KHOVANOV HOMOLOGY AND 4-MANIFOLDS

CIPRIAN MANOLESCU

Abstract. We start by defining Khovanov homology and the Rasmussen knot invariant. We will then explore a few topological applications of the Rasmussen invariant: bounds on the slice genus of knots (including the Milnor conjecture), and the construction of an exotic $\mathbb{R}^4$. We will then move on to potential constructions of exotic 4-spheres, and to generalizations of the Rasmussen invariant to knots in other 3-manifolds. Finally, we will discuss the skein lasagna module, which is an invariant of knots in the boundary of an arbitrary 4-manifold. By recent work of Ren and Willis, this can detect exotic smooth structures on some compact 4-manifolds with boundary.

Contents

1. Khovanov homology

Smooth four-manifolds are usually studied using invariants from gauge theory, i.e. some partial differential equations coming from physics (e.g. ASD Yang-Mills, Seiberg-Witten). In contrast, Khovanov homology [Kho00] is an intrinsically “combinatorial” invariant of knots. Nevertheless, it has several four-dimensional applications, among which we mention the following:

- A new proof of the Milnor conjecture (Rasmussen)
- Existence of exotic $\mathbb{R}^4$s (Rasmussen-Gompf)
- A possible approach to disprove the smooth 4-dimensional Poincaré conjecture.

1.1. Definition of Khovanov homology.

We work with an oriented link $L \subset \mathbb{S}^3$, with planar diagram $D \subset \mathbb{R}^2$. Recall that Reidemeister moves of link diagrams characterise isotopy of links.

Proposition 1.1. The outline of Khovanov homology is as follows:

(1) For each link diagram D, there is a corresponding cochain complex
$$D \rightsquigarrow C(D) = \bigoplus_{i,j\in\mathbb{Z}} C^{i,j}(D).$$
This is equipped with boundary maps
$$d : C^{i,j}(D) \to C^{i+1,j}(D), \quad d^2 = 0.$$
(2) We see that the index i gives the homological grading. On the other hand, the index j defines the “quantum” or “Jones” grading.
(3) The Khovanov homology is defined by
$$Kh^{\bullet,\bullet}(L) = H^{\bullet,\bullet}(C(D)) = \bigoplus_{i,j} Kh^{i,j}(L).$$
We show that this is invariant under Reidemeister moves, and hence an invariant of L.

Remark 1.2. In Russian, Khovanov is pronounced a little more like Hovanov. (Technically the kh is a voiceless velar fricative.) On the other hand, we see above that our theory should really be called a *cohomology* theory rather than a *homology* theory. Therefore it would be more correct for our theory to be

Hovanov Khomology.

Why do we call the j index the “Jones” index? Given a chain complex, its Euler characteristic is defined to be
$$\chi(H^\bullet(C)) = \sum_i (-1)^i \operatorname{rk} H^i(C).$$
For a bigraded complex, we modify this definition to obtain a Laurent polynomial. In particular, for the Khovanov homology,
$$\chi(Kh^{\bullet,\bullet}(L)) = \sum_{i,j} (-1)^i q^j \operatorname{rk} Kh^{i,j}(L) = \widetilde{J}_L(q) \in \mathbb{Z}[q, q^{-1}].$$
Remarkably, this Euler characteristic is an “unnormalised Jones polynomial”:
$$\widetilde{J}_L(q) = (q + q^{-1}) J_L(q^2), \quad \text{for } J_L(t) \text{ the Jones polynomial.}$$

Definition 1.3. The Jones polynomial is the polynomial invariant that transformed knot theory, characterised by the following skein relations:

- $J_0(t) = 1$.
- $t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{1/2} - t^{-1/2})J_{L_0}(t)$.

Here L_+, L_-, and L_0 correspond to the same link with a single crossing modified: L_+ has the positive oriented crossing, L_- the negative crossing, and L_0 the un-crossing.

Exercise 1.4. *Prove that the the trefoil knot has Jones polynomial*

$$J_{3_1}(t) = t + t^3 - t^4.$$

Therefore $\widetilde{J}_{3_1}(q) = (q + q^{-1})(q^2 + q^6 - q^8) = q + q^3 + q^5 - q^9$.

It turns out the Khovanov homology of the trefoil can be described as in the following table:

$j \backslash i$	0	1	2	3	χ
9				$\mathbb{Z}$	-1
7				$\mathbb{Z}/2\mathbb{Z}$	0
5			$\mathbb{Z}$		1
3	$\mathbb{Z}$				1
1	$\mathbb{Z}$				1

reading the Euler characteristic off the table, it is clear that we recover

$$\chi(Kh^{\bullet,\bullet}(3_1)) = q + q^3 + q^5 - q^9$$

as required.

Before proceeding further, we establish some notation. Hereafter M will denote a *graded* abelian group. (Think: Jones grading.) To shift the grading up by ℓ, we write $M\{\ell\}$.

Now consider a cochain complex $C^0 \to C^1 \to C^2 \to \cdots$. (Think: homological grading.) Then $C[s]$ corresponds to shifting this grading up by s. That is,

$$C[s]^k = C^{k-s}.$$

Note that this convention is the opposite of some sources. We follow [BN02].

Definition 1.5. We now define the modules in the Khovanov complex. (The boundary maps will come later.)

(1) Let D be an oriented link diagram, with n crossings. Then each crossing is either positive or negative - we write $n = n_+ + n_-$ where n_+ is the number of positive crossings, and n_- the negative crossings.
(2) Regardless of orientation, any crossing can be resolved in exactly two ways:
$$\times \xrightarrow{0})(\qquad \times \xrightarrow{1} \asymp.$$
The resolutions are labelled 0 or 1 depending on the choice. Our diagram D can have all crossings resolved in 2^n ways, each resolution corresponding to some $\alpha \in \{0,1\}^n$. This is called the *cube of resolutions*. The resolution of D corresponding to α is denoted by D_α.
(3) Any two resolutions that differ by one choice (e.g. $(0,0,1,0,1)$ and $(0,0,0,0,1)$) have an edge between them. These are formally $\xi \in \{0,1,*\}^n$, with
$$\xi = (\xi_1, \ldots, \xi_n), \quad \xi_j = * \text{ for a unique } j.$$

In the above example, the edge would be

$$\xi = (0, 0, *, 0, 1).$$

(4) Define $V = \mathbb{Z} \oplus \mathbb{Z}$, spanned by v_+ and v_-. Any $\alpha \in \{0,1\}^n$ determines a module,

$$V_\alpha(D) = V^{\otimes k}\{|\alpha|\}, \quad |\alpha| = \sum_{\alpha_i}, k = \#\text{ circles in } D_\alpha.$$

Moreover, each $v_\pm$ has Jones grading ± 1. (Thus $v_+ \otimes v_+$ has Jones grading 2, and so on.)

(5) A pre-shifted complex is defined by $[|D|]^r = \bigoplus_{\alpha, |\alpha| = r} V_\alpha(D)$. The Khovanov complex is defined by shifting this complex:

$$C^{\bullet,\bullet}(D) = ([|D|]^\bullet[-n_-]\{n_+ - 2n_-\}, d).$$

(Of course we have yet to define the boundary map d.)

Definition 1.6. Now with the "objects" of the Khovanov complex defined, we define the maps.

(1) Every edge in the cube of resolutions (oriented from $|\alpha|$ to $|\alpha| + 1$) joins two resolutions whose number of components differs by 1. If the number of components *decreases*, the map is of type m:

$$m : \begin{cases} v_+ \otimes v_+ \mapsto v_+ \\ v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_- \\ v_- \otimes v_- \mapsto 0. \end{cases}$$

If the number of components *increases*, the map is of type Δ:

$$m : \begin{cases} v_+ \mapsto v_- \otimes v_+ + v_+ \otimes v_- \\ v_- \mapsto v_- \otimes v_-. \end{cases}$$

This defines the boundary map on two components, and on the rest the map is defined to be the identity. This gives d_ξ for each edge ξ.

(2) Define $(-1)^\xi = (-1)^{\sum_{i<j} \xi_i}$, where j is the location of $*$ in ξ. For example, $*00 \rightsquigarrow 1$, $1 * 1 \rightsquigarrow -1$.

(3) The differential d^r of the complex is defined by

$$d^r = \sum_{\xi \text{ starts at } \alpha, |\alpha| = r} (-1)^\xi d_\xi.$$

1.2. **Khovanov example: the right-handed trefoil.**

Example 1. As an example, we work through the trefoil knot. We first determine the cube of resolutions in terms of diagrams (figure 1) and then the actual maps (figure 2). Based on this information, the bigraded complex forms the following table.

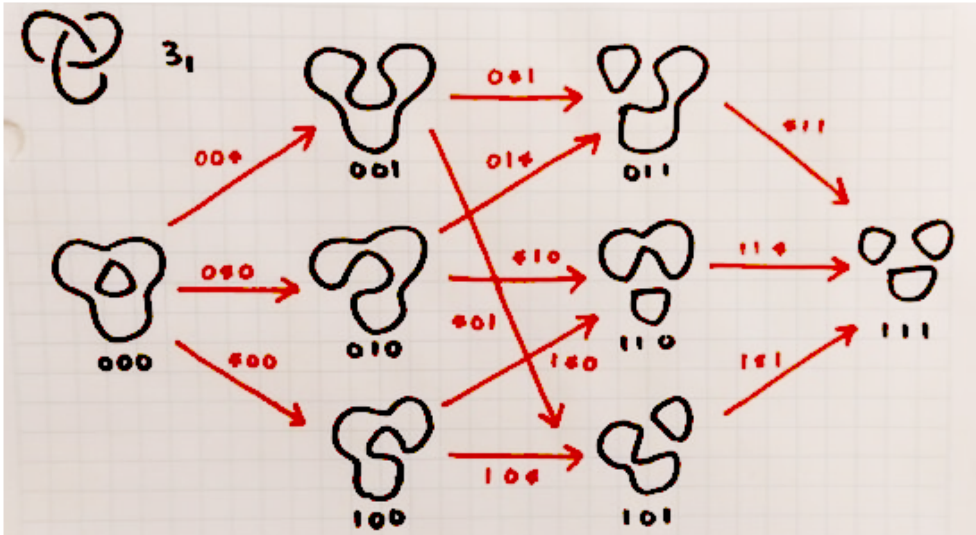


FIGURE 1. Cube of resolutions of the trefoil in terms of diagrams.

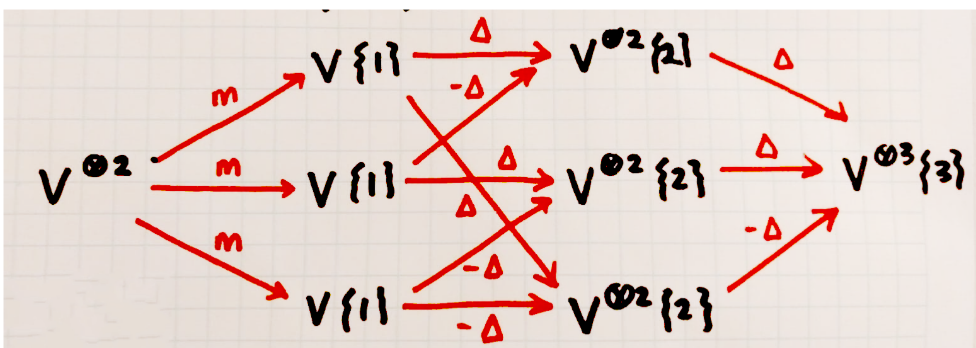


FIGURE 2. Khovanov complex of the trefoil.

j \ i	0	1	2	3
9				$v_+ \otimes v_+ \otimes v_+$
7			$v_+ \otimes v_+$ $v'_+ \otimes v'_+$ $v''_+ \otimes v''_+$	$v_+ \otimes v_+ \otimes v_-$ $v_+ \otimes v_- \otimes v_+$ $v_- \otimes v_+ \otimes v_+$
5	$v_+ \otimes v_+$	v_+ v'_+ v''_+	$v_+ \otimes v_-, v_- \otimes v_+$ $v'_+ \otimes v'_-, v'_- \otimes v'_+$ $v''_+ \otimes v''_-, v''_- \otimes v''_+$	$v_+ \otimes v_- \otimes v_-$ $v_- \otimes v_+ \otimes v_-$ $v_- \otimes v_- \otimes v_+$
3	$v_+ \otimes v_-,\ v_- \otimes v_+$	v_- v'_- v''_-	$v_- \otimes v_-$ $v'_- \otimes v'_-$ $v''_- \otimes v''_-$	$v_- \otimes v_- \otimes v_-$
1	$v_- \otimes v_-$			

Based on the above table and maps, we can compute homology groups. For example,

$$Kh^{3,9}(3_1) = Kh^{0,1}(3_1) = \mathbb{Z}, \quad Kh^{s,9}(3_1) = Kh^{t,1}(3_1) = 0, s \neq 3, t \neq 0.$$

These are immediate, since all boundary maps in the $j = 1$ and $j = 9$ gradings are trivial. We do not provide all calculations here, but we now determine the homology for the $j = 7$ grading. The potentially non-trivial homology occurs in the $(2, 7)$ and $(3, 7)$ cells, where we have a sequence isomorphic to

$$\cdots \to 0 \to \mathbb{Z}^3 \xrightarrow{d} \mathbb{Z}^3 \to 0 \to \cdots .$$

To determine the map d, we refer back to figure 2. Since each map is Δ, by also referring to the signs, we find the following:

$$\begin{aligned}
v_+ \otimes v_+ &\mapsto v_+ \otimes v_+ \otimes v_- + v_+ \otimes v_- \otimes v_+ \\
v_+' \otimes v_+' &\mapsto v_+ \otimes v_+ \otimes v_- + v_- \otimes v_+ \otimes v_+ \\
v_+'' \otimes v_+'' &\mapsto -v_+ \otimes v_- \otimes v_+ - v_- \otimes v_+ \otimes v_+.
\end{aligned}$$

Expressing this as a matrix, we have

$$d = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The second matrix is the Smith normal form of the matrix representing d. Using this change of basis, we have a sequence

$$\cdots \to 0 \to \mathbb{Z}^2 \oplus \mathbb{Z} \xrightarrow{\mathrm{id}_{\mathbb{Z}^2} \oplus 2} \mathbb{Z}^2 \oplus \mathbb{Z} \to 0 \to \cdots .$$

Therefore the homology can be read off as

$$Kh^{2,7}(3_1) = 0, \quad Kh^{3,7}(3_1) = \mathbb{Z}/2\mathbb{Z}.$$

Computing the rest of the table, we find that the Khovanov homology of the trefoil is as follows.

Khovanov homology of 3_1

j \ i	0	1	2	3
9				$\mathbb{Z}$
7				$\mathbb{Z}/2\mathbb{Z}$
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

Proposition 1.7. The Khovanov complex is genuinely a complex, that is, $d^2 = 0$.

Exercise 1.8. *Prove the above proposition by a case-by-case analysis.*

Exercise 1.9. *Compute the Khovanov homology of the Hopf link.*

Exercise 1.10. *Show that the Euler characteristic of Khovanov homology is the Jones polynomial.*

Exercise 1.11. *Show that links with an odd number of components can only have non-trivial Khovanov homology in the odd Jones degrees, while links with even components can only have non-trivial homology in the even degrees.*

1.3. **Isotopy invariance of Khovanov homology.** We have established that the Khovanov homology is truly a homology theory, but it has not yet been shown to be independent of the choice of diagram (of a given link). We must show that it is invariant under Reidemeister moves. We make use of the following lemma extensively (but first we need some definitions).

Definition 1.12. Let (C, d) be a complex, and $C' \subset C$ a subcomplex. This means that $d(C') \subset C'$. This also gives rise to a quotient complex, C/C'. We then obtain a short exact sequence

$$0 \to C' \to C \to C/C' \to 0$$

of complexes, which induces the usual long exact sequence on (co)homology

$$\cdots \to H^i(C') \to H^i(C) \to H^i(C/C') \to H^{i+1}(C') \to \cdots.$$

Lemma 1.13. If C' is acyclic, i.e. if $H^*(C') = 0$, then $H^*(C) \cong H^*(C/C')$. Similarly if $H^*(C/C') = 0$, then $H^*(C') \cong H^*(C)$.

This this notation established, we are ready to prove invariance under Reidemeister moves. Invariance under those of types 2 and 3 are left as an exercise, but we prove invariance of Khovanov homology under type 1 Reidemeister moves.

Proposition 1.14. Khovanov homology is invariant under type 1 Reidemeister moves.

Proof. Let D be a diagram with a crossing x that can be removed by a type 1 Reidemeister move. Write $[|D|]$ to denote the pre-shifted Khovanov complex of D. This factors as

$$C = [|D_0|] \xrightarrow{m} [|D_1|]\{1\}$$

where $[|D_0|]$ is a subcomplex which consists of all diagrams where x has a 0 resolution, and $[|D_1|]$ the subcomplex corresponding to x having the 1 resolution. Note that each diagram (vertex) in $[|D_0|]$ has an additional component L coming from the 0 resolution of x. On the other hand, the 1 resolution at x corresponds exactly to the type 1 Reidemeister move at x, so that $[|D_1|]$ is exactly the pre-shifted complex of D after applying a type 1 Reidemeister move.

The component L contributes two free elements v_+ and v_-. Consider the subcomplex C' of C, where the space associated to L is restricted to the span of v_+. Since the map m is defined by

$$m : v_+ \otimes w \mapsto w,$$

we have an isomorphism

$$C' = [|D_0|]_{v_+ \text{ at } L} \xrightarrow{m,\cong} [|D_1|]\{1\}.$$

The quotient complex C/C' is then given by

$$C/C' = [|D_0|]_{v_- \text{ at } L} \xrightarrow{m} 0.$$

But $[|D_0|]_{v_- \text{ at } L}$ is isomorphic to $[|D'|]\{-1\}$, where D' is D after the type 1 Reidemeister move has been applied. The shift $\{-1\}$ is to cancel the change in grading due to D' having one fewer crossing. But now by the previous lemma,

$$Kh(D) = [|D|]\{n_+ - 2n_-\} = [|D_0|]_{v_- \text{ at } L}\{n_+ - 2n_-\} = [|D'|]\{n_+ - 2n_- - 1\} = Kh(D').$$

$\square$

Proposition 1.15. The Khovanov homology is invariant under type 2 and type 3 Reidemeister moves.

Proof. These follow a similar argument. Details can be found in [BN02]. □

2. Lee homology and the Rasmussen invariant

2.1. **Generalising Khovanov homology: TQFTs.** Recall that any crossing in a link diagram can be *resolved* in two ways, giving either the 0 resolution or 1 resolution. If a link diagram is *oriented*, there is a unique way to resolve each crossing so that it agrees with the orientation.

Given a link diagram D with c components, there are 2^c possible orientations $\mathcal{O}$, each with a unique resolution $D_{\mathcal{O}}$.

Following [Kho06], let us explore the core of the invariance proofs to better understand Khovanov homology. A seemingly arbitrary choice was that each component of a resolution was associated to $\mathbb{Z} \oplus \mathbb{Z}$, and the maps m and Δ were not motivated either.

We now attempt to better understand the underlying ingredients of Khovanov homology, independent of the choices.

(1) The spaces were direct sums and tensor products of $V = \mathbb{Z} \oplus \mathbb{Z}$. These had maps $m : V \otimes V \to V$, and $\Delta : V \to V \otimes V$.
(2) $1 \in V$ is a unit for m, and $\varepsilon : V \to \mathbb{Z}$ defined by $\varepsilon(v_+) = 0$ and $\varepsilon(v_-) = 1$ is a counit for Δ.
(3) The map m itself is a commutative associative multiplication. Δ is a cocommutative coassociative comultiplication.
(4) The maps satisfy the *Frobenius law*, $\Delta \circ m = (m \otimes 1) \circ (1 \otimes \Delta)$.

These are exactly the ingredients of a commutative Frobenius algebra.

Proposition 2.1. ([Kho06]) To obtain a homological invariant of knots like Khovanov homology, we need V a commutative Frobenius algebra, free of rank 2.

Exercise 2.2. *Explain why the rank 2 condition is necessary.*

The easiest way to think about commutative Frobenius algebras is to consider (1+1)-dimensional topological quantum field theories (TQFTs).

Theorem 2.3. *There is an equivalence of groupoids*

$$\{\textit{TQFTs } 2\mathbf{Cob} \to \mathbf{Vect}_k\} \longleftrightarrow \mathbf{comFrob}_k.$$

We do not give a formal proof, but describe (1+1) dimensional TQFTs (i.e. functors $2\mathbf{Cob} \to \mathbf{Vect}_k$), and give examples of how they correspond to commutative frobenius algebras.

Remark 2.4. Here we describe TQFTs as functors into vector spaces, but in our context they are abelian groups.

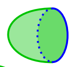
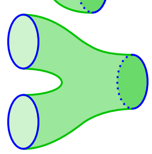
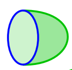
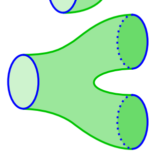
Definition 2.5. The category $2\mathbf{Cob}$ consists of (1+1) dimensional cobordisms. That is, the objects are closed one-manifolds (disjoint unions of circles), and the morphisms are cobordisms between them.

Definition 2.6. $\mathbf{Vect}_k$ is the category of vector spaces over a field k. A (1+1) dimensional TQFT is a functor that sends a 1-manifold to a vector space, and a cobordism to a homomorphism between them. Moreover, these respect the monoidal (tensor product) structure:

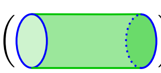
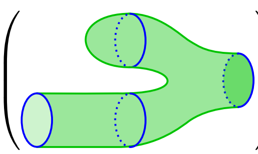
for X, Y 1-manifolds,

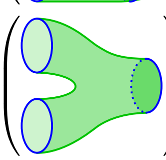
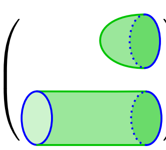
$$Z(X \sqcup Y) = Z(X) \otimes Z(Y).$$

The following table describes the four generators of $2\mathbf{Cob}$, and how they correspond to maps in a Frobenius algebra.

M	$Z(M)$	Interpretation
	$1 : k \to A$	unit
	$m : A \otimes A \to A$	multiplication
	$\varepsilon : A \to k$	counit
	$\Delta : A \to A \otimes A$	comultiplication

Properties such as associativity, commutativity, and the Frobenius law can all be verified by using the classification of surfaces. We give one example here:

$\mathrm{id}_A = Z\Big($  $\Big) = Z\Big($  $\Big)$

$= Z\Big($  $\Big) \circ Z\Big($  $\Big)$

$= m \circ (1 \otimes \mathrm{id}_A).$

Example 2. Khovanov homology can be expressed in a perhaps more intuitive form by using the perspective of Frobenius algebras. Write

$$V = \mathbb{Z}[x]/(x^2).$$

Define $m : V \otimes V \to V$ to be the usual product on $\mathbb{Z}[x]/(x^2)$. 1 is of course a unit. The map $\Delta : V \to V \otimes V$ defined by

$$\Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x$$

is a comultiplication, and $\varepsilon : V \to \mathbb{Z}$ defined by

$$\varepsilon(1) = 0, \quad \varepsilon(x) = 1$$

is a counit. This defines the Khovanov homology with the symbols $v_+ = 1$ and $v_- = x$.

Example 3. We can consider a *deformation*

$$V = \mathbb{Z}[x]/(x^2 - t),$$

over the ring $\mathbb{Z}[t]$. Let 1 and ε be as above, and m the usual multiplication on V. We define a modified comultiplication maps as follows:

$$\Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x + t(1 \otimes 1).$$

This also defines a Frobenius algebra. With the notation v_+, v_-, the multiplication and comultiplication maps can be written as

$$m : \begin{cases} v_+ \otimes v_+ & \mapsto v_+ \\ v_+ \otimes v_-, v_- \otimes v_+ & \mapsto v_- \\ v_- \otimes v_- & \mapsto tv_+ \end{cases}$$

$$\Delta : \begin{cases} v_+ & \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- & \mapsto v_- \otimes v_- + tv_+ \otimes v_+. \end{cases}$$

This gives rise to a complex $C'(D)$ of $\mathbb{Z}[t]$-modules. When $t = 0$ this is the Khovanov complex. When $t = 1$, this is the *Lee complex*, which we denote by $C_{\text{Lee}}(D)$. See [Lee05].

The corresponding integral homology theories are denoted by $Kh(K)$ and $\text{Lee}(K)$, called the Khovanov and Lee homologies respectively. We write $Kh'(K)$ to represent the Khovanov-Lee homology over $\mathbb{Z}[t]$.

2.2. **Lee homology and spectral sequences.** At the end of the previous section we introduced the Khovanov-Lee homology $Kh'(K)$, which is valued in $\mathbb{Z}[t]$. Evaluation at 0 gives the Khovanov homology, and evaluation at 1 the Lee homology.

If $C'(D)$ is the Khovanov-Lee complex, the boundary maps can be written as

$$d + t\Phi : C^i(D) \to C^{i+1}(D)$$

where the $t\Phi$ term can be read off the modified definitions of m and Δ. Here d is the usual Khovanov differential, which changes (i, j) by $(1, 0)$. On the other hand, Φ changes (i, j) by $(1, 4)$. We have not only that $d^2 = 0$, but also $(d + \Phi)^2 = 0$.

Observe that for any j, $C^{q \geq j}$ is closed under the action of $(d+\Phi)$. Therefore the Khovanov complex has a filtration

$$\cdots C^{q \geq j} \supset C^{q \geq j+1} \supset \cdots .$$

A filtered complex is exactly what gives rise to a *Spectral sequence*.

Definition 2.7. A *spectral sequence* is a collection of pages. I.e. a collection of complexes (E^r, d^r), where $d^r \circ d^r = 0$, and $E^{r+1} = H^\bullet(E^r, d^r)$.

Example 4. In our context, the filtration of the Lee complex gives a spectral sequence with

$$\begin{aligned} E^1 &= (C^\bullet, d), \\ E^2 &= (H^\bullet(E^1), \Phi^*) = (Kh(K), \Phi^*), \\ \Rightarrow E^\infty &= H^\bullet(C, d + \Phi) = \text{Lee}(K). \end{aligned}$$

The important result being used is that every filtered complex gives a spectral sequence which converges to the homology of the original complex.

Example 5. Write $Kh(K; \mathbb{Q})$ to denote $Kh(K) \otimes_{\mathbb{Z}} \mathbb{Q}$. We write out some of the pages of the rational spectral sequence corresponding to the trefoil knot.

E^1 page for 3_1

$j \backslash i$	0	1	2	3
9				$\mathbb{Q}$
7			$\mathbb{Q}^3$	$\mathbb{Q}^3$
5	$\mathbb{Q}$	$\mathbb{Q}^3$	$\mathbb{Q}^6$	$\mathbb{Q}^3$
3	$\mathbb{Q}^2$	$\mathbb{Q}^3$	$\mathbb{Q}^3$	$\mathbb{Q}$
1	$\mathbb{Q}$			

E^2 page for 3_1

$j \backslash i$	0	1	2	3
9				$\mathbb{Q}$
7				
5			$\mathbb{Q}$	
3	$\mathbb{Q}$			
1	$\mathbb{Q}$			

E^3 page for 3_1

$j \backslash i$	0	1	2	3
9				
7				
5				
3	$\mathbb{Q}$			
1	$\mathbb{Q}$			

E^∞ page for 3_1

$j \backslash i$	0	1	2	3
9				
7				
5				
3	$\mathbb{Q}$			
1	$\mathbb{Q}$			

Observe that $\mathrm{Lee}(3_1) \cong \mathbb{Q}^2 = \mathbb{Q}^{2^c}$ where c is the number of components of the trefoil knot. This is a general result.

Theorem 2.8. $\mathrm{Lee}(L;\mathbb{Q}) \cong \mathbb{Q}^{2^c}$, *where c is the number of components of L.*

Proof. To prove this, we define a new basis for V. Specifically, define a and b by

$$a = v_+ + v_-, \quad b = v_- - v_+.$$

The Lee complex boundary maps are then induced by

$$m : \begin{cases} a \otimes a & \mapsto 2a \\ a \otimes b, b \otimes a & \mapsto 0 \\ b \otimes b & \mapsto -2b \end{cases}, \quad \Delta : \begin{cases} a & \mapsto a \otimes a \\ b & \mapsto b \otimes b. \end{cases}$$

Claim: $\mathrm{Lee}(L)$ is generated by the "canonical generators" which we now construct.

(1) Let $\mathcal{O}$ be an orientation of a diagram D of L. (There are 2^c choices of orientation).
(2) There is a unique resolution $D_{\mathcal{O}}$ of D which is compatible with the orientation. This is a disjoint union of circles.
(3) Let $C \in D_{\mathcal{O}}$. Define $\tau(C) \in \mathbb{Z}/2\mathbb{Z}$ to be the number of circles separating C from infinity, plus 1 if C is oriented clockwise.
(4) Define $g_C = a$ if $\tau(C) = 0$, and $g_C = b$ if $\tau(C) = 1$. Define

$$S_{\mathcal{O}} = \bigotimes_{C \in D_{\mathcal{O}}} g_C.$$

The claim is that the $S_{\mathcal{O}}$ (of which there are exactly 2^c) are generators of $\mathrm{Lee}(L)$. We break this proof into two pieces.

Lemma 2.9. The collection of $S_{\mathcal{O}}$ forms an orthonormal set in $\mathrm{Lee}(D)$, so that $\dim \mathrm{Lee}(D) \geq 2^c$.

We first note that if any two circles have the same label (either a or b) then they *cannot* meet at a resolved vertex. It follows that each $S_{\mathcal{O}}$ is a cycle, i.e. $(d+\Phi)S_{\mathcal{O}} = 0$. Therefore $[S_{\mathcal{O}}] \in \mathrm{Lee}(D)$.

Now that it has been established that these are all elements of $\mathrm{Lee}(D)$, we equip $C_{Lee}(D)$ with an inner product by declaring that the $\{a \otimes a \otimes b \otimes \cdots\}$ is an orthonormal basis. The map $d + \Phi$ has an adjoint with respect to the inner product, namely

$$(d+\Phi)^* : \begin{cases} a \otimes a & \mapsto a \\ b \otimes b & \mapsto b \\ a & \mapsto 2a \otimes a \\ b & \mapsto -2b \otimes b \\ \text{rest} & \mapsto 0. \end{cases}$$

Then one can show that $(d+\Phi)^* S_{\mathcal{O}} = 0$. But this implies that each $S_{\mathcal{O}}$ descends to an element of $\mathrm{Lee}(D)$ while preserving pairwise orthogonality, since

$$\mathrm{Lee}(D) = H^*(S_{\mathrm{Lee}}(D)) = \ker(d+\Phi)/\operatorname{im}(d+\Phi) \cong \ker(d+\Phi) \cap \ker(d+\Phi)^*.$$

In summary this proves that $\dim \mathrm{Lee}(D) \geq 2^c$.

Exercise 2.10. *Verify that* $(d+\Phi)^* S_{\mathcal{O}} = 0$.

Lemma 2.11. In fact, $\dim \mathrm{Lee}(D) = 2^c$.

To see this, it remains to prove that $\dim \mathrm{Lee}(D) \leq 2^c$. This follows from an induction on the number of crossings of D. Let D_0 and D_1 be 0 and 1 resolutions of a single crossing x in D. Then $C_{\mathrm{Lee}}(D_1) \subset C_{\mathrm{Lee}}(D)$ is a subcomplex. This gives rise to a long exact sequence

$$\cdots \to \mathrm{Lee}(D_1) \to \mathrm{Lee}(D) \to \mathrm{Lee}(D_0) \to \mathrm{Lee}(D_1) \to \cdots.$$

There are two cases to consider. First suppose the two strands crossing at x belong to distinct components of D. Then D_0 and D_1 each have $c-1$ components each. By the inductive hypothesis,

$$\dim \mathrm{Lee}(D_0) = \dim \mathrm{Lee}(D_1) = 2^{c-1}.$$

By the long exact sequence,

$$\dim \mathrm{Lee}(D) \leq \dim \mathrm{Lee}(D_0) + \dim \mathrm{Lee}(D_1) = 2^{c-1} + 2^{c-1} = 2^c.$$

This proves the first case. For the second case, suppose the strands meeting at x belong to the same component. Then one of D_0, D_1 has c components, and the other $c+1$ components. (Assume without loss of generality that D_0 has c components, and D_1 has $c+1$ components.) The induced map

$$\mathrm{Lee}(D_0) \xrightarrow{i} \mathrm{Lee}(D_1)$$

is then injective. Therefore $\dim \mathrm{Lee}(D) = \dim \operatorname{coker} i = 2^c$. (The size of the cokernel can be verified by showing that the canonical generators of $\mathrm{Lee}(D_0)$ map to half of those of $\mathrm{Lee}(D_1)$.) The other case is formally dual, with a surjective map and so on.

This completes the proof that $\dim \mathrm{Lee}(D) = 2^c$. Therefore $\mathrm{Lee}(D) = \mathbb{Q}^{2^c}$. □

2.3. **Rasmussen's s-invariant.** Let $K \subset \mathbb{S}^3$ be a knot. The *slice genus* is the minimal genus of a surface bound by K in a 4-ball:

$$g_s(K) := \min\{g(\Sigma) : \Sigma \subset B^4 \text{ properly smoothly embedded}, \partial\Sigma = K.\}$$

Theorem 2.12 (Milnor conjecture)**.** *Let K denote the p, q-torus knot, for p, q coprime. Then $g_s(K) = (p-1)(q-1)/2$.*

- The original proof, due to Kronheimer and Mrowka in 1993, used Yang-Mills gauge theory.
- Several years later, Kronheimer and Mrowka proved the result using Seiberg-Witten gauge theory.
- In 2004, Rasmussen [Ras10] gave a "combinatorial" proof. This is what we'll discuss here.

Exercise 2.13. *Show that the slice genus is a lower bound for the unknotting number (the minimal number of crossing changes needed to turn a knot into the unknot).*

Exercise 2.14. *Show that the (p, q)-torus knot K can be unknotted in $(p-1)(q-1)/2$ moves.*

Open question 1. Is there an algorithm that computes the slice genus of a knot starting from a knot diagram?

For $K = T(p, q)$, we have

$$g_s(K) \leq u(K) \leq \frac{(p-1)(q-1)}{2}.$$

On the other hand, today we introduce Rasmussen's s-invariant $s \in 2\mathbb{Z}$. We show that

(1) $|s(K)| \leq 2g_s(K)$.
(2) $s(K) = (p-1)(q-1)$.

Therefore by combining 1 and 2,

$$\frac{(p-1)(q-1)}{2} = \frac{s(K)}{2} \leq g_s(K).$$

This will prove the Milnor conjecture.

To give a definition of the s-invariant, we consider Khovanov and Lee homology with rational coefficients. Recall that a diagram D for an arbitrary knot K determines a complex $(C(D), d)$ called the *Khovanov complex*. This in turn determines a homology theory which is invariant under Reidemeister moves, which we call the *Khovanov homology* $Kh(K)$. By perturbing the boundary maps, we obtain a different complex $(C_{\text{Lee}}(D), d + \Phi)$ called the *Lee complex*, and this also gives an invariant homology theory $\text{Lee}(K)$. Moreover,

$$Kh(K) \Rightarrow \text{Lee}(K) = \mathbb{Q} \oplus \mathbb{Q}.$$

Although $\text{Lee}(K)$ is almost trivial, the two surviving copies of $\mathbb{Q}$ have Jones (q) gradings. Let $s_{\max} \geq s_{\min}$ be the Jones gradings of the two copies. Since K is a knot, $s_{\max}, s_{\min}$ are both odd. Moreover, the isomorphism type of the spectral sequence is an invariant of K, so $s_{\max}$ and $s_{\min}$ are also invariants. It turns out that $s_{\max} = s_{\min} + 2$, so we define the *Rasmussen invariant* to be

$$s(K) = s_{\max}(K) - 1 = s_{\min}(K) + 1 \in 2\mathbb{Z}.$$

While this is the idea, we now give a formal definition of $s_{\max}(K)$ and $s_{\min}(K)$.

Definition 2.15. Let D be a diagram of a knot K. Then $C_{\text{Lee}}(D)$ has a filtration

$$C_{\text{Lee}}(D) \supset \cdots \supset C_{\text{Lee}}^{q\geq j}(D) \supset C_{\text{Lee}}^{q\geq j+1}(D) \supset \cdots \supset 0,$$

since the map $d+\Phi$ changes the bidegree (i,j) by $(1,0)$ (by d) and by $(1,4)$ (by Φ). For each j, we define

$$I_j = \text{im}(H^*(C_{\text{Lee}}^{q\geq j}(D)) \hookrightarrow H^*(C_{\text{Lee}}(D))) \subset \text{Lee}(D).$$

Note that there exists some N so that we need only consider $-N \leq j \leq N$ for j as above. Then

$$\text{Lee}(D) = I_{-N} \supset I_{-N+1} \supset \cdots \supset I_N = 0.$$

This induces a grading on $\text{Lee}(D)$, by

$$\text{Lee}(D) = \bigoplus_j I_j/I_{j+1}.$$

Now any class $[x]$ in $\text{Lee}(D)$ has a *grading*, namely

$$q([x]) = \max\{j : q(x) = j, x \in [x]\}, \quad q(x) = \max\{j : x \in C_{\text{Lee}}^{q\geq j}(D)\}.$$

In particular, we define

$$s_{\max}(K) = \max\{q([x]) : [x] \in \text{Lee}(K), [x] \neq 0\}, \quad s_{\min}(K) = \min\{q([x]) : [x] \in \text{Lee}(K), [x] \neq 0\}.$$

Given these formal definitions of the invariants $s_{\min}$ and $s_{\max}$, the definition of the Rasmussen invariant rests on the following result:

Proposition 2.16. Let K be a knot. Then $s_{\max}(K) = s_{\min}(K) + 2$.

Note that this justifies the definition of the Rasmussen invariant to be $s(K) = s_{\max} - 1 = s_{\min} + 1$.

Proof. The main idea of the proof is to study the two canonical generators $S_{\mathcal{O}}$ and $S_{\overline{\mathcal{O}}}$ of $\text{Lee}(K) = \mathbb{Q} \oplus \mathbb{Q}$ from the previous section. We use combinations of these to first show that $s_{\max} - s_{\min} \equiv 2 \mod 4$. (In particular, they differ by at least 2.) Whe they show that they differ by at most 2, to obtain the desired equality.

First note that for a knot K, we already know that C_{Lee} is supported only in odd quantum gradings. Define

$$\begin{aligned} C_{\text{Lee,even}}(D) &= \text{ generated by elements with } q = 1 \mod 4 \\ C_{\text{Lee,odd}}(D) &= \text{ generated by elements with } q = 3 \mod 4 \end{aligned}$$

Note that d preserves the q grading while Φ changes it by 4, so $d+\Phi$ preserves q modulo 4. In particular, $C_{\text{Lee}}(D) = C_{\text{Lee,even}}(D) \oplus C_{\text{Lee,odd}}(D)$, where the direct summands are preserved by $d+\Phi$. It follows that

$$\text{Lee}(K) = \text{Lee}_{\text{even}}(K) \oplus \text{Lee}_{\text{odd}}(K).$$

We now make use of this direct summand structure. Define $\iota : C_{\text{Lee}}(D) \to C_{\text{Lee}}(D)$ to act by 1 on $C_{\text{Lee,even}}$, and -1 on $C_{\text{Lee,odd}}$. Then any $x \in C_{\text{Lee}}(D)$ decomposes as

$$x = \frac{x+\iota(x)}{2} + \frac{x-\iota(x)}{2},$$

where the first term lives in $C_{\mathrm{Lee,even}}$, and the second in $C_{\mathrm{Lee,odd}}$. We further define $i : V \to V$ by $i(v_-) = v_-$ and $i(v_+) = -v_+$. Then $\iota = \pm i^{\otimes n}$. Moreover, setting $a = v_- + v_+$ and $b = v_- - v_+$ as an alternative basis, we have $i(a) = b$ and $i(b) = a$.

We now analyse $S_{\mathcal{O}}$ and $S_{\overline{\mathcal{O}}}$ more closely. These actually arise from the same diagram! Switching all orientations in a diagram and then resolving gives rise to the same resolution, but with all orientations switched. Therefore

$$i([S_{\mathcal{O}}]) = \pm[S_{\overline{\mathcal{O}}}].$$

It follows that the canonical even/odd decomposition is given by

$$[S_{\mathcal{O}}] = \frac{[S_{\mathcal{O}}] + [S_{\overline{\mathcal{O}}}]}{2} + \frac{[S_{\mathcal{O}}] - [S_{\overline{\mathcal{O}}}]}{2}.$$

This proves that the two copies of $\mathbb{Q}$ in $\mathrm{Lee}(K) = \mathbb{Q} \oplus \mathbb{Q}$ live in different gradings mod 4, as required. That is,

$$s_{\max} - s_{\min} \equiv 2 \mod 4.$$

In particular, $s_{\max}$ is at least $s_{\min} + 2$.

Finally we show that $s_{\max}$ is at most $s_{\min} + 2$. This follows from a similar calculation as showing that the Khovanov homology is invariant under Reidemeister moves. Let D' denote the diagram of K obtained by adding a crossing via a type 1 move. Then

$$C_{\mathrm{Lee}}(D') = \Big(C_{\mathrm{Lee}}(D \sqcup 0) \to C_{\mathrm{Lee}}(D)\Big).$$

This can expressed as the short exact sequence

$$0 \to C_{\mathrm{Lee}}(D) \to C_{\mathrm{Lee}}(D') \to C_{\mathrm{Lee}}(D \sqcup 0_1) \to 0$$

which induces a long exact sequence in homology

$$\cdots \to \mathrm{Lee}(K) \to \mathrm{Lee}(K) \to \mathrm{Lee}(K \sqcup 0_1) \xrightarrow{\partial} \mathrm{Lee}(K) \to \cdots$$

where $\mathrm{Lee}(K \sqcup 0_1) \cong \mathrm{Lee}(K) \otimes V$. Depending on labels near the crossing x of D' obtained from the type 1 move, we denote the two canonical generators of $C_{\mathrm{Lee}}(D)$ by s_a and s_b. Without loss of generality, $q(s_a - s_b) = s_{\max}$, and $q(s_a + s_b) = s_{\min}$. One can verify that

$$\partial([s_a - s_b] \otimes [a]) = [s_a],$$

from which it follows that

$$s_{\max} - 1 = q([s_a - s_b] \otimes [a]) \leq q([s_a]) + 1 = s_{\min} + 1.$$

Therefore $s_{\max} \leq s_{\min} + 2$ as required. Earlier we established that $s_{\max} \geq s_{\min} + 2$, so this completes the proof that $s_{\max} = s_{\min} + 2$. □

In summary the Rasumussen s-invariant is well defined.

Exercise 2.17. *If $m(K)$ is the mirror of a knot K, show that $s(m(K)) = -s(K)$.*

3. Applications

3.1. **The s-invariant bounds the slice genus.** Recall that the proof strategy for proving Milnor's conjecture is two establish the following two facts:

(1) $|s(K)| \leq 2g_s(K)$.
(2) $s(K) = (p-1)(q-1)$.

We now prove the first of these.

Proposition 3.1. For a knot K, $|s(K)| \leq 2g_s(K)$.

Proof. The idea is to use the functoriality of Khovanov-homology under link cobordisms. Let L_0 and L_1 be links, with $\Sigma \subset \mathbb{R}^3 \times [0,1]$ a cobordism between them. We induce maps $F_\Sigma : Kh(L_0) \to Kh(L_1)$, and $F_{\Sigma,\mathrm{Lee}} : \mathrm{Lee}(L_0) \to \mathrm{Lee}(L_1)$ and use their properties.

By Morse theory, Σ splits into building blocks with one critical point each, of indices 0,1, or 2. (These are with respect to the height function $\pi : \Sigma \to [0,1]$.) If D_0 is a diagram for L_0, and D_1 a diagram for L_1, then D_0 and D_1 must be related by a sequence of Reidemeister moves and *Morse moves*. By a Morse move, we mean the change in level set as we pass a critical point. Explicitly,

- Passing an index 0 critical point corresponds to taking a disjoint union with an unknot.
- Passing an index 1 critical point corresponds to locally swapping two horizontal arcs with two vertical arcs or vice versa.
- Passing an index 2 critical point corresponds to destroying a disjoint unknot.

Therefore to define a map $F_\Sigma : Kh(L_0) \to Kh(L_1)$ we must define maps corresponding to each Reidemeister or Morse move, and glue them together. We must then verify that the map F_Σ is an invariant of Σ, that is, it must not depend on the choice of Morse function/decomposition.

Explicitly, to each move, we associate the following maps:

- For each Reidemeister move D_i to D_{i+1}, there is a canonical isomorphism $F_i : Kh(D_i) \to Kh(D_{i+1})$ as used in the proof of the well-definedness of Khovanov homology.
- For an index 0 Morse move D_i to $D_{i+1} = D_i \sqcup 0_1$, define $F_i : Kh(D_i) \to Kh(D_{i+1})$ to send $1 \mapsto v_+$ on the 0_1 component, and the identity elsewhere.
- For an index 1 Morse move D_i to D_{i+1}, define F_i to be m or Δ at the location of the move depending on the change in the number of components, and the identity elsewhere.
- For an index 2 Morse move D_i to D_{i+1}, define F_i to send v_- to 1 and v_+ to 0 at the location of the move, and the identity elsewhere.

If $D_1, \dots, D_n$ are a sequence of diagrams from L_0 to L_1, the composition of the F_i defines the map $F_\Sigma : Kh(L_0) \to Kh(L_1)$. We claim without proof that the map F_Σ is well defined up to sign as an invariant of Σ. That is, the map does not depend on the decomposition of Σ. (This is a theorem of Khovanov and Jacobsson.) Note that this fact is not actually needed for the proof!

A similar construction works for the Lee homology! We obtain maps $F_{\Sigma,\mathrm{Lee}} : \mathrm{Lee}(L_0) \to \mathrm{Lee}(L_1)$ as well.

Suppose Σ is an oriented cobordism from L_0 to L_1, such that every component of Σ has a boundary component on L_0. Then by verifying each Reidemeister and Morse move, one can show that $F_{\Sigma,\mathrm{Lee}}([S_{\mathcal{O}|_{L_0}}])$ is a non-zero multiple of $[S_{\mathcal{O}|_{L_1}}]$, where $\mathcal{O}$ is an orientation of Σ. This means that if Σ is a connected cobordism between knots K_0 and K_1, then $F_{\Sigma,\mathrm{Lee}} : \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q}$ is an isomorphism.

Suppose Σ has genus $g = g_s(K)$ for a knot K. Then removing a disk D, $\Sigma' = \Sigma - D$ is a genus g cobordism from K to the unknot. Now F_Σ and $F_{\Sigma,\mathrm{Lee}}$ are maps from Khovanov and Lee homologies of K to that of the unknot. How do they change the quantum gradings? Observe that Reidemeister moves leaves q invariant, while Morse moves of index 0 and 2 change q by $+1$, and Morse moves of index 1 change q by -1. Therefore F_Σ changes q by $\chi(\Sigma')$, and $F_{\Sigma,\mathrm{Lee}}$ by at least $\chi(\Sigma')$.

Let $x \in \mathrm{Lee}(K) - \{0\}$ be a class attaining $q(x) = s_{\max} = s + 1$. Then

$$1 \geq q(F_{\Sigma'}(x)) \geq q(x) + \chi(\Sigma') = s + 1 - 2g_s(K).$$

The first inequality is because $F_{\Sigma'}(x)$ lives in $\mathrm{Lee}(0_1)$. Therefore

$$s \leq 2g_s(K)$$

as required.

Finally for the general result, consider the mirror $\overline{K}$ of K. This bounds a surface $\overline{\Sigma}$ with the same genus as Σ. But now $s(\overline{K}) = -s(K)$, so

$$-s(K) \leq 2g = 2g_s(K).$$

Combining this with the previous result, we can bound $g_s(K)$ below by $|s(K)|/2$ as required. □

Exercise 3.2. *Suppose Σ is an oriented cobordism from L_0 to L_1, such that every component of Σ has a boundary component on L_0. Show that $F_{\Sigma,\mathrm{Lee}}([S_{\mathcal{O}|_{L_0}}])$ is a non-zero multiple of $[S_{\mathcal{O}|_{L_1}}]$, where $\mathcal{O}$ is an orientation of Σ.*

3.2. **Combinatorial proof of Milnor's conjecture.** In the previous section we defined the Rasmussen s-invariant for knots, and showed that it satisfies

$$|s(K)| \leq 2g_s(K).$$

Today we show that $s(T_{p,q}) = (p-1)(q-1)$. This will be a special case of the calculation of s for *positive knots.*

Definition 3.3. A knot K is *positive* if it has an oriented diagram with only positive crossings.

For example, a torus knot is a positive knot.

If D is a positive diagram of a positive knot, then its oriented resolution D_0 is in fact the zero resolution! Our final result needed to prove the Milnor conjecture is the following:

Proposition 3.4. If K has a positive diagram D with n crossings, and D_0 consists of k circles (components), then $s(K) = n + 1 - k$.

Proof. Recall that the s-invariant has the explicit formula

$$s(K) = s = \frac{q([S_{\mathcal{O}}] + [S_{\overline{\mathcal{O}}}]) + q([S_{\mathcal{O}}] - [S_{\overline{\mathcal{O}}}])}{2}.$$

Here one of $[S_{\mathcal{O}}] \pm [S_{\overline{\mathcal{O}}}]$ has degree $s+1$, and the other has degree $s-1$. Moreover,

$$q([S_{\mathcal{O}}]) = q([S_{\overline{\mathcal{O}}}]) = s - 1.$$

Explicitly, the left side is defined to be

$$q([S_{\mathcal{O}}]) = \max\{q(x) : x \text{ is homologous to } S_{\mathcal{O}}\} = \max\{q(x) : x = S_{\mathcal{O}} + d\alpha\}.$$

But $S_{\mathcal{O}}$ lives in the lowest homological grading (since our resolution D_0 is the zero resolution). Therefore there is no non-trivial α that can map to $d\alpha$, i.e. there is a unique class homologous to $S_{\mathcal{O}}$. Hence

$$q([S_{\mathcal{O}}]) = q(S_{\mathcal{O}}), \quad S_{\mathcal{O}} = (v_+ \pm v_-) \otimes (v_+ \pm v_-) \otimes \cdots.$$

The expression on the right has k factors. But this necessarily lies in the same quantum grading as $\otimes^k v_-$. Therefore by the definition of the Khovanov homology,

$$q(S_{\mathcal{O}}) = -k + (n_+ - 2n_-) = n - k = s - 1.$$

The claimed result follows. □

Example 6. The standard diagram of the torus knot $T_{p,q}$ consists of $p(q-1)$ positive crossings, and its 0 resolution consists of q circles. Therefore $s(T_{p,q}) = p(q-1) - q + 1 = (p-1)(q-1)$.

We can now pull together a proof of Milnor's conjecture using just Rasmussen's s-invariant.

Proposition 3.5. The slice genus of the torus knot $T_{p,q}$ is

$$g_s(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

Proof. The standard diagram can me unknotted in $(p-1)(q-1)/2$ moves, giving

$$g_s(T_{p,q}) \leq u(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}.$$

Conversely, the Rasmussen s-invariant gives

$$\frac{(p-1)(q-1)}{2} = \frac{s(T_{p,q})}{2} \leq \frac{2g_s(T_{p,q})}{2} = g_s(T_{p,q}).$$

Therefore we have equality as required. □

3.3. **Combinatorial proof of the existence of exotic $\mathbb{R}^4$s.** Another application of Khovanov homology is that it gives a novel proof of the existence of exotic smooth structures on $\mathbb{R}^4$, without requiring any gauge theory. More concretely, our proof outline is as follows:

(1) Use Rasmussen's s invariant together with a result of Freedman to find knots that are *topologically slice* but not slice.
(2) Introduce the *trace embedding lemma*.
(3) Use the trace embedding lemma with manifolds obtained from a knot as in 1 to construct an open manifold which is homeomorphic to $\mathbb{R}^4$ but cannot be diffeomorphic to it. A result of Freedman states that all open 4-manifolds admit admit smooth structures, so it must then be an exotic $\mathbb{R}^4$.

We now carry out the details. First we introduce relevant definitions and results to establish point 1.

Definition 3.6. A knot K is *slice* (or *smoothly slice*) if $g_s(K) = 0$. That is, if there exists a smooth properly embedded disk $D \subset B^4$ such that $\partial D = K \subset \mathbb{S}^3$.

By replacing the notion of a smooth embedding with a topological embedding, we obtain a weaker condition.

Definition 3.7. A knot K is *topologically slice* if there exists a locally flat topologically embedded disk $D \subset B^4$ such that $\partial D = K \in \mathbb{S}^3$. This means that there is a topological embedding $\varphi : (D^2 \times D^2, \partial D^2 \times D^2) \to (B^4, \partial B^4 = \mathbb{S}^3)$ such that $\varphi(\partial D^2 \times 0) = K$. Then $\varphi(D^2 \times 0)$ is a topologically embedded disk which is *locally flat*.

Remark 3.8. The local flatness condition is necessary to obtain an "interesting" definition: without this assumption, all knots would be topologically slice by taking the embedded disk to be a cone over the knot.

We now use the following theorem of Freedman to establish the existence of topologically slice knots which aren't slice:

Theorem 3.9. *If $\Delta_K(t) = 1$, then K is topologically slice.*

Here Δ is the Alexander polynomial. One method of proliferating knots with trivial Alexander polynomials is to take the *Whitehead double* $\mathrm{Wh}(K)$ of a knot K.

Exercise 3.10. *Show that $\Delta_{Wh(K)}(t) = 1$ for any K.*

In particular,

$$\Delta_{\mathrm{Wh}(T_{2,3})} = 1.$$

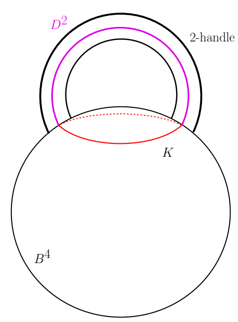
However, we can also compute the s-invariant for any given knot - this particular knot satisfies $s(\mathrm{Wh}(T_{2,3})) = 2$. Since $s/2$ is a lower bound for the slice genus, we know that $g_s(\mathrm{Wh}(T_{2,3})) \geq 1$. Therefore $\mathrm{Wh}(T_{2,3})$ is not slice, despite being topologically slice!

Remark 3.11. In fact, $s(\mathrm{Wh}(m(T_{2,3}))) = 0$. This is because the "clasp" in the Whitehead double is not mirrored, i.e. the Mirror of a Whitehead double is not the Whitehead double of a mirror. In general, it is known that all Whitehead doubles of torus knots are not slice, but such a result is not known for mirrors of torus knots!

Open question 2. Is $\mathrm{Wh}(m(T_{2,3}))$ slice?

The next ingredient in our proof of the existence of exotic smooth structures on $\mathbb{R}^4$ is the trace embedding lemma. This relates the properties of being slice (or topologically slice) to embeddings of "traces of 0 surgeries of knots".

We establish some notation. Let K be a knot, and $\mathbb{S}^3_n(K)$ the manifold obtained by n-surgery along $K \subset \mathbb{S}^3$. Let $X_n(K)$ be the manifold obtained from B^4 by attaching an n-framed 2-handle along K. Then $X_n(K)$ is called the *trace* of the n-surgery along K, and satisfies $\partial X_n(K) = \mathbb{S}^3_n(K)$. Alternatively $X_n(K)$ can be thought of a cobordism from $\mathbb{S}^3$ to $\mathbb{S}^3_0(K)$ (with the $\mathbb{S}^3$ end capped).



Exercise 3.12. *If K is the unknot, show that $\mathbb{S}^3_0(K) = \mathbb{S}^1 \times \mathbb{S}^2$, and $X_0(K) = (D^2 \times \mathbb{S}^2) - B^4$.*

The *trace embedding lemma* takes two forms for each notion of sliceness:

Proposition 3.13. $K \subset \mathbb{S}^3$ is (topologically) slice if and only if $X_0(K)$ embeds smoothly (locally flat topologically) in $\mathbb{S}^4$.

We only prove the smooth case, as the locally flat case is similar.

Proof. $\Rightarrow$. If K is slice, it bounds a disk D smoothly embedded in B^4. One can verify that

$$\mathbb{S}^4 = X_0(K) \sqcup_{\mathbb{S}^3_0(K)} (B^4 - \mathrm{int}(\mathrm{nbhd}(D))).$$

In particular, $X_0(K)$ embeds smoothly in $\mathbb{S}^4$.

$\Leftarrow$. We start by constructing an embedding $F : \mathbb{S}^2 \to X_0(K)$, so that $F(\mathbb{S}^2)$ is of the form $D \sqcup_K C$ where D is a smooth disk (and the core of the 2-handle of $X_0(K)$) and C

has a single cone point. By assumption, there is a smooth embedding $i : X_0(K) \to \mathbb{S}^4$. Therefore we have an embedding $i \circ F : \mathbb{S}^2 \to \mathbb{S}^4$ which is smooth away from the cone point. Removing a small ball around the cone point, the image of $i \circ F$ restricts to a smoothly embedded disk in B^4, whose boundary is K. □

The final step is to combine this result with the previous example of a non-slice topologically slice knot to construct an exotic $\mathbb{R}^4$.

Theorem 3.14. *There exist exotic* $\mathbb{R}^4$*s.*

Proof. Let K be a topologically slice knot which is not slice. Write

$$\mathbb{S}^4 = X_0(K) \cup (B^4 - \mathrm{nbhd}(D))$$

where D is a topologically flat disk, with boundary K. Define

$$Z = \mathbb{S}^4 - \{x\} - \mathrm{int}(X_0(K)) = \mathbb{R}^4 - \mathrm{int}(X_0(K)).$$

This is an *open* topological 4-manifold with boundary. A theorem of Freedman states that all open 4-manifolds admit smooth structures, so we equip Z with a smooth structure. In particular ∂Z is a smooth manifold.

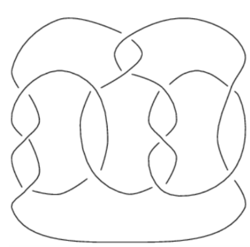
On the other hand, we already know that ∂Z is homeomorphic to $\partial X_0(K)$, which is homeomorphic to $\mathbb{S}^3_0(K)$. In dimension 3, all topological manifolds admit a unique smooth structure, so ∂Z is diffeomorphic to $\mathbb{S}^3_0(K)$. Now define

$$R = Z \sqcup_\varphi X_0(K)$$

where $\varphi : \partial_Z \to \mathbb{S}^3_0(K)$ is a diffeomorphism. This is a smooth manifold, and by Mayer-Vietoris and Seifert-van Kampen, can be shown to be homeomorphic to $\mathbb{R}^4$.

In particular, $X_0(K)$ embeds smoothly in R. Since K is *not* slice, $X_0(K)$ cannot embed smoothly in $\mathbb{R}^4$. Therefore the smooth structure on R must be distinct from that on $\mathbb{R}^4$. This completes the proof. □

3.4. **The Conway knot is not slice.** Up until 2018, it was known which knots with up to 12 crossings are slice, with one exception: *Conway's knot* C.



This is topologically slice, has $s(C) = 0$, and many other obstructions to sliceness vanish.

Piccirillo [Pic20] showed that Conway's knot C is not slice by constructing a (much larger) partner knot C' such that $X_0(C) = X_0(C')$. Then, by the Trace Embedding Lemma, C=slice $\iff$ $C' =$ slice, but a computer calculation show that $s(C') = 2 \neq 0 \Rightarrow C'$ is not slice. It follows that C is not slice.

No proof of her result is known using gauge theory.

Exercise 3.15. *Look up the knot* C' *in* [Pic20]*. Learn how to use the program SnapPy to draw the knot and get its PD or DT code. Then plug in this code into one of the programs KnotTheory' (from the Knot Atlas) or Dirk Schuetz's KnotJob and compute Rasmussen's invariant for* C'*.*

3.5. **FGMW strategy to find exotic 4-spheres.** In the proof of existence of exotic $\mathbb{R}^4$, it was crucial that Z was open. This is because Freedman's proof of the existence of smooth structures (on an arbitrary manifold) works everywhere except for a single point. Can we modify the approach to find exotic smooth structures of non-open manifolds? What about shedding light on the following famous open problem:

Open question 3. (Smooth Poincaré conjecture in dimension 4) If a closed 4-manifold X is homotopy equivalent to S^4, then is it diffeomorphic to S^4?

It is known that homotopy equivalent to S^4 implies homeomorphic to S^4 (by a result of Freedman).

We now describe an equivalent formulation of the smooth Poincaré conjecture in 4 dimensions, and show how we can attempt to understand it using Khovanov homology as we did above.

Proposition 3.16. The smooth Poincaré conjecture in dimension 4 (SPC4) is equivalent to the statement that if W^4 is smooth with $\partial W = \mathbb{S}^3$ and W contractible, then W is diffeomorphic to B^4.

The equivalence is immediate. To get from $\mathbb{S}^4$ to W, simply remove a 4-ball, and to get from W to $\mathbb{S}^4$, glue along a 3-sphere (since we know that the 3-dimensional Poincaré conjecture holds).

The Freedman-Gompf-Morrison-Walker (FGMW) strategy [FGMW10] for disproving the smooth Poincaré conjecture is as follows: find a knot K such that K bounds a smooth disk in some W contractible with $\partial W = \mathbb{S}^3$, $s(K) \neq 0$. Then K is not slice, so $W \neq B^4$. Thus W is an exotic B^4, which gives us an exotic $\mathbb{S}^4$.

Example 7. Suppose W has a handle decomposition with no 3-handles. The attaching spheres of 2-handles are in fact knots in $\mathbb{S}^3$, and moreover bound smooth disks in W (specifically the cores of the handles). Therefore if any of these K have non-trivial s-invariant, we are done. So far all such K have had trivial s-invariant.

Remark 3.17. There are invariants similar to the s-invariant that arise from Seiberg-Witten and Yang-Mills gauge theory, along with Floer homology theories. However, none of these can distinguish between sliceness in B^4 vs sliceness in homotopy B^4s, so these cannot work in a similar strategy.

Whether or not this strategy has a chance of working is an open question. More precisely, the following problem is open:

Open question 4. Let $K \subset \mathbb{S}^3 = \partial W^4$. Suppose W is smooth and contractible. Suppose $\Sigma \hookrightarrow W$ is a smooth proper embedding, with $\partial\Sigma = K$. Do we necessarily have

$$|s(K)| \leq 2g(\Sigma)?$$

This is of course true if $W = B^4$. If it is true for all W as above, then the FGMW strategy fails.

Is there any hope for the FGMW strategy? We could use the following result:

Proposition 3.18. Suppose K, K' are knots with $\mathbb{S}^3_0(K) \cong \mathbb{S}^3_0(K')$, but with K slice and K' not slice. Then SPC4 is *false*.

Proof. Recall that $\mathbb{S}^3_0(K)$ denotes the result of 0-surgery on K. The above result follows from the trace embedding lemma, which we saw in the previous section. Let $X_0(K)$ and $X_0(K')$

denote the traces of 0-surgery along K and K' respectively. Then $\partial X_0(K) = \partial X_0(K')$ as smooth manifolds. On one hand, we know that

$$\mathbb{S}^4 = X_0(K) \cup (B^4 - \mathrm{nbhd}(D)),$$

where the union glues along the boundary. Therefore we can replace $X_0(K)$ with $X_0(K')$, and consider

$$S' = X_0(K') \cup (B^4 - \mathrm{nbhd}(D)).$$

From Mayer-Vietoris, Seifert-van Kampen, and the topological Poincaré conjecture, one can show that S' is homeomorphic to $\mathbb{S}^4$. However, since K' is not slice, it cannot be diffeomorphic to $\mathbb{S}^4$ (by the trace embedding lemma). Therefore S' is an exotic $\mathbb{S}^4$, disproving SPC4. □

So far such K and K' have not been found. However, it is worth noting that an analogue of the FGMW strategy works in other 4-manifolds:

Theorem 3.19 (M.-Marengon-Piccirillo [MMP24], 2020)**.** *There exist smooth, closed, homeomorphic four-manifolds X and X' and a knot $K \subset S^3$ that bounds a null-homologous disk in $X \setminus B^4$ but not in $X' \setminus B^4$.*

For example, one can take

$$X = \#3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}, \quad X' = K3\#\overline{\mathbb{CP}^2},$$

and K be the trefoil: 

The proof of Theorem 3.19 uses gauge theory (the Seiberg-Witten equations).

Going back to the FGMW strategy in S^4, here are some ways to construct pairs of knots K, K' with the same 0-surgery:

- dualizable patterns (Akbulut, Lickorish, Brakes; 1977-80);
- annulus twisting (Osoinach, 2006);
- some satellites (Yasui, 2015).

The author and Piccirillo [MP23] give a general construction of *all* zero-surgery homeomorphisms $\phi : S^3_0(K) \to S^3_0(K')$ based on certain 3-component links called *RBG links.*

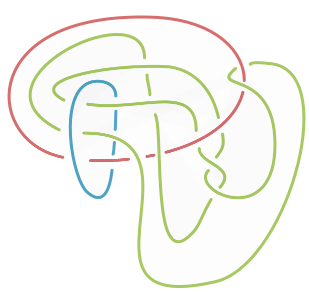
We will describe a special case.

Definition 3.20. *A* special RBG link *is a 3-component link $L = R \cup B \cup G$ such that there are isotopies*

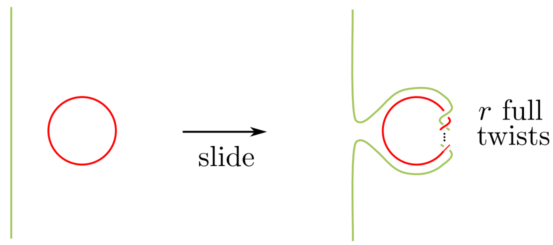
$$R \cup B \cong R \cup \mu_R \cong R \cup G$$

and R is r-framed such that the linking number $l = lk(B, G)$ satisfies $l = 0$ or $rl = 2$. (Here, μ_R is a meridian for R.)

Example:



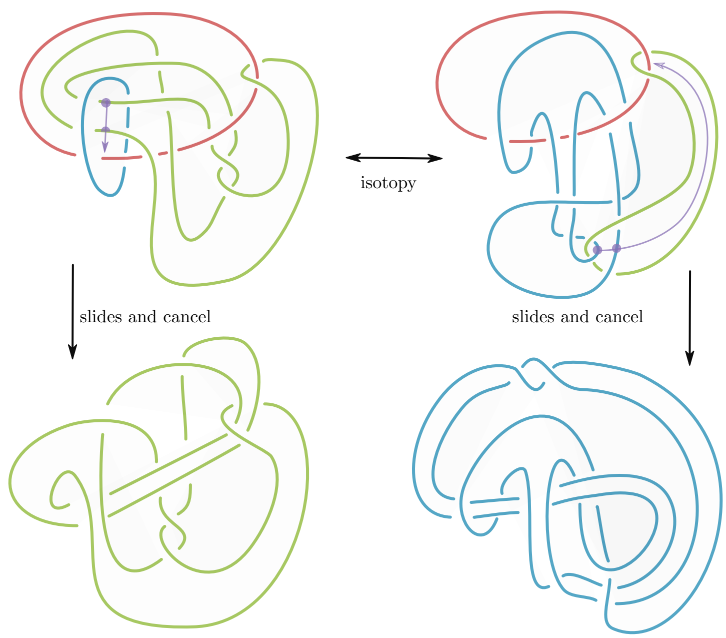
From a special RBG link L we obtain a knot K_G by sliding G over R until no geometric linking of B and G remains. Similarly, we obtain a knot K_B by sliding B over R until no geometric linking of B and G remains.



Theorem 3.21 ([MP23])**.** *If L is a (special) RBG link, there is an associated homeomorphism*

$$\phi_L : S^3_0(K_B) \to S^3_0(K_G).$$

An example:



Goal: Find an example where K_B is slice and $s(K_G) \neq 0$ (or vice versa). If V is the complement of a slice disk for K_B, then the homotopy 4-sphere

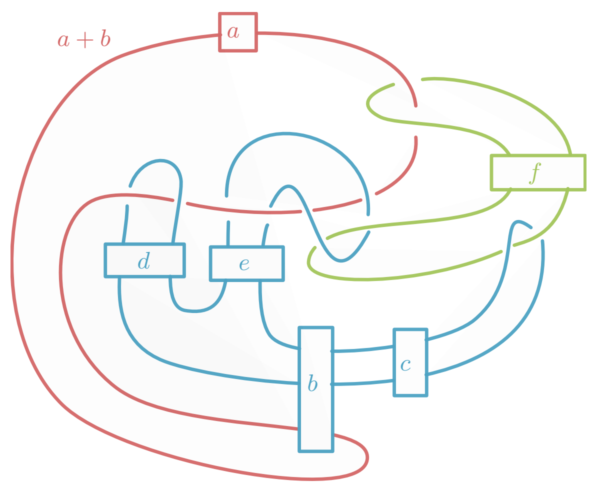
$$W = V \cup_{S^3_0(K)} (-X(K_G))$$

would be exotic, and we would disprove SPC4.

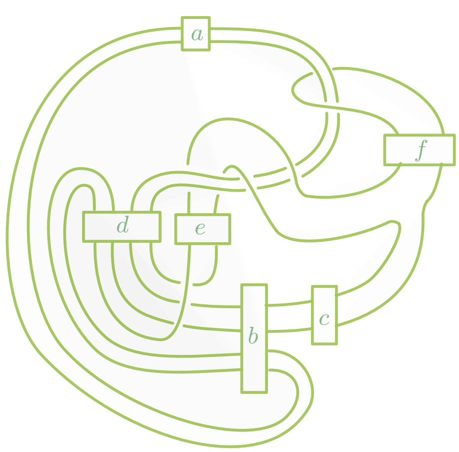
Using a computer program, in [MP23] we studied a 6-parameter family consisting of 3375 special RBG links. We found no examples as above. In most cases, the knots in the same pair have the same s-invariant.

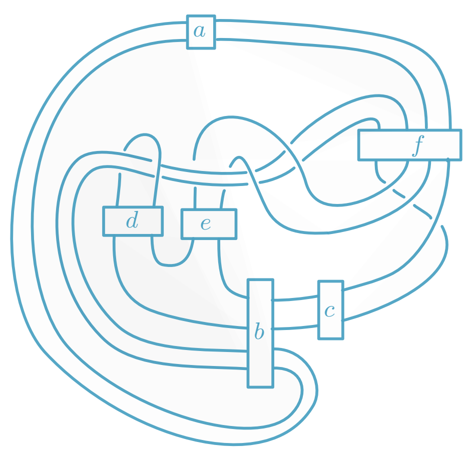
However, in about 1% of cases, the s-invariants differ. In 5 of those examples, one knot has $s \neq 0$, and we could not immediately determine if the other knot was slice.

Here is our family, where the boxes denote the number of full twists.

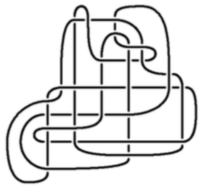


The resulting knots K_B and K_G are:



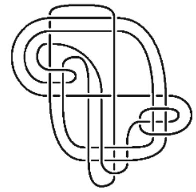


Here are 5 topologically slice knots, whose companions (with the same 0-surgery) have $s \neq 0$. If any of them had been slice, then SPC4 would have been false:

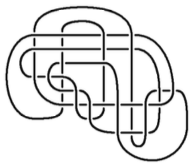


K_1

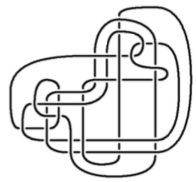
K_2



K_3



K_4



K_5

However, Nakamura [Nak23] later showed they are not slice, and in fact that the s-invariant cannot help disprove SPC4 using special RBG links where R is the unknot.

One can still hope to either:

- Consider other RBG links (e.g. special RBG links where R is a nontrivial knot) and search for slice/non-slice pairs using the s invariant; or
- Consider special RBG links where R is the unknot, but distinguish them using other invariants (e.g refined versions of the s-invariant from the Steenrod squares on Khovanov homology, or from other knot homologies).

3.6. **The FGMW strategy fails for Gluck twists.** A popular way of constructing (potentially exotic) homotopy 4-spheres is by a Gluck twist.

Definition 3.22. Let $\Sigma \cong \mathbb{S}^2 \to B^4$ be an embedding. Then there is a neighbourhood N of Σ diffeomorphic to $\mathbb{S}^2 \times D^2$. The *Gluck twist* of B^4 by Σ is

$$W = B^4_\Sigma = (B^4 - N) \sqcup_\varphi N$$

where $\varphi : \partial N = \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ is the map

$$\varphi : (z, e^{i\theta}) \mapsto (\mathrm{rot}_\theta(z), e^{i\theta}).$$

It is known that a Gluck twist of B^4 is homeomorphic to B^4, but not if it is diffeomorphic.

Definition 3.23. G_Σ denotes the Gluck twist of $\mathbb{S}^4$ by an embedding $\Sigma \hookrightarrow \mathbb{S}^4$, with $\mathbb{S}^2 \cong \Sigma$. By the following remark, there is no ambiguity in writing G_Σ.

Remark 3.24. The diffeomorphism $\varphi \in \mathrm{Aut}(\partial N)$ is a generator of $\pi_1(\mathbb{RP}^3) = \pi_1(\mathrm{SO}(3)) = \{\mathbb{S}^1 \to \mathrm{rot}(\mathbb{S}^2)\} = \mathbb{Z}/2\mathbb{Z}$. If two maps in $\mathrm{Aut}(\partial N)$ are homotopic, they give the same Gluck twists.

In the previous section we introduced the Freedman-Gompf-Morrison-Walker strategy to disprove the smooth Poincaré conjecture. It is an open question whether or not the strategy can be carried out. However, we now give a proof outline to show that the strategy fails for Gluck twists.

Theorem 3.25 (Manolescu, Marengon, Sarkar, Willis [MMSW23])**.** *The inequality $|s(K)| \leq 2g(\Sigma)$ holds as above if W is a Gluck twist of B^4.*

This means that if W is a smooth manifold homeomorphic to B^4 obtained via a Gluck twist, and K is a knot bounding a disk in W, we cannot show that K is not slice (and hence W is not diffeomorphic to B^4) by using the s-invariant. In other words, the FGMW strategy fails for such W.

It is interesting that such a result can be proven, since we expect to only know information about cylinders $\mathbb{S}^3 \times [0,1]$ based on the definition of the Khovanov homology.

The proof outline for the MMSW theorem is as follows:

(1) Prove a special case with $W = \overline{\mathbb{CP}^2} - B^4$.
(2) Prove a special case with $W = \mathbb{CP}^2 - B^4$.
(3) Use Kirby diagrams to prove a result analogous to the "stable diffeomorphism" classification of 4-manifolds. Concretely, we show that $G_\Sigma \# \mathbb{CP}^2$ is diffeomorphic to $\mathbb{CP}^2$, and $G_\Sigma \# \overline{\mathbb{CP}^2}$ is diffeomorphic to $\overline{\mathbb{CP}^2}$.
(4) We combine the three results to prove the general result.

We now state and prove the first special case:

Proposition 3.26. Let $W = \overline{\mathbb{CP}^2} - B^4$, and $K \subset \partial W = \mathbb{S}^3$. Let $\Sigma \subset W$ be smoothly properly embedded, with $\partial \Sigma = K$. Suppose $[\Sigma] = 0 \in H_2(W, \partial W) = H_2(\overline{\mathbb{CP}^2}) = \mathbb{Z}$. Then $s(K) \leq 2g(\Sigma)$.

Proof. The goal is to reduce the problem further to a surface in a cylinder. In that case we obtain a map corresponding to the surface.

Consider the data of $W = \overline{\mathbb{CP}^2} - B^4$, $K \subset \partial W = \mathbb{S}^3$, and $\Sigma \subset W$ smoothly properly embedded, with $\partial \Sigma = K$ and $[\Sigma] = 0 \in H_2(W, \partial W)$. Note that $H_2(W, \partial W)$ is generated by $[\overline{\mathbb{CP}^1}]$.

Let N be a regular neighbourhood of $\overline{\mathbb{CP}^1}$. Then $\partial N = \mathbb{S}^3$. Moreover, the "radial" projection $\partial N \to \overline{\mathbb{CP}^1} \cong \mathbb{S}^2$ is the (negative) Hopf fibration. Decomposing along the boundary of N, we then have

$$\overline{\mathbb{CP}^2} = N \sqcup_{\partial N} (\mathbb{S}^3 \times [0,1]) \sqcup_{\partial W} B^4.$$

We also assume that $[\Sigma] = 0 \in H_2(W, \partial W)$. Therefore $[\Sigma] \cdot [\overline{\mathbb{CP}^1}] = 0$. That is, assuming transversality, Σ and $\overline{\mathbb{CP}^1}$ intersect at $2p$ points, p positively signed and p negatively signed. Therefore Σ intersects N along $2p$ disks, and intersects ∂N along $2p$ circles. Each of these circles is a fibre of the negative Hopf fibration mentioned above.

The collection of fibres forms a link $L_{p,p} \subset \mathbb{S}^3$ in the total space of the Hopf fibration. In fact, this is a torus link $T_{2p,2p}$ with p strands oriented in one direction and p the other way.

One can define Rasmussen's s-invariant for *links* rather than just knots. Recall that $\dim \mathrm{Lee}(L) = 2^\ell$ where L has ℓ components, and $Kh(L) \Rightarrow \mathrm{Lee}(L)$. This time there are many generators, but our link has a given orientation, so there exist canonical generators $S_{\mathcal{O}}$ and $S_{\overline{\mathcal{O}}}$. We can define the s-invariant to be

$$s(L) = \frac{q([S_{\mathcal{O}}] + [S_{\overline{\mathcal{O}}}]) + q([S_{\mathcal{O}}] - [S_{\overline{\mathcal{O}}}])}{2}.$$

By the definition of Σ, its restriction to $\mathbb{S}^3 \times [0,1]$ is a cobordism inside $\mathbb{S}^3 \times [0,1]$ from K to $L_{p,p}$, of genus $g(\Sigma)$. By functoriality of the Khovanov homology under cobordisms (as in Rasmussen's proof of the Milnor conjecture), we find that

$$s(K) - 2g(\Sigma) + 1 - 2p \leq s(L_{p,p}).$$

We can compute $s(L_{p,p})$. (This takes some work and is the main content of the paper by MMSW; see Section 4.4 for more details), but these turn out to be $1 - 2p$. Therefore the inequality above gives the desired result. $\square$

It is now straight forward to prove the result for $\mathbb{CP}^2$ instead of $\overline{\mathbb{CP}^2}$. Explicitly, we have the following proposition:

Proposition 3.27. Let $W = \mathbb{CP}^2 - B^4$, and $K \subset \partial W = \mathbb{S}^3$. Let $\Sigma \subset W$ be smoothly properly embedded, with $\partial\Sigma = K$. Suppose $[\Sigma] = 0 \in H_2(W, \partial W) = H_2(\mathbb{CP}^2) = \mathbb{Z}$. Then $-s(K) \leq 2g(\Sigma)$.

Proof. This follows from Proposition 3.25 by working with the mirror of K. $\square$

The final ingredient for proving the general theorem 3.25 is a result reminiscent of stable diffeomorphisms.

Proposition 3.28. For any $\Sigma \hookrightarrow \mathbb{S}^4$, $G_\Sigma \# \mathbb{CP}^2 \cong \mathbb{CP}^2$, and $G_\Sigma \# \overline{\mathbb{CP}^2} \cong \overline{\mathbb{CP}^2}$, where G_Σ is the Gluck twist of $\mathbb{S}^4$ by Σ.

Proof. The proof makes use of Kirby diagrams. (If you are not familiar with Kirby calculus, you can safely ignore it, as it is independent of the rest of the notes.) Given $\Sigma \hookrightarrow \mathbb{S}^4$, we can write Kirby diagrams for $\mathbb{S}^4$ and G_Σ are as in figure 3 (where the component labelled with a 0 is a 2-handle determined by Σ). We now briefly explain the origins of these Kirby

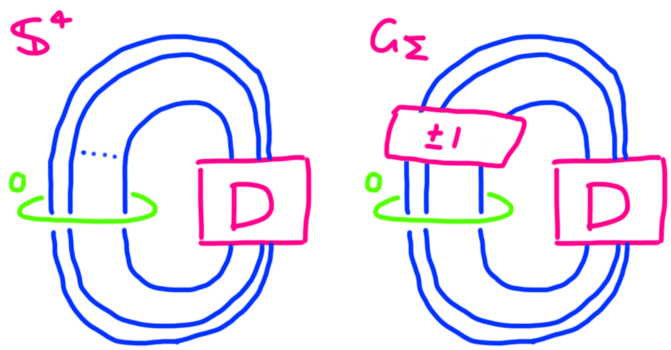


FIGURE 3. Kirby diagrams for $\mathbb{S}^4$ and G_Σ.

diagrams. We can write

$$\mathbb{S}^4 = (\mathbb{S}^4 - N) \cup N, \quad G_\Sigma = (\mathbb{S}^4 - N) \sqcup_\varphi N,$$

where φ is the twisting map, and N is a regular neighbourhood of Σ. We now choose a Morse function $f : \mathbb{S}^4 \to \mathbb{R}$ such that $N = f^{-1}(-\infty, 0]$, and let $h : \mathbb{S}^2 \to \mathbb{R}$ be the standard height function. Next let $\pi : \mathbb{S}^2 \times D^2 \cong N \to \mathbb{S}^2$ be the usual projection map.

Finally we update f so that $f|_N$ is defined by

$$f|_N(x, z) = (h \circ \pi)(x, z) + |z|^2.$$

The Kirby diagram for $\mathbb{S}^4$ shown in figure 3 is with respect to this Morse function f, and applying a Gluck twist gives the diagram on the right.

Next we prove using Kirby calculus that

$$G_\Sigma \# \mathbb{CP}^2 \cong \mathbb{CP}^2 \# \mathbb{S}^4 \cong \mathbb{CP}^2.$$

We use the above diagrams, making only local changes at the 0-framed 2 handle shown in green. The proof is contained in figure 4. The proof of

$$G_\Sigma \# \overline{\mathbb{CP}^2} \cong \overline{\mathbb{CP}^2} \# \mathbb{S}^4 \cong \overline{\mathbb{CP}^2}$$

is similar, and not included. $\square$

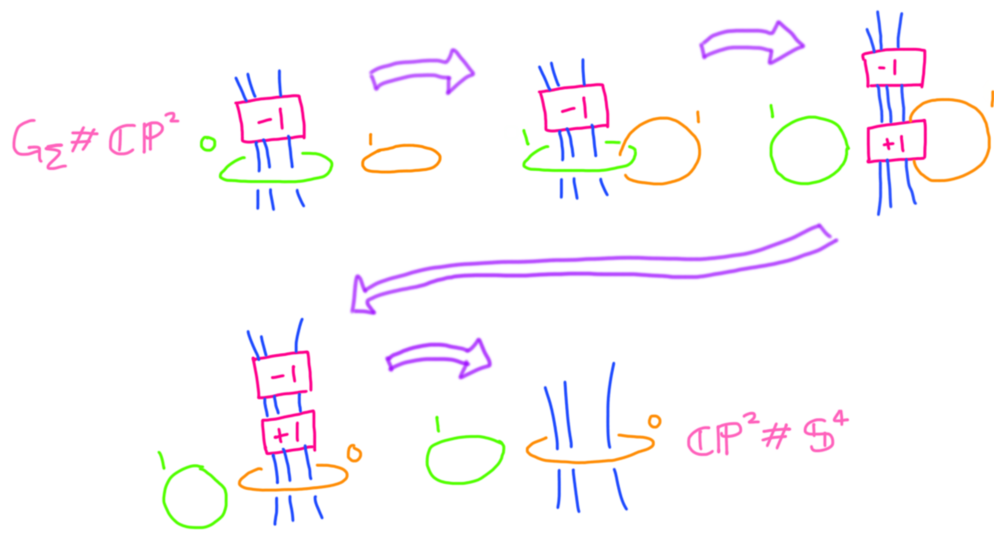


FIGURE 4. Proof that $G_\Sigma \# \mathbb{CP}^2 \cong \mathbb{CP}^2 \# \mathbb{S}^4$.

We now have all of the necessary ingredients to prove theorem 3.25 of MMSW (which we repeat here for clarity).

Theorem 3.29 ([MMSW23])**.** *Let $K \subset \mathbb{S}^3 = \partial W$, where W is obtained as a Gluck twist of B^4. Suppose $\Sigma \hookrightarrow W$ is a smooth embedding, with $\partial\Sigma = K$. Then $|s(K)| \leq 2g(\Sigma)$.*

Proof. Let W be a Gluck twist of B^4, and $\Sigma \subset W$ such that $\partial\Sigma = K$. For some surface S, we have $W = G_S - B^4$. By the above result, $G_S \# \mathbb{CP}^2 = \mathbb{CP}^2$, so in particular

$$W \# \mathbb{CP}^2 = \mathbb{CP}^2 - B^4.$$

By Proposition 3.27, it follows that

$$-s(K) \leq 2g(\Sigma).$$

Moreover, we also know that $G_S \# \overline{\mathbb{CP}^2} = \overline{\mathbb{CP}^2}$, from which it follows that $W \# \overline{\mathbb{CP}^2} = \overline{\mathbb{CP}^2} - B^4$, so by Proposition 3.26,

$$s(K) \leq 2g(\Sigma).$$

Combining these two results, we find that

$$|s(K)| \leq 2g(\Sigma)$$

as required. □

4. Extensions to other three-manifolds

4.1. **Extensions of Khovanov homology.** The original Khovanov homology is for links in S^3. It can be generalized to other 3-manifolds:

(1) Asaeda-Przytycki-Sikora (2004): for links in I-bundles over surfaces; in particular, in $S^1 \times D^2$ (annular Khovanov homology) and in $\mathbb{RP}^2 \tilde{\times} I = \mathbb{RP}^3 \setminus \{*\}$ (but only with $\mathbb{Z}/2$ coefficients);
(2) Manturov (2005): for virtual links;
(3) Rozansky (2010): for links in $S^1 \times S^2$;
(4) Willis (2018): for links in $\#^n(S^1 \times S^2)$;
(5) Manturov (2006) and Gabrovšek (2018): for links in $\mathbb{RP}^3$, with $\mathbb{Z}$ coefficients;
(6) Morrison-Walker-Wedrich (2019): for links in any $Y^3 = \partial X^4$, we have the *skein lasagna module* $\mathcal{S}_0(X; L)$. When $X = B^4$, we recover Kh. This will be described in more detail in Section 5.

4.2. **Extensions of Rasmussen's invariant.** Here are some still for knots in S^3:

(1) Mackaay-Turner-Vaz (2005): from the Bar-Natan deformation of Kh (over a field characteristic p, for any prime p);
(2) Lobb (2010): from $\mathfrak{sl}(N)$ link homology;
(3) Lipshitz-Sarkar (2012), Sarkar-Scaduto-Stoffregen (2019): using the Steenrod squares from (even and odd) Khovanov stable homotopy;
(4) Sano-Sato (2022), Schütz (2022), Dunfield-Lipshitz-Schütz (2024), Lewark (2024): from the deformations of (even and odd) Kh over $\mathbb{Z}$;

There is also an extension of s to virtual knots, due to Dye-Kaestner-Kauffman (2014).

However, in the next few subsections we will not discuss any of the above. Rather, we will focus on some generalizations of s to links in 3-manifolds other than S^3.

4.3. **In $S^1 \times D^2$.** Let $A = S^1 \times I$ be the annulus. Khovanov homology for annular links ($K \subset A \times I \cong S^1 \times D^2$) can be defined similarly to that for links in S^3, except that now we use projections onto the annulus (cf. Asaeda-Przytycki-Sikora). We now have two types of circles in the resolutions: those that bound disks and those that go around the annulus. By keeping track of the latter, we get a third grading on annular Khovanov homology.

In 2016, Grigsby-Licata-Wehrli [GLW17] computed the homology of its Lee deformation, and defined a family of Rasmussen-type invariants

$$d_t(K) \in \mathbb{R}, \ t \in [0, 2],$$

such that $d_t = d_{2-t}$, $d_0 = d_2 = s$, and d_t is piecewise linear. We denote by m_t its right-hand slope.

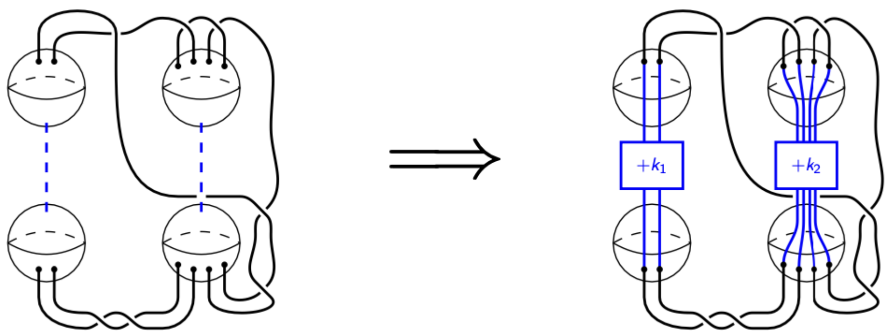
The invariants d_t give bounds on the genus of knot cobordisms in $I \times S^1 \times D^2$.

Furthermore, for closures $\hat{\sigma}$ of braids $\sigma \in B_n$, they showed:

- If σ is quasi-positive, then $m_t(\hat{\sigma}) = n$ for all $t \in [0, 1)$.
- If $m_t(\hat{\sigma}) = n$ for some $t \in [0, 1)$, then σ is right-veering.

4.4. **In $\#^r S^1 \times S^2$.** Rozansky [Roz10] defined Kh for links in $S^1 \times S^2$. This was generalized to $\#^r(S^1 \times S^2)$ by Willis [Wil21]. We assume that $[K] = 0 \in H_1(M; \mathbb{Z})$.

We represent a link $K \subset \#^r(S^1 \times S^2)$ by a diagram D as on the left. From here we get a link $D(\vec{k}) \subset S^3$ by inserting $\vec{k} = (k_1, \dots, k_r)$ full twists:



Then take the limit of $Kh(D(\vec{k}))$ as $k_i \to \infty$, after suitable grading shifts.

Marengon-M.-Sarkar-Willis [MMSW23] computed the Lee deformation of this theory, and defined an invariant $s(K)$ for null-homologous $K \subset \#^r S^1 \times S^2$. This gives bounds on the genus of link cobordisms in $I \times \#^r S^1 \times S^2$.

Moreover, denote by $g_{S^1 \times B^3}(K)$ the minimum genus of a surface $\Sigma \subset \natural^r(S^1 \times B^3)$ with $\partial\Sigma = K$. We define $g_{B^2 \times S^2}(K)$ similarly, but using surfaces $\Sigma \subset \natural^r(B^2 \times S^2)$. If K is a knot, we have:

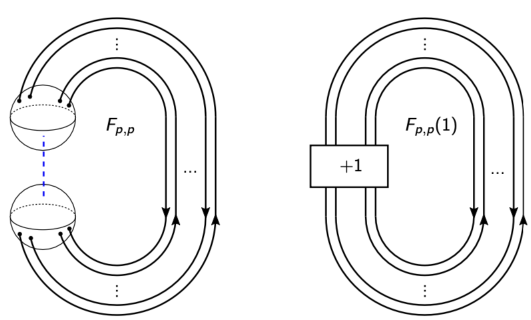
$$\boxed{s(K)/2 \leq g_{B^2 \times S^2}(K), \quad -s(-K)/2 \leq g_{S^1 \times B^3}(K).}$$

In a roundabout way, $s(K)$ for $K \subset \#^r S^1 \times S^2$ can also be used to say something new about the s-invariant of knots in S^3. This is based on the following fact: if D is a diagram of K with n_+ positive crossings, then

$$s(K) = s(D(\vec{k}))$$

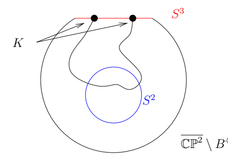
where $\vec{k} = (k, \cdots, k)$ with $k \geq \left\lceil \frac{n_+ + 2}{2} \right\rceil$.

In particular, consider the null-homologous link $F_{p,p} \subset S^1 \times S^2$, which is the union of $2p$ fibers $S^1 \times \{x_i\}$. It has a diagram with 0 crossings, so its s invariant is the same as that of the $(2p, 2p)$ balanced torus link $F_{p,p}(1) \subset S^3$:



Using Hochschild homology, we computed $s(F_{p,p}(1)) = s(F_{p,p}) = 1 - 2p$.

One application is a genus bound in $\#^r \overline{\mathbb{CP}^2}$.



Theorem 4.1 (MMSW, 2019)**.** *If a knot $K \subset S^3$ bounds a null-homologous surface $\Sigma \subset \#^r\overline{\mathbb{CP}^2} \setminus B^4$, then $s(K)/2 \leq g(\Sigma)$.*

Sketch of proof: A surface $\Sigma \subset \overline{\mathbb{CP}^2} \setminus B^4$ intersects $S^2 = \overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$ in p positive and p negative points. This gives a cobordism $C \subset S^3 \times [0,1]$ between K and the torus link $F_{p,p}(1)$: We then apply the genus bounds for surfaces in $S^3 \times [0,1]$. The case of $\#^r\overline{\mathbb{CP}^2} \setminus B^4$ is similar.

A more general genus bound in $\#^r\overline{\mathbb{CP}^2}$: Ren [Ren24a] gave a more elementary proof of the formula $s(F_{p,p}(1)) = 1 - 2p$, and in fact computed s for all torus links $T(n,m)$, equipped with any orientation (where p strands are oriented one way and q the other way):

$$s(T(n,m)_{p,q}) = (\tfrac{n}{\gcd(n,m)}|p-q| - 1)(\tfrac{m}{\gcd(n,m)}|p-q| - 1) - 2\min(p,q).$$

A corollary is the general adjunction inequality:

Theorem 4.2 ([Ren24a])**.** *If a knot $K \subset S^3$ bounds a surface $\Sigma \subset \#^r\overline{\mathbb{CP}^2} \setminus B^4$, then*

$$s(K) \leq 1 - \chi(\Sigma) - [\Sigma]^2 - |[\Sigma]|.$$

This is similar to the adjunction inequality for the τ invariant in Heegaard Floer homology, previously proved by Ozsváth and Szabó (2003).

Open question 5. Does the adjunction inequality for the s-invariant hold for surfaces in any negative-definite 4-manifold, instead of just $\#^r\overline{\mathbb{CP}^2}$?

Theorem 4.2 has an application to k-sliceness, as follows.

Recall that the trace embedding lemma says that: a knot $K \subset S^3$ is slice $\iff$ the trace of 0-surgery $X_0(K)$ embeds in S^4. Here, $\partial X_0(K) = S^3_0(K)$. This was used by Piccirillo to show that the Conway knot C is not slice, even though $s(C) = 0$, by finding a knot C' with $s(C') \neq 0$ and $X_0(C) = X_0(C')$.

More generally, a knot $K \subset S^3$ is called k-slice in a 4-manifold X if K bounds a disk $D \subset X \setminus B^4$ such that $[D]^2 = -k$.

Exercise 4.3. *Prove a variant of the trace embedding lemma: K is k-slice iff $-X_k(K)$ smoothly embeds in X.*

If a simply connected, closed 4-manifold has negative definite intersection form, it is homeomorphic to $\#^r\overline{\mathbb{CP}^2}$ for some r. Currently, there are no known examples of exotic $\#^r\overline{\mathbb{CP}^2}$.

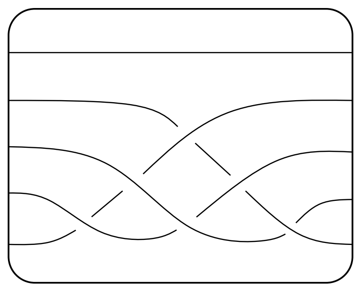
Here is a variant of the FGMW strategy: In principle, one could attempt to find exotic S^4 (resp. exotic $\#^r\overline{\mathbb{CP}^2}$) by exhibiting knots $K, K' \subset S^3$ with $S^3_0(K) = S^3_0(K')$ (resp. $S^3_k(K) = S^3_k(K')$) such that K is slice (resp. k-slice in $\#^r\overline{\mathbb{CP}^2}$) and K' is not.

As previously mentioned, M.-Piccirillo [MP23] studied a family of pairs of knots, where the knots in each pair have the same 0-surgery. We found some examples of pairs (K, K') where K' is not slice, and K was of unknown sliceness. Nakamura [Nak23] showed that those examples of K are not slice (and, in fact, not 0-slice in $\#^r\overline{\mathbb{CP}^2}$), by using the null-homologous adjunction inequality in $\#^r\overline{\mathbb{CP}^2}$ and generalizing Piccirillo's method.

Using Ren's general adjunction inequality, Qin [Qin23] found new obstructions to knots being k-slice in $\#^r\overline{\mathbb{CP}^2}$.

4.5. **In $\mathbb{RP}^3$.** For links in $\mathbb{RP}^3$, we distinguish between **class-0 links** and **class-1 links**, according to their homology class in $H_1(\mathbb{RP}^3;\mathbb{Z}) = \mathbb{Z}/2$. In particular, we have the *class-0 unknot* $U_0 \subset \mathbb{R}^3 \subset \mathbb{RP}^3$ and the *class-1 unknot* $U_1 = \mathbb{RP}^1 \subset \mathbb{RP}^3$.

Note that $\mathbb{RP}^3 \setminus *$ is an I-bundle over $\mathbb{RP}^2$. We represent links through their projections to $\mathbb{RP}^2$. The antipodal points on the boundary of the disk are identified to form $\mathbb{RP}^2$. For example, this is a class-1 knot K_1:



Using this kind of projections, Khovanov homology for links in $\mathbb{RP}^3$ was defined by Asaeda-Przytycki-Sikora (2004) over $\mathbb{Z}/2$, and by Manturov (2006) and Gabrovšek (2018) over $\mathbb{Z}$.

M.-Willis [MW23] construct a Lee deformation, and use it to define a Rasmussen-type invariant $s(K)$ for $K \subset \mathbb{RP}^3$. This bounds the genus of (oriented) link cobordisms in $I \times \mathbb{RP}^3$.

In particular, if $K \subset \mathbb{RP}^3$ is a class-α knot, $\alpha \in \{0, 1\}$, we define the *slice genus* $g_s(K)$ to be the minimal genus of a compact, oriented cobordism $\Sigma \subset I \times \mathbb{RP}^3$ from K to the class-α unknot U_α.

Theorem 4.4 ([MW23])**.** *If $K \subset \mathbb{RP}^3$ is a knot, then $|s(K)|/2 \leq g_s(K)$.*

Theorem 4.5 ([MW23])**.** *There exist knots K_0, K_1 in $\mathbb{RP}^3 = S^3/\tau$ that are not concordant (they do not co-bound an annulus in $I \times \mathbb{RP}^3$), but such that their lifts to S^3 are concordant (co-bound an annulus in $I \times S^3$).*

Idea of proof: Consider the lifts $\widetilde{K}$ of K and $\widetilde{K}'$ of $K' = (-K)\#\widetilde{K}$, where $K \subset \mathbb{RP}^3$ is any class-1 knot such that $s(\widetilde{K}) \neq 2s(K)$. Then $s(K') = s(-K) + s(\widetilde{K}) = -s(K) + s(\widetilde{K}) \neq s(K)$, but $\widetilde{K}$ and $\widetilde{K}' = (-\widetilde{K})\#\widetilde{K}\#\widetilde{K}$ are concordant.

An example is the knot K_1 with lift $\widetilde{K}_1 = 12n403$, which has $s(K_1) = 0$ and $s(\widetilde{K}_1) = 2$.

Note: The lifts of class 1-knots in $\mathbb{RP}^3$ are *freely periodic* knots in S^3. A concordance in $I \times \mathbb{RP}^3$ is the same as a standardly equivariant concordance of the lifts in $I \times S^3$.

Open question 6. Is there a (class-1) non-slice knot in $\mathbb{RP}^3$ whose lift to S^3 is slice?

Let DTS^2 be the disk bundle associated to the tangent bundle to S^2. Its boundary is $\mathbb{RP}^3$.

Theorem 4.6 (Ren [Ren24b]). *If $K \subset \mathbb{RP}^3$ bounds an oriented surface $\Sigma \subset DTS^2$ with $[\Sigma] = d \in H_2(DTS^2, \mathbb{RP}^3) \cong \mathbb{Z}$, then*

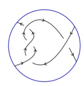
$$g(\Sigma)/2 \geq -s(K) - \frac{d^2}{2}$$

provided $d \in \{0, \pm 1, \pm 2, \pm 3\}$. (Conjecturally, for all d.)

Idea: The surface Σ gives a cobordism in $I \times \mathbb{RP}^3$ relating K to a link $T(d; p, q) \subset \mathbb{RP}^3$. The proof is based on calculating $s(T(d; p, q))$ for some values of d, p, q.

Chen (2023) studied a Bar-Natan deformation of the Khovanov complex in $\mathbb{RP}^3$ with $\mathbb{Z}/2$ coefficients. He defines an invariant $s^{BN}(K)$ for class-0 knots $K \subset \mathbb{RP}^3$.

Theorem 4.7 ([Che25]). *Let $K \subset \mathbb{RP}^3$ be a class-0 knot, and $\Sigma \subset \mathbb{RP}^3 \times I$ be a surface with $\partial\Sigma = K \subset \mathbb{RP}^3 \times \{1\}$, such that Σ is* **twisted orientable**, *i.e. its lift to $S^3 \times I$ has an orientation that is reversed under the deck transformation. Then, $|s^{BN}(K)| \leq -\chi(\Sigma)$.*

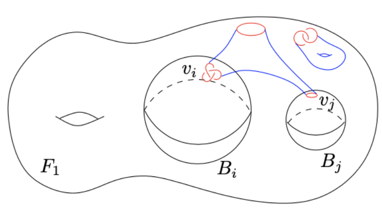
Application: The knot  has $s^{BN} = 2 \neq s$. It bounds an orientable slice surface of genus 1, but no such orientable surface that is also twisted orientable.

5. Skein lasagna modules

5.1. **The skein lasagna module.** Morrison-Walker-Wedrich [MWW22] defined an invariant of (framed, oriented) links L in any 3-manifold Y presented as the boundary of a 4-manifold X.

A *lasagna filling* $F = (\Sigma, \{(B_i, L_i, v_i)\})$ of X with boundary L consists of

- A finite collection of disjoint 4-balls B_i (called *input balls*) embedded in the interior or X;
- A framed oriented surface Σ properly embedded in $X \setminus \cup_i B_i$, meeting ∂X in L and meeting each ∂B_i in a link L_i; and
- for each i, a homogeneous label $v_i \in Kh(L_i)$.



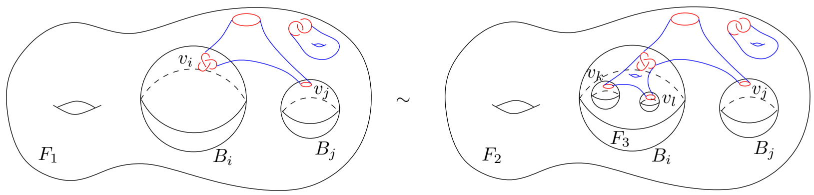
We define

$$\mathcal{S}_0(X; L) := \mathbb{Z}\{\text{lasagna fillings } F \text{ of } X \text{ with boundary } L\}/\sim$$

where $\sim$ is the transitive and linear closure of the following relations:

- Linear combinations of lasagna fillings are set to be multilinear in the labels v_i;
- F_1 and F_2 are set to be equivalent if F_1 has an input ball B_i with label v_i, and F_2 is obtained from F_1 by replacing B_i with another lasagna filling F_3 of a 4-ball such

that $v_i = Kh(F_3)(v_k \otimes v_l \otimes \cdots)$, followed by an isotopy rel ∂X (where the isotopy is allowed to move the input balls):



In other words: roughly, $\mathcal{S}_0(X; L)$ is the group generated by the Khovanov homology of links in the boundaries of all balls in X, modulo all cobordism relations between these.

Let us investigate its properties.

First, note that $\mathcal{S}_0(X; L)$ is bigraded (just like Kh) and decomposes as a direct sum of $\mathcal{S}_0(X; L; \alpha)$ according to the relative homology classes α of the fillings:

$$\mathcal{S}_0(X : L) = \bigoplus_{i,j \in \mathbb{Z}} \bigoplus_{\alpha \in \partial^{-1}([L]) \subset H_2(X,L)} \mathcal{S}_{0,i,j}(X, L : \alpha).$$

Second, note that it is functorial under cobordisms, by construction. If Z is a four-dimensional cobordism from Y to Y" and $\Sigma \subset Z$ is an embedded surface with boundaries $L \subset Y$ and $L' \subset Y'$¡ we get a map

$$\Psi_{Z,\Sigma} : \mathcal{S}_0(X, L) \to \mathcal{S}_0(X \cup Z; L')$$

given by attaching Σ to the lasagna fillings in X.

The following exercise shows that $\mathcal{S}_0$ is indeed a generalization of Khovanov homology.

Exercise 5.1. *When $X = B^4$, we have $\mathcal{S}_0(X; L) = Kh(L)$ (up to a grading shift).* Hint: *Embed all input balls into a larger ball.*

$\mathcal{S}_0$ is, however, a different kind of extension than the ones considered in Section 4. It cannot be computed directly from a knot projection of some sort. Indeed, the definition makes it at first look uncomputable. Nevertheless, in some cases computations are possible. These are based on formulas for $\mathcal{S}_0(X; L)$ in terms of a handle decomposition of X.

In general, by Morse theory, a smooth manifold can be decomposed into k-handles for various k: attachments of $D^k \times D^{n-k}$ along $\partial D^k \times D^{n-k-1}$. For a 4-manifold, a 0-handle is just introducing a ball, and a 4-handle is attaching a B^4 to an S^3 boundary.

Exercise 5.2. *Show that attaching a 4-handle to $(X, \emptyset)$ does not change $\mathcal{S}_0(X, \emptyset)$.*

Exercise 5.3. *Show that the map on $\mathcal{S}_0$ given by attaching a 3-handle is surjective.*

Formulas for attaching 1- and 3-handles were developed by M.-Walker-Wedrich [MWW23].

Let us focus on 2-handles, which are the main source of complexity in 4-manifolds. When X is made of 2-handles attached to B^4 along a link K, M.-Neithalath [MN22] showed that $\mathcal{S}_0(X; L)$ is a direct limit of the Khovanov homologies of the union of L and certain cables of K. Specifically, let us assume that K is a knot for simplicity, and let $K(r_1, r_2)$ be the link obtained by following K with r_1 strands oriented in one direction and r_2 in the other direction. We have a cobordism Z from $K(r_1, r_2) \sqcup L \sqcup$unknot to $K(r_1+1, r_2+1) \sqcup L$, given by an annulus between two strands, from which we removed a small disk with boundary the unknot.

Theorem 5.4 (M.-Neithalath [MN22]). *We have*

$$\mathcal{S}_0(W', L) \cong \bigoplus_{n_1, n_2 \in \mathbb{N}} \mathcal{S}_0(W, K(r_1, r_2) \cup L)/\sim$$

where the equivalence relation $\sim$ *is generated by the following:*

- *permuting the strands of* $K(r_1, r_2)$*, preserving orientations;*
- $\Psi_Z(v \otimes v_-) \sim 0, \quad \forall v$*;*
- $\Psi_Z(v \otimes v_+) \sim v, \quad \forall v$*.*

Exercise 5.5. *Use this theorem to compute the skein lasagna module* $\mathcal{S}_0(S^2 \times D^2; \emptyset)$*, noting that* $S^2 \times D^2$ *is obtained by attaching a 2-handle along the unknot.*

The handle decomposition formulas allowed for computations of $\mathcal{S}_0(X) := \mathcal{S}_0(X; \emptyset)$ for manifolds such as $S^1 \times S^3$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, disk bundles over S^2. For example, we have $\mathcal{S}_0(\mathbb{CP}^2) = 0$ but $\mathcal{S}_0(\overline{\mathbb{CP}^2}) \neq 0$. Also, $\mathcal{S}_0(S^2 \times S^2) = 0$. See [MWW23], [MN22] as well as the works of Sullivan-Zhang (2024) and Ren-Willis (2024).

5.2. **Extensions of the Rasmussen invariant: in any** $Y^3 = \partial X^4$**.** Morrison-Walker-Wedrich [MWW24] and Ren-Willis [RW24] computed the Lee deformation of $\mathcal{S}_0(X; L)$. Ren and Willis also identified the generators of Lee homology, and used these to define invariants

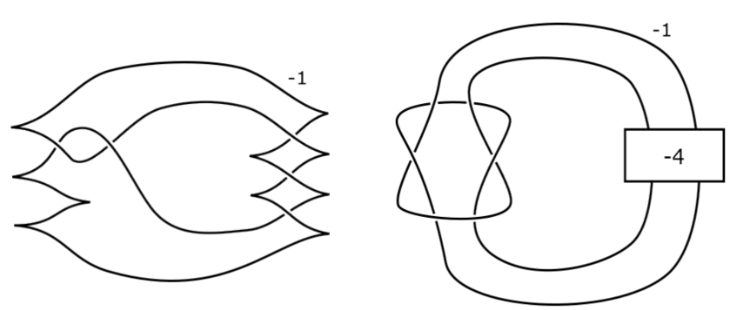
$$s(X; L; \alpha) \in \mathbb{Z} \cup \{-\infty\}.$$

When $X = B^4$, we recover the Rasmussen invariant up to a sign and shift: $s(B^4; L) = -s(-L) - w(L) + 1$.

Theorem 5.6 (Ren-Willis [RW24]). *If* $\Sigma \subset X$ *is smoothly embedded with* $\partial\Sigma = L$*, then* $2g(\Sigma) \geq s(X; L; [\Sigma]) + [\Sigma]^2 - \#L$*.*

In general, there is no algorithm for computing $s(X; L; \alpha)$. Furthermore, $s(X; L; \alpha)$ can be $-\infty$ (e.g. for $X = S^2 \times S^2$ or $\mathbb{CP}^2$ and $L = \emptyset$), in which case we get no genus bounds.

5.3. **Detection of exotic smooth structures.** Nevertheless, using the formula for $\mathcal{S}_0$ of 2-handlebodies X, Ren and Willis compute $s(X; \alpha) = s(X, \emptyset; \alpha)$ in many cases. In particular, consider the traces of -1 surgeries on the knots $K_1 = -5_2$ and $K_2 = P(3, -3, -8)$:



we have $s(X_1; 1) = 3$ and $s(X_2; 1) = 1$. Hence X_1 and X_2 are not diffeomorphic, even though they are homeomorphic.

5.4. **Detection of exotic smooth structures.** Akbulut (1991) was the first to show that X_1 and X_2 is an exotic pair, using gauge theory.

Ren and Willis give the first *analysis-free proof.* Here are the main ideas in their computation:

- For $X_1 = X_{-1}(K_1)$, they use that $K_1 = -5_2$ is a positive knot with $s = 2$. For such knots, they prove that $s(X_n(K); 1) = s(K) - n$ by calculating Kh of cables of K in the highest homological degree, generalizing an argument of Stošić (2006).
- For $X_2 = X_{-1}(K_2)$, they use that $K_2 = P(3, -3, -8)$ is slice, i.e. concordant to the unknot. Hence, the cables of K_2 are concordant to the cables of the unknot, which implies that $s(X_{-1}(K_2); 1) = s(\overline{\mathbb{CP}^2}; 1) = 1$.

Ren and Willis also have some *new* exotic examples: detectable using s, but not (immediately) using gauge theory or Heegaard Floer homology.

Open question 7. Construct Khovanov homology and the Rasmussen invariant for links in other 3-manifolds (e.g. lens spaces beyond S^3 and $\mathbb{RP}^3$) in a more elementary way; i.e., without skein lasagna modules.

Open question 8. Can skein lasagna modules detect exotic smooth structures on some *closed* 4-manifolds?

Acknowledgement. The first part of these notes consists of lecture notes for a class at Stanford taken by Shintaro Fushida-Hardy.

References

[BN02] Dror Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. **2** (2002), 337–370.

[Che25] Daren Chen, *Bar-Natan homology for null homologous links in* $\mathbb{RP}^3$, Pacific J. Math. **334** (2025), no. 2, 233–268.

[FGMW10] Michael H. Freedman, Robert E. Gompf, Scott Morrison, and Kevin Walker, *Man and machine thinking about the smooth 4-dimensional Poincaré conjecture*, Quantum Topology **1** (2010), no. 2, 171–208.

[GLW17] J. Elisenda Grigsby, Anthony M. Licata, and Stephan M. Wehrli, *Annular Khovanov-Lee homology, braids, and cobordisms*, Pure Appl. Math. Q. **13** (2017), no. 3, 389–436.

[Kho00] Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.

[Kho06] ———, *Link homology and Frobenius extensions*, Fundamenta Mathematicae **190** (2006), no. 1, 179–190.

[Lee05] Eun Soo Lee, *An endomorphism of the Khovanov invariant*, Adv. Math. **197** (2005), no. 2, 554–586.

[MMP24] Ciprian Manolescu, Marco Marengon, and Lisa Piccirillo, *Relative genus bounds in indefinite four-manifolds*, Math. Ann. **390** (2024), no. 1, 1481–1506.

[MMSW23] Ciprian Manolescu, Marco Marengon, Sucharit Sarkar, and Michael Willis, *A generalization of Rasmussen's invariant, with applications to surfaces in some four-manifolds*, Duke Math. J. **172** (2023), no. 2, 231–311.

[MN22] Ciprian Manolescu and Ikshu Neithalath, *Skein lasagna modules for 2-handlebodies*, J. Reine Angew. Math. **788** (2022), 37–76.

[MP23] Ciprian Manolescu and Lisa Piccirillo, *From zero surgeries to candidates for exotic definite 4-manifolds*, J. Lond. Math. Soc. (2) **108** (2023), no. 5, 2001–2036.

[MW23] Ciprian Manolescu and Michael Willis, *A Rasmussen invariant for links in* $\mathbb{RP}^3$, 2023, arXiv:2301.09764.

[MWW22] Scott Morrison, Kevin Walker, and Paul Wedrich, *Invariants of 4-manifolds from Khovanov-Rozansky link homology*, Geom. Topol. **26** (2022), no. 8, 3367–3420.

[MWW23] Ciprian Manolescu, Kevin Walker, and Paul Wedrich, *Skein lasagna modules and handle decompositions*, Adv. Math. **425** (2023), Paper No. 109071, 40.

[MWW24] Scott Morrison, Kevin Walker, and Paul Wedrich, *Invariants of surfaces in smooth 4-manifolds from link homology*, 2024, arXiv:2401.06600.

[Nak23] Kai Nakamura, *Trace embeddings from zero surgery homeomorphisms*, J. Topol. **16** (2023), no. 4, 1641–1664.

[Pic20] Lisa Piccirillo, *The Conway knot is not slice*, Ann. of Math. (2) **191** (2020), no. 2, 581–591.

[Qin23] Qianhe Qin, *An RBG construction of integral surgery homeomorphisms*, 2023, arXiv:2308.04681.

[Ras10] Jacob Rasmussen, *Khovanov homology and the slice genus*, Inventiones Mathematicae **182** (2010), no. 2, 419–447.

[Ren24a] Qiuyu Ren, *Lee filtration structure of torus links*, Geom. Topol. **28** (2024), no. 8, 3935–3960.

[Ren24b] ———, *Slice genus bound in DTS^2 from s-invariant*, Algebr. Geom. Topol. **24** (2024), no. 7, 4115–4125.

[Roz10] Lev Rozansky, *A categorification of the stable SU(2) Witten-Reshetikhin-Turaev invariant of links in $S^2 \times S^1$*, 2010, arXiv:1011.1958.

[RW24] Qiuyu Ren and Michael Willis, *Khovanov homology and exotic 4-manifolds*, 2024, arXiv:2402.10452.

[Wil21] Michael Willis, *Khovanov homology for links in $\#^r(S^2 \times S^1)$*, Michigan Math. J. **70** (2021), no. 4, 675–748.

Department of Mathematics, Stanford University, 450 Jane Stanford Way, Building 380, Stanford, CA 94305, USA

Email address: cm5@stanford.edu