4) If H is an algebra then $H - mod$ is the category whose objects are H-modules and morphisms are k-linear maps which commute with the action of H.
(5) Bim_H its objects are algebra objects in H_mod and bimodules are bimodule objects in $H - mod$. The composition is given by the tensor product over the algebra action (as above). Bimodules are considered up to isomorphism. Exercice: why does one need to consider bimodules up to isomorphism?

9.2. **Monoidal categories and functors.**

Definition 9.4 (Monoidal category)**.** A monoidal category is a category $\mathcal{C}$ equipped with a tensor product bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object denoted $\mathbf{1}$ such that :

(1) For each Σ there exists natural isomorphisms $\phi^L_\Sigma : \Sigma \to \Sigma \otimes \mathbf{1}$ and $\phi^R_\Sigma : \Sigma \to \mathbf{1} \otimes \Sigma$;
(2) For each objects $\Sigma, \Sigma', \Sigma"$ there exists natural isomorphisms $\psi^{\Sigma,\Sigma',\Sigma"} : \Sigma \otimes (\Sigma' \otimes \Sigma") \to (\Sigma \otimes \Sigma') \otimes \Sigma"$.

(Here naturality means that for all morphisms $f \in Mor(\Sigma_0, \Sigma), g \in Mor(\Sigma_0', \Sigma'), h \in Mor(\Sigma_0'', \Sigma")$ it holds $\phi^R \circ f = (Id \otimes f) \circ \phi^R, \phi^L \circ f = (f \otimes Id) \circ \phi^L$, and $\psi^{\Sigma,\Sigma',\Sigma"} \circ (f \otimes (g \otimes h)) = ((f \otimes g) \otimes h) \circ \psi^{\Sigma_0,\Sigma_0',\Sigma_0''}$.) Such that $\phi^R_{\mathbf{1}} = \phi^L_{\mathbf{1}}$ and for all objects the following pentagon diagrams commute :

$$\begin{tikzcd}
& \Sigma_1 \otimes (\Sigma_2 \otimes (\Sigma_3 \otimes \Sigma_4)) \arrow[r, "Id \otimes \psi^{\Sigma_2,\Sigma_3,\Sigma_4}"] \arrow[dl, "\psi^{\Sigma_1,\Sigma_2,(\Sigma_3 \otimes \Sigma_4)}"'] & \Sigma_1 \otimes ((\Sigma_2 \otimes \Sigma_3) \otimes \Sigma_4) \arrow[dd, "\psi^{\Sigma_1,\Sigma_2 \otimes \Sigma_3,\Sigma_4}"] \\
(\Sigma_1 \otimes \Sigma_2) \otimes (\Sigma_3 \otimes \Sigma_4) \arrow[dr, "\psi^{\Sigma_1 \otimes \Sigma_2,\Sigma_3,\Sigma_4}"] & \text{Pentagon equation} & \\
& ((\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3) \otimes \Sigma_4 & (\Sigma_1 \otimes (\Sigma_2 \otimes \Sigma_3)) \otimes \Sigma_4 \arrow[l, "\psi^{\Sigma_1,\Sigma_2,\Sigma_3} \otimes Id"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\Sigma\otimes\mathbf{1})\otimes\Sigma' \\
\Sigma\otimes\Sigma' \arrow[ur, "\phi^L_\Sigma\otimes Id"] \arrow[dr, "Id\otimes\phi^R_{\Sigma'}"] & \\
 & \Sigma\otimes(\mathbf{1}\otimes\Sigma') \arrow[uu, "\psi^{\Sigma,\mathbf{1},\Sigma'}"']
\end{tikzcd}$$

The category is *strict* if $\mathbf{1}\otimes\Sigma = \Sigma = \Sigma\otimes\mathbf{1}$ and $\phi^L_\Sigma = \phi^R_\Sigma = Id_\Sigma$ for all $\Sigma \in Ob(\mathcal{C})$, and finally for each three objects $\Sigma, \Sigma', \Sigma''$ it holds $\Sigma\otimes(\Sigma'\otimes\Sigma") = (\Sigma\otimes\Sigma')\otimes\Sigma"$ and $\psi^{\Sigma,\Sigma',\Sigma"} = Id_{\Sigma\otimes\Sigma'\otimes\Sigma"}$.

Definition 9.5 (Monoidal functors)**.** A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories is a functor such that there exist a natural isomorphism $d : F(\mathbf{1}) \to \mathbf{1}$ and for all objects Σ, Σ' there exist natural isomorphisms $i_{\Sigma,\Sigma'} : F(\Sigma)\otimes F(\Sigma') \to F(\Sigma\otimes\Sigma')$ which commute with all the associators and identity morphisms, i.e. $\forall\Sigma, \Sigma',\ \forall f \in \mathrm{Mor}(\Sigma,\Sigma), f' \in \mathrm{Mor}(\Sigma',\Sigma')$ the following holds:

$$\begin{tikzcd}
F(\Sigma) & F(\Sigma)\otimes\mathbf{1} \arrow[l, "(\phi^L)^{-1}"'] & F(\Sigma) & \mathbf{1}\otimes F(\Sigma) \arrow[l, "(\phi^R)^{-1}"'] \\
F(\Sigma\otimes\mathbf{1}) \arrow[u, "F((\phi^L)^{-1})"] & F(\Sigma)\otimes F(\mathbf{1}) \arrow[l, "i"] \arrow[u, "Id_{F(\Sigma)}\otimes d"'] & F(\mathbf{1}\otimes\Sigma) \arrow[u, "F((\phi^L)^{-1})"] & F(\mathbf{1})\otimes F(\Sigma) \arrow[l, "i"] \arrow[u, "d\otimes Id_{F(\Sigma)}"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(\Sigma)\otimes(F(\Sigma')\otimes F(\Sigma")) \arrow[r, "\psi'"] \arrow[d, "Id\otimes i"] & (F(\Sigma)\otimes F(\Sigma'))\otimes F(\Sigma") \arrow[r, "i\otimes Id"] & F(\Sigma\otimes\Sigma')\otimes F(\Sigma") \arrow[d, "i"] \\
F(\Sigma)\otimes F(\Sigma'\otimes\Sigma") \arrow[r, "i"] & F(\Sigma\otimes(\Sigma'\otimes\Sigma")) \arrow[r, "F(\psi)"] & F((\Sigma\otimes\Sigma')\otimes\Sigma")
\end{tikzcd}$$

where we denoted ψ (resp. ψ') the associator in $\mathcal{C}$ (resp. in $\mathcal{D}$). A monoidal functor F is a *strict* if $F(\mathbf{1}) = \mathbf{1}$ and for each object Σ, Σ' of $\mathcal{C}$ it holds $F(\Sigma\otimes\Sigma') = F'(\Sigma)\otimes F(\Sigma')$ and the corresponding maps d, i are Id.

Definition 9.6 (Natural transformations of monoidal functors)**.** Let $\mathcal{C}, \mathcal{D}$ be two monoidal categories and $F, F' : \mathcal{C} \to \mathcal{D}$ be two monoidal functors. A natural tensor transformation $n : F \to F'$ is a natural transformation $n : F \to F'$ such that the following diagrams commute for every couple of objects $U, V \in \mathcal{C}$:

$$\begin{tikzcd}
F'(\mathbf{1}) \arrow[dr, "d"] & & F(U)\otimes F(V) \arrow[r, "i"] \arrow[d, "n\otimes n"] & F(U\otimes V) \arrow[d, "n"] \\
F(\mathbf{1}) \arrow[u, "n"] \arrow[r, "d'"] & \mathbf{1} & F'(U)\otimes F'(V) \arrow[r, "i"] & F'(U\otimes V)
\end{tikzcd}$$

A natural tensor transformation $n : F \to F'$ is a natural tensor isomorphism if it is a natural isomorphism (see the end of Definition 9.2). A tensor equivalence F between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ such that there exists a monoidal functor $G : \mathcal{D} \to \mathcal{C}$ and natural tensor isomorphisms $n : G\circ F \to Id_{\mathcal{C}}$ and $n' : F\circ G \to Id_{\mathcal{D}}$.

From now on, when speaking of functors between monoidal categories we will always mean monoidal ones.

A fundamental theorem allowing to replace monoidal categories with strict monoidal ones is the following:

Theorem 9.7 (Maclane, see [26] 2.8.5)**.** *Let $\mathcal{C}$ be a monoidal category. Then there exists a a monoidal equivalence $F : \mathcal{C} \to \mathcal{C}^{st}$ where $\mathcal{C}^{st}$ is a strict monoidal category.*

Because of the above theorem, we will often replace a category $\mathcal{C}$ with a strict category equivalent to it, without even changing the notation when this will cause no confusion.

9.3. **Braidings.**

Definition 9.8 (Braided category)**.** A brading on a monoidal category $\mathcal{C}$ is the datum of natural isomorphisms for every pair of objects $\Sigma, \Sigma' \in Ob(\mathcal{C})$ $b_{\Sigma,\Sigma'} : \Sigma \otimes \Sigma' \to \Sigma' \otimes \Sigma$ such that the following diagrams (known as "Hexagon equations") commute :

$$
\begin{tikzcd}
\Sigma \otimes (\Sigma' \otimes \Sigma'') \arrow[r, "b_{\Sigma,(\Sigma'\otimes\Sigma'')}"] \arrow[d, "\psi"] & (\Sigma' \otimes \Sigma'') \otimes \Sigma \arrow[r, "\psi^{-1}"] & \Sigma' \otimes (\Sigma'' \otimes \Sigma) \arrow[d, "Id\otimes b_{\Sigma'',\Sigma}"] \\
(\Sigma \otimes \Sigma') \otimes \Sigma'' \arrow[r, "b_{\Sigma,\Sigma'}\otimes Id"] & (\Sigma' \otimes \Sigma) \otimes \Sigma'' \arrow[r, "\psi^{-1}"] & \Sigma' \otimes (\Sigma \otimes \Sigma'')
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\Sigma \otimes \Sigma') \otimes \Sigma'' \arrow[r, "b_{(\Sigma\otimes\Sigma''),\Sigma"}"] \arrow[d, "\psi^{-1}"] & \Sigma" \otimes (\Sigma \otimes \Sigma') \arrow[r, "\psi"] & (\Sigma" \otimes \Sigma) \otimes \Sigma' \arrow[d, "b_{\Sigma'',\Sigma}\otimes Id_{\Sigma'}"] \\
\Sigma \otimes (\Sigma' \otimes \Sigma'') \arrow[r, "Id_\Sigma\otimes b_{\Sigma',\Sigma''}"] & \Sigma \otimes (\Sigma'' \otimes \Sigma') \arrow[r, "\psi"] & (\Sigma \otimes \Sigma'') \otimes \Sigma'.
\end{tikzcd}
$$

A braided category is a monoidal category equipped with a braiding. If for each pair of objects $\Sigma, \Sigma' \in \mathcal{C}$ it holds $b_{\Sigma',\Sigma} \circ b_{\Sigma,\Sigma'}$ then the braiding is also called a *symmetry* and $\mathcal{C}$ is a *symmetric monoidal category.*

Remark 9.9. As proved in [18], Proposition XIII 1.2, the following diagrams always commute in a braided category :

$$
\begin{tikzcd}
\Sigma \otimes \mathbf{1} \arrow[r, "b_{\Sigma,\mathbf{1}}"] & \mathbf{1} \otimes \Sigma \\
\Sigma \arrow[u, "\phi^L"] \arrow[r, "Id"] & \Sigma \arrow[u, "\phi^R"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \otimes \Sigma \arrow[r, "b_{\mathbf{1},\Sigma}"] & \Sigma \otimes \mathbf{1} \\
\Sigma \arrow[u, "\phi^R"] \arrow[r, "Id"] & \Sigma \arrow[u, "\phi^L"']
\end{tikzcd}
$$

.

Furthermore when $\mathcal{C}$ is strict the commutativity of the hexagon diagrams is equivalent to the following equalities :

$$b_{\Sigma,\Sigma'\otimes\Sigma''} = (Id_{\Sigma'} \otimes b_{\Sigma,\Sigma"}) \circ (b_{\Sigma,\Sigma'} \otimes Id_{\Sigma''}) \qquad b_{\Sigma'\otimes\Sigma'',\Sigma} = (b_{\Sigma',\Sigma} \otimes Id_{\Sigma''}) \circ (Id_{\Sigma'} \otimes b_{\Sigma",\Sigma}).$$

Definition 9.10 (Braided functors)**.** A braided functor $F : \mathcal{C} \to \mathcal{D}$ between braided monoidal categories is a lax monoidal functor F such that for all the objects of $\mathcal{C}$ the following diagram commutes :

$$
\begin{tikzcd}
F(\Sigma) \otimes F(\Sigma') \arrow[r, "b_{F(\Sigma),F(\Sigma')}"] \arrow[d, "i"] & F(\Sigma') \otimes F(\Sigma) \arrow[d, "i"] \\
F(\Sigma \otimes \Sigma') \arrow[r, "F(b_{\Sigma,\Sigma'})"] & F(\Sigma' \otimes \Sigma)
\end{tikzcd}
$$

The following is an enhanced version of Maclane's coherence theorem. See [18] Proposition XI.5.1 and Exercice XIII.6.5 or [47] Chapter XI, Remark 1.4:

Theorem 9.11. *Let $\mathcal{C}$ be a braided category. Then there exists a strict braided category $\mathcal{C}^{str}$ and a monoidal equivalence $F : \mathcal{C} \to \mathcal{C}^{str}$ which is also a braided functor.*

Example 9.12. The prototypical braided category is Br whose objects are integers $\mathbb{N}$ and morphisms are $\mathrm{Mor}_{Br}(m,n) = \emptyset$ if $m \neq n$ and $\mathrm{Mor}_{Br}(n,n) = B_n$ where B_n is the group of braids with n-strands. The composition is given by vertical stacking. It was proved by Joyal and Street that "Br is the free strict braided monoidal category generated by a single object".

9.4. **Rigid categories.** Because of Theorem 9.11 we will from now on assume that all the monoidal categories are strict.

Definition 9.13 (Left and right duality)**.** A left dual of an object Σ of a strict monoidal category $\mathcal{C}$ is the datum of a *left dual object* Σ^* and morphisms $\overleftarrow{ev}_\Sigma : \Sigma^* \otimes \Sigma \to \mathbf{1}$, $\overleftarrow{coev} : \mathbf{1} \to \Sigma \otimes \Sigma^*$ such that the following "triangular equalities" hold :

$$(Id_\Sigma \otimes \overleftarrow{ev}_\Sigma) \circ (\overleftarrow{coev}_\Sigma \otimes Id_\Sigma) = Id_\Sigma \qquad (\overleftarrow{ev}_\Sigma \otimes Id_{\Sigma^*}) \circ (Id_{\Sigma^*} \otimes \overleftarrow{coev}_\Sigma) = Id_{\Sigma^*}.$$

If Σ_1, Σ_2 are left dualisable objects and $f \in \mathrm{Mor}(\Sigma_1, \Sigma_2)$ the *left adjoint* of f, denoted $f^* \in \mathrm{Mor}(\Sigma_2^*, \Sigma_1^*)$ is the morphism defined as:

$$f^* := (\overleftarrow{ev}_{\Sigma_2} \otimes Id_{\Sigma_1^*}) \circ (Id_{\Sigma_2^*} \otimes f \otimes Id_{\Sigma_1^*}) \circ (Id_{\Sigma_2^*} \otimes \overleftarrow{coev}_{\Sigma_1}).$$

Similarly a right dual of Σ is the datum of a *right dual object* ${}^*\Sigma$ and morphisms $\overrightarrow{ev}_\Sigma : \Sigma \otimes ({}^*\Sigma) \to \mathbf{1}$, $\overrightarrow{coev} : \mathbf{1} \to ({}^*\Sigma) \otimes \Sigma$ such that the following "triangular equalities" hold:

$$(\overrightarrow{ev}_\Sigma \otimes Id_\Sigma) \circ (Id_\Sigma \otimes \overrightarrow{coev}_\Sigma) = Id_\Sigma \qquad (Id_{({}^*\Sigma)} \otimes \overrightarrow{ev}_\Sigma) \circ (\overrightarrow{coev}_\Sigma \otimes Id_{({}^*\Sigma)}) = Id_{({}^*\Sigma)}.$$

The right adjoint of $f \in \mathrm{Mor}(\Sigma_1, \Sigma_2)$ is the morphism $({}^*f) \in \mathrm{Mor}({}^*\Sigma_2, {}^*\Sigma_1)$ defined as:

$$({}^*f) := (Id_{({}^*\Sigma_1)} \otimes \overrightarrow{ev}_{\Sigma_2}) \circ (Id_{({}^*\Sigma_1)} \otimes f \otimes Id_{({}^*\Sigma_2)}) \circ (\overrightarrow{coev}_{\Sigma_1} \otimes Id_{({}^*\Sigma_2)}).$$

If all the objects of $\mathcal{C}$ have both left and right duals, then $\mathcal{C}$ is called *autonomous* or *rigid*.

Remark 9.14. It can be proven (exercise!) that the left (resp. right) dual object, if it exists, is unique up to unique isomorphism. Therefore, in what follows for each object V admitting a left (or right) dual we shall implicitly **choose** one and call it V^* (resp. ${}^*V$); if these coincide we'll denote them both V^*.

Furthermore it is important to observe that the existence of a dual object for $\Sigma \in \mathcal{C}$ is a property of V and not an additional structure one defines on $\mathcal{C}$. Finally it can be proven that if $\mathcal{C}$ is autonomous then, each $V \in \mathcal{C}$ is isomorphic to both ${}^*(V^*)$ and $({}^*V)^*$. But in general it is not true that $(V^*)^*$ is isomorphic to V.

Let $\mathcal{C}^{op}$ be the category whose objects are those of $\mathcal{C}$ and morphisms are $Mor^{op}(\Sigma_1, \Sigma_2) = Mor(\Sigma_2, \Sigma_1)$. Equip it with a strict monoidal structure given by $V \otimes^{op} W := W \otimes V$. Then if $\mathcal{C}$ all the objects of $\mathcal{C}$ have a left dual ("$\mathcal{C}$ has a left duality"), the "left dual functor" : $L : \mathcal{C} \to \mathcal{C}^{op}$ associating to each object its left dual and to each morphism its left adjoint is a monoidal functor indeed the map $i_{\Sigma_1,\Sigma_2} : L(\Sigma_1) \otimes^{op} L(\Sigma_2) = \Sigma_2^* \otimes \Sigma_1^* \to L(\Sigma_1 \otimes \Sigma_2) = (\Sigma_1 \otimes \Sigma_2)^*$ is given by:

$$i_{\Sigma_1,\Sigma_2} := (\overleftarrow{ev}_{\Sigma_2} \otimes Id_{(\Sigma_1 \otimes \Sigma_2)^*}) \circ (Id_{\Sigma_2^*} \otimes \overleftarrow{ev}_{\Sigma_1} \otimes Id_{\Sigma_2 \otimes (\Sigma_1 \otimes \Sigma_2)^*}) \circ (Id_{\Sigma_2^* \otimes \Sigma_1^*} \otimes \mathrm{coev}_{\Sigma_1 \otimes \Sigma_2}).$$

Similarly for the right dual functor $R : \mathcal{C} \to \mathcal{C}^{op}$.

Definition 9.15 (Pivotal categories)**.** An autonomous category is pivotal if the left and right duality functors are isomorphic. In this case we will denote $i : Id \to {}^{**}$ the natural monoidal isomorphism induced by the identity of left and right functors. Explicitly, $i_X = (\overrightarrow{ev}_X \otimes Id_{X^{**}}) \circ (Id_X \otimes \overleftarrow{coev}_{X^*})$

The categorical *left trace* and *right trace* of any endomorphism $f : X \to X$ of a pivotal category $\mathcal{C}$ are defined by

$$\mathrm{tr}_l(f) = \overleftarrow{\mathrm{ev}}_X(\mathrm{id}_{X^*} \otimes f)\,\overrightarrow{\mathrm{coev}}_X \quad \text{ and } \quad \mathrm{tr}_r(f) = \overrightarrow{\mathrm{ev}}_X(f \otimes \mathrm{id}_{X^*})\,\overleftarrow{\mathrm{coev}}_X\ .$$

Both take values in the commutative monoid $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ of endomorphisms of the monoidal unit $\mathbb{1}$ and share a number of properties of the standard trace of matrices such as cyclicity (i.e., symmetry). More generally, the *left partial trace* of a morphism $g : X \otimes Y \to X \otimes Z$ is the morphism

$$\mathrm{ptr}_l^X(g) = (\overleftarrow{\mathrm{ev}}_X \otimes \mathrm{id}_Z)(\mathrm{id}_{X^*} \otimes g)(\overrightarrow{\mathrm{coev}}_X \otimes \mathrm{id}_Y) : Y \to Z,$$

and the *right partial trace* of a morphism $h : X \otimes Y \to Z \otimes Y$ is the morphism

$$\mathrm{ptr}_r^Y(h) = (\mathrm{id}_Z \otimes \overrightarrow{\mathrm{ev}}_Y)(h \otimes \mathrm{id}_{Y^*})(\mathrm{id}_X \otimes \overleftarrow{\mathrm{coev}}_Y) : X \to Z.$$

If $\mathcal{C}$ and $\mathcal{D}$ are pivotal categories and $F : \mathcal{C} \to \mathcal{D}$ is a strong monoidal functor then for each $X \in \mathcal{C}$ $(F(X^*), F(\mathrm{ev}_X), F(\mathrm{coev}_X))$ is a dual of $F(X)$ therefore it comes with a unique isomorphism $u_X : F(X^*) \to F(X)^*$, natural in X.

Definition 9.16. We say that F is a pivotal functor if the following diagram commutes for every $X \in \mathcal{C}$:

$$\begin{tikzcd}
FX \arrow[r, "i_{FX}"] \arrow[d, "Fi_X"] & (FX)^{**} \arrow[d, "u_X^*"] \\
F(X^{**}) \arrow[r, "u_{F(X^*)}"] & F(X^*)^*
\end{tikzcd}$$

9.5. **A key example.** Let H be a Hopf algebra (i.e. an associative unital algebra H over a field k endowed with a morphism of algebras $\Delta : H \to H \otimes H$ which is coassociative and with a counit $\epsilon : H \to k$ and with an antimorphism of algebras $S : A \to A$ called the antipode satisfying $S \otimes Id \circ \Delta = Id \otimes S \circ \Delta = \epsilon$). Then the category of finite dimensional H modules is automatically endowed with a monoidal structure given by $V \otimes W = V \otimes_k W$ endowed with the action given by first mapping $H \to H \otimes H$ via Δ and then acting. It is also rigid where the left dual of $V = \mathrm{Hom}(V, k)$ is endowed with the action given by $h \cdot f(\cdot) = f(S(h)\cdot)$. The right dual of V is $\mathrm{Hom}(V, k)$ with the action given by $h \cdot f(\cdot) = f(S^{-1}(h)\cdot)$ (when S is invertible which is always the case for us).

Exercise 9.17. Convince yourself that the category of H-modules is indeed a rigid monoidal category. A priori it is not automatically pivotal, although I don't know an example of a non pivotal case.

It is fair to say that a Hopf algebra is exactly the kind of structure needed to make sure that its category of finite dimensional modules is not just a category but a rigid monoidal one.

9.5.1. *The Hopf algebra* $U_q(\mathfrak{sl}_2)$*.* The following example is a special case of the general construction of modular categories from quantum groups due to Drinfeld and Jimbo and resumed for instance in [47] (Chapter 6). Let q be a non-zero complex number (different from 1) and $U_q(\mathfrak{sl}_2)$ be the Hopf algebra generated by $K^{\pm 1}, E, F$ with relations:

$$KK^{-1} = K^{-1}K = 1, KE = q^2 EK, KF = q^{-2} FK \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Its bialgebra structure is given by :

$$\Delta(K) = K \otimes K, \Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

Its antipode and counit are :

$$\epsilon(K) = 1, \epsilon(E) = \epsilon(F) = 0, S(K) = K^{-1}, S(E) = -K^{-1}E, S(F) = -FK.$$

Exercise 9.18. Check that $U_q(\mathfrak{sl}_2)$ is a Hopf algebra.

Exercise 9.19. Let $q \in \mathcal{C} \setminus \{0\}$ be not a root of unity. Prove that a simple module S on $U_q(\mathfrak{sl}_2)$ is generated by a "highest weight vector namely a vector v_0 such that $Kv_0 = \xi^n v_0, Ev_0 = 0$. Prove that the the eigenvalues of K on such an s are of the form $\pm q^i, i \in \{-n, -n+2, \cdots, n-2, n\}$ where $dim(S) = 2n+1$ (for some $n \in \mathbb{Z}/2$). From now on such an S will be called $S^{\pm}_{2n}$.

It is usual to restrict to the category of S^{+}_{2n}. The tensor product of two such modules decomposes as a direct sum of such by the Clebsch-Gordan formula:

$$S_a \otimes S_b = P \oplus \left(\bigoplus_{c=|a-b| \ by \ 2}^{(a+b)} S_c \right)$$

. (one says that the category of $U_q(\mathfrak{sl}_2)$-modules is "semi-simple" when q is not a root of unity). It turns out that the following operator, although it is not an element of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ acts on each $S_a \otimes S_b$ (because the sum becomes finite on it and because one can make sense of the term $q^{H \otimes H}(v \otimes w)$ as $q^{nm} v \otimes w$ if $Kv = q^n v, Kw = q^m w$):

$$R = q^{\frac{H \otimes H}{2}} \sum_n \frac{1}{[n]!} E^n \otimes F^n$$

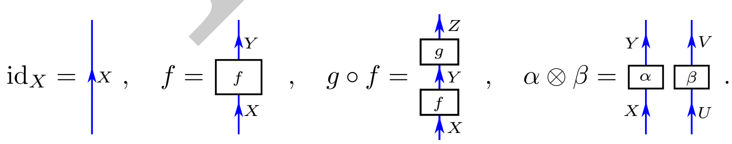
where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]! = \prod_{i=0}^{n} [i]$. More importantly the action of R followed by the flip $v \otimes w \to w \otimes v$ implements a braiding on the category of $U_q(\mathfrak{sl}_2)$-modules.

Exercise 9.20. Compute the action of the braiding on $S_1 \otimes S_1$. Relate it to the Kauffman relations.

Exercise 9.21. Prove that for each $x \in U_q(\mathfrak{sl}_2)$ it holds $S^2(x) = KxK^{-1}$. Then prove that for each $U_q(\mathfrak{sl}_2)$-modules V the map $\psi : V \to V^{**}$ given by $v \to K^{-1}v$ is an isomorphism of modules over $U_q(\mathfrak{sl}_2)$. Prove that $\psi : U_q(\mathfrak{sl}_2) - mod \to U_q(\mathfrak{sl}_2) - mod$ is a natural transformation from the identity functor to the functor **.

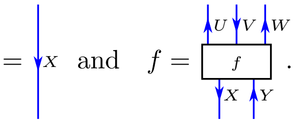
By the previous exercise $U_q(\mathfrak{sl}_2)$-modules is a pivotal category.

9.6. **Penrose graphical calculus.** We represent morphisms in a pivotal category $\mathcal{C}$ by plane diagrams to be read from the bottom to the top. Diagrams are made of oriented arcs colored by objects of $\mathcal{C}$ and of boxes colored by morphisms of $\mathcal{C}$. The arcs connect the boxes and have no mutual intersections or self-intersections. The identity id_X of an object X, a morphism $f : X \to Y$, the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$, and the monoidal product of two morphisms $\alpha : X \to Y$ and $\beta : U \to V$ are represented as follows:



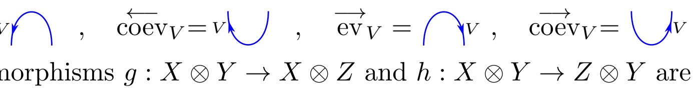
A box whose lower/upper side has no attached strands represents a morphism with source/target $\mathbb{1}$. If an arc colored by X is oriented downward, then the corresponding object in the source/target of morphisms is X^*. For example, id_{X^*} and a morphism $f : X^* \otimes Y \to U \otimes V^* \otimes W$ may be depicted as:

$\mathrm{id}_{X^*} =$ and $f =$.



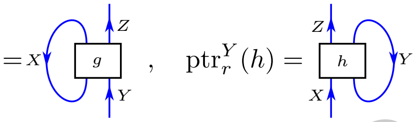
The duality morphisms are depicted as

$\overleftarrow{\mathrm{ev}}_V =$, $\overleftarrow{\mathrm{coev}}_V =$, $\overrightarrow{\mathrm{ev}}_V =$, $\overrightarrow{\mathrm{coev}}_V =$.



The partial traces of morphisms $g : X \otimes Y \to X \otimes Z$ and $h : X \otimes Y \to Z \otimes Y$ are depicted as

$\mathrm{ptr}_l^X(g) =$, $\mathrm{ptr}_r^Y(h) =$



Note that the morphisms represented by the diagrams are invariant under isotopies of the diagrams in the plane keeping fixed the bottom and top endpoints (see [JS, TVi]).

9.7. **Ribbon categories.**

Definition 9.22. A strict, braided category $\mathcal{C}$ is *ribbon* if it is endowed with a natural family of isomorphisms $\theta_\Sigma : \Sigma \to \Sigma$, $\forall \Sigma \in Ob(\mathcal{C})$ such that for all $\Sigma_1, \Sigma_2 \in \mathcal{C}$ it holds :

$$\theta_{\Sigma_1 \otimes \Sigma_2} = (\theta_{\Sigma_1} \otimes \theta_{\Sigma_2}) \circ b_{\Sigma_2,\Sigma_1} \circ b_{\Sigma_1,\Sigma_2}$$

and, if Σ is left-dualisable, $\theta_{\Sigma^*} = (\theta_\Sigma)^*$. (The naturality of the isomorphisms means that for each $f \in \mathrm{Mor}(\Sigma_1, \Sigma_2)$ it holds $\theta_{\Sigma_2} \circ f = f \circ \theta_{\Sigma_1}$.)

Remark 9.23. Often one requires a ribbon category to be rigid by definition. We will not here for reasons which will be clear later on. But the reader should notice that in most of the results we will cite, we will make this additional hypothesis explicitly.

In a ribbon category $\mathcal{C}$, if Σ has a left dual Σ^* then it also has a right dual which is still Σ^* and $\overleftarrow{ev}_\Sigma := \overrightarrow{ev}_\Sigma \circ b_{\Sigma,\Sigma^*} \circ (\theta_\Sigma \otimes Id_{\Sigma^*})$ and $\overleftarrow{coev}_\Sigma := (Id_{\Sigma^*} \otimes \theta_\Sigma) \circ b_{\Sigma,\Sigma^*} \circ \overrightarrow{coev}_\Sigma$ (for a proof that these morphisms do indeed define a right duality on $\mathcal{C}$ see [18] Proposition XIV.3.5). Hence each ribbon category in which all objects are left-dualisable is autonomous; it can actually be proven that it is also pivotal.

Example 9.24. Let *Rib* be the category whose objects are finite sequences of $\pm$ and morphisms between two such sequences s and s' are isotopy classes of oriented ribbon tangles in $\mathbb{R}^2 \times [0,1]$ with negative boundary $s \subset \mathbb{R} \times \{0\} \times \{-1\}$ (where s is realised as a finite sequence of signed points in $\mathbb{R}$) and positive boundary $s' \subset \mathbb{R} \times \{0\} \times \{1\}$. The composition is given by vertical stacking (exercice: the result is independent on the choice of the position of the points realising the sequences of signs in $\mathbb{R}$). The monoidal product is given by "horizontal stacking" i.e. concatenation in $\mathbb{R}$ (exercice : it is well defined).

Example 9.25. If $\mathcal{C}$ is a rigid ribbon category, then one can consider the category $Rib_{\mathcal{C}}$, whose objects are finite sequences of objects of $\mathcal{C}$ with signs. The morphisms are isotopy classes of $\mathcal{C}$-colored ribbon tangles, i.e. ribbon tangles (as before) whose edges are oriented and "colored" by an object of $\mathcal{C}$ and possibly containing some "coupons" which are rectangles with a distinguished "lower base" and opposite "upper base" decorated with a morphism

$$f : V_1^{\epsilon_1} \otimes \cdots \otimes V_k^{\epsilon_k} \to W_1^{\eta_1} \otimes \cdots \otimes W_h^{\eta_h}$$

where $V_1, \dots V_k$ are the objects decorating the edges attached to the lower base read from left to right and W_i those to the upper one (read from left to right), $\epsilon_i = 1$ iff i is outgoing the coupon while $\eta_i = 1$ iff the corresponding edge i is incoming the coupon, and finally we use the notation $V^1 = V$, $V^{-1} = V^*$. As before, the composition is given by vertical stacking and the monoidal product is given by horizontal stacking. Exercice: show that $Rib_{\mathcal{C}}$ is a strict rigid ribbon category (who is the twist?).

The following is fundamental:

Theorem 9.26 (Reshetikhin-Turaev)**.** *There exists a monoidal functor $RT : Rib_{\mathcal{C}} \to \mathcal{C}$ uniquely defined by imposing $RT((V, \pm)) = V^{\pm}$, $\forall V \in \mathcal{C}$ and the value on the "basic morphisms" consisting of maxima, minima, crossings and their tensor products (horizontal stacking) as depicted in Figure* **??**. put figure

Exercise 9.27. Given $V \in \mathcal{C}$, compute $F((V, \epsilon))^*$ and $F((V, \epsilon)^*)$ for $\epsilon \in \{\pm\}$ and deduce the natural transformation $u : F(X)^* \to F(X^*)$ for each $X \in Rib_{\mathcal{C}}$. Compute the natural transformation i for the pivotal structure of $Rib_{\mathcal{C}}$ and that for $\mathcal{C}$. Is the Reshetikhin-Turaev functor pivotal ?

The following is a follow-up of Example 9.3.

Example 9.28. The following will be the most interesting categories for us:

1. Cob. The monoidal structure is given by the disjoint union. The dual of an object M is $\overline{M}$ (it is both a right and left dual). The evaluation and coevaluations are both given by the cobordism $M \times [-1, 1]$ but whose boundary is either all ∂_- or ∂_+.
2. $Vect$. The tensor product is the standard k-linear one. The dualisable objects are exactly the finite dimensional vector spaces. Exercice: $Vect$ is ribbon and the full subcategory $Vect^{fin}$ of finite dimensional vector spaces is rigid ribbon.
3. Bim. Its monoidal structure is given by the standard tensor product. (Exercice : formalise this structure !)
4. If H is an algebra then $H - mod$ is a monoidal category if H is a bialgebra and it is rigid if H is a Hopf algebra (exercice: prove it!). It is braided if H is quasi-triangular; ribbon if H is ribbon. All these facts are going to be recalled later on.
5. Bim_H. As an exercice, the reader can try to understand under which condition it is monoidal (spoiler: $H - mod$ should be braided). Understanding when it is even braided is a whole other story we just clarified together with M. Faitg [9].

Exercise 9.29. Formalize the definition of Cob, for instance the requirement on the diffeomorphisms for the category Cob. Then define properly the composition of two morphisms. Verify that this composition is indeed associative and that the identity on an object M is $M \times [-1, 1]$ with the identity parametrizations. Show that the group of orientation preserving self-diffeomorphisms of an object M is mapped to a subgroup of $Hom_{\mathrm{Cob}}(M, M)$ via $f \to (M \times [-1, 1], Id, f)$. Se later for the answer. Show that each object is dualisable.

Exercise 9.30. Let $Vect^{fin}$ be the category of finite dimensional vector spaces and End : $Vect^{fin} \to Bim$ be the functor which associates to V its endomorphism algebra $End(V)$ and to each linear map $f : V \to W$ the bimodule $Hom_k(V, W)$. Show that End is a functor.

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