Equivariant deformation Retractions of Teichmüller Space
Lecture 1

June 2025

Some groups

$F_n :=$ free group with n generators

$\mathrm{Inn}(F_n) :=$ automorphisms of F_n obtained by conjugating with elements of F_n

$\mathrm{Aut}(F_n) :=$ group automorphisms of F_n

Exercise Show $\mathrm{Inn}(F_n) \triangleleft \mathrm{Aut}(F_n)$

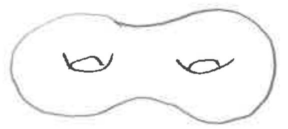
$$\mathrm{Out}(F_n) := \frac{\mathrm{Aut}(F_n)}{\mathrm{Inn}(F_n)}$$

Abelianisation map $F_n \to \mathbb{Z}^n$ induces homomorphism $\mathrm{Out}(F_n) \to GL(n, \mathbb{Z})$

$\mathbb{Z}^2 = \pi_1(T^2)$

torus

$S_g :=$ connected, orientable surface of genus g

e.g.  S_2

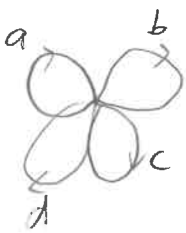
$\Gamma_g^{\pm} := \mathrm{Out}(\pi_1(S_2))$

1-relator group

Γ_g mapping class group Orientation preserving automorphisms of S_g up to isotopy

Examples

Element of F_4



$a \mapsto a$

$b \mapsto ab$

$c \mapsto c\,a\,b\,a^{-1}b^{-1}d$

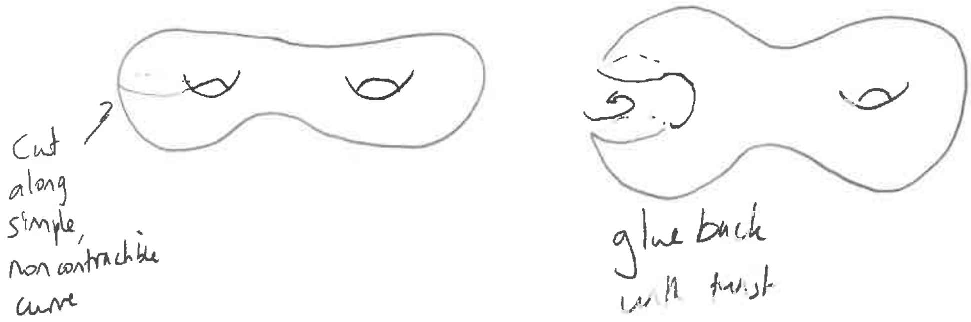
$d \mapsto d\,b\,c\,d$

Reference for $\mathrm{Out}(F_n)$

Bestvina "The topology of $\mathrm{Out}(F_n)$"

Dehn twist

Element of Γ_g



Dehn, Lickorish - Γ_g generated by Dehn twists

Reference for mapping class groups - Farb & Margalit "A primer on mapping class groups"

Actions of these groups on spaces:

For $GL(n, \mathbb{Z})$ there is an action (change of coordinates) on +ve definite quadratic forms (These are inner products on $\mathbb{R}^n$, & hence determine unit volume flat metrics on T^n)

A marking on T^n determines an ordered basis for $\pi_1(T^n)$, & determines how a metric is mapped to T^n

Defn - A marking on a Euclidean torus with metric g is a homotopy class of homeomorphisms $\rho: T^n \to (T^n, g)$, up to equivalence

Two markings ρ_1 & ρ_2 are equivalent if $\exists$ an isometry $i: (T^n_1, g_1) \to (T^n_2, g_2)$ for which $\rho_2^{-1} \circ i \circ \rho_1$ is homotopic to the identity

$X :=$ space of equiv. classes of marked, unit volume Euclidean tori

$GL(n, \mathbb{Z})$ acts on X by changing the marking

Similarly, Teichmüller space $\mathcal{T}_g$, $g \geq 2$

The space of marked, unit volume hyperbolic surfaces of genus g.

Γ_g acts on $\mathcal{T}_g$ by changing the marking

$$\Gamma_g \times \mathcal{T}_g \to \mathcal{T}_g$$
$$\gamma \times \mathcal{T}_g \mapsto (S_g, \rho \circ \gamma^{-1})$$

$Out(F_n)$ acts on a space called Outer Space

for all these group actions, the action is "nice", in particular

1) properly discontinuous, i.e. every pt has a nbhd U_x for which $\exists$ at most finitely many elements of the group $\{g_{1,x}, \dots g_{k,x}\}$ for which $g_{i,x}(U_x) \cap U_x \neq \emptyset$

2) the space on which the group acts is contractible

3) for each of the spaces, there is an equivariant deformation retraction onto a complex of the smallest possible dimension

for $GL(n,\mathbb{Z})$, this is $\frac{n(n-1)}{2}$ (Proven by Ash)

for $Out(F_n)$ this is $2n-3$ (Proven by Culler Vogtmann)

for Γ_g this is $4g-5$ (Punctured surface case Harer, Penner; Closed surface, Thurston)

Thurston's Γ_g-equivariant deformation retraction is the focus of this mini course

Teichmüller space T_g

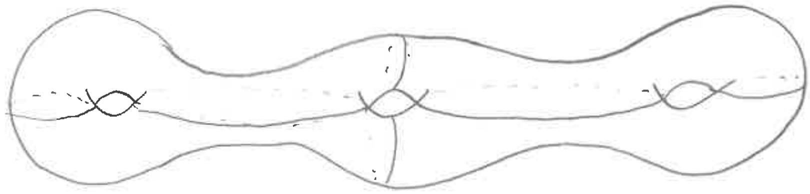
This mini course will use the definition of T_g as the space of marked hyperbolic structures on S_g

Alternative defn - the space of complex structures \ conformal structures modulo the action of homeos isotopic to the identity

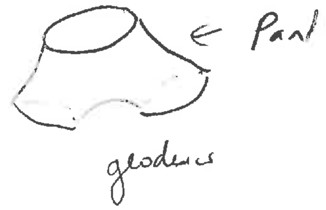
Moduli space $\mathrm{Mod}_g = T_g / T_g$ is an orbifold

Fenchel-Nielsen coordinates

A system of $6g-6$ coordinates on T_g



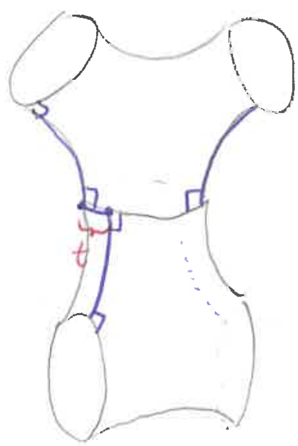
choose a pants decomposition of S_g.



The lengths of the $3g-3$ curves making up the pants decomposion give $3g-3$ parameters

The remaining $3g-3$ parameters are twist parameters

Twist parameters take values in $\mathbb{R}$.



Thm - Any tuple in $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ determines a unique point in T_g

Corollary - T_g is a cell of dimension $6g-6$

Lecture 2

Length functions

c simple closed geodesic on hyperbolic surface S_g

$L(c): \mathcal{T}_g \to \mathbb{R}_+$, $x \mapsto$ length of c in hyperbolic metric corresponding to the point x.

$L(c)$ is an analytic function

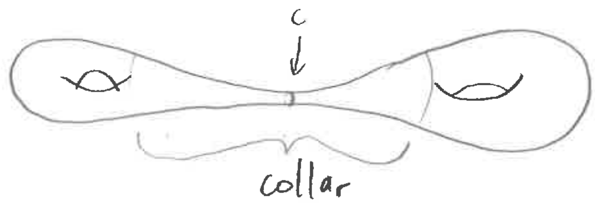
C is a finite set of simple closed geodesics

A length function is a map $\mathcal{T}_g \to \mathbb{R}_+$ of the form

$$\sum_{c_i \in C} a_i L(c_i)$$

Thm (Wolpert)
Length functions are convex functions on $\mathcal{T}_g$ w.r.t. a specific Γ_g-equivariant metric called the Weil-Petersson metric.

Keen's collar lemma $\Rightarrow$ short geodesics have long collars



when $L(c)$ is small, the width of the collar $\sim \dfrac{1}{L(c)}$

At a point $x \in \mathcal{T}_g$, the systoles are the set of shortest geodesics in the hyperbolic metric corresponding to x

There is a constant ε_M "Margulis constant"
The "thick" part of $\mathcal{T}_g$ is defined as $f_{sys}^{-1}[\varepsilon_M, \infty)$
The "thin" part of $\mathcal{T}_g$ is the complement of the thick part
Thm (Margulis) – the systoles are disjoint in the thin part of $\mathcal{T}_g$
Thm (Mumford) – the thick part of Mod_g is compact

Exercise - show that 2 systoles can intersect in at most 1 point

Local finiteness. for any $l \in \mathbb{R}_+$, on a hyperbolic surface S_g, the number of curves of length $\leq l$ is bounded from above by $n(l)$, where $n(l)$ can be chosen indendently of the point in T_g.

Local finiteness $\Rightarrow$ if C is the set of systoles at a point $x \in T_g$, $\exists$ nbhd $N(x)$ in T_g with the property that at every point of $N(x)$, the systoles are contained in the set C.

Systole function $f_{sys} : T_g \to \mathbb{R}_+$, $x \mapsto$ length of systole

f_{sys} piecewise smooth

Thurston's Lipschitz Maps - Reference Papadopoulos + Théret "Shortening all the simple closed geodesics on surfaces with boundary"

Purpose of these maps is to construct a map $T_g \to T_g$ with the property that this map increases the lengths of a specific set of curves.

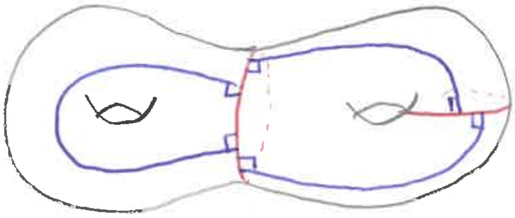
Input m, geodesic multicurve on S_g

A, a set of disjoint geodesic arcs with endpts on m

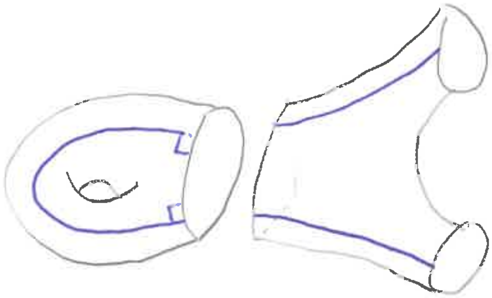
A is required to have nonzero geometric intersection number with every boundary component of a tubular neighbourhood of m.

Suppose S_g is a hyperbolic surface corresponding to a point in T_g

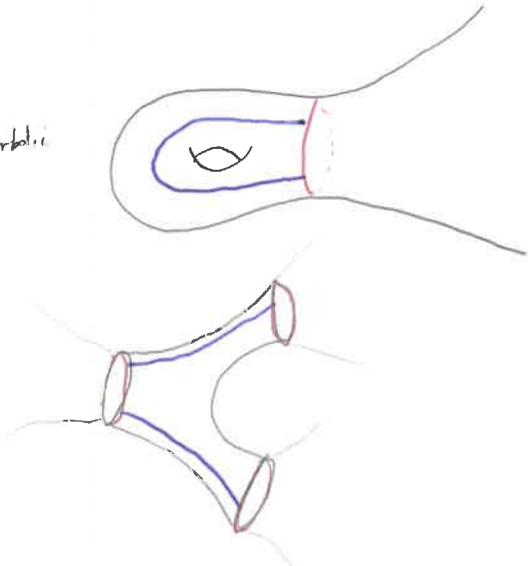
— m — A



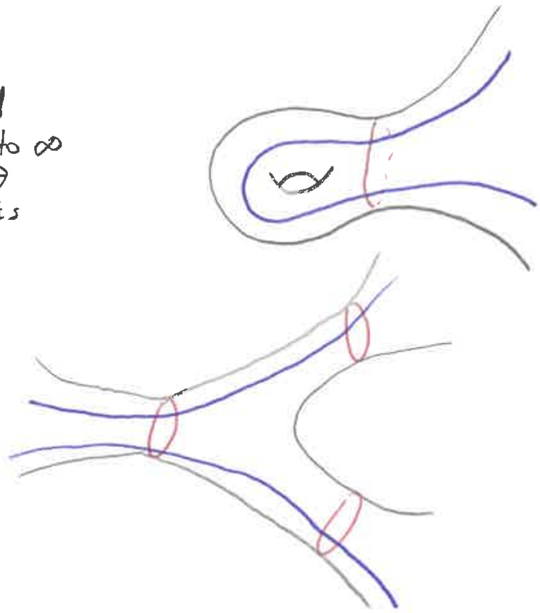
Cut along $\xrightarrow{m}$



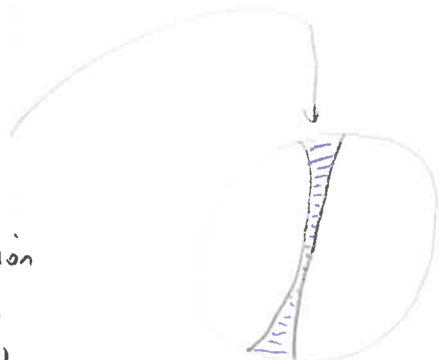
Glue on hyperbolic flaring ends $\longrightarrow$



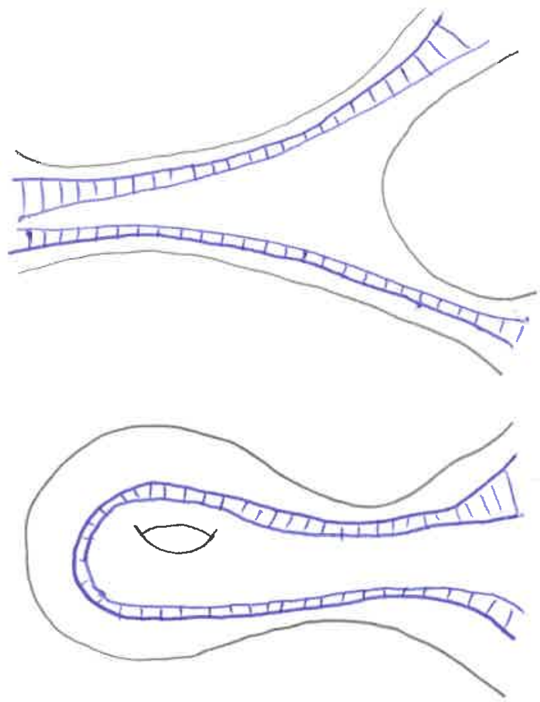
extend arcs to ∞ $\longrightarrow$ geodesics



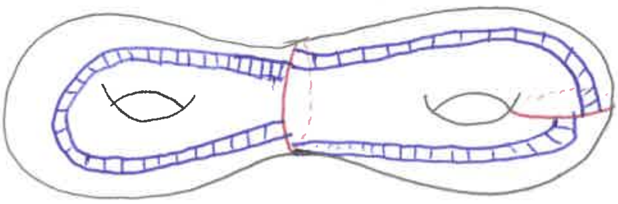
Cut along arcs and glue in strips $\longrightarrow$



A "strip" is a rectangular region in the hyperbolic plane, bounded by 2 disjoint ∞ geodesics



Cut and $\longrightarrow$ glue



The relative widths of the strips must be chosen such that the geodesics representing the two different "sides" of m have the same lengths and can be glued together

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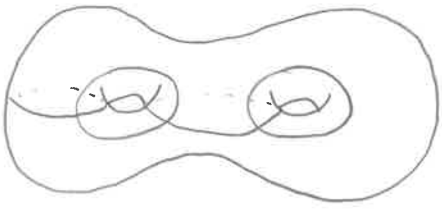
A set of geodesics C fills S_g if the complement is a set of polygons

Example



Proposition (Thurston, Bers, ...)
Suppose C is a set of geodesics that do not fill. At every point $x \in \mathcal{T}_g$ $\exists$ an open cone of directions in $T_x \mathcal{T}_g$ of directions in which the lengths of curves in C are strictly increasing.

Proof - In Thurston's Lipschitz map construction, choose m to be the boundary of the subsurface filled by C, and choose A so that every geodesic in C intersects at least 1 arc in A.
If the widths of the strips approach zero linearly, the resulting 1-parameter family gives a vector in the interior of the cone $\square$

A spine for $\mathcal{T}_g$ is a CW complex that is the image of a deformation retraction

A continuous map $f : \mathcal{T}_g \times [0,1] \to \mathcal{T}_g$ is a deformation retraction onto a CW complex A if, for every $x \in \mathcal{T}_g$ and every a in A
$f(x,0) = x$, $f(x,1) \in A$ and $f(a,1) = a$

We are interested in Γ_g-equivariant deformation retractions
i.e deformation retractions that commute with the action of Γ_g
action of Γ_g
These give deformation retractions of Mod_g

The Thurston spine, P_g, is the set of points in $\mathcal{T}_g$ at which the systoles fill

Exercise - Bers showed that there exists a constant (Bers' constant) depending only on genus, such that every closed surface of genus g has a systole of length bounded from above by Bers constant
Show that P_g is not empty

In the next lecture, Thurston's proposition will be used to show that P_g is a Γ_g-equivariant spine for $\mathcal{T}_g$.

P_g is made up of strata. A stratum of P_g is the set of points at which the systoles are given by a fixed set of curves

Each stratum is defined by finite system of analytic equalities (the lengths of the set of systoles are all equal) and inequalities (the systoles are shorter than other curves). By local finiteness, we only need to consider finitely many inequalities
There is a theorem of Lojasiewicz that implies the existence of a triangulation of P_g compatible with the stratification

P_g is a CW complex

Exercise - Using local finiteness and compactness of the thick part of Mod_g, show that P_g has only finitely many strata

Lecture 3

This lecture is based on Thurston's notes. Additional comments and more details can be found in the reference

Irmer "Thurston's deformation retraction of Teichmüller space"

Thurston makes a claim "Every compact set [of T_g] is eventually carried inside $P_{B\varepsilon}$"

This claim will be discussed in Lecture 4

Lecture 4

Gap function $g_{sys} : T_g \to \mathbb{R}_+$ is given by $x \mapsto \inf \left\{ r \in \mathbb{R}_+ \;\middle|\; \text{the closed geodesics of length at most } f_{sys}(x)+r \text{ fill} \right\}$

$g_{sys} > 0$ away from P_g

Thurston's claim that every compact set is eventually carried inside P_g can be proven by showing that, on a neighbourhood of P_g, Thurston's flow decreases g_{sys}

If all one wants is to show the existence of an f_{sys}-increasing flow that decreases g_{sys} near P_g, the following lemma contains most of the ideas needed

Lemma - Suppose C is a filling set of curves, and $C \setminus \{c\}$ is not filling for some $c \in C$. Then for any $x \in T_g$, $\exists$ an open cone of directions in $T_x T_g$ in which $L(c)$ is decreasing and every $L(c')$ for $c' \in C \setminus \{c\}$ is increasing

Proof - Uses Thurston's Lipschitz map construction. If m is a geodesic that intersects c but not any curve in $C \setminus \{c\}$. If we choose A to contain the arcs $c \cap (S_g \setminus m)$ and to intersect every curve in $C \setminus \{c\}$, it is possible to choose the ratios of the widths of the strips to give the required open cone. For details, see Lemma 3.3 of Irmer "The Morse-Smale Property of the Thurston spine" $\square$

To show that Thurston's choice of VF gives a flow that does everything claimed of it, P_g can be described using Morse theory.

Thm (Schmutz, Akrout) - f_{sys} is a topological Morse function

A continuous function $f : M \to \mathbb{R}$ on a topological manifold M is a topological Morse function if the points of M can be classified as regular or critical. On a neighbourhood of a regular point, $\exists$ a homeomorphic parameterisation, with one of the parameters being f. On the neighbourhood of a critical point p, $\exists$ a homeomorphism ψ and a set of parameters $x_1, \ldots, x_n$ for which

$$f(\psi(x)) - f(p) = -x_1^2 - \ldots - x_r^2 + x_{r+1}^2 + \ldots + x_n^2$$

for a (smooth) Morse function, the unstable manifold of a critical point p is the union of p and all flowlines γ of the vector field $-\nabla f$ with $\lim_{t \to -\infty} \gamma(t) = p$

The stable manifold of p is the union of p & all flowlines γ of the vector field $-\nabla f$ with $\lim_{t \to +\infty} \gamma(t) = p$

For topological Morse functions, Morse defined stable/unstable manifolds, but these are not necessarily unique

(5)

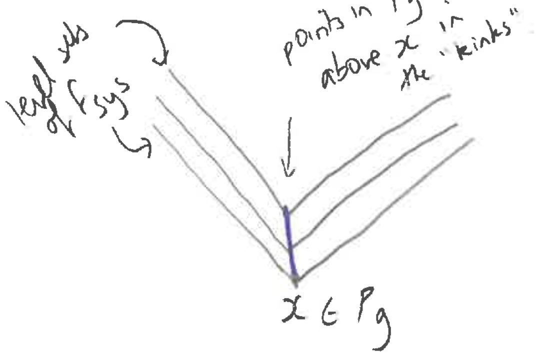
Thm (II) - P_g contains a (choice of) the unstable manifolds of the topological morse function f_{sys}

(Note: this implies Thurston's claim)

Important idea - study properties of "generic points" and then generalise to nongeneric points

For example, at a generic point $x \in T_g$, if $C' \subset C$ are sets of curves, for which $\mathrm{span}\{\nabla L(c)(x) \mid c \in C'\} = \mathrm{Span}\{\nabla L(c)(x) \mid c \in C\}$

Then C fills $\Rightarrow$ C' fills. If this were not so, $\exists$ curve c^* disjoint from all the curves in C' but intersecting curves in C. Changing the twist parameter around c^* would change the length of curves in C but not in C', giving a contradiction



Suppose $x \in P_g$ is not a critical point

The level sets of f_{sys} through x and hence above x have "kinks" in them

The points of P_g above x are contained in the kinks. These are places at which the gradients of the lengths of the systoles have the same span as the gradients of the lengths of the filling sets of systoles at x

Question - where are the stable manifolds of the critical points?

These can be described in terms of Schmutz Schaller's "sets of minima"

A set of closed geodesics C will be called eutactic at a point $x \in \mathcal{T}_g$ if for every derivation v in $T_x \mathcal{T}_g$, either

- $v\,L(c)(x) = 0 \;\; \forall \; c \in C$, or
- $\exists\, c_1, c_2 \in C$ s.t. $v\,L(c_1)(x) > 0$ and $v\,L(c_2)(x) < 0$

Note that when C eutactic at $x \Leftrightarrow \exists$ a length function $\sum_{c_i \in C} a_i L(c_i)$, $a_i > 0$ with minimum at x. This follows from convexity of length functions.

$\mathrm{Min}(C) := \{ x \in \mathcal{T}_g \mid C \text{ is eutactic at } x \}$

Exercise - Show that $\mathrm{Min}(C)$ is empty if C does not fill.

Thm (Schmutz Schaller) - C fills $\Leftrightarrow$ $\mathrm{Min}(C)$ is not empty
$\mathrm{Min}(C)$ is a cell iff the dimension of $\{ \nabla L(c) \mid c \in C \}$ stays constant over $\mathrm{Min}(C)$
Points on the boundary of $\mathrm{Min}(C)$ are in $\mathrm{Min}(C')$ for $C' \subsetneq C$

Reference - Schmutz Schaller "Systoles and topological Morse functions"

Exercise - Show that there can be at most one point in $\mathrm{Min}(C)$ at which the set of systoles is given by C

Exercise - Show that a point x in $\mathrm{Min}(C)$ at which the set of systoles is given by C is a critical point of f_{sys} of index r given by the dimension of $\mathrm{Span}\{ \nabla L(c)(x) \mid c \in C \}$
This amounts to showing that on a neighbourhood of x, the level sets are homeomorphic to level sets of $-x_1^2 - \dots - x_r^2 + x_{r+1}^2 + \dots + x_n^2$

Thm (Akrout) - $x \in \mathcal{T}_g$ is a critical point of f_{sys} iff $x \in \mathrm{Min}(C)$ where C is the set of systoles at x.